PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 98, Number 4, December 1986

THE RESTRICTED TANGENT BUNDLE OF A RATIONAL CURVE ON A QUADRIC IN $\mathbf{P}^3$

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ABSTRACT. Let $\psi^* T_{\mathbf{P}^3}$ be the pull-back of the tangent bundle to $\mathbf{P}^3$ via a parametrization ψ of a rational, reduced, irreducible curve C in $\mathbf{P}^3$ contained in an irreducible quadric surface. Since C is rational, the bundle $\psi^* T_{\mathbf{P}^3}$ splits into the direct sum of three line bundles.

In this paper we study the relationship between the degrees of the line bundles of the splitting of $\psi^* T_{\mathbf{P}^3}$ and the geometry of the curve C.

0. Introduction. Let $T_{\mathbf{P}^r}$ be the tangent bundle to the r-dimensional projective space $\mathbf{P}^r$ over an algebraically closed field. Throughout this paper C will denote a rational, reduced, irreducible curve in $\mathbf{P}^r$ of degree d_C not contained in any hyperplane and $\psi\colon \mathbf{P}^1 \to \mathbf{P}^r$ will be a parametrization of C. We will drop the subscript C anytime this will not lead to confusion.

We consider the vector bundle $\psi^* T_{\mathbf{P}^r}$ on $\mathbf{P}^1$ and, by abuse of language, we will refer to it as the tangent bundle *restricted* to the *curve* C. By a well-known theorem of Grothendieck $\psi^* T_{\mathbf{P}^r}$ splits into the direct sum of r line bundles.

The aim of this paper is to determine the decomposition of the tangent bundle restricted to a singular curve lying on a quadric surface in $\mathbf{P}^3$.

The motivation for studying $\psi^* T_{\mathbf{P}^r}$ comes from the relationship (suggested in [**EV**]) between it and the geometry of the embedded curve, and because of the information about the normal bundle, the Hilbert function, and the regularity of the curve one can derive from the splitting of $\psi^* T_{\mathbf{P}^r}$.

We distinguish between the cases where the quadric surface Q is singular from the ones where it is not. If C lies on a quadric cone then we will denote by a the intersection number of C with the lines of the ruling on Q outside the vertex; if C lies on a smooth quadric then $(a, d-a)$ will be its divisor class. In both cases we can assume that $a \le \lfloor d/2 \rfloor$.

Our main results are the following

THEOREM (0.1). *If C lies on a quadric cone, it passes through its vertex and one of the following holds*:

(i) $a \le \lfloor d/3 \rfloor$,

(ii) $\lfloor d/3 \rfloor < a$ *and there exist two points on C of multiplicity a*,

(iii) $\lfloor d/3 \rfloor < a$ *and there exists a point on C of multiplicity a such that the total number of inflections at the point is a. Then*

$$\psi^* T_{\mathbf{P}^e} \cong \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(2d-2a).$$

Received by the editors November 4, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14H45.

This paper is part of the author's Ph.D. thesis prepared under the supervision of Dr. D. Eisenbud (Brandeis University, May 1985).

THEOREM (0.2). *If C is singular, it lies on a smooth quadric and if $a \leq \lfloor d/3 \rfloor$ then*

$$\psi^* T_{\mathbf{P}^3} \cong \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(2d-2a).$$

These results mean that $\psi^* T_{\mathbf{P}^3} \cong \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(2d-2a)$ as soon as C has "a big singularity" which is equivalent, in the case where Q is smooth, to saying that the divisor class of C is not very different from that of a smooth rational curve.

The philosophy behind those results seems to be that the nature and the distribution of the singularities of the curve determine the decomposition of the restricted tangent bundle. The results about the decomposition of the tangent bundle restricted to a plane curve which will be presented in a forthcoming paper will confirm this idea.

In §1 we describe the method used for the investigation of $\psi^* T_{\mathbf{P}^3}$ which can be used for studying $\psi^* T_{\mathbf{P}^r}$, $r \neq 3$, as well. Furthermore, we give an idea of the proofs of the main theorems and we discuss some minor results and open problems.

