THE RESOLVENT OF SINGULAR INTEGRAL EQUATIONS*

BY

M. H. WILLIAMS

Princeton University

1. Introduction. Many two-dimensional mixed boundary-value problems in mathematical physics lead to the singular integral equation:

$$\fint_0^1 K(t - x)\phi(t)\, dt = w(x) \tag{1}$$

for a function $\phi(x)$ which is integrable over the interval $[0, 1]$. The inhomogeneous term $w(x)$ is specified on $[0, 1]$, while the function $K(x)$, called the "kernel", is specified on $(-\infty, \infty)$. The kernel is assumed here to be bounded everywhere except at the origin, where it has a generalized Cauchy singularity:

$$K(x) \sim C/x\, |x|^\mu, \qquad 0 \le \mu < 1 \quad \text{as} \quad x \to 0. \tag{2}$$

The integral in (1) is interpreted as a Cauchy principal value. Integral equations of the above type have been discussed extensively in [1, 2], principally for kernels with simple poles, $\mu = 0$.

Because the kernel has a strong singularity at the origin, Eq. (1) possesses a nontrivial, integrable homogeneous solution. That is, there exists a function, $h(x)$, for which $\int_0^1 h(x)\, dx$ exists, such that:

$$\fint_0^1 K(t - x)h(t)\, dt = 0 \quad \text{for all} \quad x \in (0, 1). \tag{3}$$

It may be shown, by a local expansion about the end points, that this eigensolution behaves like:

$$h(x) \sim \eta^{\nu-1} \quad \text{as} \quad \eta = x \quad \text{or} \quad 1 - x \to 0 \tag{4}$$

where $\nu = \frac{1}{2} + \mu/2$. Thus the solution $\phi(x)$ of (1) is indeterminate within the addition of an arbitrary multiple of $h(x)$.

To make the solution of (1) unique we shall impose the constraint:

$$\phi(1) = 0. \tag{5}$$

In aerodynamics this is referred to as the Kutta condition (cf. [3, 4]). It will be noted that if $\phi(x)$ satisfies (5), then, in general, $\phi(x)$ will be singular at $x = 0$, behaving like $x^{\nu-1}$.

The particular solution of (1) determined by (5) can, in principle, be written in the

form:

$$\phi(x) = \fint_0^1 \Gamma(t, x) w(x)\, dx \qquad (6)$$

where $\Gamma(t, x)$, which does not depend on w, is called the "resolvent" of the kernel K. The general solution of (1) is, of course, obtained by adding an arbitrary multiple of $h(x)$ to (6).

The purpose of the investigation described herein is the construction of the resolvent for any given kernel function. When $w(x)$ is smooth, however, the solution of (1) can usually be found quite simply by any one of numerous direct numerical techniques (e.g. collocation or least squares, or other methods described in [5]). But when $w(x)$ is discontinuous or otherwise ill-behaved—as, for instance, in the aerodynamic problem of flow over a flapped airfoil—such direct numerical methods become very difficult. Thus the practical application of the solution by resolvent described below is to problems with ill-behaved inhomogeneous terms.

2. Solution of the adjoint equation for polynomial $w(x)$.

We begin by constructing a sequence of solution pairs $(q_n(x), \xi_n(t))$ of the adjoint equation:

$$\fint_0^1 K(t - x) q_n(x)\, dx = \xi_n(t); \qquad q_n(0) = 0, \qquad n = 0, 1, \cdots \qquad (7)$$

where the $\xi_n(t)$ are nth-order polynomials. To do this we suppose that $q_0(x)$ and the homogeneous solution $h_0(x)$ are known and defined by:

$$\fint_0^1 K(t - x) q_0(x)\, dx = 1; \qquad q_0(0) = 0, \qquad (8)$$

$$\fint_0^1 K(t - x) h_0(x)\, dx = 0; \qquad \int_0^1 h_0(x)\, dx = 1. \qquad (9)$$

