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The Restricted Singular Value Decomposition: Properties and Applications

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Abstract

The *restricted singular value dccomporition* (RSVD) is the factorization of a given matrix, relative to two other given matrices. It can be interpreted as the *ordinary singular value decomposition* with different inner products in row and column spaces. Its properties and structure are investigated in detail as well as its connection to generalized eigenvdue problems, canonical correlation analysis and other generalizationz of the singular value decomposition.
Applications that are discussed include the analysis of the extended shorted operator, unitarily invariant norm minimization with rank constraints, rank minimization in matrix balls, the analysis and solution of linear matrix equations, rank minimization of *a* partitioned matrix and the connection with generalized Schur complements, constrained linear and total linear least squares problems, with mixed exact and noisy data, including *a* generalized Gauss-Markov estimation scheme. Two constructive proofs of the RSVD in terms of other generalizations of the ordinary singular value decomposition are provided as well.

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1 Introduction

The *ordinary singular value decomposition* (OSVD) has a long history with contributions of Sylvester (1889), Autonne (1902) [1], Eckart and Young (1936) [9] and many others. It has become an important tool in the analysis and numerical solution of numerous problems arising in such diverse applications as psychometrics, statistics, signal processing, system theory, etc.... Not only does it allow for an elegant problem formulation, but at the same time it provides geometrical and algebraic insight together with an immediate numerically robust implementation [12].
Recently, several generalizations to the OSVD have been proposed and their properties analysed. The best known one is the ***generalized** SVD* as introduced by Paige and Saunders in 1981 [20], which we propose to rename as the *Quotient SVD* (QSVD) [7]. Another example is the ***Product** SVD* (PSVD) as proposed by Fernando and Hammarling in [11] and further analysed in [8]. The third one is the ***Restricted** SVD* (RSVD), introduced in its explicit form by Zha in [28] and further developed and discussed in this paper.
A common feature of these generalizations is that they are related to the OSVD on the one hand and to certain generalized eigenvalue problems on the other hand. Many of their properties and structures can be established by exploiting these connections. However, in all cases, the explicit generalized SVD formulation possesses a richer structure than is revealed in the corresponding generalized eigenvalue problem. We conjecture that numerical algorithms that obtain the decomposition in a direct way, without conversion to the generalized eigenvalue problem, will be better behaved numerically. The main reason is that the generalized SVDs are related to their corresponding generalized eigenvalue problem or OSVD via Gramian-type or normal equations like squaring operations as for instance in AA^*, the explicit formation of which results in a well known non-trivial loss of accuracy.

In this paper, we propose and analyse a new generalization of the singular value decomposition: ***the Restricted Singular Value Decomposition* (RSVD)**, which applies for a given triplet of (possibly complex) matrices A, B, C of compatible dimensions (Theorem 4). In essence, the RSVD provides a factorization of the matrix A, relative to the matrices B and C. It could be considered as the OSVD of the matrix A, but with different (possibly nonnegative definite) inner products applied in its column and in it8 row space.

It will be shown that the RSVD not only allow8 for an elegant treatment of algebraic and geometric problem8 in a wide variety of applications, but that it8 structure provide8 a powerful tool in simplifying proofs and derivation8 that are algebraically rather complicated.

This paper is **organised** as follows:

- In section 2, the main structure of the decomposition of a triplet of matrices is **analysed** in term8 of the rank8 of the concatenation of certain matrices. The factorization is related to a generalized eigenvalue problem (section **2.2.1)** and a variational characterization is provided in section 2.2.2. A generalized dyadic decomposition is explored in section 2.2.3 together with a geometrical interpretation.
  It is shown how the RSVD contain8 other generalization8 of the OSVD, such as the PSVD and the QSVD (see below) a8 special case8 in section 2.2.4. In section 2.2.5, it is demonstrated how a special case leads to **canonical correlation** analysis. In section 2.2.6, we investigate the relation of the RSVD with **some** expressions that involve pseudo-inverses.

- In section 3, several application8 are **discussed:**
  - *Rank minimization* and the ***extended*** shorted ***operator*** are the subject of section 3.1, **as** *well* **as** ***unitarily invariant norm*** *minimization* ***with*** *rank* ***constraints*** and the relation with ***matrix balls.*** We al80 investigate *a* certain linear matrix equation.
  - The rank *reduction* of *a partitioned* ***matrix*** when only one of it8 block8 can be modified, is explored in section 3.2 together with ***total least squares with mixed exact and noisy data and linear constraints.*** While the role of the Schur complement and it8 close **connection** to **least** squares estimation **is** well understood, it will be shown in this section, that there exists a similar relation between constrained total linear **least** squares solutions and a ***generalized Schur compkment.***
  - ***Generalized Gauss-Markov*** mode&, possibly with constraints, are discussed in section 3.3 and it **is** shown how the RSVD simplifies

the solution of linear least squares problem8 with constraints.

- In section 4, the main conclusions are presented together with some perspectives.

Notations, Conventions, Abbreviations

Throughout the paper, matrices are denoted by capitals, vector8 by lower case letter8 other than $i, j, k, l, m, n, p, q, r$, which are nonnegative integers. Scalars (complex) are denoted by Greek letters.
A ($m \times n$), B ($m \times p$), C ($q \times n$) are given complex matrices. Their rank will be denoted by r_a, r_b, r_c. D is a $p \times q$ matrix. M is the matrix with A, B, C, D^* as it8 blocks: $M = \begin{pmatrix} A & B \\ C & D^* \end{pmatrix}$. We **shall** also frequently use the following ranks:

$$r_{ac} = rank \begin{pmatrix} A \\ C \end{pmatrix} \qquad r_{abc} = rank \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$
$$r_{ab} = rank(\; A \;\; B \;) \qquad .$$

The matrix A^+ is the unique **Moore-Penrose** pseudo-inverse of the matrix A, A^t is the transpose of a (possibly complex) matrix A and $\overline{A}$ is the complex conjugate of A. A^* denotes the complex conjugate transpose of a (complex) matrix: $A^* = \overline{A}^t$. The matrix A^{-*} represent8 the inverse of A^*. I_k is the $k \times k$ identity matrix. The subscript **is** omitted when the dimensions are clear from the context. e_i ($m \times 1$) and f_i ($n \times 1$) are identity vectors: all component8 are 0 except the i-th one, which is 1. The matrices U_a ($m \times m$), V_a ($n \times n$), V_b ($p \times p$), U_c ($q \times q$) **are unitary:**

$$U_a U_a^* = I. = U_a^* U_a \qquad V_a V_a^* = I_n = V_a^* V_a$$
$$V_b V_b^* = I_p = V_b^* V_b \qquad U_c U_c^* = I_q = U_c^* U_c$$

The **matrices** P ($m \times m$), Q ($n \times n$) are square non-singular. The **non**-zero **elements** of the matrices S_a, S_b and S_c, which appear in the theorems, are **denoted** by α_i, β_i and γ_i. The vector a_i denote8 the i-th column of the matrix A. The range **(columnspace)** of the matrix A **is** denoted by $R(A) R(A) = \{y | y = Ax\}$. The row space of A **is** denoted by $R(A^*)$. The null space of the matrix A **is** represented a8 $N(A) N(A) = \{x | Ax = 0)$. $\bigcap$ denote8 the intersection of two **vectorspaces.**
We **shall** frequently use the following **well** known:

Lemma 1

$$\begin{aligned} dim(R(A)\bigcap R(B)) &= r_a + r_b - r_{ab} \\ dim(R(A^*)\bigcap R(C^*)) &= r_a + r_c - r_{ac} \end{aligned}$$

$\|A\|$ is any unitarily invariant matrix norm while $\|A\|_F$ is the Frobenius norm: $\|A\|_F^2 = trace(AA^*)$. The norm of the vector a is denoted by $\|a\|_2$ where $\|a\|_2^2 = a^*a$. Moreover, we will adopt the following convention for block matrices: Any (possibly rectangular) block of zeros is denoted by 0, the precise dimensions being obvious from the block dimensions. The symbol I represents a matrix block corresponding to the square identity matrix of appropriate dimensions. Whenever a dimension indicated by an integer in a block matrix is zero, the corresponding block row or block column should be omitted and all expressions and equations in which a block matrix of that block row or block column appears, can be disregarded. An equivalent formulation would be that we allow $0 \times n$ or $n \times 0$ $(n \neq 0)$ blocks to appear in matrices. This allows an elegant treatment of several cases at once.

Before starting the main subject of this paper, the exploration of the properties of the RSVD, let us first recall for completeness the theorems for the OSVD and its generalizations, namely the PSVD and the **QSVD**.[1]

Theorem 1
The Ordinary Singular Value **Decomposition:** The **Autonne-Eckart-**Young theorem
Every *m* x *n* ***matrix*** *A can be factorized as* ***follows:***

$$A = U_a S_a V_a^*$$

[1]Recently, we have proposed a **standardized** nomenclature and format for the singular value **decomposition** and its generalizations [7]. We propose to refer to the generalized SVD of Paige and Saunders [20] as the quotient SVD (QSVD) because the Product SVD (PSVD) and the **Restricted** SVD **(RSVD)** can also be considered as 'generalizations' of the Ordinary SVD (OSVD). **This set** of names has the additional advantage of being .... and ***mnemonic:*** **O-P-Q-R-SVD** !

where U_a and V_a are unitary matrices and S_a is a real m x n diagonal matrix with r_a = rank(A) positive diagonal entries:

$$S_a = \begin{matrix} \\ r_a \\ m - r_a \end{matrix} \begin{matrix} r_a & n - r_a \\ \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} \end{matrix}$$

where $D_a = diag(\sigma_i)$, $\sigma_i > 0, i = 1, \ldots, r_a$.

The columns of U_a are the left singular vectors while the columns of V_a are the right singular vectors. The diagonal elements of S_a are the so-called singular values and by convention they are ordered in non-increasing order. A proof of the OSVD and numerous properties can be found in e.g. [12]. Applications include rank reduction with unitarily invariant norms, linear and total linear least squares, computation of canonical correlations, pseudo-inverses and canonical forms of matrices [24].

The product *singular* value *&composition* (PSVD) was introduced by Fernando and Hammarling [11] in 1987.

Theorem 2
The Product Singular Value Decomposition
Every pair of matrices A, m x n and B, m x p can be factorized as:

$$\begin{aligned} A &= P^{-*} S_a V_a^* \\ B &= P S_b V_b^* \end{aligned}$$

where V_a, V_b are unitary and P is square nonsingular. S_a and S_b have the following structure:

$$S_a = \begin{matrix} \\ r_1 \\ r_a - r_1 \\ r_b - r_1 \\ m - r_a - r_b + r_1 \end{matrix} \begin{pmatrix} r_1 & r_a - r_1 & n - r_a \\ D_a & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_b = \begin{matrix} \\ r_1 \\ r_a - r_1 \\ r_b - r_1 \\ m - r_a - r_b + r_1 \end{matrix} \begin{pmatrix} r_1 & r_b - r_1 & p - r_b \\ D_b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $D_a = D_b$ are square diagonal matrices with positive diagonal elements and $r_1 = rank(A^*B)$.

A constructive proof based on the **OSVDs** of A and B, can be found in [8], where also all possible sources of non-uniqueness are explored.
The name PSVD originates in the fact that the OSVD of A^*B is a direct consequence of the PSVD of the pair A, B. The matrix $D_a^2 = D_b^2$ contains the **nonzero** singular values of A^*B. The column vectors of P *are* the **eigen**-vectors of the eigenvalue problem $(BB^*AA^*)P = P\Lambda$. The column vectors of V_a are the eigenvectors of the **eigenvalueproblem** $(A^*BB^*A)V_a = V_a\Lambda$ while those of V_b are eigenvectors of $(B^*AA^*B)V_b = V_b\Lambda$. The precise connection between V_a, V_b and P is **analysed** in [8]. The pairs of diagonal elements of S_a and S_b are called the ***product singular value pairs*** while their products are called the ***product singular values.*** Hence, there are zero and **nonzero** product singular values. By convention, the diagonal elements of S_a and S_b are ordered-such that the product singular values are non-increasing.
Applications include the orthogonal Procrustes problem, balancing of state space models and computing the **Kalman** decomposition (see [8] for references.)

The ***quotient singular value decomposition*** was introduced by Van Loan in [27](' the BSVD ') in 1976 although the idea had been around for a number of years, albeit implicitly (disguised as a generalized **eigenvalue** problem). Paige and Saunders extended Van Loan's idea in order to handle all possible **cases** [20] (they called it the generalized **SVD**).

Theorem 3
The Quotient Singular Value **Decomposition**
Every pair of matrices** A, $m \times n$ and B, $m \times p$ can be factorized **as:

$$\begin{aligned} A &= P^{-*} S_a V_a^* \\ B &= P^{-*} S_b V_b^* \end{aligned}$$

*where V_a and V_b **are unitary*** and P ***is square nonsingular.** The matrices S_a and S_b have the **following** structure:*

$$S_a = \begin{array}{r} \\ r_{ab} - r_b \\ r_a + r_b - r_{ab} \\ r_{ab} - r_a \\ m - r_{ab} \end{array} \begin{array}{c} \begin{array}{ccc} r_{ab} - r_b & r_a + r_b - r_{ab} & n - r_a \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & D_a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$S_b = \begin{array}{c} \\ r_{ab}-r_b \\ r_a+r_b-r_{ab} \\ r_{ab}-r_a \\ m-r_{ab} \end{array} \begin{array}{c} \begin{array}{ccc} p-r_b & r_a+r_b-r_{ab} & r_{ab}-r_a \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & D_b & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{array} \right) \end{array}$$

where D_a and D_b are square diagonal matrices *with positive diagonal elements, satisfying:*

$$D_a^2 + D_b^2 = I_{r_a+r_b-r_{ab}}$$

The *quotient* singular values are defined as the ratios of the diagonal elements of S_a and S_b. Hence, there are zero, non-zero, infinite and arbitrary (or undefined) quotient singular values. By convention, the non-trivial quotient singular value pairs are ordered such that the quotient singular values are non-increasing.

The name QSVD originates in the fact that under certain conditions, the QSVD provides the OSVD of A^+B, which could be considered as a matrix quotient. Moreover, in most application8 the quotient singular values are relevant (not the diagonal elements of S_a and. S_b as such).

The column vectors of P axe the eigenvectors of the generalized eigenvalue problem $AA^*P = BB^*P\Lambda$.

Applications include rank reductions of the form $A + BD$ with minimization of any unitarily invariant norm of D, least squares (with constraints) [21] and total least squares (with exact columns), signal processing and system identification, etc . . . [24][12].

2 The Restricted Singular Value Decomposition (RSVD)

The idea of a generalization of the OSVD for three matrices is implicit in the S, T-singular value decomposition of Van Loan [27] via its relation to a generalized eigenvalue problem. An explicit formulation and derivation of the *restricted singular value decomposition* was introduced by Zha in 1988 [28] who derived a constructive proof via a sequence of OSVDs and QSVDs, which can be found in appendix A. Another proof via a sequence of OSVDs and PSVDs, which is more elegant though, was derived by the authors and can be found also in appendix A.

In this section, we first state the main theorem (section **2.1)**, which describes the structure of the RSVD, followed by a discussion of the main properties, including the connection to generalized eigenvalue problems, a generalized dyadic decomposition, geometrical insights and the demonstration that the RSVD contains the OSVD, the PSVD and the QSVD as special cases.