In §2 we explain how from the splitting of the restricted tangent bundle we can get information about the regularity of the curve; furthermore, we compute the regularity of the curves which satisfy the hypothesis of either Theorem (0.1) or Theorem (0.2).

1. The decomposition of $\psi^* T_{\mathbf{P}^3}$ can be written as follows:

$$\psi^* T_{\mathbf{P}} \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{P}^1}(d+d_i)$$

where $1 \leq d_1 \leq d_2 \leq \cdots \leq d_r$ and $\sum_{i=1}^{r} d_i = d$. Let k be the ground field. We denote the free $k[t,s]$-module whose generator has degree $(-l)$ by $S(l)$. The method used in this paper is based on the following results.

LEMMA (1.1). *The sheaf associated to the graded S-module whose elements are the first syzygies among the ψ_i's is isomorphic to*

$$(\psi^* T_{\mathbf{P}^r})^* \otimes_{\mathcal{O}_{\mathbf{P}^1}} \mathcal{O}_{\mathbf{P}^1}(d) \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbf{P}^1}(d_i).$$

PROOF. We consider the short exact sequence of vector bundles on $\mathbf{P}^1$ [**H**, II, Theorem (8.13)]

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1} \xrightarrow{\psi} \mathcal{O}_{\mathbf{P}^1}^{(r+1)}(d) \xrightarrow{\phi} \psi^* T_{\mathbf{P}^r} \longrightarrow 0. \tag{1.2}$$

We apply the functor $F(-) = \sum_\nu H^0(\mathbf{P}^1, - \otimes \mathcal{O}_{\mathbf{P}^1}(\nu))$ to the short exact sequence dual to (1.2). By Proposition (5.13) [**H**, Chapter II] we obtain the following exact sequence of graded S-modules

$$0 \longrightarrow \bigoplus_{i=1}^{r} S(-d-d_i) \xrightarrow{F(\phi)} S^{(r+1)}(-d) \xrightarrow{F(\psi)} S \tag{1.3}$$

where $F(\psi) = (\psi_0, \ldots, \psi_r)$. Therefore,

$$\operatorname{Ker} F(\psi) = \operatorname{Im} F(\phi) = \left(\bigoplus_{i=1}^{r} S(-d-d_i) \right)(d).$$

Since Ker $F(\psi)$ is the S-module whose elements are the first syzygies among the ψ_i's, the lemma follows.

LEMMA (1.4). *Let f_1, $i = 1, \dots, n$, $n \geq 2$, be polynomials of the same degree contained in the same polynomial ring A. Suppose there exist in A n polynomials $a_1, \dots, a_n$ not all zero, all of the same degree e such that $\sum_{i-1}^{n} a_i f_i \equiv 0$. If F is the subset of $f_1, \dots, f_n$ such that*

$$f_i \in F \Longleftrightarrow a_i \not\equiv 0$$

then

$$e \geq \max\{\deg \gcd(F \backslash \{f_i\})\} - \deg \gcd(F).$$

PROOF. One can assume that $\gcd(f_1, \dots, f_n) = 1$ and that $\operatorname{card}(F) = n$. Hence $\sum_{i=1}^{n} a_i f_i \equiv 0$ implies

$$\rho\left(\sum_{i=1}^{n} a_i f_i'\right) \equiv -a_n f_n$$

where $\rho = \gcd(f_1, \dots, f_{n-1})$, $f_i = \rho f_i'$. Since $\gcd(f_1, \dots, f_n) = 1$ it follows that f_n has to divide $\sum_{i=1}^{n-1} a_i f_i'$. Therefore, for all $i \in \{1, \dots, n-1\}$, we have

$$e \geq \deg(f_i') - \deg(f_n) = \deg(\rho).$$

We conclude that

$$e \geq \max_i\{\deg \gcd(F \backslash \{f_i\})\},$$

as desired.

Idea of the proof of Theorems (0.1) *and* (0.2). Because of Theorem (0.1), the idea is to study the first syzygies of the S-module S/I_C where I_C is the ideal generated by a suitable parametrization $\psi = [\psi_0, \psi_1, \psi_2, \psi_3]$ of the curve. In this analysis we make use of Lemma (1.4). We show which parametrizations for C turn out to be suitable.