Note that $h(t) = h_0(1 - t)$ satisfies (3), i.e., it is the homogeneous solution of the original equation (1). Moreover we note that $h_0(x)$ may be constructed from $q_0(x)$ and its "conjugate" $\hat{q}_0(x)$ defined by:

$$\fint_0^1 K(t - x) \hat{q}_0(x)\, dx = 1; \qquad \hat{q}_0(1) = 0. \qquad (10)$$

In terms of q_0 and $\hat{q}_0$ the homogeneous solution is:

$$h_0(x) = (q_0(x) - \hat{q}_0(x)) \bigg/ \int_0^1 [q_0(x) - \hat{q}_0(x)]\, dx. \qquad (11)$$

For the important special case of antisymmetric kernel functions $\hat{q}_0(x) = -q_0(1 - x)$, and therefore:

$$h_0(x) = (q_0(x) + q_0(1 - x)) \bigg/ 2 \int_0^1 q_0(t)\, dt, \qquad K(-x) = -K(x). \qquad (12)$$

Normally the functions $q_0(x)$ and $h_0(x)$ must be constructed numerically. This presents no intrinsic difficulty, however, since the "forcing" terms involved are smooth functions.

To construct higher-order pairs of the sequence (q_n, ξ_n) we employ the following two properties:

(i) $\quad \dfrac{\partial}{\partial t} K(t - x) = -\dfrac{\partial}{\partial x} K(t - x),$ \hfill (13)

(ii) $d\xi_{n+1}/dt$ is a linear combination of $(\xi_j , j = 0, n).$ \hfill (14)

We shall choose the polynomials ξ_n in such a way that dq_{n+1}/dx is a linear combination of the $(q_j , j = 0, n)$ and h_0. There are many ways of doing this, the simplest being to set:

$$\xi_n(t) = (t^n/n!) + \xi_n(0) - \delta_{n0} , \tag{15}$$

where δ_{n0} is a Kronecker delta, so that:

$$d\xi_{n+1}/dt = \xi_n(t) + \delta_{n0} - \xi_n(0). \tag{16}$$

The constants $\xi_n(0)$ will be determined later.

Differentiating (7), we find (using (13) and a formal integration by parts):

$$\fint_0^1 K(t - x)(dq_{n+1}/dx) \, dx = K(t - 1)q_{n+1}(1) + (d\xi_{n+1}/dt). \tag{17}$$

This is a formal step since in general $q_{n+1}(1)$ is infinite. Clearly, though, to obtain the desired recursion relation between the q_j we must impose the additional constraint:

$$q_{n+1}(1) = 0, \qquad n = 0, 1, \cdots \tag{18}$$

on all pairs above the lowest order. If this condition is met then the general solution of (17) is (by (7), (16) and (9)):

$$dq_{n+1}/dx = q_n(x) + [\delta_{n0} - \xi_n(0)]q_0(x) - C_{n+1}h_0(x) \tag{19}$$

for an arbitrary constant C_{n+1}. But, since $q_{n+1}(0) = q_{n+1}(1) = 0$, the solution of this equation is:

$$q_{n+1}(x) = \int_0^x q_n(t) \, dt + [\delta_{n0} - \xi_n(0)] \int_0^x q_0(t) \, dt - C_{n+1} \int_0^x h_0(t) \, dt \tag{20}$$

where

$$C_{n+1} = \int_0^1 q_n(t) \, dt + [\delta_{n0} - \xi_n(0)] \int_0^1 q_0(t) \, dt. \tag{21}$$

The constants $\xi_n(0)$ are determined by the defining relation (7):

$$\xi_n(0) = \fint_0^1 K(-x)q_n(x) \, dx. \tag{22}$$

Now, with $q_0 , \xi_0 ,$ and h_0 known, $q_1(x)$ is fully defined by (20–21). Thus $\xi_1(0)$, in turn, is determined by (22) and $\xi_1(t)$ by (15). Clearly this process can be continued indefinitely, thereby defining the entire sequence (q_n , ξ_n).