2.1 The RSVD theorem

With the notations and conventions of section 1, we have the following:

Theorem 4
The **Restricted** Singular Value **Decomposition**
Every triplet of matrices *A (mx n), B (mx p) and C (q x n) can be* ***factorized as:***

$$
\begin{aligned}
A &= P^{-*} S_a Q^{-1} \\
B &= P^{-*} S_b V_b^* \\
C &= U_c S_c Q^{-1}
\end{aligned}
$$

where P $(m \times m)$ and Q $(n \times n)$ are square ***nonsingular,*** *V_b $(p \times p)$ and U_c $(q \times q)$ are* ***unitary.*** *S_a $(m \times n)$, S_b $(m \times p)$ and S_c $(q \times n)$ are* ***real*** *pseudo-diagonal* ***matrices*** *with nonnegative elements and the following block* ***structure:***

$$
S_a = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
S_1 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

$$
S_b = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{pmatrix}
I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & S_2 \\
0 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

$$S_c = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S_3 & 0 \end{pmatrix} \end{array}$$

The block dimensions of the **matrices** S_a, S_b, S_c *are:*

Block columns *of* S_a and S,:

1. $r_{abc} + r_a - r_{ac} - r_{ab}$
2. $r_{ab} + r_c - r_{abc}$
3. $r_{ac} + r_b - r_{abc}$
4. $r_{abc} - r_b - r_c$
5. $r_{ac} - r_a$
6. $n - r_{ac}$

Block columns of S_b:

1. $r_{abc} + r_a - r_{ac} - r_{ab}$
2. $r_{ac} + r_b - r_{abc}$
3. $p - r_b$
4. $r_{ab} - r_a$

Block rows of S_a **and** S_b:

1. $r_{abc} + r_a - r_{ab} - r_{ac}$
2. $r_{ab} + r_c - r_{abc}$
3. $r_{ac} + r_b - r_{abc}$
4. $r_{abc} - r_b - r_c$
5. $r_{ab} - r_a$
6. $m - r_{ab}$

Block rows of S_c:

1. $r_{abc} + r_a - r_{ab} - r_{ac}$

2. $r_{ab} + r_c - r_{abc}$

3. $q - r_c$

4. $r_{ac} - r_a$

The matrices S_1, S_2, S_3 are square nonsingular diagonal with positive diagonal elements.

For two constructive proofs, which are straightforward, the reader is referred to appendix A. The **first** one is based on the properties of the OSVD and the PSVD. The second one is borrowed from [28] and exploits the properties of the OSVD and the **QSVD.**

We propose-to **call *restricted singular* *ualue* *triplets,*** $(\alpha_i$, pi, $\gamma_i)$, the following triplets of numbers:

- $r_{abc} + r_a - r_{ab} - r_{ac}$ triplets of the form $(\alpha_i, 1, 1)$ with $\alpha_i > 0$. By convention, they will be ordered as:

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{r_{abc}+r_a-r_{ab}-r_{ac}} > 0$$

- $r_{ab} + r_c - r_{abc}$ triplets of the form (1,O, 1).
- $r_{ac} + r_b - r_{abc}$ triplets of the form (1, 1,O).
- $r_{abc} - r_b - r_c$ triplets of the form $(1,0,0)$.
- $r_{ab} - r_a$ triplets of the form $(0, \beta_j, 0)$, $\beta_j > 0$ (elements of S_2).
- $r_{ac} - r_a$ triplets of the form $(0, 0, \gamma_k)$, $\gamma_k > 0$ (elements of S_3).
- $min(m - r_{ab}, n - r_{ac})$ trivial triplets $(0, 0, 0)$.

Formally, the *restricted singular* ***values*** are the numbers:

$$\sigma_i = \frac{\alpha_i}{\beta_i \gamma_i}$$

Hence, there are zero, infinite, **nonzero** and **undefined** (arbitrary, trivial) restricted singular values. However, obviously the triplets themselves contain much more structural information than the ratios, as will also be evidenced by the geometrical interpretation. Nevertheless, there are:

- $r_{ab} + r_{ac} - r_{abc}$ infinite restricted singular **values.**
- $r_a + r_{abc} - r_{ab} - r_{ac}$ finite **nonzero** restricted singular values.
- $min(r_{ab} - r_a, r_{ac} - r_a)$ zero restricted singular values (the reason for considering them to be 0 is explained in section 3.1.4).
- $min(m - r_{ab}, n - r,)$ undefined (trivial) restricted singular values.

It will be shown in section **3.1.**, how unitarily invariant norms in restricted problems can be expressed as a function of the restricted singular values, just **as** unitarily invariant norms in the unrestricted case are a function of the ordinary singular values as was shown by Mirsky in **[17].**

Some algorithmic issues are discussed in **[10][29][26][25],** though a full portable and documented algorithm for the RSVD is still to be developed.

The reasons for **chosing** the name of the factorization of a matrix triplet as described in Theorem 4, to be the *restricted singular* ***value decomposition,*** **are** the following:

- It will be shown in section **3** how the **RSVD** allows us to analyse matrix problems that can be stated in terms of:

  $$A + BDC \qquad\qquad M = \begin{pmatrix} A & B \\ C & D^* \end{pmatrix}$$

  where typically, one is interested in the ranks of these matrices as the matrix D is modified. In both cases, the matrices B and C represent certain ***restrictions*** **as** to the nature of the allowed modifications. The rank of the matrix $A + BDC$ can only be reduced by modifications that belong to the column space of B and the row space of C. It will be shown how the rank of M can be **analysed** via a generalized **Schur** complement, which is of the form $D^* - CA^-B$, where again, C and B represent certain restrictions and A' is an inner inverse of A **(definition** 1 in section 2.2.7).
- The **RSVD** allows to obtain the restriction of the linear operator A to the column space of B and the row space of C.

- Finally, the RSVD can be interpreted as an OSVD but with certain restrictions on the inner products. to be used in the column and row space of the matrix A (see section 2.2.1).

2.2 Properties

The OSVD as well as the PSVD and the QSVD, can all be related to a certain (generalized) eigenvalue problem [7]. It comes as no surprise that this is also the case for the RSVD. First, the generalized eigenvalue problem that applies for the RSVD will be analysed (section 2.2.1), followed by a variational characterization in section 2.2.2. A generalized dyadic decomposition and some geometrical properties are investigated in section 2.2.3. In section 2.2.4, it is shown how the OSVD, PSVD and QSVD are special cases of the RSVD while the connection between the RSVD and canonical correlation analysis is explored in section 2.2.5. Finally, some interesting results connecting the RSVD to pseudo-inverses are derived in section 2.2.6.

2.2.1 Relation to a generalized eigenvalueproblem

From Theorem 4, it follows that:

$$\begin{aligned} P^*(BB^*)P &= S_b S_b^t \\ Q^*(C^*C)Q &= S_c^t S_c \end{aligned}$$

Hence, the column vectors of P are orthogonal with respect to the inner product provided by the nonnegative definite matrix BB^*. A similar remark holds for the column vectors of the matrix Q. Consider the generalized eigenvalue problem:

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} BB^* & 0 \\ 0 & C^*C \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \lambda \qquad (1)$$

Observe that, whenever $BB^* = I_{,,}$ and $C^*C = I_{,,}$ the eigenvalues λ are given by $\pm$ the singular values of the matrix A.
Assume that the vectors p and q form a solution to the generalized eigenvalue problem (1), then from the RSVD it follows that:

$$\begin{aligned} S_a(Q^{-1}q) &= (S_b S_b^t)(P^{-1}p)\lambda \\ S_a^t(P^{-1}p) &= (S_c^t S_c)(Q^{-1}q)\lambda \end{aligned}$$

Call $p' = P^{-1}p$ and $q' = Q^{-1}q$. Using a obvious partitioning of p' and q' (according to the block diagonal structure of S_a, S_b, S_c as in Theorem 4), one finds that:

$$\begin{pmatrix} S_1 q'_1 \\ q'_2 \\ q'_3 \\ q'_4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p'_1 \\ 0 \\ p'_3 \\ 0 \\ (S_2 S_2^t) p'_5 \\ 0 \end{pmatrix} \lambda \quad and \quad \begin{pmatrix} S_1^t p'_1 \\ p'_2 \\ p'_3 \\ p'_4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} q'_1 \\ q'_2 \\ 0 \\ 0 \\ (S_3^t S_3) q'_5 \\ 0 \end{pmatrix} \lambda.$$

The generalized eigenvalue problem (1) can have 4 types of eigenvalues:

- **1.** λ **is** a diagonal element of S_1

  It is easy to verify that the vectors p' and q' have the form:

  $$p' = \begin{pmatrix} p'_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ p'_6 \end{pmatrix} \quad q' = \begin{pmatrix} q'_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ q'_6 \end{pmatrix}$$

  with $S_1 q'_1 = p'_1 \lambda$ and $S_1^t p'_1 = q'_1$ A. p'_1 and q'_1 have only one **nonzero** element, which is 1, if all diagonal elements of S_1 are distinct.
  Observe that the vectors p'_6 and q'_6 are completely arbitrary. They correspond to the trivial restricted singular triplets.

- *2*. $\lambda = 0$

  From (1) it follows immediately that:

  $$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 0.$$

  However, not every pair of vectors p, q satisfying this relation, satisfies the BB^* and C^*C orthogonality conditions.
  The corresponding vectors p' and q' *axe of* the form:

  $$p' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p'_5 \\ p'_6 \end{pmatrix} \quad q' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ q'_5 \\ q'_6 \end{pmatrix}.$$

- **3.** $\lambda = \infty$
  From (1) it follows that:

$$\begin{pmatrix} BB^* & 0 \\ 0 & C^*C \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 0$$

  and the corresponding vectors p' and q' are of the form:

$$p' = \begin{pmatrix} 0 \\ p'_2 \\ 0 \\ p'_4 \\ 0 \\ p'_6 \end{pmatrix} \quad q' = \begin{pmatrix} 0 \\ 0 \\ q'_3 \\ q'_4 \\ 0 \\ q'_6 \end{pmatrix} .$$

- 4. A= **arbitrary** (not one of the **preceeding)**
  Only the components corresponding to p'_6 and q'_6 are **nonzero** and arbitrary.

This characterization of the **RSVD as** a 'generalized eigenvalue problem may be important **in statistical applications**, where typically the matrices BB^* and C^*C **are** noise **covariance matrices.** Especially when these **co**variance matrices are (almost) singular, the **RSVD** might provide a robust computational implementation.

2.2.2 A variational characterization

The variational characterization of the vectors p_i and q_i is the following:

Let

$$\phi(x, y) = x^* A y$$

be a bilinear **from** of 2 vectors x and y. We wish to **maximize** $\phi(x, y)$ over all vectors x, y subject to

$$\begin{aligned} x^* BB^* x &= 1 \ , \\ y^* C^* C y &= 1 \ . \end{aligned}$$

It follows directly from the **RSVD** that a solution exists only if one of the following situations occurs:

- $r_{abc}+r_a-r_{ab}-r_{ac} \neq$ 🗀📫 In this case, the maximum is equal to the largest diagonal element of S_1 and the optimizing vectors are $x = p_1$, $y = q_1$ so that $\phi(p_1, q_1) = a1.$

- $r_{abc} + r_a - r_{ab} - r_{ac} =$ 🗀📫 The norm constraints on x and y can only be satisfied if:

$$r_{ac} + r_b - r_{abc} > 0 \quad \text{or} \quad r_{ab} - r_a > 0$$
$$\text{and}$$
$$r_{ab} - r_c - r_{abc} > 0 \quad \text{or} \quad r_{ac} - r_a > 0$$

  In either case, the maximum is 0.

- If none of these conditions is satisfied, there is ***no*** solution.

Assume that the maximum is achieved for the vectors $x_1 = p_1$ and $y_1 = q_1$. Then, the other **extrema** of the objective function $\phi(x, y) = x^*Ay$ with the BB^*- **and** C^*C**-orthogonality** conditions, can be found in an obvious recursive manner. The **extremal** solutions x **and** y are simply the appropriate columns of the matrices P and Q.

2.2.3 A generalized dyadic decomposition and geometrical properties.

Denote $P' = P^{-*}$ **and** $Q^{-1} = Q'^*$. Then, with an obvious partitioning of the matrices P', Q', U_c and V_b, corresponding to the diagonal structure of the matrices S_a, S_b, S_c of Theorem 4, it is **straigtforward** to obtain the following:

$$\begin{aligned} A &= P_1'S_1Q_1'^* + P_2'Q_2'^* + P_3'Q_3'^* + P_4'Q_4'^* \\ B &= P_1'V_{b1}^* + P_3'V_{b2}^* + P_5'V_{b4}^* \\ C &= U_{c1}Q_1'^* + U_{c2}Q_2'^* + U_{c4}Q_5'^* \quad . \end{aligned}$$

Hence,

$$\begin{aligned} R(P_1') + R(P_3') &= R(A) \bigcap R(B) \\ R(Q_1'^*) + R(Q_2'^*) &= R(A^*) \bigcap R(C^*) \ . \end{aligned}$$

One could consider the decomposition of A **as** a decomposition relative to $R(B)$ and $R(C^*)$:

	in $R(B)$	not in $R(B)$
in $R(C^*)$	$P_1' S_1 Q_1'^*$	$P_2' Q_2'^*$
not in $R(C^*)$	$P_3' Q_3'^*$	$P_4' Q_4'^*$

Obviously, the term $P_1' S_1 Q_1'^*$ represents the ***restriction*** of the linear operator represented by the matrix A to the column space of the matrix B and the row space of the matrix C, while the term $P_4' Q_4'^*$ is the restriction of A to the orthogonal complements of $R(B)$ and $R(C^*)$.
Also, one finds that:

$$\begin{aligned} R(B^*) &= R(V_{b1}^*) + R(V_{b2}^*) + R(V_{b3}^*) \\ R(C) &= R(U_{c1}) + R(U_{c2}) + R(U_{c4}) \end{aligned}$$

and:

$$\begin{aligned} BV_{b3} &= 0 \implies N(B) = R(V_{b3}) \\ U_{c3}^* C &= 0 \implies N(C^*) = R(U_{c3}) . \end{aligned}$$

Finally, some of the block dimensions in the RSVD of the matrix triplet (A, B, C) can be related to some **geometrical interpretations** as the following:

$$\begin{aligned} dim\left[R\begin{pmatrix} A \\ C \end{pmatrix} \bigcap R\begin{pmatrix} B \\ 0 \end{pmatrix} \right] &= r_{ac} + r_b - r_{abc} , \\ dim[\, R(A\ B)^* \bigcap R(C\ 0)^* \,] &= r_{ab} + r_c - r_{abc} , \\ dim[\, R(A) \bigcap R(B) \,] &= r_a + r_b - r_{ab} , \\ dim[\, R(A^*) \bigcap R(C^*) \,] &= r_a + r_c - r_{ac} . \end{aligned}$$

Also, it is easy to show that:

$$\begin{aligned} R(Q_6') &= N(A) \bigcap N(C) \\ R(P_6') &= N(A^*) \bigcap N(B^*) . \end{aligned}$$

Hence Q_6' **provides** a basis for the common null space of A and C, which is of dimension $n - r_{ac}$, while P_6' provides a basis for the common null space of A^* and B^*, which is of dimension $m - r_{ab}$.

2.2.4 Relation to **(generalized) SVDs**

The RSVD reduces to the OSVD, the PSVD or the **QSVD** for special choices of the matrices A, B and/or C.

