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The response of framed structures on elastic foundations to ground motion

L.A. Dalguer, C.L. Sahlit, P.W. Partridge
Departamento de Eng. Civil, Universidade de Brasília, 70910-900 Brasília DF, Brazil

Abstract

The study of framed structures on elastic foundations subjected to ground motion is considered using a model which combines conventional frame elements with elements obtained using the Winkler hypothesis. A series of different base excitations, including earthquake response spectra, are considered. Results are compared with those obtained for similar structures not including the elastic foundations.

1 Introduction

The formulation to be presented is based on the beam on a continuous elastic foundation which is a classical model of soil-structure interaction.

The first detailed studies of the differential equation of the beam on a continuous spring considering the Winkler hypothesis of soil reaction proportional to deflection are due to Hetényi [1]. The equation in its usual form for a beam with a uniform cross section is

$$EI\frac{\partial^4 y}{\partial x^4} + ky = q \tag{1}$$

where EI is the flexural rigidity, k is the stiffness modulus of the foundation, q is the variable transverse load per unit length and y is the deflection.

This work was followed by many applications to structural analysis. Early finite element models of the beam on a elastic foundation used the equation

$$u = \mathbf{A}\boldsymbol{\alpha} \tag{2}$$

as a starting point. In (2), $\mathbf{A}$ is a set of four functions which appear in the complementary solution to (1), [2,3], $\boldsymbol{\alpha}$ consists of four unknown coefficients and u is a generalized displacement function. Following the usual finite element procedure (2) is written for each degree of freedom

$$\mathbf{u}^n = \mathbf{C}\boldsymbol{\alpha} \tag{3}$$

in which $\mathbf{C}$ is a coefficient matrix. Thus $u = \mathbf{A}\mathbf{C}^{-1}\mathbf{u}^n$. This leads to the following relationship for the stiffness matrix

$$\mathbf{K} = (\mathbf{C}^{-1})^T EI \int_0^L (\mathbf{A}'')^T \mathbf{A}'' dx \mathbf{C}^{-1} \tag{4}$$

where $(\mathbf{A}'')$ is the second derivative of the basis functions $\mathbf{A}$ with respect to x, L is the length of the element and T signifies transpose. Equation (4) does not require the calculation of a shape function, [2]. The mass matrix can be found in a similar way, [3]. This process is described in Zienkiewicz [4] for plate problems.

Another approach is that due to Bowles [5] in which the elastic foundation is modeled using discrete springs at the extremities of the elements, the accuracy of the results thus depending on the number of elements employed. Later, Ting & Mockry [6] calculated the matrix (4), directly using the stiffness method, thus avoiding inverting the coefficient matrices $\mathbf{C}$. The same approach was also adopted by Wang, [7].

Recently a shape function for the beam element on an elastic foundation element was developed by Lai et al [8] who then applied this element to free vibration problems. Here forced vibration problems will be considered in which the shape functions for beam elements on an elastic foundation with ends clamped/clamped and clamped/hinged are used to obtain the stiffness and mass matrices and the equivalent nodal load vector in each case. Some examples involving ground motion will be presented.

2 Beam Elements on an Elastic Foundation

A beam on an elastic foundation is shown in fig. 1.

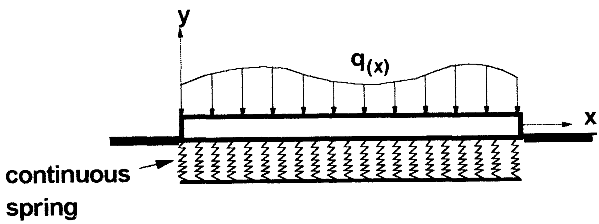

Fig. 1 Beam on an Elastic Foundation

Equation (1) can be used as a starting point for the development of a shape function for a beam element on an elastic foundation, from which the mass and stiffness matrices can then be obtained, as done by Lai et al [8] for the case of both ends of the element being clamped. For the case of one end clamped and the other hinged, similar expressions can be obtained [9], the shape function being given by

$$\phi_{i(x)} = A_i \cos\left(\frac{w}{L}x\right) \cosh\left(\frac{w}{L}x\right) + B_i \cos\left(\frac{w}{L}x\right) \sinh\left(\frac{w}{L}x\right) + C_i \sin\left(\frac{w}{L}x\right) \cosh\left(\frac{w}{L}x\right) \tag{5}$$

where $A_i = a_i - d_i(a_2/d_2)$, $B_i = b_i - d_i(b_2/d_2)$ and $C_i = c_i - d_i(c_2/d_2)$, and