That the sequence so defined is, in fact, a sequence of solution pairs can be proved by a simple induction argument, without recourse to the differentiation used to obtain the result. We presume that the pair (q_n , ξ_n) satisfies (7), and write, for brevity:

$$q_{n+1}(x) = \int_0^x B_n(t) \, dt \tag{23}$$

where

$$B_n(t) = q_n(t) + [\delta_{n0} - \xi_n(0)]q_0(t) - C_{n+1}h_0(t). \tag{24}$$

The constants C_{n+1} and $\xi_n(0)$ are defined by (21) and (22).

Now we apply the kernel to the function (23), inverting the order of integration:

$$\fint_0^1 K(t - x)q_{n+1}(x)\,dx = \int_0^1 dx \int_0^x ds\, K(t - x)B_n(s)$$

$$= \int_0^1 ds \int_s^1 dx\, K(t - x)B_n(s). \tag{25}$$

But it may easily be shown that:

$$\int_s^1 K(t - x)\,dx = \int_0^t K(\eta - s)\,d\eta + \int_0^1 K(-\eta)\,d\eta - \int_{1-t}^1 K(\eta)\,d\eta, \tag{26}$$

by which (25) becomes:

$$\fint_0^1 K(t - x)q_{n+1}(x)\,dx = \int_0^t dx \fint_0^1 K(x - s)B_n(s)\,ds$$

$$+ \int_0^1 ds\, B_n(s) \int_s^1 d\eta\, K(-\eta) - \int_{1-t}^1 K(\eta)\,d\eta \int_0^1 B_n(s)\,ds. \tag{27}$$

Each of the three terms on the right-hand side of (27) may be evaluated. Thus:

(i) $\displaystyle \int_0^1 B_n(s)\,ds = q_{n+1}(1) = 0$ by definition ((23)–(24)).

(ii) $\displaystyle \int_0^1 ds\, B_n(s) \int_s^1 d\eta\, K(-\eta) = \fint_0^1 K(-x)q_{n+1}(x) = \xi_{n+1}(0),$ by   (25)

and the definition of ξ_{n+1} .

(iii) $\displaystyle \fint_0^1 K(\eta - s)B_n(s)\,ds = \fint_0^1 K(\eta - s)(q_n(s) + [\delta_{n0} - \xi_n(0)]q_0(s) - C_{n+1}h_0(s))\,ds$

$$= \xi_n(\eta) + \delta_{n0} - \xi_n(0)$$

since (q_n, ξ_n) is a solution. Thus the first term in (27) is (by (16)):

$$\int_0^t [\xi_n(\eta) + \delta_{n0} - \xi_n(0)]\,d\eta = \int_0^t \frac{d\xi_{n+1}}{d\eta}\,d\eta = \xi_{n+1}(t) - \xi_{n+1}(0).$$

Combining (i)–(iii), then, (27) becomes:

$$\fint_0^1 K(t - x)q_{n+1}(x)\,dx = [\xi_{n+1}(t) - \xi_{n+1}(0)] + [\xi_{n+1}(0)] - 0 = \xi_{n+1}(t).$$

Thus, if (q_n, ξ_n) is a solution of (7), then $(q_{n+1}, \xi_{n+1},)$ is also, which is the desired proof.

For the simplest Cauchy kernel, $K(x) = -(1/\pi x)$, the sequence defined here is closely related to the set of Chebyshev polynomials of the first and second kind, familiar from the aerodynamic theory of subsonic airfoils (see [3, 4]). The reduction of the general results presented in this paper to the classical solution of the airfoil equation is demonstrated in Appendix 1.

3. Simplification of the recursion relations. The definition of the sequence (q_n, ξ_n) given in the preceeding section is somewhat cumbersome. It may be considerably simplified by the use of the following two properties:

(i) $\qquad \int_0^1 \xi_n(t) h_0(1 - t)\, dt = \delta_{n0} \quad \text{for all} \quad n,$ $\qquad\qquad$ (28)

(ii) $\qquad \int_0^1 q_n(x)\, dx = \int_0^1 \xi_n(x) q_0(1 - x)\, dx \quad \text{for all} \quad n.$ $\qquad$ (29)

These relations are obtained by multiplying (7) by $h_0(1 - t)$ (for (i)) or $q_0(1 - t)$, (for (ii)) and integrating.