Theorem S
Special cases of the **RSVD**

1. **RSVD of** (A, I_m, I_n) ***gives the*** **OSVD** ***of*** A
2. **RSVD** ***of*** (I_m, B, C) ***gives the*** **PSVD** ***of*** (B^*, C)
3. **RSVD** ***of*** (A, B, I_n) ***gives the*** **QSVD** ***of*** (A, B)
4. **RSVD** ***of*** (A, I_m, C) ***gives the*** **QSVD** ***of*** (A, C)

Proof!

Case **1:** $B = I_m, C = I_n$: Consider the **RSVD** of (A, I_m, I_n). Obviously

$$\begin{aligned} I_m &= P^{-*} S_b V_b^* \\ I_n &= U_c S_c Q^{-1} \end{aligned}$$

and this implies

$$\begin{aligned} P^{-*} &= V_b S_b^{-1} \\ Q^{-1} &= S_c^{-1} U_c^* \end{aligned} .$$

Hence,

$$A = V_b (S_b^{-1} S_a S_c^{-1}) U_c^*$$

which is an OSVD of A.

Case 2: $A = I_m$: Consider the **RSVD** of (I_m, B, C) then obviously

$$I_m = P^{-*} S_1 Q^{-1}$$

which **implies**

$$Q^{-1} = S_1^{-1} P^* .$$

hence,

$$\begin{aligned} B^* &= V_b S_b^t P^{-1} \\ C &= U_c(S_c S_1^{-1}) P^* \end{aligned}$$

which is nothing else than a **PSVD** of (B^*, C).

Case 3: $C = I_n$: Consider the RSVD of (A, B, I_n). Then

$$I_n = U_c S_c Q^{-1}$$

which implies

$$Q^{-1} = S_c^{-1} U_c^* .$$

Then,

$$\begin{aligned} A &= P^{-*}(S_a S_c^{-1}) U_c^* \\ B &= P^{-*} S_b V_b^* \end{aligned}$$

which **is** (up to a diagonal scaling of the diagonal matrices) a **QSVD** of the matrix pair (A, B).

Case 4: $B = I_m$: The proof is similar to **case** 3.

□

2.2.s Relation with canonical correlation analysis.

In the case that the **matrices** BB^* and C^*C are nonsingular, it can be shown that the generalized **eigenvalue** problem (1) is equivalent to singular value decomposition. In **[10]**, an algorithmic derivation along these lines is given.

Let p_i and q_i be the **i-th** column of P, resp. Q, then it follows from (1) that

$$\begin{aligned} Aq_i &= BB^* p_i \lambda_i \\ A^* p_i &= C^* C q_i \lambda_i \quad . \end{aligned}$$

If BB^* and C^*C are both **nonsingular,** then there exist nonsingular matrices W_b and W_c (for example the Cholesky decomposition) such that

$$\begin{aligned} BB^* &= W_b^* W_b \ , \\ C^*C &= W_c^* W_c \ . \end{aligned}$$

Then,

$$\begin{aligned} (W_b^{-*} A W_c^{-1})(W_c q_i) &= (W_b p_i)\lambda_i \\ (W_c^{-*} A W_b^{-1})(W_b p_i) &= (W_c q_i)\lambda_i \quad . \end{aligned}$$

Then, the BB^* orthogonality of the vectors p_i and the C^*C-orthogonality of the vectors q_i, implies that the vectors $W_b p_i$ and $W_c q_i$ are (multiples of) the left and right singular vectors of the matrix $W_b^{-*} A W_c^{-1}$.

It can be shown (see e.g. [2]) that the principal angles θ_k and the principal vectors u_k, v_k between the column spaces of a matrix A and B are given by:

$$\begin{aligned} cos(\theta_k) &= \sigma_k \\ u_k &= A p_k \\ v_k &= B q_k \qquad k = 1,2,\ldots \end{aligned}$$

where σ_k are the **eigenvalues** and p_k, q_k properly normalized **eigenvectors** to the generalized eigenvalue problem

$$\begin{pmatrix} 0 & A^*B \\ B^*A & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} A^*A & 0 \\ 0 & B^*B \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \sigma \ .$$

The σ_k are also the canonical correlations.- Comparing this to the generalized eigenvalue problem (1) that corresponds to the RSVD, one can see immediately that the canonical correlation eigenvalue problem is a special case of the RSVD eigenvalue problem (1). The canonical correlations are the restricted singular values of the matrix triplet (A^*B, A^*, B) and the principal vectors follow from the column vectors of the unitary matrices in the RSVD.
There exist however applications where the matrices BB^* and C^*C are (almost) singular (see e.g. [10] [15] [25] [26] and the references therein). It is in these situations that the **RSVD** may provide essential insight into the geometry of the singularities and at the same time yield a numerically robust and elegant implementation of the solution.

2.2.8 The RSVD and expressions with pseudo-inverses.

The **RSVD** can also be used to obtain the OSVD, PSVD and **QSVD** of certain matrix expressions containing pseudo-inverses. Hereto we need the following definitions and lemmas, which will be used also in section 3 (see [19] for references):

Definition 1
$A(i,j,\ldots)$-inverse of a matrix

A matrix X is called an A(i, j, . . .)-inverse of the matrix A if it satisfies equation i, j, . . . of the following:

1. $AXA = A$
2. $XAX = X$
3. $(AX)^* = AX$
4. $(XA)^* = XA$

An A(1) inverse is also called an inner inverse and denoted by A'. The A(1,2,3,4) inverse is the Moore-Penrose pseudo-inverse A^+ and it is unique.

We shall also need the following lemmas:

Lemma 2
Inner inverse of a factored matrix
Every inner-inverse A- of the matrix A, which is factored as follows:

$$A = P^{-*} \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}$$

where D_a is square $r_a \times r_a$ nonsingular, can be written as

$$A^- = Q \begin{pmatrix} D_a^{-1} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} P^* \qquad (2)$$

where Z_{12}, Z_{21}, Z_{22} are arbitrary matrices. Conversely, every matrix A- of this form is an inner inverse of A.

Proof: The proof follows immediately from definition 1. □

Lemma 3
Moore-Penrose pseudo-inverse of a **factored** matrix.
Let P and *Q* be *partitioned as follows:*

$$P = (P_1\ P_2) \qquad (Q_1\ Q_2)$$

where P_1 and Q_1 have r_a columns. Then the Moore-Penrose pseudo-inverse of A is given by:

$$A^+ = ((I - Q_2(Q_2^*Q_2)^{-1}Q_2^*)Q_1\ Q_2) \begin{pmatrix} D_a^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_1^*(I - P_2(P_2^*P_2)^{-1}P_2^*) \\ P_2^* \end{pmatrix} \qquad (3)$$

Proof: Obviously, the Moore-Penrose pseudo-inverse is a unqiue element of the set of inner inverses described in Lemma 2. The expression for A^+ follows from substituting the expression for A^- of Lemma 2 in the equations defining the $A(1,2,3,4)$ inverse and calculating the matrices Z_{12}, Z_{21}, Z_{22} that satisfy these 4 conditions.

Hence, the Moore-Penrose pseudo-inverse A^+ is a uniquely determined element among all the inner inverses of the matrix A, obtained from orthogonalization of P_1 and Q_1 with respect to P_2 and Q_2.
An immediate consequence is the following:

Corollary 1 *Let A be a rank r_a matrix that is factorized as:*

$$A = P^{-*} S_a Q^{-1} = P^{-*} \begin{pmatrix} D_a & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}$$

where D_a is $r_a \times r_a$ nonsingular diagonal and P and Q, which are square nonsingular, are partitioned a8 follows:

$$P = (\ P_1\ P_2\ ) \qquad Q = (\ Q_1\ \ Q_2\ )$$

where P_1 and Q_1 have r_a columns. Then,

$$A^+ = Q S_a^+ P^* = Q \begin{pmatrix} D_a^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^*$$

if and only if:

$$P_1^* P_2 = 0 \text{ and } Q_1^* Q_2\ .$$

Returning now to the **RSVD, assume** that, whenever we need the pseudo-inverse of a matrix, it follows that:

$$\begin{aligned} A^+ &= Q S_a^+ P^* && (4) \\ B^+ &= V_b S_b^+ P^* && (5) \\ C^+ &= Q S_c^+ U_c^*\ . && (6) \end{aligned}$$

For instance,. each of these is true when the matrices are square and nonsingular. The expression for B^+ is true if B is of Ml row rank while that for C^+ holds for C being of full column rank.

Then, we have

Theorem 6
On the RSVD and pseudo-inverse+
Assume that conditions (4-6) hold true as needed. Then,

1. $CA^+B = U_c(S_cS_a^+S_b)V_b^*$ *is an OSVD of* CA^+B.
2. $B^+AC^+ = V_b(S_b^+S_aS_c^+)U_c^*$ *is an OSVD of* B^+AC^+.
3. $$\begin{aligned}(A^+B)^* &= V_b(S_a^+S_b)^tQ^* \\ C &= U_cS_cQ^{-1}\end{aligned}$$
   is a PSVD *of the matrix pair* $((A^+B)^*, C)$.
4. *Similarly,*
   $$\begin{aligned}CA^+ &= U_c(S_cS_a^+)P^* \\ B^* &= V_bS_b^tP^{-1}\end{aligned}$$
   is a PSVD *of* (CA^+, B^*).
5. $$\begin{aligned}B^+A &= V_b(S_b^+S_a)Q^{-1} \\ C &= U_cS_cQ^{-1}\end{aligned}$$
   is a **QSVD** *of the matrix pair* (B^+A, C).
6. *Similarly*
   $$\begin{aligned}(AC^+) &= U_c(S_aS_c^+)^tP^{-1} \\ B &= V_bS_b^tP^{-1}\end{aligned}$$
   is a **QSVD** *of the matrix pair* $((AC^+)^*, B)$.

Proof: The proof is merely an exercise in substitution and invoking the conditions (46). □

- In case that A is square and **nonsingular,** the singular values of $CA^{-1}B$ axe the reciprocals of the restricted singular values. These are the singular values of $B^{-1}AC^{-1}$ if both B and C are square and nonsingular.

- The conditions (46) are **sufficient** for the Theorem to hold, but may be relaxed. Indeed, take, for **instance**, the expression $B^+A = V_b(S_b^+ S_a)Q^{-1}$. The necessary conditions for this to be true are less restrictive than expressed in (4-6). This can be investigated using the formula (2) for the inner inverse. However, we shall not pursue this any further here.

3 Applications

The RSVD typically provides a lot of insight in applications where its structure can be exploited in order to convert the problem to a simpler one (in terms of the diagonal matrices S_a, S_b, S_c) such that the solution of the simpler problem is straightforward. The general solution can then be found via backsubstitution. In another **type** of applications, it is the unitarity of the matrices U_c and V_b that is essential.

In this section, we shall **first** explore the use of the RSVD in the analysis of problems related to expressions of the form $A + BDC$ (section 3.1). The connection with Mitra'e concept of the extended shorted operator [18] and with matrix balls will be discussed as **well as** the solution of the matrix equation $B\,DC = A$, which led Penrose to rediscover the **pseudo-inverse** of a matrix [22] [23]. In section 3.2, it is shown how the RSVD can be used to solve constrained total linear least squares problems with exact rows and columns and the close connection to the generalized **Schur** complement [3] is **emphazised.** In section 3.3, we discuss the application of the RSVD in the analysis and solution of generalized Gauss-Markov models, with and without constraints.

Throughout this section, we shall use a matrix E, defined as

$$E = V_b^* D U_c \tag{7}$$

with a block partitioning derived **from** the block structure of S_b and S_c as follows:

$$\begin{array}{c} \\ r_{abc}+r_a-r_{ab}-r_{ac} \\ r_{ac}+r_b-r_{abc} \\ p-r_b \\ r_{ab}-r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{abc}+r_a-r_{ab}-r_{ac} & r_{ab}+r_c-r_{abc} & q-r_c & r_{ac}-r_a \end{array} \\ \left(\begin{array}{cccc} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{array}\right) \end{array} . \tag{8}$$

3.1 On the structure of $A + BDC$

The RSVD provides geometrical insight into the structure of a matrix A relative to the column space of a matrix B and the row space of a matrix C. As will now be shown, it is an appropriate tool to analyse expressions of the form:

$$A + BDC$$

where D is an arbitrary $p \times q$ matrix.

In this section, it will be shown that the RSVD allows us to analyse and solve the following questions:

1. What is the range of ranks of $A + BDC$ over all possible $p \times q$ matrices D (section 3.1)?
2. When is the matrix D that minimizes the rank of $A + BDC$, unique (section 3.2)?
3. When is the term BDC that *minimizes* $rank(A + BDC)$, unique? It will be shown how this corresponds to Mitra's extension of the shorted operator [18] in section 3.3.
4. In case of non-uniqueness, what is the minimum norm solution (for unitarily invariant norms) D that *minimizes* $rank(A + BDC)$ (section 3.4)?
5. The reverse question is the following: Assume that $\|D\| \leq \delta$ where δ is a given positive real scalar. What is the minimum rank of $A + BDC$? This can be linked to rank minimization problems in so called matrix balls (section 3.5).
6. An extreme case occurs if one looks for the (minimum norm) solution D to the linear matrix equation $BDC = A$. The RSVD provides the necessary and sufficient conditions for consistency and allows us to parametrize all solutions (section 3.6).

3.1.1 The range of ranks of $A + BDC$

The range of ranks of $A + BDC$ for all possible matrices D is described in the following theorem:

Theorem 7
On the rank of $A + BDC$

$$r_{ab} + r_{ac} - r_{abc} \leq rank(A + BDC) \leq min(r_{ab}, r_{ac})$$

For every number r in between these bounds, there exists a matrix D such that rank(A + BDC) = r.

Proof: The proof is straightforward using the RSVD structure of Theorem 4:

$$\begin{aligned} A + BDC &= P^{-*}S_aQ^{-1} + P^{-*}S_bV_b^*DU_cS_cQ^{-1} \\ &= P^{-*}(S_a + S_bES_c)Q^{-1} \end{aligned}$$

where $E = V_b^*DU_c$. Because of the nonsingularity of P, Q, U_c, V_b, we have that:

$$rank(A + BDC) = rank(S_a + S_bES_c)$$

and the analysis is simplified because of the diagonal structure of S_a, S_b, S_c. Using elementary row and column operations and the block partitioning of E as in (8), it is easy to show that:

$$rank(A + BDC) = rank \begin{pmatrix} S_1 + E_{11} & 0 & 0 & 0 & E_{14}S_3 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ S_2E_{41} & 0 & 0 & 0 & S_2E_{44}S_3 & 0 \end{pmatrix} \quad (9)$$

the block dimensions of which are the same as these of S_a in Theorem 4. Obviously, a lower bound is achieved for $E_{11} = -S_1$, $E_{14} = 0$, $E_{41} = 0$, $E_{44} = 0$. The upper bound is generically achieved for any arbitrary ('random') choice of $E_{11}, E_{14}, E_{41}, E_{44}$. □

Observe that, if $r_a = r_{ab} + r_{ac} - r_{abc}$, then there is no S_1 block in S_a and the minimal rank of $A + BDC$ will be r_a. Also observe that the minimal achievable rank, $r_{ab} + r_{ac} - r_{abc}$, is precisely the number of infinite restricted singular values. This is no coincidence as will be clarified in section 3.1.4.