$$a_1 = 1 \quad a_2 = 0 \quad a_3 = 0 \quad a_4 = 0$$

$$b_1 = \frac{sc + SC}{s^2 - S^2} \quad b_2 = \frac{Ls^2}{w(s^2 - S^2)} \quad b_3 = -\frac{cS + sC}{s^2 - S^2} \quad b_4 = \frac{LsS}{w(s^2 - S^2)}$$

$$c_1 = -b_1 \quad c_2 = -\frac{LS^2}{w(s^2 - S^2)} \quad c_3 = -b_3 \quad c_4 = -b_4$$

$$d_1 = \frac{(cS)^2 + (sC)^2}{s^2 - S^2} \quad d_2 = \frac{L(SC - cs)}{w(s^2 - S^2)} \quad d_3 = -2\frac{w}{L}b_4 \quad d_4 = \frac{L(sC - cS)}{w(s^2 - S^2)}$$

where $s = \sin(w)$, $S = \sinh(w)$, $c = \cos(w)$ and $C = \cosh(w)$. In the above

$$w = L\left(\frac{k}{4EI}\right)^{0.25} \tag{6}$$

is a constant which relates the soil and the structure. From equation (5) the mass and stiffness matrices, and for forced vibrations the equivalent nodal load vector, can be calculated for the clamped/hinged element:

$$\mathbf{K} = \int_0^L EI(\phi'')^T\phi'' dx \tag{7}$$

$$\mathbf{M} = \int_0^L \rho A\phi^T\phi dx$$

$$\mathbf{Q}(t) = \int_0^L \phi^T q(t)dx$$

where ρ represents the mass density of the material and A is the area of the transverse section of the element.

3 Equations of Motion for Support Excitation

The equations of motion in matrix form for the general dynamic problem are:

$$\mathbf{M}\ddot{\mathbf{U}}^t + \mathbf{C}\dot{\mathbf{U}}^t + \mathbf{K}\mathbf{U}^t = \mathbf{Q} \tag{8}$$

where $\ddot{\mathbf{U}}^t$, $\dot{\mathbf{U}}^t$ and $\mathbf{U}^t$ are vectors of total acceleration, velocity and displacement, respectively. The matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are assembled using both conventional beam elements and the elements including the elastic foundation described in the previous section, in order to model the superstructure and the base for a given case.

For base excitation problems, equation (8) may be rewritten as [10]

$$\mathbf{M}_{ff}\ddot{\mathbf{U}}_f + \mathbf{C}_{ff}\dot{\mathbf{U}}_f + \mathbf{K}_{ff}\mathbf{U}_f = \mathbf{P}(t) \tag{9}$$

where $\ddot{\mathbf{U}}$, $\dot{\mathbf{U}}$ and $\mathbf{U}$ are now vectors of relative acceleration, velocity and displacement and

$$\mathbf{P}(t) = -(\mathbf{M}_{fr} + \mathbf{M}_{ff}\mathbf{R})\ddot{\mathbf{U}}_r - (\mathbf{C}_{fr} + \mathbf{C}_{ff}\mathbf{R})\dot{\mathbf{U}}_r \tag{10}$$

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is the vector of effective base forces. In equations (9) and (10) f and r denote free and restricted degrees of freedom respectively, and $\mathbf{R}$ collects the pseudo-static influence coefficients which are the displacements caused by a unit motion of one support with all other supports fixed. As the contribution of damping to the effective base forces is, in general, relatively small, the second term in (10) will be ignored.

Alternatively, for seismic excitation the spectral modal analysis technique may be employed. In this case the maximum displacements for each vibration mode are given by

$$\mathbf{U}_n^{\max} = \boldsymbol{\chi}_n \frac{\boldsymbol{\chi}_n^T \mathbf{M} \mathbf{i} (S_a)_n}{\boldsymbol{\chi}_n^T \mathbf{M} \boldsymbol{\chi}_n \omega_n^2} \tag{11}$$

where $\mathbf{U}_n^{\max}$ is the vector of maximum displacements for mode n, $\boldsymbol{\chi}_n$ is the mode shape associated with frequency ω_n, $\mathbf{i}$ is the vector of pseudo-static coefficients and $(S_a)_n$ is the pseudo-acceleration related to the frequency ω_n and to the damping ratio ξ_n.