By the definition (15) of $\xi_n(t)$, (28) and (29) may be written as:

$$\xi_n(0) = 2\delta_{n0} - \frac{1}{n!} \int_0^1 t^n h_0(1 - t)\, dt,$$ $\qquad\qquad$ (30)

$$\int_0^1 q_n(x)\, dx = \frac{1}{n!} \int_0^1 t^n q_0(1 - t)\, dt + (\xi_n(0) - \delta_{n0}) \int_0^1 q_0(t)\, dt.$$ $\qquad$ (31)

Thus the constants C_{n+1} defined in (21) are:

$$C_{n+1} = \frac{1}{n!} \int_0^1 t^n q_0(1 - t)\, dt.$$ $\qquad\qquad$ (32)

Eqs. (30) and (32) relate the constants $\xi_n(0)$ and C_{n+1} directly to the known functions q_0 and h_0, thereby making (20) a simple one-step recursion relation between q_n and q_{n+1}. This recursion will now be solved.

4. Solution of the recursion relation (20). We write (20) in the form:

$$q_{n+1} = I \cdot q_n + I \cdot P_n,$$ $\qquad\qquad$ (33)

where

$$I \cdot \psi = \int_0^x \psi(t)\, dt,$$ $\qquad\qquad$ (34)

$$P_n(t) = [\delta_{n0} - \xi_n(0)] q_0(t) - C_{n+1} h_0(t).$$ $\qquad\qquad$ (35)

Repeated application of (33) implies that:

$$q_{n+1}(x) = I^{n+1} \cdot q_0 + \sum_{i=0}^{n} I^{i+1} \cdot P_{n-i}.$$ $\qquad\qquad$ (36)

We now define the functions:

$$Q_n(x) = I^{n+1} \cdot q_0 = \frac{1}{n!} \int_0^x (x - t)^n q_0(t)\, dt,$$ $\qquad\qquad$ (37)

$$H_n(x) = I^{n+1} \cdot h_0 = \frac{1}{n!} \int_0^x (x - t)^n h_0(t)\, dt.$$ $\qquad\qquad$ (38)

Thus, by (30) and (32), we may write $P_n(t)$ as:

$$P_n(t) = (H_n(1) - \delta_{n0}) q_0(t) - Q_n(1) h_0(t),$$ $\qquad\qquad$ (39)

and, therefore:

$$I^{i+1} \cdot P_{n-i} = (H_{n-i}(1) - \delta_{n,i})Q_i(x) - Q_{n-i}(1)H_i(x). \tag{40}$$

Substituting (40) into (36), we find that:

$$q_{n+1}(x) = \sum_{i=0}^{n} [H_{n-i}(1)Q_i(x) - H_i(x)Q_{n-i}(1)]. \tag{41}$$

But, from the definitions (37–38) of Q_i and H_i, it is clear that this sum can be performed, using the binomial theorem, with the result that:

$$q_{n+1}(x) = \frac{1}{n!} \int_0^1 dt \int_0^x ds\, R(s, t)(1 + x - t - s)^n \tag{42}$$

where $R(s, t) = q_0(s)h_0(t) - q_0(t)h_0(s)$.

Finally, we note that the integrand in (42) is antisymmetric in the interchange of s and t, so that the integral over the square of side x vanishes. The general element in the sequence, therefore, can be written in the form:

$$q_{n+1}(x) = \frac{1}{n!} \int_x^1 dt \int_0^x ds\, R(s, t)(1 + x - t - s)^n$$

$$\xi_{n+1}(x) = \frac{x^{n+1}}{(n+1)!} - \frac{1}{(n+1)!} \int_0^1 t^{n+1}h_0(1-t)\, dt \tag{43}$$

where $R(s, t) = q_0(s)h_0(t) - q_0(t)h_0(s)$.