3.1.2 The unique rank minimking matrix D

When is the matrix D that minimizes the rank of $A + BDC$, unique? The answer is given in the following theorem:

Theorem 8

Let D be such that $\text{rank}(A + BDC) = r_{ab} + r_{ac} - r_{abc}$ and assume that $r_a > r_{ab} + r_{ac} - r_{abc}$. Then the matrix D that minimizes the rank of $A + BDC$ is unique iff:

1. $r_c = q$
2. $r_b = p$
3. $r_{abc} = r_{ab} + r_c = r_{ac} + r_b$

In the case where these conditions are satisfied, the matrix D is given as

$$D = V_b \begin{pmatrix} -S_1 & 0 \\ 0 & 0 \end{pmatrix} U_c^* \ .$$

Proof: It can be verified from the matrix in (9) that the rank of $A + BDC$ is independent of the block matrices $E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{24}, E_{31}, E_{32}, E_{33}, E_{34}, E_{42}, E_{43}$. Hence, the rank minimizing matrix D will not be unique, whenever one of the corresponding block dimensions is not zero, in which case it is parametrized by the blocks E_{ij} in:

$$D = V_b \begin{pmatrix} -S_1 & E_{12} & E_{13} & 0 \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ 0 & E_{42} & E_{43} & 0 \end{pmatrix} U_c^* \ . \qquad (10)$$

Setting the expressions for these block dimensions equal to zero, results in the necessary conditions. The unique optimal matrix D is then given by $D = V_b E U_c^*$ where

$$E = \begin{matrix} \\ p + r_a - r_{ab} \\ r_{ab} - r_a \end{matrix} \begin{matrix} \begin{matrix} q + r_a - r_{ac} & r_{ac} - r_a \end{matrix} \\ \begin{pmatrix} E_{11} & E_{14} \\ E_{41} & E_{44} \end{pmatrix} \end{matrix} = \begin{pmatrix} -S_1 & 0 \\ 0 & 0 \end{pmatrix} \ .$$

□

- Observe that the expression for the matrix D in Theorem 8 is nothing else than an OSVD!
- In case one of the conditions of Theorem 8 is not satisfied, the matrix D that minimizes the rank of $A + BDC$ *is* not unique. It can be parametrized by the blocks E_{ij} as in (10). It will be shown in section **3.1.4** how to select the minimum norm matrix D.

3.1.3 On the uniqueness of *BDC:* The extended shorted operator

A related question concerns the uniqueness of the product term *BDC* that minimizes the rank of $A + BDC$. As a matter of fact, this problem has received a lot of attention in the literature where the term *BDC* is called the ***extended shorted operator*** and was introduced in [18]. It is an extension to rectangular matrices, of the shorting of an operator considered by Krein, Anderson and Trapp only for positive operators (see [18] for references). It will now be shown how the **RSVD** provides an utmost elegant analysis tool for analysing questions related to shorted operators.

Definition *2*
The extended shorted operator [2]
Let A ($m \times n$), B ($m \times p$) and C ($q \times n$) be given matrices. A shorted matrix $S(A|B,C)$ is any $m \times n$ matrix that satisfies the following conditions:

1.

$$R(S(A|B,C)) \subseteq R(B)$$
$$R(S(A|B,C)^*) \subseteq R(C^*)$$

2. If F is an $m \times n$ matrix satisfying $R(F) \subseteq R(B)$ and $R(F^*) \subseteq R(C^*)$, then,

$$rank(A - F) \geq rank(A - S(A|B,C)) \quad .$$

Hence, the shorted operator is a matrix for which the column space belongs to the column space of *B*, the row space belongs to the row space of C and it minimizes the rank of *A-F* over all matrices *F*, satisfying these conditions. From this, it follows that the shorted operator can be written as:

$$S(A|B,C) = BDC$$

for a certain $p \times q$ matrix *D*. This establishes the direct connection of the concept of extended shorted operator with the RSVD.

The shorted operator is not always unique as can be seen from the following example. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad and \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

[2] We have slightly changed the notation that is used in [18].

Then, all matrices of the form

$$S = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

minimize the rank of $A - S$, which equals 2, for arbitrary α and β.

Necessary conditions for uniqueness of the shorted operator can be found in a straightforward way **from the RSVD.**

Theorem 9
On the **uniqueness** of the extended shorted operator
Let the RSVD *of the matrix triplet (A, B, C) be given as in Theorem 1. Then,*

$$S(A|B,C) = P^{-*} S(S_a \; S_c) Q^{-1} .$$

The extended shorted operator $S(A|B,C)$ is unique iff

1. $r_{abc} = r_c + r_{ab}$
2. $r_{abc} = r_b + r_{ac}$

and is given by

$$S(A|B,C) = P^{-*} \begin{pmatrix} -S_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} Q^{-1} .$$

Proof: It follows from Theorem 7 that the minimal rank of $A + BDC$ is $r_{ab} + r_{ac} - r_{abc}$ and that in this case

$$E_{11} = -S_1 \quad E_{14} = 0 \quad E_{41} = 0 \quad E_{44} = 0 .$$

A short computation shows

$$BDC = P^{-*} \begin{pmatrix} -S_1 & E_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & E_{24}S_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S_2E_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} Q^{-1}$$

Hence, the matrix BDC is unique iff the blocks $E_{12}, E_{22}, E_{42}, E_{21}, E_{22}$ and E_{24} do not appear in this decomposition. Setting the corresponding block dimensions equal to zero, proves the theorem. □

- Observe that the conditions for uniqueness of the extended shorted operator BDC are less restrictive than the uniqueness conditions for the matrix D (Theorem 8).
- As a consequence of Theorem 9, we also obtain a parametrization of all shorted operators in the case where the uniqueness conditions are not satisfied. All possible shorted operators are then parametrized by the matrices $E_{12}, E_{21}, E_{22}, E_{24}, E_{42}$. Observe that the shorted operator is independent of the matrices $E_{13}, E_{23}, E_{31}, E_{33}, E_{34}, E_{43}$.
- The result of Theorem 9, derived via the RSVD, corresponds to Theorem 4.1 and Lemma 5.1 in [18]. Some connections with the generalized **Schur** complement and statistical applications of the shorted operator can also be found in [18].

3.1.4 The minimum norm solutions D that minimize $rank(A + BDC)$

In Theorem 7, we have described the set of matrices D that minimize the rank of $A + BDC$. In this section, we investigate how to select the ***minimum norm*** matrix D **that achieves** this task.

Before **examining matrices** D **that** *minimize* the rank of $A + BDC$, note that, whenever $min(r_{ab}, r_{ac}) - r_a > 0$, there exist many matrices that will ***increase*** the rank of $A + BDC$. *In* this case:

$$inf_{\epsilon} \{ \epsilon = \|D\| \, | rank(A + BDC) > r_a \} = 0 \tag{11}$$

which implies that there exist arbitrarily 'small' matrices D that will increase **the** rank.

Consider the problem of **finding** the matrix D of minimal (unitarily invariant) norm $\|D\|$ such that:

$$rank(A + BDC) = r < r_a$$

where r is a prescribed nonnegative integer.

0 bserve that:

- It follows from Theorem 7 that necessarily

$$r \geq r_{ab} + r_{ac} - r_{abc}$$

  for a solution to exist.

- Observe that if $r_a = r_{ab} + r_{ac} - r_{abc}$, no solution exists. In this case, there is no diagonal matrix S_1 in S_a of Theorem 4. Hence, it will be assumed that

$$r_a > r_{ab} + r_{ac} - r_{abc} \quad .$$

- Assume that the required rank r *equals* the minimal achievable: $r = r_{ab} + r_{ac} - r_{abc}$. Then, if the conditions of Theorem 8 are satisfied, the optimal D is unique and follows directly from the RSVD. The interesting case occurs whenever the rank minimizing D is not unique.

The general solution is straightforward from the RSVD. In addition to the nonsingularity of U_c, V_b, P, Q, *we* will also *exploit* the unitarity of U_c and V_b.

Theorem **10**
Assume that

$$r_{ab} + r_{ac} - r_{abc} \leq r = rank(A + BDC) < r_a$$

where r is a given integer and $\|.\|$ is any unitarily invariant norm. A matrix D of minimal norm $\|D\|$ is given by:

$$D = -V_b \begin{pmatrix} S_1^r & 0 \\ 0 & 0 \end{pmatrix} U_c^*$$

where S_1^r is a singular diagonal matrix

$$S_1^r = \begin{matrix} r + r_{abc} - r_{ab} - r_{ac} \\ r_a - r \end{matrix} \overset{\begin{matrix} r + r_{abc} - r_{ac} - r_{ab} & r_a - r \end{matrix}}{\begin{pmatrix} 0 & 0 \\ 0 & S_d \end{pmatrix}} .$$

S_d contains the $r_a - r$ *smallest diagonal elements of S_1.*

Proof: From the RSVD of the matrix triplet *A, B,* C it follows that

$$\begin{aligned} A + BDC &= P^{-*}(S_a + S_b(V_b^* D U_c) S_c) Q^{-1} \\ &= P^{-*}(S_a + S_b E S_c) Q^{-1} \end{aligned}$$

with $\|E\| = \|V_b^* D U_c\| = \|D\|$. The result follows immediately from the partitioning of E as in (8) and from equation (9) cl

Obviously, the minimum norm follows immediately from the restricted singular values, because every unitarily invariant norm of D can be expressed in terms of the restricted singular values.
As a matter of fact, one could use this property to **define** the *restricted singular values* σ_k.

$$\sigma_k = inf_\epsilon\{ \epsilon = \|D\|_\sigma \mid rank(A + BDC) = k - 1 \}$$

where $\|.\|_\sigma$ denotes the maximal ordinary singular value.

- Because the *rank* of *A+BDC can* not be reduced below $r_{ab}+r_{ac}-r_{abc}$, there will be $r_{ab} + r_{ac} - r_{abc}$ **infinite** restricted singular values.
- Obviously, there are $r_a + r_{abc} - r_{ab} - r_{ac}$ finite restricted singular values, corresponding to the diagonal elements of S_1.
- It can easily be seen from (9) that the diagonal elements of S_2 and S_3 can be used to increase the rank of $A + BDC$ to $min(r_{ab}, r_{ac})$ (Theorem 7). However, from (11) it is obvious that $min(r_{ac} - r_a, r_{ab} - r_a)$ restricted singular values will be zero.
- It follows from Theorem 7 that $min(m - r_{ab}, n - r_{ac})$ restricted singular values are undetermined.
- Theorem 10 is a central result in the analysis and solution of the Restricted Total Least Squares problem, which is studied in [26] where also an algorithm is presented.

3.1.5 The reverse problem: Given $\|D\|$, what is the minimal rank of $A + BDC$?

The results of section **3.1.3** and 3.1.4. allow us to obtain in a simple fashion, the answer to the reverse question:

Assume that we are given a positive real number δ such that $\|D\| \leq 6$. What is the minimum rank r_{min} of $A + BDC$?

The answer is an immediate consequence of Theorem 10. Note that the optimal matrix D is given as the product of three matrices, which form its OSVD! Hence,

$$\|D\| = \|S_1^r\|$$

and the integer r_{min} can be determined immediately from:

$$r_{min} = r_a - (maz_{\ i}\ \{size(S_i)\ \text{such that}\ \|S_i\| \leq 6))\ . \tag{12}$$

where S_i is an $i \times i$ diagonal matrix containing the i smallest elements of S_1.
It is interesting to note that expressions of the form $A + BDC$ with restrictions on the norm of D can be related to the notion of ***matrix balls***, which show up in the analysis of so-called completion problems [5].

Definition 3
Matrix ball
For given matrices A (m x n), B (m x p) and C (q x n), the closed matrix ball $R(A|B,C)$ with center A, left semi-radius B and right semi-radius C is defined by:

$$R(A|B,C) = \{\ X\ |\ X = A + BDC\ \text{where}\ \ \|D\|_2 \leq 1\}$$

Using Theorem 10 and (12), we can find all matrices of least rank within a certain given matrix ball by simply requiring that:

$$\sigma_{max}(D) \leq 1$$

and observing that $\sigma_{max}(D)$ is a unitarily invariant norm. The solution is obtained from the appropriate truncation of S_1^r in Theorem 10. The conclusion is that the RSVD allows to detect the matrices of minimal rank within a given matrix ball. Since the solution of the completion problems investigated in [5] are described in terms of matrix balls, it follows that we can find the minimal rank solution in the matrix ball of all solutions, using the **RSVD**.

3.1.6 The matrix equation $BDC = A$

Consider the problem of investigating the consistency, and, if consistent, **finding** a (minimum norm) solution to the linear equation in the unknown matrix D:

$$BDC = A.$$

This equation has an historical significance because it led Penrose to rediscover what is now called the Moore-Penrose pseudo-inverse [19] [22]. Of course, this problem can be viewed as an extreme case of Theorem 8 and 10, with the prescribed integer r = 0.

Theorem **11**

The matrix equation BDC = A in the unknown matrix D is consistent iff

$$\begin{aligned} r_{ab} &= r_b \\ r_{ac} &= r_c \\ r_{abc} &= r_b + r_c \,. \end{aligned}$$

AU solutions are then given by

$$D = V_b \begin{pmatrix} S_1 & E_{13} & 0 \\ E_{31} & E_{33} & E_{34} \\ 0 & E_{43} & 0 \end{pmatrix} U_c^*$$

and the minimum norm solution corresponds to $E_{13} = 0$, $E_{31} = 0$, $E_{33} = 0$, $E_{34} = 0$, $E_{43} = 0$.

Proof: Let $E = V_b^* D U_c$ and partition E as in (8). The consistency of $BDC = A$ depends on whether the following is satisfied with equality

$$\begin{pmatrix} E_{11} & E_{12} & 0 & 0 & E_{14}S_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & E_{24}S_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ S_2E_{41} & S_2E_{42} & 0 & 0 & S_2E_{44}S_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ? = ? \begin{pmatrix} S_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Comparing the diagonal blocks, the conditions for consistency follow immediately as

$$\begin{aligned} r_{abc} &= r_{ab} + r_c \\ &= r_{ac} + r_b \\ &= r_b + r_c \,, \end{aligned}$$

which implies

$$
\begin{aligned}
r_{ab} &= r_b \\
r_{ac} &= r_c \,.
\end{aligned}
$$

These conditions express the fact that the column space of A should be contained in the column space of B and that the row space of A should be contained in the row space of C.
If these conditions are satisfied, the matrix equation $BDC = A$ is consistent and the matrix $E = V_b^* D U_c$ is given by

$$
E = \begin{array}{c} \\ r_a \\ p - r_b \\ r_b - r_a \end{array} \begin{array}{c} \begin{array}{ccc} r_a & q - r_c & r_c - r_a \end{array} \\ \begin{pmatrix} E_{11} & E_{13} & E_{14} \\ E_{31} & E_{33} & E_{34} \\ E_{41} & E_{43} & E_{44} \end{pmatrix} \end{array} .
$$

The equation $BDC = A$ is equivalent to

$$
\begin{pmatrix} E_{11} & E_{14}S_3 & 0 \\ S_2E_{41} & S_2E_{44}S_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .
$$

This is solved as

$$
E_{11} = S_1 \quad E_{14} = 0 \;\; E_{41} = 0 \quad E_{44} = 0 \,.
$$

Observe that the solution is independent of the blocks $E_{13}, E_{31}, E_{33}, E_{34}, E_{43}$. Hence, all solutions can be parametrized as:

$$
D = \begin{pmatrix} V_{b1} & V_{b3} & V_{b4} \end{pmatrix} \begin{pmatrix} S_1 & \mathrm{E13} & 0 \\ E_{31} & E_{33} & E_{34} \\ 0 & E_{43} & 0 \end{pmatrix} \begin{pmatrix} U_{c1}^* \\ U_{c3}^* \\ U_{c4}^* \end{pmatrix}
$$

Obviously, the minimum norm solution is given by:

$$
D = V_b \begin{pmatrix} S_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} U_c^*
$$

□

- Observe that the result of Theorem 11 could also be obtained directly from Theorem 10 with $r =$ ..