In Theorem (0.1) the curve C lies on a quadric cone Q. It is not restrictive to assume that Q is given by the equation $X_0^2 = X_1 X_2$ where X_0, X_1, X_2, X_3 are homogeneous coordinates in $\mathbf{P}^3$. Therfore a parametrization for C is give by

$$\begin{cases} \psi_0 = \sigma\phi_1\phi_2, \\ \psi_1 = \sigma\phi_1^2, \\ \psi_2 = \sigma\phi_2^2, \\ \psi_3, \end{cases}$$

where $\sigma = \gcd(\psi_0, \psi_1, \psi_2)$, $\deg\sigma = \mu_\nu C$, $\deg(\sigma, \psi_3) = 1$. If $a \leq \lfloor d/3 \rfloor$ then we choose for C the previous parametrization. If $a > \lfloor d/3 \rfloor$, instead, it is necessary to specialize the previous parametrization further. We notice that, without loss of generality, any two given points $p_1, p_2 \in C$ not collinear with the vertex of the quadric cone can be supposed to have homogeneous coordinates $[0,0,1,0]$, $[0,1,0,0]$, respectively. Therefore, we can assume that

$$\gcd(\psi_i, \psi_3) \neq 1, \qquad \psi_3 = \psi_{13}\psi_{23}\psi_3, \qquad \psi_i = \psi_{i3}\psi_i,$$

where $\psi_{i3} = \gcd(\psi_i, \psi_3)$. It follows that

$$\deg \psi_{i3} = \mu_{p_i} C \quad \text{for } i = 1, 2.$$

In the cases where there exist two points of multilplicity equal to a, we take them as p_1 and p_2. In the cases where there exists a point of both multiplicity and total inflection equal to a we take it as p_1 and it turns out that it is not necessary to specify p_2.

In the proof of Theorem (0.2) it is not restrictive to assume that $X_0X_1 = X_2X_3$ is an equation for the smooth quadric Q on which the curve C lies. Let $(a, d-a)$ be the divisor class of C on Q. We choose for C the parametrization

$$\begin{cases} \psi_0 = \psi_{02}\psi_{03}, \\ \psi_1 = \psi_{13}\psi_{12}, \\ \psi_2 = \psi_{02}\psi_{12}, \\ \psi_3 = \psi_{13}\psi_{03}, \end{cases}$$

where we take

$$\begin{aligned} &\psi_{ij} = \gcd(\psi_i, \psi_j) \quad \text{for } i, j \in \{0,1,2,3\},\ i < j, \\ &\deg \psi_{02} = \deg \psi_{13} = a, \\ &\deg \gcd(\psi_{02}, \psi_{03}) = \deg \gcd(\psi_{03}, \psi_{12}) = 0, \\ &0 \leq \deg \gcd(\psi_0, \psi_1) \leq a \quad \text{for } = 2, 3. \end{aligned}$$

Among some minor results we can prove that for curves on a quadric surfacae whose multiplicities at singular points are smaller that a, the splitting of the restricted tangent bundle is different from the one obtained in Theorems (0.1) and (0.2). Furthermore, it is not true that for all triples of integers (d_1, d_2, d_3) with $d_i \geq 1$ and $\sum_{i=1}^{3} d_i = d$ there exists a rational curve C on a quadric surface whose restricted tangent bundle is isomorphic to $\bigoplus_{i=1}^{3} \mathcal{O}_{\mathbf{P}^1}(d+d_i)$. This follows from the fact that if one of the d_i's is equal to $(d-2a)$ then the other two have to be equal to a.

Furthermore, the condition

$$\psi^* T_{\mathbf{P}^3} \cong \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(2d-2a),$$

where $a = (d - \mu_p C)/2$ for some point $p \in C$ with $\mu_p C \geq \lfloor d/3 \rfloor$, is not sufficient to imply that C lies on a quadric surface. We prove this by producing the following counterexample. We consider the quintic curve C parametrized by the equations

$$\begin{cases} \psi_0 = (t-s)t^2 s(t-2s), \\ \psi_1 = (t-s)t^2(t+s)(t-2s), \\ \psi_2 = (t-s)(t+s)^4, \\ \psi_3 = t^2 s^3. \end{cases}$$

It turns out that

$$\psi^* T_{\mathbf{P}^3} \cong (\mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(d+a) \oplus \mathcal{O}_{\mathbf{P}^1}(2d-2a))_{a=2}.$$

Since the arithmetic genus is equal to two the curve cannot lie on a quadric surface.