We now show how this set of functions can be used to construct the resolvent of the kernel.

5. Construction of the resolvent.

We suppose that the right-hand side of (1) can be expanded in the polynomials ξ_n:

$$w(x) = \sum_{n=0}^{\infty} a_n\xi_n(1 - x). \tag{44}$$

The solution ϕ which satisfies (5), then, must be:

$$\phi(x) = \sum_{n=0}^{\infty} a_n q_n(1 - x). \tag{45}$$

Now we note that q_n and ξ_n are related by:

$$q_n(x) = \int_x^1 dt \int_0^x ds\, R(s, t)\xi_n'(1 + x - t - s), \qquad n = 1, 2, \cdots \tag{46}$$

which follows from (43). Thus the relation between ϕ and w expressed by (44–45) can clearly be written as:

$$\phi(1 - x) = a_0 q_0(x) - \int_x^1 dt \int_0^1 ds\, R(s, t)w'(s + t - x), \tag{47}$$

where, by the orthogonality property (28),

$$a_0 = \int_0^1 h_0(x)w(x)\, dx. \tag{48}$$

Eq. (47) constitutes the principal result of this investigation. From it, however, may be derived several alternative forms, including the canonical form (6).

For instance, by a double application of Leibnitz's role, the differentiation may be taken outside the integrals, giving the expression:

$$\phi(1 - x) = \frac{d}{dx} \int_x^1 dt \int_0^x ds\, R(s,\, t) w(s + t - x) + h_0(x) \int_0^1 q_0(t) w(t)\, dt. \qquad (49)$$

It will be noted that since the double integral vanishes at both end points, this equation implies that:

$$\int_0^1 \phi(x)\, dx = \int_0^1 q_0(t) w(t)\, dt, \qquad (50)$$

which is a generalization of (29), and the first in an infinite series of moment relations to be discussed below.

A third form of the solution (47) is obtained by making the transformation:

$$\eta = s + t - x, \qquad \xi = (t - s)/2,$$

under which (47) becomes:

$$\phi(x) = a_0 q_0 (1 - x) + \int_0^1 d\eta\, w'(\eta) G(\eta,\, x) \qquad (51)$$

where

$$G(\eta,\, x) = -\int_0^\eta d\xi\, R(\xi,\, 1 - x + \eta - \xi) = -\int_0^{1-x} d\xi\, R(\xi,\, 1 - x + \eta - \xi). \qquad (52)$$

The equality of the two forms in (52) follows from the antisymmetry of $R(s,\, t)$ which implies that

$$\int_0^z R(\xi,\, z - \xi)\, d\xi = 0. \qquad (53)$$

The canonical form (6) of the solution is obtained by an integration by parts in (51). Thus:

$$\phi(x) = \int_0^1 dt\, w(t) \Gamma(t,\, x)$$

where

$$\Gamma(t,\, x) = h_0(t) q_0 (1 - x) - \frac{\partial}{\partial t} G(t,\, x). \qquad (54)$$

In deriving (54) the properties $G(1,\, x) = G(0,\, x) = 0$, which follow from (52) and (53), were used. It will be noted that $G(\eta,\, 1) = G(\eta,\, 0) = 0$ also and, therefore, that $\Gamma(t,\, 1) = 0$. This, of course, is necessary for $\phi(x)$ to satisfy the presumed boundary condition, $\phi(1) = 0$. Note, however, that $\phi(0)$ is bounded if and only if w is orthogonal to the homogeneous solution.

Strictly speaking, since there is no general guarantee of the convergence of the series (44) and (45), the above derivation shows only that the result (54) is valid for any finite polynomial $w(x)$. A rigorous demonstration that (54) is the correct resolvent for any $w(x)$ is given in Appendix 2.