- Penrose originally proved [19][22], that it is a necessary and sufficient condition for $BDC = A$ to have a solution, that

$$BB^{-}AC^{-}C = A \tag{13}$$

where B^{-} and C^{-} are inner-inverses of B and C (see definition 1). All solutions D can then be written as:

$$D = B^{-}AC^{-} + Z - BB^{-}ZC^{-}C \tag{14}$$

where Z is an arbitrary $p \times q$ matrix. It requires a tedious though straightforward calculation to verify that our solution of Theorem 11 coincides with (14). In order to verify this, consider the RSVD of A, B, C and use Lemma 2 to obtain an expression for the inner-inverses of B and C, which will contain arbitrary matrices. Using the block dimensions of S_a, S_b, S_c as in Theorem 4, it can be shown that the consistency conditions of Theorem 11, coincide with the consistency condition (13).

Before concluding this section, it is worth mentioning that all results of this section can be specialized for the case where either B or C equals the identity matrix. In this case, the RSVD specializes to the **QSVD** (Theorem 3 and 5) and ***mutatis mutandis***, the same type of questions, now related to 2 matrices, can be formulated and solved using the **QSVD** such as shorted operators, minimum norm rank minimization, solution of the matrix equation $DC = A$ etc...

3.2 On the rank reduction of a partitioned matrix.

In **this** section, the RSVD will be used to analyse and solve problems that can be stated in terms of the matrix [3]

$$M = \begin{pmatrix} A \\ C \end{pmatrix} \tag{15}$$

[3] In order to keep the notation consistent with that of section 3.1, we use the matrix which is the complex conjugate transpose of in section 3.1, as the lower right block of M. This allows us for instance to use the same matrix E as defined in (7) and (8.

where A, B, C, D are given matrices.
The main results include:

1. The analysis of the (generalized) Schur complement [3] in terms of the RSVD (section 3.2.1).

2. The range of ranks of the matrix M as D is modified and the analysis of the (non)-unique matrix D that minimizes the rank of M (section 3.2.2.).

3. The solution of constrained total least squares problem with exact and noisy data by imposing additional norm constraints on D (section 3.2.3.)

3.2.1 (Generalized) Schur complements and the RSVD

The notion of a Schur complement S of the matrix A in M (which is $S = D^* - CA^{-1}B$ when A is square nonsingular), can be generalized to the case where the matrix A is rectangular and/or rank deficient [3]:

Deflnition **4**
(Generalized) Schur complement
A Schur complement of A *in*

$$M = \begin{pmatrix} A & B \\ C & D^* \end{pmatrix}$$

is any matrix

$$S = D^* - CA - B \tag{16}$$

where A' ***is an inner inverse of*** A.

In general there are many of these Schur complements, because from lemma 2, we know that there are many inner inverses. However, the RSVD allows us to investigate the dependency of S on the choice of the inner inverse.

Theorem **12**
The Schur ***complement*** $S = D^* - CA^- B$ ***is independent of*** A- ***iff***

$$r_a = r_{ab} = r_{ac} \; .$$

In this case, S ***is given by***

$$S = U_c \begin{pmatrix} E_{11}^* - S_1^{-1} & E_{21}^* & E_{31}^* \\ E_{12}^* & E_{22}^* & E_{32}^* \\ E_{13}^* & E_{23}^* & E_{33}^* \end{pmatrix} V_b^* \; .$$

Proof: Consider the factorization of A as in the RSVD. From Lemma 2, every inner inverse of A can be written. as:

$$A^- = Q \begin{pmatrix} S_1^{-1} & 0 & 0 & 0 & X_{15} & X_{16} \\ 0 & I & 0 & 0 & X_{25} & X_{26} \\ 0 & 0 & I & 0 & X_{35} & X_{36} \\ 0 & 0 & 0 & I & X_{45} & X_{46} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} \end{pmatrix} P^*$$

for certain block matrices X_{ij}, where the block dimensions of the middle factor correspond to the block dimensions of the matrix S_a^* of Theorem 4. It is straightforward to show that:

$$CA^-B = U_c \begin{pmatrix} S_1^{-1} & 0 & 0 & X_{15}S_2 \\ 0 & 0 & 0 & X_{25}S_2 \\ 0 & 0 & 0 & 0 \\ S_3X_{51} & S_3X_{53} & 0 & S_3X_{55}S_2 \end{pmatrix} V_b^* .$$

Hence, this product is dependent on the blocks $X_{15}, X_{25}, X_{51}, X_{53}, X_{55}$. The corresponding block dimensions are zero if and only if $r_a = r_{ab} = r_{ac}$.
□

Observe that the theorem is equivalent with the statement, that the (generalized) Schur complement $S = D^* - CA^-B$ is independent of the precise choice of A^- if and only if

$$R(B) \subset R(A) \qquad R(C^*) \subset R(A^*) .$$

This corresponds to Carlson's statement of the result (Proposition 1 in [3]). In case these conditions are not satisfied, all possible generalized Schur complements are parametrized by the blocks $X_{51}, X_{53}, X_{15}, X_{25}$ and X_{55} as

$$S = U_c \begin{pmatrix} E_{11}^* - S_1^{-1} & E_{21}^* & E_{31}^* & E_{41}^* - X_{15}S_2 \\ E_{12}^* & E_{22}^* & E_{32}^* & E_{42}^* - X_{25}S_2 \\ E_{13}^* & E_{23}^* & E_{33}^* & E_{43}^* \\ E_{14}^* - S_3X_{51} & E_{24}^* - S_3X_{53} & E_{34}^* & E_{44}^* - S_3X_{55}S_2 \end{pmatrix} V_b^* . \quad (17)$$

3.2.2 How does the rank of M change with changing D?

Define the matrix $M(\tilde{D})$ as:

$$M(\tilde{D}) = \begin{pmatrix} A & B \\ C & D^* - \tilde{D}^* \end{pmatrix}$$

We **shall** also use $\hat{D} = D - \tilde{D}$. What is the precise relation between the rank of $M(\tilde{D})$ and $\tilde{D}$? Before answering this question, we need to state the following (well known) lemma.

Lemma 4

Rank of a partitioned matrix and the **Schur** complement

If A is square and nonsingular,

$$rank\begin{pmatrix} A & B \\ C & D^* \end{pmatrix} = rank(A) + rank(D^* - CA^{-1}B)$$

Proof: Observe that:

$$\begin{pmatrix} A & B \\ C & D^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D^* - CA^{-1}B \end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$$

□

Thus we have,

Theorem **13**

$$rank\begin{pmatrix} A & B \\ C & D^* \end{pmatrix} = r_{ab} + r_{ac} - r_a + rank\begin{pmatrix} E_{11}^* - S_1^{-1} & E_{21}^* & E_{31}^* \\ E_{12}^* & E_{22}^* & E_{32}^* \\ E_{13}^* & E_{23}^* & E_{33}^* \end{pmatrix} .$$

Proof: From the RSVD, it follows immediately that the required rank is equal to the rank of the matrix

$$\begin{pmatrix} S_a & S_b \\ S_c & E^* \end{pmatrix} = \begin{pmatrix} S_1 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & E_{11}^* & E_{21}^* & E_{31}^* & E_{41}^* \\ 0 & I & 0 & 0 & 0 & 0 & E_{12}^* & E_{22}^* & E_{32}^* & E_{42}^* \\ 0 & 0 & 0 & 0 & 0 & 0 & E_{13}^* & E_{23}^* & E_{33}^* & E_{43}^* \\ 0 & 0 & 0 & 0 & S_3 & 0 & E_{14}^* & E_{24}^* & E_{34}^* & E_{44}^* \end{pmatrix} .$$

From the nonsingularity of S_2 and S_3, it follows that the rank **is** independent of $E_{41}, E_{42}, E_{43}, E_{14}, E_{24}, E_{34}, E_{44}$. The result then follows immediately **from**

Lemma 4, taking into account the block dimensions of the matrices. □

A consequence of Lemma 3 is the following result:

Corollary **2** *The mnge of ran&s r of M attainable by an appropriate choice of $\tilde{D}$ in*

$$M = \begin{pmatrix} A & B \\ C & D^* - \tilde{D}^* \end{pmatrix}$$

is

$$r_{ab} + r_{ac} - r_a \le r \le min(p + r_{ac}, q + r_{ab}) \quad .$$

The minimum is attained for

$$\tilde{D}^* = U_c \begin{pmatrix} E_{11}^* - S_1^{-1} & E_{21}^* & E_{31}^* & \tilde{E}_{41}^* \\ E_{12}^* & E_{22}^* & E_{32}^* & \tilde{E}_{42}^* \\ E_{13}^* & E_{23}^* & E_{33}^* & \tilde{E}_{43}^* \\ \tilde{E}_{14}^* & \tilde{E}_{24}^* & \tilde{E}_{34}^* & \tilde{E}_{44}^* \end{pmatrix} V_b^* \qquad (18)$$

where the ***matrices*** $\tilde{E}_{14}, \tilde{E}_{24}, \tilde{E}_{34}, \tilde{E}_{41}, \tilde{E}_{42}, \tilde{E}_{43}$ ***and*** $\tilde{E}_{44}$ ***are arbitrary*** *matrices.*

Compare the expression of $\tilde{D}$ of Corollary **2** with the expression for the generalized Schur complement of A in M **as** given by (17). Obviously, the set of matrices $\tilde{D}$ contains all generalized Schur complements; it are those matrices $\tilde{D}$ for which:

$$\tilde{E}_{34} = E_{34} \qquad \tilde{E}_{43} = E_{43} \ .$$

If these blocks are not present in E, there are no other matrices than generalized Schur complements, **that** minimize the rank of M.
Hence, we have proved the following

Theorem 14
The rank of *M(d)* ***is minimized for*** $\tilde{D}$ ***equal to each generalized*** *Schur complement of A in M. The rank of M(d)* ***is minimized*** only ***for*** $\tilde{D} = D^* - CA\text{-}B$ ***iff:***

$$r_{ab} = r_a \ \ or \ \ r_c = q$$
$$and$$
$$r_{ac} = r_c \ \ or \ \ r_b = p \ .$$

If $r_a = r_{ab} = r_{ac}$, ***then the minimizing*** $\tilde{D}$ ***is unique.***

Proof: The fact that each generalized Schur complement minimizes the rank of M(d) follows directly from the-comparison of $\tilde{D}$ in Corollary 2 with the expression for the generalized Schur complement in (17). The rank conditions follow simply from setting the block dimensions of E_{34} and E_{43} in (8) equal to 0. The condition for uniqueness of $\tilde{D}$ follows from Theorem 12. □

This theorem can also be found as theorem 3 in [3], where it is proved via a different approach.

3.2.3 Total Linear Least Squares with exact rows and columns

The nomenclature ***total linear least squares*** was introduced in [13] as an extension of least squares fitting in the case where there are errors in both the observation vector b and the data matrix A for overdetermined equations $As \approx b$. The analysis and solution is given completely in terms of the **OSVD** of the **concatenated** matrix $(A\ b)$. *In* the case where some of the columns of A are noisefree while the others contain errors, a mixed least squares - total least squares strategy was developed in [14]. The problem where also some rows are errorfree, was analysed via a Schur-complement based approach in [6]. However, one of the key canonical decompositions (Lemma 2 in [6]) and related results concerning rank minimization, were described earlier in [3].

We shall now show how the **RSVD** allows us to treat the general situation in an elegant way.

Again, let the **data** matrix be given as

$$M = \begin{pmatrix} A & B \\ C & D^* \end{pmatrix}$$

whem A, B, C are free of error and only D is contaminated by noise. It is assumed that the data matrix is of full row rank.

The ***constrained*** total ***linear least squares problem*** is equivalent to the following.

Find the matrix $\hat{D}$ and the nonzero vector x such that

$$\begin{pmatrix} A & B \\ C & \hat{D}^* \end{pmatrix} x = 0 \quad ,$$

and $\|D - \hat{D}\|_F$ is minimized.

A slightly more general problem is the following.

Find the matrix $\hat{D}$ such that $\|D - \hat{D}\|_F$ is minimal and

$$rank \begin{pmatrix} A & B \\ C & \hat{D}^* \end{pmatrix} \leq r \quad . \tag{19}$$

The error matrix $D - \hat{D}$ will be denoted by $\tilde{D}$.

$$\tilde{D} = D - \hat{D}$$

Assume that a solution x is found. By partitioning x conformally to the dimensions of A and B, one finds that the vector x satisfies:

$$\begin{aligned} Ax_1 + Bx_2 &= 0 \\ Cx_1 + \hat{D}^* x_2 &= 0 \quad . \end{aligned}$$

Hence, the total least squares problem can be interpreted as follows: The rows of A and B correspond to linear constraints on the solution vector x. The **columns** of the matrix C contain error-free (noiseless) data while those of the matrix D are corrupted by noise. In order to find a solution, one has to modify the matrix D with minimum effort, as measured by the **Frobenius** norm of the 'error matrix' $\tilde{D}$, into the matrix $\hat{D}$.
Without the constraints, the problem reduces to a mixed linear - total linear least squares problem as is analysed and solved in [14].

From the results in section **3.2.2.**, we already know that a necessary condition for a solution to exist is $r \geq r_{ab} + r_{ac} - r_a$ (Corollary 2). When $r = r_{ab} + r_{ac} - r_a$, then,

- The class of rank minimizing matrices $\tilde{D}$ is described by Corollary 2. Theorem **14** shows how the generalized **Schur** complements of A in M form a subset of this Set.

- **From** Corollary 2, it is straightforward to find the minimum norm matrix $\tilde{D}$ that reduces the rank of $M(\tilde{D})$ to $r = r_{ab} + r_{ac} - r_a$. It is given by:

$$\tilde{D}^* = U_c \begin{pmatrix} E^*_{11} - S_1^{-1} & E^*_{21} & E^*_{31} & 0 \\ E^*_{13} & E^*_{32} & E^*_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V_b^* .$$

- The ***minimum norm generalized Schur complement*** that minimizes the rank of M is given by

$$S = U_c \begin{pmatrix} E^*_{11} - S_1^{-1} & E^*_{21} & E^*_{31} & 0 \\ E^*_{12} & E^*_{22} & E^*_{32} & 0 \\ E^*_{13} & E^*_{23} & E^*_{33} & E^*_{43} \\ 0 & 0 & E^*_{34} & 0 \end{pmatrix} V_b^* .$$

  This corresponds to a choice of inner inverse in (17) given by

$$\begin{aligned} X_{15} &= E^*_{41} S_2^{-1} \\ X_{25} &= E^*_{42} S_2^{-1} \\ X_{51} &= S_3^{-1} E^*_{14} \\ X_{53} &= S_3^{-1} E^*_{24} \\ X_{55} &= S_3^{-1} E^*_{44} S_2^{-1} . \end{aligned}$$

We shall now investigate two solution strategies, both of which are based on the RSVD. The **first** one is an immediate consequence of Theorem 10, but, while elegant and extremely simple, might be considered as suffering from some 'overkill'. It is a direct application of the insights obtained in **analysing** the sum $A + BDC$. The second one is less elegant but is more in the line of results reported in [3] and [6]. It exploits the insights obtained from analysing the partitioned matrix $M = \begin{pmatrix} A & B \\ C & D^* \end{pmatrix}$.