It would be interesting to know what is the minimal condition which needs to be added to the one on the splitting of the restricted tangent bundle in order to ensure that the curve lies on a quadric surface.

2. Regularity. We will see how from the splitting of the restrictd tangent bundle we can derive information about the regularity of the curve; furthermore, we will show that the curves satisfying the hypothesis of either Theorem (0.1) or Theorem (0.2) are $(d-d_1)$-regular and $(d-d_1-1)$-irregular.

DEFINITION (2.1). A coherent sheaf $\mathcal{F}$ on $\mathbf{P}^r$ is said to be n-regular if $H^1(\mathbf{P}^r, \mathcal{F}(n-i)) = 0$ for $i > 0$.

DEFINITION (2.2). A curve in $\mathbf{P}^r$ is n-regular if its ideal sheaf is, and n-irregular otherwise.

We have the following

PROPOSITION (2.3). *Let C be an irreducible rational curve in $\mathbf{P}^3$ not contained in a plane and let $\psi\colon \mathbf{P}^1 \to \mathbf{P}^3$ be a morphism of degree d whose image is C. The curve C is $(d-d_1)$-regular where $(d+d_1)$ is the smallest degree of a line bundle which appears in the decomposition of $\psi^* T_{\mathbf{P}^3}$.*

PROOF. We consider the bundle $M = \psi^*(T_{\mathbf{P}^3}(-1))^*$ which can be rewritten in our notation as $M = \bigoplus_{i=1}^3 \mathcal{O}_{\mathbf{P}^1}(-d_i)$. Since

$$\Lambda^2 M \cong \bigoplus_{1\le i<j\le 3} \mathcal{O}_{\mathbf{P}^1}(-d_i - d_j)$$

and $d_1 \le d_2 \le d_3$, it follows that $\mathcal{O}_{\mathbf{P}^1}(d-d_1-1)$ is the line bundle of smallest degree such that

$$H^1(\mathbf{P}^1, \Lambda^2 M \otimes_{\mathcal{O}_{\mathbf{P}^1}} \mathcal{O}_{\mathbf{P}^1}(d-d_1-1)) = 0.$$

By Proposition (1.2) in [**GLP**] we conclude that the curve is $h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d-d_1-1))$-regular, that is, $(d-d_1)$-regular.

Let C be a curve on a quadric Q. We recall that if the quadric is singular then we denote by a the intersectin number of C with the general line of the ruling outside the vertex; if the quadric is smooth then $(a, d-a)$ is the divisor class of C.

REMARK (2.4). A curve on a quadric has a pencil of $(d-a)$-secant lines. Hence it is $(d-a-1)$-irregular.

PROPOSITION (2.5). *Let C be a curve on a quadric Q. If $a \le \lfloor d/3 \rfloor$ then C is $(d-a)$-regular and $(d-a-1)$-irregular. If one of the following holds:*

(i) *$a > \lfloor d/3 \rfloor$, Q is singular and there exist two points on C of multiplicity equal to a,*

(ii) *$a > \lfloor d/3 \rfloor$, Q is singular and there exists a point on C of both multiplicity and total inflection equal to a. Then C is $2a$-regular and $(2a-1)$-irregular.*

PROOF. Let $a \le \lfloor d/3 \rfloor$. By Theorems (0.1) and (0.2) we have $d_1 = a$. Thus Proposition (2.3) and Remark (2.4) imply that the curve is $(d-a)$-regular and $(d-a-1)$-irregular, respectively. Let $a > \lfloor d/3 \rfloor$. Theorem (0.1) says that $d_1 = d-2a$. Hence, by Proposition (2.3), the curve is $2a$-regular. Furthermore, the curve has a $2a$-secant line; hence it is $(2a-1)$-irregular.

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