It should be noted here that the resolvent $\Gamma(t, x)$, like the kernel $K(t - x)$, will be singular at $t = x$. Recalling the local singular behavior, (4), of the homogeneous solution near the end points for a kernel which behaves like (2), it is possible to show that the corresponding singularity in the resolvent is of the form:

$$\Gamma(t, x) \sim |\eta|^{\mu}/\eta \quad \text{as} \quad \eta = t - x \to 0.$$

Thus when $\mu = 0$, Γ, like K, has a simple pole. When $0 < \mu < 1$ then Γ has only a weak singularity. That the singularity in Γ should be weaker than that of K is not surprising since these two functions represent mutually inverse operators.

6. Moments of ϕ. In many applications the moments

$$M_n = \frac{1}{n!} \int_0^1 x^n \phi(x) \, dx \tag{55}$$

are of particular importance. For instance, in aerodynamic theory, M_0 and M_1 are proportional, respectively, to the total lift and pitching moment (about the leading edge) of a two-dimensional airfoil.

These moments follow directly from the sequence (q_n, ξ_n) developed earlier. Multiplying (1) by $q_n(x)$ and integrating, we find that:

$$\int_0^1 q_n(x)w(x) \, dx = \int_0^1 \xi_n(t)\phi(t) \, dt. \tag{56}$$

Clearly, then, the moments (55) are:

$$M_n = \int_0^1 q_n(x)w(x) \, dx + M_0(\delta_{n0} - \xi_n(0)). \tag{57}$$

The first of these relations,

$$M_0 = \int_0^1 q_0(x)w(x) \, dx, \tag{58}$$

will be recognized as Eq. (50).

It is also worthy of note that the expression for M_1 becomes particularly simple for antisymmetric kernels. In this case it is readily shown that:

$$\xi_1(0) = -1/2,$$

$$q_1(x) = \frac{1}{2} \int_0^x [q_0(t) - q_0(1 - t)] \, dt,$$

and, therefore, that

$$M_1 = \frac{1}{2} \int_0^1 dx \, q_0(x) \int_x^{1-x} w(t) \, dt + \tfrac{1}{2}M_0 . \tag{59}$$

7. Solution for a discontinuous forcing term. Let us suppose that $w(x)$ has a finite discontinuity of strength b at $x = x_0$. Then $w(x) = \bar{w}(x) + bH(x - x_0)$, where $H(\xi)$ is a unit step function and $\bar{w}(x)$ is continuous. The corresponding solution will be of the form $\phi(x) = \bar{\phi}(x) + b\phi_1(x)$. The function $\bar{\phi}$ corresponding to the continuous part of w can usually be obtained most easily by direct numerical solution. We therefore consider

here only the part ϕ_1 , defined by:

$$\fint_0^1 K(t - x)\phi_1(t) \, dt = H(x - x_0); \qquad \phi_1(1) = 0. \tag{60}$$

Noting that the derivative of a step function is a delta function, we see at once from Eq. (51) that the solution is:

$$\phi_1(x) = q_0(1 - x) \int_{x_0}^1 h_0(t) \, dt + G(x_0 , x) \tag{61}$$

The moments of $\phi_1(x)$ are easily computed by the methods of the preceding section. We note in particular that:

$$\int_0^1 \phi_1(x) \, dx = \int_{x_0}^1 q_0(x) \, dx, \tag{62}$$

$$\int_0^1 x\phi_1(x) \, dx = \int_{x_0}^1 [q_1(x) - \xi_1(0)q_0(x)] \, dx, \tag{63}$$

which, for an anti-symmetric kernel, becomes:

$$\int_0^1 x\phi_1(x) \, dx = \tfrac{1}{2}[Q_1(1 - x_0) - Q_1(1) + x_0Q_0(1) - Q_0(x_0)], \tag{64}$$

where $Q_n(x)$ is defined in (37).

The validity of the solution (61) of (60) is established in Appendix 2 as part of the general demonstration of the result (54).

8. Conclusions. We have found that the general solution of the singular integral equation $\int_0^1 K(t - x)\phi(t) \, dt = w(x)$ for an arbitrary $w(x)$ can be written explicitly in terms of two independent solutions for $w(x) = 1$. When $w(x)$ is sufficiently ill-behaved that standard numerical methods cannot be used, or when solutions must be computed for a large number of different forcing terms, the approach outlined here should be the method of choice.