3.3.3.1. Constrained total linear least squares directly via the RSVD

It is straightforward to show that the constrained total least squares problem can be recast as a minimum norm problem as discussed in Theorem 10.

Consider the following problem:

Find the matrix $\tilde{D}$ of minimum norm $\|\tilde{D}\|$ such that

$$rank\left(\begin{pmatrix} A & B \\ C & D^* \end{pmatrix} + \begin{pmatrix} 0_{m\times q} \\ I_q \end{pmatrix} \tilde{D}^* \begin{pmatrix} 0_{p\times n} & I_p \end{pmatrix}\right) \leq r$$

The solution follows as an immediate consequence of Theorem 10.

Corollary 3 *The solution of the constrained total linear least squares probkm follows from the application of Theorem 10 to the matrix triplet A', B', C' where*

$$A' = \begin{pmatrix} A & B \\ C & D^* \end{pmatrix}, \quad B' = \begin{pmatrix} 0_{m\times q} \\ I_q \end{pmatrix} \quad and \quad C' = \begin{pmatrix} 0_{p\times n} & I_p \end{pmatrix}$$

Hence, all what is needed is the RSVD of the matrix triplet (A', B', C') and the truncation of the matrix S_1 as described in Theorem 10. It is interesting to apply also Theorem 7 to the matrix triplet (A', B', C'):

$$\begin{aligned} r_{a'b'} &= rank\begin{pmatrix} A & B & 0 \\ C & D^* & I_q \end{pmatrix} = r_{ab} + q \\ r_{a'c'} &= rank\begin{pmatrix} A & B \\ C & D^* \\ 0 & I_p \end{pmatrix} = r_{ac} + p \\ r_{a'b'c'} &= rank\begin{pmatrix} A & B & 0 \\ C & D^* & I_q \\ 0 & I_p & 0 \end{pmatrix} = r_a + p + q \end{aligned}$$

Hence, from Theorem 7, the minimum achievable rank is:

$$r_{a'b'} + r_{a'c'} - r_{a'b'c'} = r_{ab} + r_{ac} - r_a$$

which corresponds precisely to the result from Corollary 2.

As a special&e, consider. the Golub-Hoffman-Stewart result [14] for the total linear least squares solution of

$$(A \;\; B)x \approx 0$$

where A is noise free and B is contaminated with errors. Instead of applying the QR-SVD-Least Squares solution as discussed in [14], one could as well achieve the mixed linear / total linear least squares solution from:

Minimize $\|\tilde{B}\|$ *such that*

$$rank((A\ B) - \tilde{B}(0_{p\times n}\ I_p)) \leq r$$

where r is a prespecified integer. This can be done directly via the QSVD of the matrix pair $((A\ B), (0_{p\times n}\ I_p))$ and it is not too difficult to provide another proof of the Golub-Hoffman-Stewart result derived in [14], now in terms of the properties of the QSVD.
As a matter of fact, the RSVD of the matrix triplet of Corollary 3, allows us to provide a geometrical proof of constrained total linear least squares, in the line of the Golub-Hoffman-Stewart result, taking into account the structure of the matrices B' and C'. We shall however not consider this any further in this paper.

3.2.3.2. Solution via **RSVD** - OSVD

While the solution to the constrained total least squares problem as presented in Corollary 3 is extremely simple, one might object it because of the apparent 'overkill' in computing the RSVD of the matrix triplet (A', B', C') where B' and C' have an extremely simple structure (zeros and the identity matrix).
It will now be shown that the RSVD, combined with the OSVD may lead to a computationally simpler solution, which more closely follows the lines of the solution as presented in [6].

Using the RSVD, we find that:

$$\begin{pmatrix} A & B \\ C & D^* \end{pmatrix} = \begin{pmatrix} P^{-*} & 0 \\ 0 & U_c \end{pmatrix} \begin{pmatrix} S_a & S_b \\ S_c & U_c^* D^* V_b \end{pmatrix} \begin{pmatrix} Q^{-1} & 0 \\ 0 & V_b^* \end{pmatrix}$$

Let $E^* = U_c^* D^* V_b$. Since U_c and V_b are unitary matrices, the problem can be restated as follows:
Find $\hat{E}$ *such that* $\|E - \hat{E}\|_F$ *is minimal and:*

$$rank\begin{pmatrix} S_a & S_b \\ S_c & \hat{E}^* \end{pmatrix} \leq r \quad .$$

The constrained total least squares problem can now be solved as follows.

Theorem 1b
RSVD-OSVD *solution of constrained total least squares.*

- *Consider the* OSVD:

$$\begin{pmatrix} E_{11} - S_1^{-1} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} = \sum_{i=1}^{r_e} u_i^e \sigma_i^e (v_i^e)^*$$

  where r_e *is the mnk of this matrix.*

- *The modification of minimal Frobenius norm follows immediately from the* **OSVD** *of this matrix by truncating its dyadic decomposition after* $r - r_{ab} - r_{ac} + r_a$ *terms. Let*

$$\hat{E} = \sum_{i=1}^{r-r_{ab}-r_{ac}+r_a} u_i^e \sigma_i^e (v_i^e)^* .$$

  Then the optimal $\hat{D}$ *is given by*

$$\hat{D} = V_b \begin{pmatrix} \hat{E} & 0 \\ 0 & 0 \end{pmatrix} U_c^* .$$

Proof: **From** Theorem **13,** it follows that the rank of $\begin{pmatrix} A & B \\ C & D^* \end{pmatrix}$ can be reduced by reducing the rank of the matrix

$$\begin{pmatrix} E_{11} - S_1^{-1} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} .$$

The matrix $\tilde{D}$ is then obtained from (18) by setting the blocks $\tilde{E}_{14}$, $\tilde{E}_{24}$, $\tilde{E}_{34}$, $\tilde{E}_{41}$, $\tilde{E}_{42}$, $\tilde{E}_{43}$, $\tilde{E}_{43}$ to 0 in order to minimize the Frobenius norm and then truncating the OSVD of the matrix above. □

We conclude this section by pointing out that more results and also algorithms to solve total least squares problems with and without constraints and given covariance matrices, can be found in [6][25][26].

3.3 Generalized Gauss-Markov models with constraints.

Consider the problem of finding x, y and z while minimizing $\|y\|^2 + \|z\|^2 = y^*y + z^*z$ *in:*

$$\begin{aligned} b &= Ax + By \\ z &= Cx \end{aligned}$$

where A, B, C, b are given.

This formulation is a generalization of the conventional least squares problem where $B = I_m$ and $C = 0$. The above formulation is more general because it allows us for singular or ill-conditioned matrices B and C, corresponding to singular or ill-conditioned covariance matrices in a statistical formulation of this generalized Gauss-Markov model. The problem formulation as presented here could be considered as a 'square root' version of the problem:

Find x such that:

$$\|b - Ax\|_{W_b} \quad \text{and} \quad \|x\|_{W_c}$$

are minimized, where $\|u\|_{W_b} = u^*W_bu$ *and* W_b *and* W_c *are nonnegative definite symmetric matrices.*

In case that BB^* is nonsingular, one can put $W_b = (BB^*)^{-1}$ and $W_c = C^*C$. The solution can then be obtained asfollows:

Minimize $\|y\|^2 + \|z\|^2$ *where:*

$$\begin{aligned} y^*y &= (b - Ax)^*W_b(b - Ax) \\ z^*z &= x^*C^*Cx \end{aligned}$$

Setting the derivative with respect to x equal to 0, results in

$$x = (A^*W_bA + C^*C)^{-1}A^*W_bb \quad .$$

In case that $W_b = I_m$ and $C = 0$, this is easily seen to be the classical least squares expression. However, for this more general case, one can see a connection with so-called regularization problems. Consider the case $C \neq 0$ and $B = I_m$. If the matrix A is ill-conditioned (because of so-called collinearities, which are (almost) linear dependencies among the columns of A), the addition of the term C^*C may possibly make the sum better suited for numerical inversion than the original product A^*A, hence stabilizing the solution x.

The matrix B acts as a 'static' **noise filter:** Typically, it is assumed that the vector y is normally distributed with the covariance matrix $E(yy^*)$ being a multiple of the identity. The error vector By for the first equation can only be in a direction which is present in the column space of B. If the observation vector b has some component in a certain direction not present in the column space of B, this component should be considered as errorfree. The matrix C represents a weighting on the components of x. It reflects possible a priori information concerning the unknown components of x or may reflect the fact that certain components of x (or linear combinations thereof) are more 'likely' or less costly than others. The fact that one tries to minimize $y^*y + z^*z$ reflects the intention that one tries to explain as much as possible (i.e. min y^*y) in terms of the data (columns of the matrix A), taking into account a priori knowledge of the geometrical distribution of the noise (the weighting W_b). The matrix C reflects the cost per component, expressing the preference (or prejudice?) of the **modeller** to use more of one variable in explaining the phenomenon than of another.
In applications, however, typically, the matrix A contains much more rows than columns, which corresponds to the fact that better results are to be expected if there are more equations (measurements) than unknowns. However, the condition that BB^* is nonsingular requires quite some a priori knowledge concerning the statistics of the noise. Because typically this knowledge is rather limited, B will have less columns than rows, implying that BB^* is singular such that the explicit solution of (3.3) does not hold.

In this case, the **RSVD** can be applied in order to convert the problem to an easier one, while at the same time providing important geometrical insight and results on the sensitivity. Using the RSVD, the problem can be rewritten as:

$$\begin{aligned}(P^*b) &= S_a(Q^{-1}x) + S_b(V_b^*y)\\ (U_c^*z) &= S_c(Q^{-1}x)\end{aligned}$$

Define $b' = P^*b, x' = Q^{-1}x, y' = V_b^*y, z' = U_c^*z$ then with obvious partitionings of b', x', y', z' it follows that:

$$b'_1 = S_1x'_1 + y'_1$$

$$\begin{aligned} b'_2 &= x'_2 \\ b'_3 &= x'_3 + y'_2 \\ b'_4 &= x'_4 \\ b'_5 &= S_2 y'_4 \\ b'_6 &= 0 \end{aligned}$$

and

$$\begin{aligned} z'_1 &= x'_1 \\ z'_2 &= x'_2 \\ z'_3 &= 0 \\ z'_4 &= S_3 x'_5 \ . \end{aligned}$$

Observe that $b'_6 = 0$ is a ***consistency*** condition. It reflects the fact that b is not allowed to have a component in a direction that is not present in the column space of $(A\ B)$. x'_2 and x'_4 can be estimated without error while the fact that $b'_5 = S_2 y'_4$ could be exploited to estimate the variance of the noise.

Most terms in the object function $y^*y + z^*z$ can now be expressed with the subvectors x'_i, $(i = 1, \ldots, 6)$,

$$\begin{aligned} y^*y + z^*z = \ & b'^*_1 b'_1 + x'^*_1 S_1^2 x'_1 - 2b'^*_1 S_1 x'_1 + b'^*_3 b'_3 + x'^*_3 x'_3 - 2b'^*_3 x'_3 \\ & + y'^*_3 y'_3 + b'^*_5 S_2^{-2} b'_5 + x'^*_1 x'_1 + x'^*_5 S_3^2 x'_5 + b'^*_2 b'_2 \end{aligned}$$

The minimum solution follows from **differentation** with respect to these vectors and results in

$$\begin{aligned} x'_1 &= (I + S_1^2)^{-1} S_1 b'_1 & y'_1 &= (I + S_1^2)^{-1} b'_1 & z'_1 &= (I + S_1^2)^{-1} S_1 b'_1 \\ x'_2 &= b'_2 & y'_2 &= 0 & z'_2 &= b'_2 \\ x'_3 &= b'_3 & y'_3 &= 0 & z'_3 &= 0 \\ x'_4 &= b'_4 & y'_4 &= S_2^{-1} b'_5 & z'_4 &= 0 \\ x'_5 &= 0 \\ x'_6 &= \text{arbitrary} \end{aligned}$$

Statistical properties, such as **(un)biasedness** and consistency, can be analysed in the same spirit as in **[21],** where Paige has related the Gauss Markov model without the x-equation, to the QSVD. Similarly, the RSVD also allows **us** to analyse the sensitivity of the solution. If for instance S_2 is **ill**-conditioned, then the minimum of the object function will tend to be high, whenever b_5' has strong components among the 'weak' singular vectors of S_2, because of the term $b_5'^{*} S_2^{-2} b_5'$.

A related problem is the following:

Minimize $y^* y$ in

$$b = A\,x + B\,y$$

subject to

$$c\,x = c \quad .$$

This is a Gauss-Markov linear estimation problem as in [21], but with constraints. The solution is again straightforward from the RSVD. With $b' = P^* b$, $x' = Q^{-1} x$, $y' = V_b^* y$, $c' = U_c^* c$ and an appropriate partitioning, one finds

$$\begin{array}{ll}
x_1' = c_1' & y_1' = b_1' - S_1 c_1' \\
x_2' = c_2' = b_2' & y_2' = o \\
x_3' = b_3' & y_3' = 0 \\
x_4' = b_4' & y_4' = S_2^{-1} b_5' \\
x_5' = S_3^{-1} c_4' & \\
x_6' = arbitrary &
\end{array}$$

Observe that $c_2' = b_2'$ represents a ***consistency*** condition.

4 Conclusions and perspectives.

In this paper, we have derived a generalization of the OSVD, the ***restricted singular value decomposition*** (RSVD), which has the OSVD, PSVD and QSVD as special cases. Besides a constructive proof, we have also analysed in detail its structural and geometrical properties and its relations to generalized eigenvalue problems and canonical correlation analysis.
It was shown how it is a valuable tool in the analysis and solution of rank minimization problems with restrictions. First, we have shown how to study expressions of the form $A + BDC$ and find matrices D of minimum norm that minimize the rank. It was demonstrated how this problem is connected to the concept of shorted operators and matrix balls. Second, we have analysed in detail the rank reduction of a partitioned matrix, when only one of its blocks can be modified. The close relation with generalized Schur

complements was discussed and it was shown how the RSVD allows us to solve constrained total linear least squares problems with mixed exact and noisy data. Third, it was demonstrated how the RSVD provides an elegant solution to Gauss-Markov models with constraints and can be used to study and compute canonical correlations.

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Appendix A: Two constructive proofs of the RSVD.

The analysis and the constructive proofs of the RSVD will be performed using the $(m+q) \times (n+p)$ matrix T:

$$T = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} . \tag{20}$$

With the notation of section 1, we have:

$$rank(T) = r_{abc}$$

Obviously, from the RSVD theorem, it follows that:

$$\begin{pmatrix} P^* & 0 \\ 0 & U_c^* \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & V_b \end{pmatrix} = \begin{pmatrix} S_a & S_b \\ S_c & 0 \end{pmatrix}$$

Therefore, we shall derive expressions for P, Q, U_c and V_b via a factorization approach, in which the matrix T will be transformed into matrices $T^{(k)}$ via a recursive procedure of the form:

$$\begin{aligned} T^{(k+1)} &= \begin{pmatrix} (P^{(k)})^* & 0 \\ 0 & (U_c^{(k)})^* \end{pmatrix} T^{(k)} \begin{pmatrix} (Q^{(k)}) & 0 \\ 0 & V_b^{(k)} \end{pmatrix} \\ &= \begin{pmatrix} A^{(k)} & B^{(k)} \\ C^{(k)} & 0 \end{pmatrix} \end{aligned}$$

with $T^{(0)} = T$. In each step, the matrices $P^{(k)}, Q^{(k)}$ are square non-singular while $U_c^{(k)}, V_b^{(k)}$ are unitary. Hence the important observation that:

Lemma 5
Rank preservation
For all k:

- $rank(T^{(k)}) = rank(T) = r_{abc}$
- $rank(A^{(k)}) = rank(A) = r_a$
- $rank(B^{(k)}) = rank(B) = r_b$
- $rank(C^{(k)}) = rank(C) = r_c$

At each recursion, we get closer to the required canonical structure from Theorem 4. The final matrices P, Q, U_c, V_b are then simply obtained by multiplication of the matrices $P^{(i)}, Q^{(i)}, U^{(i)}, V^{(i)}$.