4 Applications

Problem 1

This example is included in order to compare results obtained using the present formulation with those obtained by Lai et al [8]. A frame on an elastic foundation is considered, as shown in fig. 2, having the following properties: cross section of the girder $0.305m \times 0.61m$, of the columns $0.51m \times 0.51m$, of the footings $1.22m \times 0.406m$ and of the foundation slab $1.22m \times 0.305m$; $E = 2.486 \times 10^{10} Pa$, $\rho = 2397 kg/m^3$ and $k = 4.3 \times 10^7 Pa$.

The first three natural frequencies given by this formulation are listed together with those given by Lai et al [8] in table 1. It can be seen that the results are very similar; in addition, one can observe that the differences in these natural frequencies change very little when 22 elements are employed instead of 11, demonstrating the independence of the results with respect to the number of elements used in the discretization.

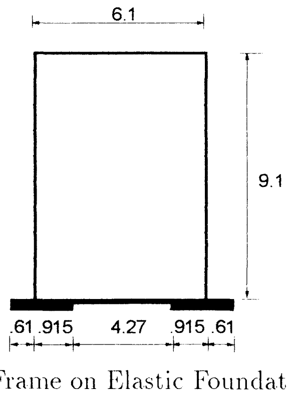

Fig. 2 Frame on Elastic Foundation

Problem 2

In this example the frame, shown in fig. 3a, is subjected to a horizontal and vertical seismic excitations represented by the pseudo-acceleration response spectrum given in fig. 3b. The properties of the structure are as follows: cross section of the beams $0.20m \times 0.60m$, of the columns

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Table 1: Comparison of Natural Frequencies for Problem 1

Frequencies	Present Formulation		Lai et al [8]	
(Hz)	11 Elements	22 Elements	11 Elements	22 Elements
ω_1	2.8327	2.8364	2.8042	2.8036
ω_2	11.8566	11.8194	11.7677	11.7187
ω_3	16.3096	16.3096	16.2757	16.2542

$0.20m \times 0.40m$, of the footings $1.20m \times 0.40m$ and of the foundation slab $1.20m \times 0.30m$; $E_{\text{superstructure}} = 3.0 \times 10^{10} Pa$, $E_{\text{foundation}} = 2.486 \times 10^{10} Pa$, $\rho_{\text{columns}} = \rho_{\text{foundation}} = 2450 kg/m^3$, $\rho_{\text{beams}} = 25483 kg/m^3$, $k = 4.3 \times 10^7 Pa$.

This problem was studied be Moller & Rubinstein [11] without considering the elastic foundation. Stress resultants for horizontal seismic excitation are shown in fig. 4a for this case and compare very well with those given in the reference. In fig. 4b similar results are presented for the same structure including an elastic foundation. For both cases, the stress resultants have been reduced by using a ductility factor [10] of 3.5.

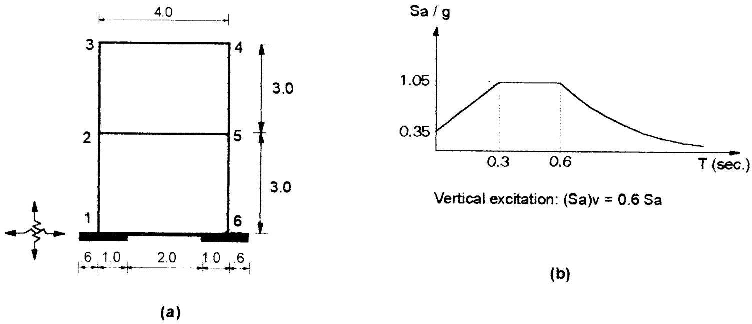

Fig. 3 Frame for Problem 2 and Response Spectrum

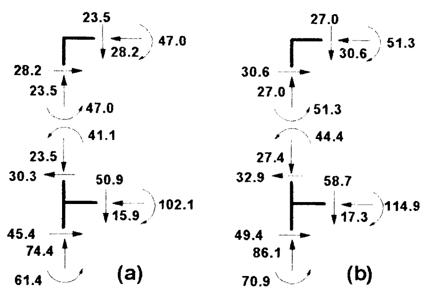

Fig. 4 Stress Resultants for Horizontal Excitation for Problem 2
a) without foundation, b) with foundation

Corresponding results for vertical excitation, considering the first four vibration modes, are given in figures 5a and 5b. It may be seen that the inclusion of the elastic foundation modifies considerably the response obtained. This is corroborated by the results shown in figure 6, where the variation of the maximum vertical displacement at node 3 in relation to the stiffness modulus k of the foundation is plotted. It may be observed that for large values of k, this displacement approaches the corresponding value for the structure without the elastic foundation.