The method is particularly simple to apply when the basic functions q_0 , h_0 have been approximated by linear series in some appropriately chosen set of mode functions ψ_i :

$$q_0(x) = \sum a_i\psi_i(x), \qquad h_0(x) = \sum b_i\psi_i(x). \tag{65}$$

When this is done the resolvent $\Gamma(s, t)$ becomes a bilinear series in the coefficients (a_i , b_i):

$$\Gamma(s, t) = \sum \sum a_ib_i\Gamma_{ij}(s, t) \tag{66}$$

where the $\Gamma_{ij}(s, t)$ are determined uniquely by the choice of mode functions. For a broad class of kernel functions with simple pole singularities the modes ψ_i may be taken to be the Chebeshev polynomials of the second kind. For these modes the $\Gamma_{ij}(s, t)$ can be evaluated once and for all and the construction of the resolvent function for any kernel (within the class for which these modes are reasonable) reduces to simply finding the constants a_i , b_i .

Appendix 1: The solution for $K(x) = -1/\pi x$. The airfoil equation:

$$\frac{1}{\pi} \fint_0^1 \frac{\phi(t)}{x - t} \, dt = w(x), \tag{A(1)}$$

has the well-known general solution [1, 2]:

$$\phi(x) = ah_0(x) - \frac{q_0(1 - x)}{\pi} \fint_0^1 \frac{q_0(t)w(t)}{x - t} \, dt \tag{A(2)}$$

where

$$q_0(x) = \left(\frac{x}{1 - x}\right)^{1/2}, \tag{A(3)}$$

$$h_0(x) = \frac{1}{\pi(x(1 - x))^{1/2}}, \tag{A(4)}$$

and a is an arbitrary constant.

The particular solution satisfying $\phi(1) = 0$ is given by A(2) with $a = 0$. We now show that the solution (54) reduces to A(2) for this kernel.

From (43) and A(3–4) we have:

$$R(s, t) = \frac{s - t}{\pi(s(1 - s)t(1 - t))^{1/2}}, \tag{A(5)}$$

and, therefore:

$$R(z - \xi, z + \xi) = \frac{2\xi}{\pi((z^2 - \xi^2)((1 - z)^2)^{1/2} - \xi^2)}. \tag{A(6)}$$

Now by (52):

$$G(\eta, x) = -\int_0^\eta d\xi \, R(\xi, 1 - x + \eta - \xi)$$

$$= -\int_{z-\eta}^z d\xi \, R(z - \xi, z + \xi) \tag{A(7)}$$

where $z \equiv (1 - x + \eta)/2$. Defining the new integration variable:

$$u = \left(\frac{z^2 - \xi^2}{(1 - z)^2 - \xi^2}\right)^{1/2}, \tag{A(8)}$$

in $A(7)$, we find, after some algebra, that:

$$R(z - \xi, z + \xi) \, d\xi = -\frac{2}{\pi} \frac{du}{u^2 - 1}. \tag{A(9)}$$

Furthermore, we note that:

$$u = 0 \quad \text{at} \quad \xi = z, \qquad u = q_0(\eta)/q_0(x) \quad \text{at} \quad \xi = z - \eta. \tag{A(10)}$$

Combining A(7, 9, 10), we find that

$$G(\eta, x) = \frac{2}{\pi} \int_0^{q_0(\eta)/q_0(x)} \frac{du}{1 - u^2} = \frac{1}{\pi} \ln \left|\frac{q_0(\eta) + q_0(x)}{q_0(\eta) - q_0(x)}\right|. \tag{A(11)}$$

It is then easily shown that:

$$\partial G(\eta, x)/\partial \eta = \frac{1}{\pi} \frac{h_0(\eta)}{h_0(x)} \frac{1}{x - \eta}, \tag{A(12)}$$

which, by (54), implies that:

$$\Gamma(t, x) = h_0(t)q_0(1 - x) - \frac{\partial}{\partial t} G(t, x)$$

$$= -\frac{1}{\pi} \frac{q_0(1 - x)q_0(t)}{x - t}.$$

This is identical to the classical result A(2), confirming the analysis. It follows that the sequence (q_n, ξ_n) defined by (43) can be linearly combined to form the set of Chebeshev polynomials $(\sin n \cos^{-1} (1 - 2x), \cos n \cos^{-1} (1 - 2x))$, commonly used in airfoil theory.