We will now present 2 constructive approaches. The first one is based upon the properties of OSVD and the PSVD and the second one is based upon the properties of the OSVD and the QSVD.

Constructive proof 1: OSVD and PSVD

The construction proceeds in **4** steps:

1. First the data in the matrix T are compressed via three OSVDs.
2. Then the **Schur** complement Lemma **4** is invoked to eleminate some matrices.
3. A **PSVD** is performed which delivers at once the structure as in Theorem 4.
4. The last step is a simple scaling and reordening.

Compared to the second constructive proof based on the **QSVD**, the proof with the **PSVD** is algebraically more elegant.

Step 1: An orthogonal reduction

The first step consists of an orthogonal reduction, based upon three **OSVDs.** The idea can be found in [6] though a similar reduction can also be found in [3].

Lemma 6
There exist unitary matrices $P^{(1)}, U^{(1)}, Q^{(1)}, V^{(1)}$ *such that:*

$$\begin{pmatrix} (P^{(1)})^* & 0 \\ 0 & (U^{(1)})^* \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} (Q^{(1)}) & \\ 0 & A \end{pmatrix}$$

$$
\begin{array}{r} \\ r_a \\ r_{ab}-r_a \\ = m-r_{ab} \\ r_{ac}-r_a \\ q+r_a-r_{ac} \end{array}
\begin{array}{c} \begin{array}{ccccc} r_a & r_{ac}-r_a & n-r_{ac} & r_{ab}-r_a & p+r_a-r_{ab} \end{array} \\ \begin{pmatrix} A_{11}^{(1)} & 0 & 0 & B_{11}^{(1)} & B_{12}^{(1)} \\ 0 & 0 & 0 & B_{21}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_{11}^{(1)} & C_{12}^{(1)} & 0 & 0 & 0 \\ C_{21}^{(1)} & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}
$$

where each of $A_{11}^{(1)}, B_{21}^{(1)}, C_{12}^{(1)}$ is either square and non-singular or null. (If one is null, delete the corresponding rows and *columns.)*

Proof: The proof consists of a straightforward sequence of *3* OSVDs. From the OSVD of A it follows that there exists unitary matrices U_{a1} and V_{a1} such that

$$U_{a1}^* A V_{a1} = \begin{pmatrix} A_{11}^{(1)} & 0 \\ 0 & 0 \end{pmatrix}$$

where $A_{11}^{(1)}$ is square nonsingular diagonal containing the non-zero singular **values** of A. With $r(A_{11}^{(1)}) = r_a$, one **finds** that

$$\begin{pmatrix} U_{a1}^* & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} V_{a1} & 0 \\ 0 & I_p \end{pmatrix} = \begin{pmatrix} A_{11}^{(1)} & 0 & B_1^{(1)} \\ 0 & 0 & B_2^{(1)} \\ C_1^{(1)} & C_2^{(1)} & 0 \end{pmatrix}$$

From the OSVDs of $B_2^{(1)}$ and $C_2^{(1)}$, obtain $U_{b1}, V_{b1}, U_{c1}, V_{c1}$ such that

$$U_{b1}^* B_2^{(1)} V_{b1} = \begin{pmatrix} B_{21}^{(1)} & 0 \\ 0 & 0 \end{pmatrix} \qquad U_{c1}^* C_2^{(1)} V_{c1} = \begin{pmatrix} C_{12}^{(1)} & 0 \\ 0 & 0 \end{pmatrix}$$

where $B_{21}^{(1)}$ and $C_{12}^{(1)}$ are square nonsingular containing the non-zero singular values of $B_2^{(1)}$ and $C_2^{(1)}$. Then

$$\begin{pmatrix} I_{r_a} & 0 & 0 \\ 0 & U_{b1}^* & 0 \\ 0 & 0 & U_{c1}^* \end{pmatrix} \begin{pmatrix} A_{11}^{(1)} & 0 & B_1^{(1)} \\ 0 & 0 & B_2^{(1)} \\ C_1^{(1)} & C_2^{(1)} & \end{pmatrix} \begin{pmatrix} I_{r_a} & 0 & 0 \\ 0 & V_{c1} & 0 \\ 0 & 0 & V_{b1} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11} & 0 & 0 & B_{11}^{(1)} & B_{12}^{(1)} \\ 0 & 0 & 0 & B_{21}^{(1)} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_{11}^{(1)} & C_{12}^{(1)} & 0 & 0 & 0 \\ C_{21}^{(1)} & 0 & 0 & 0 & 0 \end{pmatrix}$$

with obvious definitions for $B_{11}^{(1)}, B_{12}^{(1)}, C_{11}^{(1)}, C_{21}^{(1)}$. It is straightforward to show using Lemma 1, that

$$rank(B_{21}^{(1)}) = r_{ab} - r_a \tag{21}$$

$$rank(C_{12}^{(1)}) = r_{ac} - r_a \, . \tag{22}$$

Also it follows that

$$rank(B_{12}^{(1)}) = r_a + r_b - r_{ab} \tag{23}$$

$$rank(C_{21}^{(1)}) = r_a + r_c - r_{ab} \tag{24}$$

because obviously

$$rank(B) = r_b = rank(B_{12}^{(1)}) + rank(B_{21}^{(1)})$$

$$rank(C) = r_c = rank(C_{12}^{(1)}) + rank(\&)) \, .$$

Then letting

$$(P^{(1)})^* = \begin{pmatrix} I_{r_a} & 0 \\ 0 & U_{b1}^* \end{pmatrix} U_{a1}^* \quad (U^{(2)})^* = U_{c1}^* \, ,$$

$$(Q^{(1)}) = V_{a1} \begin{pmatrix} I_{r_a} & 0 \\ 0 & V_{c1} \end{pmatrix} \, , \quad V^{(2)} = V_{b1}$$

proves the Lemma. *0*

The matrix $T^{(1)}$ takes the form

$$T^{(1)} =$$

$$\begin{array}{c|ccccc|} & r_a & r_{ac} - r_a & n - r_{ac} & r_{ab} - r_a & p - r_{ab} \; t \; r_a \\ \hline r_a & 11 & 0 & 0 & & \\ r_{ab} - r_a & 0 & 0 & 0 & B_{21}^{(1)} & 0 \\ m - r_{ab} & 0 & 0 & 0 & 0 & 0 \\ r_{ac} - r_a & C_{11}^{(1)} & C_{12}^{(1)} & 0 & 0 & 0 \\ q - r_{ac} + r_a & C_{21}^{(1)} & 0 & 0 & B_{11}^{(1)} & B_{12}^{(1)} \end{array}$$

•

Step 2: Elimation of $C_{11}^{(1)}$ and $B_{11}^{(1)}$

Recall that the matrices $B_{21}^{(1)}$ and $C_{12}^{(1)}$ are square nonsingular diagonal. Because of the nonsingularity of $C_{12}^{(1)}$, the matrix $C_{11}^{(1)}$ can be eliminated by a non-singular transformation using Lemma 3, as follows

$$\begin{pmatrix} C_{11}^{(1)} & C_{12}^{(1)} \\ C_{21}^{(1)} & 0 \end{pmatrix} \begin{pmatrix} I_{r_a} & 0 \\ -(C_{12}^{(1)})^{-1} C_{11}^{(1)} & I_{r_{ac}-r_a} \end{pmatrix} = \begin{pmatrix} 0 & C_{12}^{(1)} \\ C_{21}^{(1)} & 0 \end{pmatrix}$$

Similarly for the matrix $B_{11}^{(1)}$ from the nonsingularity of $B_{21}^{(1)}$

$$\begin{pmatrix} I_{r_a} & -B_{11}^{(1)}(B_{21}^{(1)})^{-1} \\ 0 & I_{r_{ab}-r_a} \end{pmatrix} \begin{pmatrix} B_{11}^{(1)} & B_{12}^{(1)} \\ B_{21}^{(1)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_{12}^{(1)} \\ B_{21}^{(1)} & 0 \end{pmatrix} .$$

Define $P^{(2)}$ as

$$(P^{(2)})^* = \begin{pmatrix} I_{r_a} & -B_{11}^{(1)}(B_{21}^{(1)})^{-1} & 0 \\ 0 & I_{r_{ab}-r_a} & 0 \\ 0 & 0 & I_{m-r_{ab}} \end{pmatrix}$$

and $Q^{(2)}$ as

$$(Q^{(2)}) = \begin{pmatrix} I_{r_a} & 0 & 0 \\ -(C_{12}^{(1)})^{-1} C_{11}^{(1)} & I_{r_{ac}-r_a} & 0 \\ 0 & 0 & I_{n-r_{ac}} \end{pmatrix} .$$

At the same time we will permute the block rows and columns with

$$(U^{(2)})^* = \begin{pmatrix} 0 & I_{q-r_{ac}+r_a} \\ I_{r_{ac}-r_a} & 0 \end{pmatrix}$$

and

$$(V^{(2)}) = \begin{pmatrix} 0 & I_{r_{ab}-r_a} \\ I_{p-r_{ab}+r_a} & 0 \end{pmatrix}$$

The resulting matrix $T^{(2)}$ is then given by

$$T^{(2)} = \tag{25}$$

$$
\begin{array}{c}
 \\ r_{ab} - r_a \\ m - r_{ab} \\ q - r_{ac} + r_a \\ r_{ac} - r_a
\end{array}
\begin{array}{c}
\begin{array}{ccccc} r_a & r_{ac} - r_a & n - r_{ac} & p - r_{ab} + r_a & r_{ab} - r_a \end{array} \\
\left(\begin{array}{ccccc}
0 & 0 & 0 & 0 & B_{21}^{(1)} \\
0 & 0 & 0 & 0 & 0 \\
C_{21}^{(1)} & 0 & 0 & 0 & 0 \\
A_{11}^{(1)} & C_{12}^{(1)} & 0 & B_{12}^{(1)} & 0
\end{array} \right)
\end{array} .
$$

Let's now first determine the rank of the matrix $T^{(2)}$.

$$
\begin{aligned}
rank(T^{(2)}) &= r_{abc} \\
&= rank(T) \\
&= r \begin{pmatrix}
A_{11}^{(1)} & 0 & 0 & B_{12}^{(1)} & 0 \\
0 & 0 & 0 & 0 & B_{21}^{(1)} \\
0 & 0 & 0 & 0 & 0 \\
C_{21}^{(1)} & 0 & 0 & 0 & 0 \\
0 & C_{12}^{(1)} & 0 & 0 & 0
\end{pmatrix} \\
&= rank \begin{pmatrix} A_{11}^{(1)} & B_{12}^{(1)} \\ C_{21}^{(1)} & 0 \end{pmatrix} + r(C_{12}^{(1)}) + r(B_{21}^{(1)}) \\
&= r(A_{11}^{(1)}) + r(-C_{21}^{(1)}(A_{11}^{(1)})^{-1}B_{12}^{(1)}) + r(C_{12}^{(1)}) + r(B_{21}^{(1)}) \\
&= r_a + (r_{ab} - r_a) + (r_{ac} - r_a) + r(-C_{21}^{(1)}(A_{11}^{(1)})^{-1}B_{12}^{(1)}) .
\end{aligned}
$$

The second step follows **from** the non-singularity of $C_{12}^{(1)}$ and $B_{21}^{(1)}$ while step **3** follows from the **Schur** complement argument (see Lemma **4** in section *3.2.2.)* and the **nonsingularity** of $A_{11}^{(1)}$.
Hence:

$$
r_{abc} = rank(T) = rank(T^{(2)}) = r_{ab} + r_{ac} - r_a + rank(C_{21}^{(1)}(A_{11}^{(1)})^{-1}B_{12}^{(1)}) \quad (26)
$$

Step 3: The PSVD step

Let's **first** concentrate on the submatrix

$$
\begin{pmatrix} A_{11}^{(1)} & B_{12}^{(1)} \\ C_{21}^{(1)} & 0 \end{pmatrix} .
$$

Recall that A_{11} is square nonsingular diagonal, containing the **nonzero** singular values of A. Consider the PSVD of the **matrix pair**

$(C_{21}^{(1)}(A_{11}^{(1)})^{-1/2}, (B_{12}^{(1)})^*(A_{11}^{(1)})^{-1/2})$:

$$\begin{aligned} C_{21}^{(1)}(A_{11}^{(1)})^{-1/2} &= U_{c3}S_{c3}X_3^* \\ (B_{12}^{(1)})^*(A_{11}^{(1)})^{-1/2} &= V_{b3}S_{b3}X_3^{-1} \ . \end{aligned}$$

Define r_{a4} as

$$r_{a4} = rank(C_{21}^{(1)}(A_{11}^{(1)})^{-1}B_{12}^{(1)}) \ .$$

Then, from (26) we find immediately that

$$r_{a4} = r_{abc} + r_a - r_{ab} - r_{ac} \ . \tag{27}$$

and from the PSVD (Theorem 2 in section 1), it follows that the matrices S_{c3} and S_{b3} have the following structure:

$$S_{c3} =$$

$$\begin{array}{l} \\ r_{abc}+r_a-r_{ab}-r_{ac} \\ r_{ab}+r_c-r_{abc} \\ q-r_c \end{array} \begin{array}{c} \begin{array}{cccc} r_{abc}+r_a-r_{ab}-r_{ac} & r_{ab}+r_c-r_{abc} & r_{ac}+r_b-r_{abc} & r_{abc}-r_b-r_c \end{array} \\ \left(\begin{array}{cccc} S_{a4}^{1/2} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

$$S_{b3} =$$

$$\begin{array}{l} \\ r_{abc}+r_a-r_{ab}-r_{ac} \\ r_{ac}+r_b-r_{abc} \\ p-r_b \end{array} \begin{array}{c} \begin{array}{cccc} r_{abc}+r_a-r_{ab}-r_{ac} & r_{ab}+r_c-r_{abc} & r_{ac}+r_b-r_{abc} & r_{abc}-r_b-r_c \end{array} \\ \left(\begin{array}{cccc} S_{\theta 4}^{1/2} & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

Now, use the PSVD to **define:**

$$(P^{(3)})^* = \begin{pmatrix} X_3^*(A_{11}^{(1)})^{-1/2} & 0 & 0 \\ 0 & I_{r_{ab}-r_a} & 0 \\ 0 & 0 & I_{m-r_{ab}} \end{pmatrix}$$

and

$$(Q^{(3)}) = \begin{pmatrix} & 0 & \\ (A_{11}^{(1)})^{-1/2}X_3^{-*} & I_{r_{ac}-r_a} & 0 \\ 0 & 0 & I_{n-r_{ac}} \end{pmatrix}$$

and

$$(U^{(3)})^* = \begin{pmatrix} U_{c3}^* & 0 \\ 0 & I_{r_{ac}-r_a} \end{pmatrix}$$

$$(V^{(3)}) = \begin{pmatrix} V_{b3} & 0 \\ 0 & I_{r_{ab}-r_a} \end{pmatrix}.$$

It is straightforward to show that

$$T^{(3)} = \begin{pmatrix} I_{r_a} & 0 & 0 & S_{b3}^t & 0 \\ 0 & 0 & 0 & 0 & B_{21}^{(1)} \\ 0 & 0 & 0 & 0 & 0 \\ S_{c3} & 0 & 0 & 0 & 0 \\ 0 & C_{12}^{(1)} & 0 & 0 & 0 \end{pmatrix}.$$