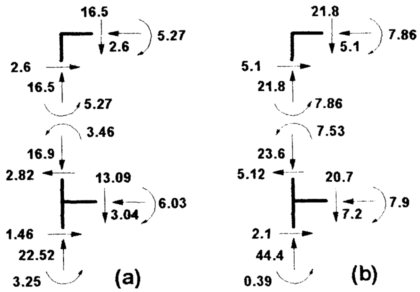

Fig. 5 Stress Resultants for Vertical Excitation for Problem 2
a) without foundation, b) with foundation

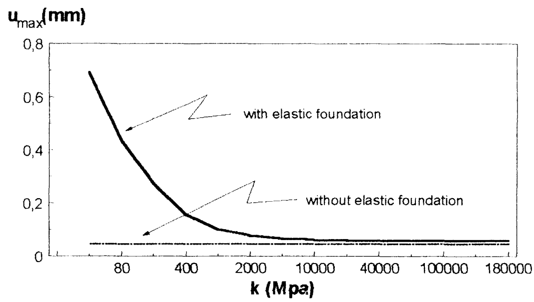

Fig. 6 Maximum Displacement v. Stiffness Modulus of Foundation

Problem 3

In this example the structure shown in figure 7 will be considered for which a vertical acceleration

$$\ddot{v}_g(t) = 0.5\pi^2 \sin(5\pi t) \tag{12}$$

is applied to the left footing. The properties of the frame are: $E = 2.05 \times 10^{11} Pa$, $A = 0.0062 m^2$, $I = 1.739 \times 10^{-4} m^4$ and $\rho = 7800 kg/m^3$. The properties of the foundation are $E = 2.5 \times 10^{10} Pa$ and $\rho = 2200 kg/m^3$, $I = 0.008 m^4$ and $A = 1.5m \times 0.4m$ for the footings, and $I = 0.00195 m^4$ and $A = 1.5m \times 0.25m$ for the foundation slab. The Winkler constant k is $25 \times 10^6 Pa$.

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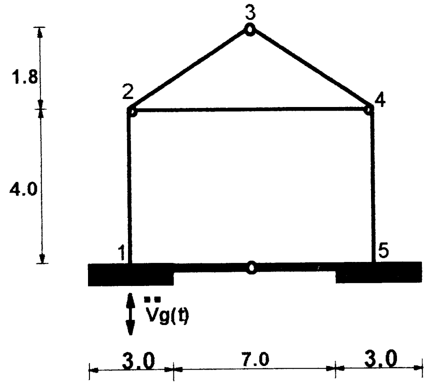

Fig. 7 Frame for Example 3

The objective of this example is to show the potential of the proposed method. Here a framed structure on an elastic foundation is considered for which both the frame and the foundation include internal hinges. The structure is analysed with and without the elastic foundation. The horizontal and vertical displacements at node 2 for the two cases considered are shown in figures 8a and 8b, respectively. It can be seen that, for the structure with the elastic foundation, the vertical displacement is much larger than the corresponding one for the conventional frame.

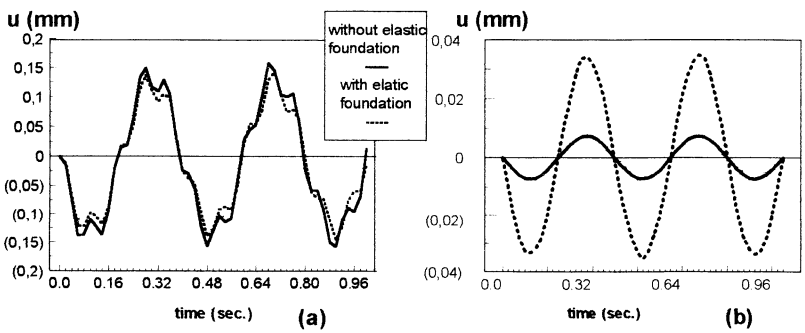

Fig. 8 (a) Horizontal Displacement v. Time;
(b) Vertical Displacement v. Time

5 Conclusions

The beam on elastic foundation elements employed here for several examples have shown themselves to be satisfactory for modelling the response of framed structures on an elastic foundation to dynamic excitation including ground motion.

The results show important differences in modelling the structures with and without the elastic foundation, indicating the importance of the inclusion of the latter in the model. The degree of influence depends clearly on the type of soil and the structural foundation.

This study is being extended through the use of the boundary element method to model the soil as an elastic medium.

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