Appendix 2. Proof of Eq. (54). We shall demonstrate that the function defined in (54) solves the original integral equation (1) for general $w(x)$. This is so if K and Γ satisfy the reciprocal relations:

$$\int_0^1 dt\ K(t - x)\Gamma(s, t) = \int_0^1 dt\ K(s - t)\Gamma(t, x) = \delta(s - x), \tag{B1}$$

which are obtained by the substitution of (54) into (1) and vice versa.

The proof of (B1) will be given in two parts:
i) that (61) is the solution of (60); ii), that (B1) follows from i).

i) *Proof of (60) given (61).* By definition (61):

$$\phi_1(x, s) = q_0(1 - x) \int_s^1 h_0(t)\ dt + G(s, x). \tag{B2}$$

Thus, for x in $(0, 1)$,

$$\int_0^1 dt\ K(t - x)\phi_1(t, s) = \int_s^1 dt\ h_0(t) + I_2 , \tag{B3}$$

where

$$I_2 = \int_0^1 dt\ K(t - x)G(s, t). \tag{B4}$$

Substituting the first of the forms (52) for G and changing variables, we find that I_2 is:

$$I_2 = -\int_0^s d\xi \int_{s-\xi}^{1+s-\xi} dt\ K(1 + s - x - \xi - t)R(\xi, t). \tag{B5}$$

But R vanishes, by definition, outside the unit square, so that the upper limit on the t integration can be replaced by 1. Furthermore, the integrand in (B5) is antisymmetric in the interchange of ξ and t. Consequently I_2 may be written in either of the alternative forms:

$$I_2 = -\int_0^s d\xi \int_0^1 dt\ K(1 + s - x - \xi - t)R(\xi, t) \tag{B6a}$$

$$= \int_s^1 d\xi \int_0^1 dt\ K(1 + s - x - \xi - t)R(\xi, t). \tag{B6b}$$

Now, by the definition of $R(\xi, t)$ in (43) and the properties of q_0, h_0, it is easily shown

that:

$$\int_0^1 dt\, K(y - t)R(\xi, t) = -h_0(\xi) \quad \text{if } y \text{ is in } (0, 1). \tag{B7}$$

Thus, since $1 + s - x - \xi$ is in $(0, 1)$ when $s < x$ in (B6a) and when $s > x$ in (B6b), we find that:

$$I_2 = \int_0^s h_0(\xi)\, d\xi, \qquad s < x$$

$$= -\int_s^1 h_0(\xi)\, d\xi, \qquad s > x. \tag{B8}$$

Substituting this result into (B3) and recalling that h_0 is normalized, we find the desired result:

$$\int_0^1 dt\, K(t - x)\phi_1(t, s) = H(x - s), \tag{B9}$$

which is Eq. (60) in the text.

ii) *Proof of* (B1).   From the definitions (B2) and (54) of ϕ_1 and Γ, it is clear that

$$\Gamma(s, x) = -\frac{\partial}{\partial s}\phi_1(x, s). \tag{B10}$$

Thus, by differentiating (B9) with respect to s we obtain:

$$\int_0^1 dt\, K(t - x)\Gamma(s, t) = \delta(s - x), \tag{B11}$$

which is the first of the relations (B1). The second relation follows immediately from (B11) and the identity:

$$\Gamma(1 - t, 1 - s) = \Gamma(s, t), \tag{B12}$$

which can be shown to result from the definitions of Γ and G. Substitution of (B12) into (B11), with the transformations $t \to 1 - t$, $s \to 1 - x$, $x \to 1 - s$ yields the desired result.

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