Inserting the structure of S_{b3} and S_{c3} results in the following structure for the matrix $T^{(3)}$

$$T^{(3)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 8 \\ 10 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 6 & 6 & 7 & 8 & 8 & 10 \end{array} \\ \left(\begin{array}{cccccccccc} I & 0 & 0 & 0 & 0 & 0 & S_{a4}^{1/2} & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & I & O & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{21}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ S_{a4}^{1/2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{12}^{(1)} & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}.$$

The block dimensions of $T^{(3)}$ are the following

	block rows	block columns
1	$r_{abc} + r_a - r_{ab} - r_{ac}$	$r_{abc} + r_a - r_{ab} - r_{ac}$
2	$r_{ab} + r_c - r_{abc}$	$r_{ab} + r_c - r_{abc}$
3	$r_{ac} + r_b - r_{abc}$	$r_{ac} + r_b - r_{abc}$
4	$r_{abc} - r_b - r_c$	$r_{abc} - r_b - r_c$
5	$r_{ab} - r_a$	$r_{ac} - r_a$
6	$m - r_{ab}$	$n - r_{ac}$
7	$r_{abc} + r_a - r_{ab} - r_{ac}$	$r_{abc} + r_a - r_{ab} - r_{ac}$
8	$r_{ab} + r_c - r_{abc}$	$r_{ac} + r_b - r_{abc}$
9	$q - r_c$	$p - r_b$
10	$r_{ac} - r_a$	$r_{ab} - r_a$

Step 4: Scaling and Permutation

The final scaling step in order to find the canonical structure of Theorem 4, is easily derived from the following observation

$$\begin{pmatrix} S_{a4}^{-1/2} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & S_{a4}^{1/2} \\ S_{a4}^{1/2} & 0 \end{pmatrix}\begin{pmatrix} S_{a4}^{-1/2} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} S_{a4}^{-1} & I \\ I & 0 \end{pmatrix}$$

Moreover, we shall do a permutation of block rows 3 and 4 and block columns 3 and 4. Hence the matrices $P^{(4)}$ and $Q^{(4)}$ are determined by:

$$(P^{(4)})^* = \begin{pmatrix} S_{a4}^{-1/2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

$$(Q^{(4)}) = \begin{pmatrix} S_{a4}^{-1/2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

where the block dimensions of the identity matrices are obvious from the block dimensions in $T^{(3)}$. It is now easily found that:

$$T^{(4)} = \begin{pmatrix} S_a & S_b \\ S_c & 0 \end{pmatrix}$$

which proves the Theorem. □

Constructive proof 2: the OSVD and the QSVD

Instead of **using** the structure and properties of the **PSVD**, it is feasible to derive a constructive proof of the RSVD using the structure and properties of the **QSVD**. The idea is borrowed from [28]. The resulting proof is a little less elegant than the one via the **PSVD** and consists of 7 steps:

1. First, an orthogonal reduction based upon 3 **OSVDs** is performed.
2. A Schur complement elimination is the second step.
3. Then a **QSVD** is required of a certain matrix pair . . .
4. . . . followed by a second **QSVD.**
5. Some blocks can again be eliminated by a Schur complement factorization.
6. An additional OSVD is required.
7. Finally, there is a diagonal scaling.

Step 1: Orthogonal reduction

The first step is nothing else than the orthogonal reduction described in Lemma 7.

Step 2: Elimination of C_{11} and B_{11}

The second step corresponds to step **2** described in the first constructive proof, resulting in the matrix $T^{(2)}$.

Step 3: QSVD of the pair $(C_{21}^{(1)}, A_{11}^{(1)})$

Consider the matrix $T^{(2)}$ and let the **QSVD** of the matrix pair $(C_{21}^{(1)}, AI',))$ be given as

$$\begin{aligned} C_{21}^{(1)} &= U_{c2} \begin{pmatrix} C_{c2} & 0 \\ 0 & 0 \end{pmatrix} X_2^{-1} \\ A_{11}^{(1)} &= U_{a2} \begin{pmatrix} S_{a2} & 0 \\ 0 & I_{r_{ac}-r_c} \end{pmatrix} X_2^{-1} . \end{aligned}$$

Matrices S_{a2} and C_{c2} are $(r_a + r_c - r_{ac}) \times (r_a + r_c - r_{ac})$ square nonsingular **diagonal** matrices with positive diagonal elements, satisfying

$$S_{a2}^2 + C_{c2}^2 = I_{r_a+r_c-r_{ac}} .$$

Observe that there are no zero elements in the diagonal matrix of the decomposition for $A_{11}^{(1)}$ because $A_{11}^{(1)}$ is square nonsingular. Now, define

$$(P^{(3)})^* = \begin{pmatrix} U_{a2}^* & 0 & 0 \\ 0 & I_{r_{ab}-r_a} & 0 \\ 0 & 0 & I_{m-r_{ab}} \end{pmatrix} \quad ;$$

$$(Q^{(3)}) = \begin{pmatrix} X_2 & 0 & 0 \\ 0 & I_{r_{ac}-r_a} & 0 \\ 0 & 0 & I_{n-r_{ac}} \end{pmatrix} \begin{pmatrix} C_{c2}^{-1} & 0 & 0 & 0 \\ 0 & I_{r_{ac}-r_c} & 0 & 0 \\ 0 & 0 & I_{r_{ac}-r_a} & 0 \\ 0 & 0 & 0 & I_{n-r_{ac}} \end{pmatrix} \quad ;$$

$$U_3^* = \begin{pmatrix} U_{c2}^* & 0 \\ 0 & I_{r_{ac}-r_a} \end{pmatrix} \quad ; \quad v3 = I_q \ .$$

This results in a matrix $T^{(3)}$ as:

$$T^{(3)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} S_{a2}(C_{c2})^{-1} & 0 & 0 & 0 & B_{11}{}^{(3)} & 0 \\ 0 & I_{r_a-r_{c2}} & 0' & 0 & B_{21}{}^{(3)} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{21}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ I_{r_{c2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{12}^{(1)} & 0 & 0 & 0 \end{pmatrix} \end{array} .$$

Here we have that

$$\begin{pmatrix} B_{11}{}^{(3)} \\ B_{21}{}^{(3)} \end{pmatrix} = U_{a2}^* B_{12}^{(1)} \ .$$

The block dimensions of $T^{(3)}$ are

	block rows	block columns
1	$r_a + r_c - r_{ac}$	$r_a + r_c - r_{ac}$
2	$r_{ac} - r_c$	$r_{ac} - r_c$
3	$r_{ab} - r_a$	$r_{ac} - r_a$
4	$m - r_{ab}$	$n - r_{ac}$
5	$r_a + r_c - r_{ac}$	$p + r_a - r_{ab}$
6	$q - r_c$	$r_{ab} - r_a$
7	$r_{ac} - r_a$	

Step 4: QSVD of $({B_{11}}^{(3)}, S_{a2}C_{c2}^{-1})$

Let the QSVD of the matrix pair $({B_{11}}^{(3)}, S_{a2}C_{c2}^{-1})$ be given as

$$
\begin{aligned}
{B_{11}}^{(3)} &= X_3^{-*}\begin{pmatrix} C_{b3} & 0 \\ 0 & 0 \end{pmatrix} {U_{b3}}^* \quad ; \\
S_{a2}C_{c2}^{-1} &= X_3^{-*}\begin{pmatrix} S_{a3} & 0 \\ 0 & I_{r_a+r_c-r_{ac}-r_{b3}} \end{pmatrix} {U_{a3}}^*
\end{aligned}
$$

where S_{a3} and C_{b3} are $r_{b3} x\, r_{b3}$ diagonal matrices with positive diagonal elements, satisfying

$$S_{a3}^2 + C_{b3}^2 = I_{r_{b3}}$$

and

$$r_{b3} = rank({B_{11}}^{(3)}) .$$

An expression for r_{b3} in terms of $r_a, r_b, r_c, r_{ab}, r_{ac}, r_{abc}$ will now be determined. Choose:

$$
(P^{(4)})^* = \begin{pmatrix} X_3^* & 0 & 0 & 0 \\ 0 & I_{r_{ac}-r_c} & 0 & 0 \\ 0 & 0 & I_{r_{ab}-r_a} & 0 \\ 0 & 0 & 0 & I_{m-r_{ab}} \end{pmatrix} \quad ;
$$

$$
(Q^{(4)}) = \begin{pmatrix} 0 & I_{r_{ac}-r_c} & 0 & 0 \\ U_{a3} & 0 & I_{r_{ac}-r_a}\ 0 & I_{n-r_{ac}}\ 0 \end{pmatrix} \quad ;
$$

$$
(U^{(4)})^* = \begin{pmatrix} U_{a3}^* & 0 & 0 \\ 0 & I_{q-r_c} & 0 \\ 0 & 0 & I_{r_{ac}-r_a} \end{pmatrix} \quad ; \quad V_4 = \begin{pmatrix} U_{b3} & 0 \\ 0 & I_{r_{ab}-r_a} \end{pmatrix}
$$

Then we have that

$$
T^{(4)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\left(\begin{array}{cccccccc}
S_{a3} & 0 & 0 & 0 & 0 & C_{b3} & 0 & 0 \\
0 & I_{r_a+r_c-r_{ac}-r_{b3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{r_{ac}-r_c} & 0 & 0 & B_{31}^{(4)} & B_{32}^{(4)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{21}^{(1)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{r_{b3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{r_a+r_c-r_{ac}-r_{b3}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_{12}^{(1)} & 0 & 0 & 0 & 0
\end{array} \right) \end{array}
$$

where

$$
(B_{31}^{(4)} \;\; B_{32}^{(4)}) = B_{21}^{(3)} U_{b3} \quad .
$$

The rank of $T^{(4)}$ can now be determined as follows:

$$
\begin{aligned}
rank(T^{(4)}) &= rank(T) \\
&= r_{abc} \\
&= r_{ac} + r_{ab} - r_a + r_{b3} \, .
\end{aligned}
$$

The third line follows **from** the **Schur** complement rank property (Lemma 4 in section 3.2.2.). Hence

$$
r_{b3} = r_{abc} + r_a - r_{ab} - r_{ac} \quad . \tag{28}
$$

The block dimensions of the matrix $T^{(4)}$ are the following

	block rows	block columns
1	$r_{abc} + r_a - r_{ab} - r_{ac}$	$r_{abc} + r_a - r_{ab} - r_{ac}$
2	$r_{ab} + r_c - r_{abc}$	$r_{ab} + r_c - r_{abc}$
3	$r_{ac} - r_c$	$r_{ac} - r_c$
4	$r_{ab} - r_a$	$r_{ac} - r_a$
5	$m - r_{ab}$	$n - r_{ac}$
6	$r_{abc} + r_a - r_{ab} - r_{ac}$	$r_{abc} + r_a - r_{ab} - r_{ac}$
7	$r_{ab} + r_c - r_{abc}$	$p + r_{ac} - r_{abc}$
8	$q - r_c$	$r_{ab} - r_a$
a	$r_{ac} - r_a$	

Step **5:** Elimination of $B_{31}^{(4)}$

It is easy to verify that $B_{31}^{(4)}$ can be eliminated by choosing (we have omitted the subscripts of the identity matrices):

$$(P^{(5)})^* = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ -B_{31}^{(4)}C_{b3}^{-1} & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} ;$$

and

$$(Q^{(5)}) = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ B_{31}^{(4)} & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} ;$$

with

$$U_5 = I_p \;\; ; \;\; V_5 = I_q \; .$$

The result is:

$$T^{(5)} = \begin{pmatrix} S_{a3} & 0 & 0 & 0 & 0 & C_{c3} & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & B_{32}^{(4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{21}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{12}^{(1)} & 0 & 0 & 0 & 0 \end{pmatrix}$$

having the same block dimensions as the matrix $T^{(4)}$.

Step **6:** Elimination of C_{c3} and $B_{32}^{(4)}$

Consider the **OSVD** of $B_{32}^{(4)}$ **as:**

$$B_{32}^{(4)} = U_{b4}\begin{pmatrix} S_{b4} & 0 \\ 0 & 0 \end{pmatrix} V_{b4}^*$$

where S_{b4} is $r_{b4}\, x\, r_{b4}$ diagonal with positive diagonal elements and

$$r_{b4} = rank(B_{32}^{(4)}) \ .$$

We shall now derive an expression for r_{b4}. Hereto choose (the block dimensions follow from those of $T^{(4)}$)

$$(P^{(6)})^* = \begin{pmatrix} C_{c3}^{-1} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & U_{b4}^* & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \quad ;$$

$$(Q^{(6)}) = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & U_{b4} & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \quad ;$$

$$(U^{(6)})^* = I_q \quad ; \qquad V^{(6)} = \begin{pmatrix} I & 0 & 0 \\ 0 & V_{b4} & 0 \\ 0 & 0 & I \end{pmatrix} \quad .$$

The result **is** the **matrix** $T^{(6)}$:

$$T^{(6)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 8 \\ 10 \end{array} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ S_{a3}C_{c3}^{-1} & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & S_{b4} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{21}^{(1)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{12}^{(1)} & 0 & 0 & 0 & 0 & 0 \end{array}$$

Observe that **from** the block dimensions of $T^{(4)}$ it follows that

$$r_b = r_{abc} - r_{ac} + r_{b4} \quad .$$

Hence,

$$r_{b4} = r_{ac} + r_b - r_{abc} \quad .$$

Hence, the block dimensions of $T^{(6)}$ *axe*

	block rows	block columns
1	$r_{abc}+r_a-r_{ab}-r_{ac}$	$r_{abc}+r_a-r_{ab}-r_{ac}$
2	$r_{ab}+r_c-r_{abc}$	$r_{ab}+r_c-r_{abc}$
3	$r_{ac}+r_b-r_{abc}$	$r_{ac}+r_b-r_{abc}$
4	$r_{abc}-r_b-r_c$	$r_{abc}-r_b-r_c$
5	$r_{ab}-r_a$	$r_{ac}-r_a$
6	$m-r_{ab}$	$n-r_{ac}$
7	$r_{abc}+r_a-r_{ab}-r_{ac}$	$r_{abc}+r_a-r_{ab}-r_{ac}$
8	$r_{ab}+r_c-r_{abc}$	$r_{ac}+r_b-r_{abc}$
9	$q-r_c$	$p-r_b$
10	$r_{ac}-r_a$	$r_{ab}-r_a$

Step 7: Diagonal Scaling

Pre- and postmultiplication of $T^{(6)}$ with $(P^{(7)})^*$ and $(Q^{(7)})$ results in the desired diagonal forms where

$$(P^{(7)})^* = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & S_{b4}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

$$(Q^{(7)})^{-t} = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & S_{b4} & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} .$$

The block dimensions of the identity matrices are obvious from the block dimensions of $T^{(6)}$ and it is straightforward to verify that

$$T^{(7)} = \begin{pmatrix} S_a & S_b \\ S_c & 0 \end{pmatrix}$$

which proves the Theorem. □