THE RETURN ON INVESTMENT FROM PROPORTIONAL PORTFOLIO STRATEGIES

Sid Browne [*]

Columbia University

Abstract

Dynamic asset allocation strategies that are continuously rebalanced so as to always keep a fixed constant proportion of wealth invested in the various assets at each point in time play a fundamental role in the theory of optimal portfolio strategies. In this paper we study the *rate of return on investment*, defined here as the net gain in wealth divided by the cumulative investment, for such investment strategies in continuous time. Among other results, we prove that the limiting distribution of this measure of return is a gamma distribution. This limit theorem allows for comparisons of different strategies. For example, the mean return on investment is maximized by the same strategy that maximizes logarithmic utility, which is also known to maximize the exponential rate at which wealth grows. The return from this policy turns out to have other stochastic dominance properties as well. We also study the *return on the risky investment alone*, defined here as the present value of the gain from investment divided by the present value of the cumulative investment in the risky asset needed to achieve the gain. We show that for the log-optimal, or optimal growth policy, this return tends to an exponential distribution. We compare the return from the optimal growth policy with the return from a policy that invests a constant amount in the risky stock. We show that for the case of a single risky investment, the constant investor's expected return is twice that of the optimal growth policy. This difference can be considered the cost for insuring that the proportional investor does not go bankrupt.

Key words: Portfolio Theory; Diffusions; Stationary Distributions; Convergence in Distribution; Limit Theorems; Stochastic Order Relations; Logarithmic Utility; Optimal Growth Policy.

AMS 1991 Subject Classification: Primary: 90A09, 60F99. Secondary: 90A10, 60J60, 60H10.

1   Introduction

Constant proportions investment strategies play a fundamental role in portfolio theory. Under these policies, an investor follows a dynamic trading strategy that continuously rebalances the portfolio so as to always allocate fixed constant proportions of the investor's wealth across the investment opportunities. These strategies are quite widely used in practice and are also sometimes referred to as constant mix, or continuously rebalanced, strategies (see e.g., Perold and Sharpe [25]). Furthermore, for certain objectives and under some specific assumptions about the stochastic behavior of the investment opportunities, it is well known that these policies have many *optimality* properties associated with them as well. These properties are reviewed in the next section.

Given the fundamental nature of such policies in theoretical as well as actual portfolio practice, it is of interest to know what the stochastic behavior of the *rate of return on investment* (RROI) — defined here as the net gain divided by the cumulative investment — is for such policies. In this paper we study this dimension of the portfolio problem. We take as our setting the continuous time financial model introduced in Merton [22] and used in Black and Scholes [3]. For this model we obtain some limit theorems for the RROI which allow us to compare and derive some specific optimality properties for certain portfolio strategies.

A summary of our main results and the organization of the paper is as follows: in the next section, we review the continuous-time model and some well known optimality properties associated with constant proportion investment policies. In Section 3 we provide our main result (Theorem 3.1): that the return on investment for such policies converges to a limiting distribution which is a *gamma* distribution. This result provides a basis upon which to compare different strategies and to explore and identify various optimality criteria. For example, with this distributional limit theorem in hand we show in Section 3 that the policy that maximizes *logarithmic utility* of wealth generates a RROI that has some stochastic dominance properties over other policies. The logarithmic utility function has certain other objective optimality properties that are reviewed in Section 2. In Section 4 we prove Theorem 3.1. We show in particular that the limiting behavior of the RROI is in fact determined by the limiting behavior of a related diffusion process, which is completely analyzed. In Section 5 we move on to consider the excess return from investment above the risk-free rate. We call this the *rate of return on risky investment* (RRORI). We show that this measure converges to a different gamma distribution, and in particular, for the case of logarithmic utility, to an *exponential* distribution. However, for the RRORI measure, the logarithmic utility function has only limited stochastic dominance properties over other policies, and we show that the mean RRORI is in fact maximized by a different class of strategies, namely, by strategies that invest only

1

a constant *amount* (as opposed to a constant proportion) in the risky assets. For such strategies the RRORI follows a Gaussian process that is independent of the constant amount invested in the risky assets. Furthermore, it turns out that that the mean RRORI for such constant amount investment strategies is *twice* the mean RRORI for a logarithmic utility function. (These last two statements are specific to the case with a single risky stock, and do not hold for the more general case treated in Section 6.) Since bankruptcy is possible under such strategies, this halved return can be considered the price the constant proportional investor must pay for the insurance of never going bankrupt, since in continuous time bankruptcy is impossible under a proportional investment strategy (in the absence of any withdrawals and other constraints). The precise distributional results obtained in Section 5 allow us to compute explicitly various comparative probabilities. Finally, in Section 6 we extend all our results to the multiple risky stock case.

Our study was motivated by the stimulating paper of Ethier and Tavare [9] who studied the return on investment in a discrete-time gambling model, where the return on the individual gambles is assumed to follow a random walk. They showed that the *asymptotic* distribution of the return, as the mean increment in the random walk goes to zero, is a gamma distribution. Since there is only one investment opportunity in the model of Ethier and Tavare [9], their results have counterparts in our treatment of the RRORI, but they did not obtain the discrete-time analogs of our more general results for the RROI, and hence of the optimality of the logarithmic utility policy, discussed in Sections 3 and 4.

2 Optimal Properties of Proportional Investment

We recall here some basic facts about certain optimal properties associated with investment policies that invest a fixed proportion (possibly greater than 1) of current wealth in the risky asset. These policies are commonly referred to as *constant proportions* or *constant mix* strategies. While for expositional purposes we concentrate on the case of a single stock in Sections 3-5, we introduce here the model with an arbitrary number of risky stocks since we will return to the multiple stock case later in Section 6.

The model we treat is that of a complete market with constant coefficients, with k (correlated) risky stocks generated by k independent Brownian motions. The prices of these stocks will be denoted by $\{S_t^{(i)} : i = 1, \ldots, k\}$, where it is assumed that the prices evolve according to

$$dS_t^{(i)} = \mu_i S_t^{(i)} dt + \sum_{j=1}^{k} \sigma_{ij} S_t^{(i)} dW_t^{(j)}, \ \text{for} \ i = 1, \ldots, k \tag{2.1}$$

where $\{\mu_i : i = 1, \ldots, k\}$ and $\{\sigma_{ij} : i, j = 1, \ldots, k\}$ are constants, and $W_t^{(j)}$ denotes a standard

independent Brownian motion, for $j = 1, \ldots, k$. There is also a risk-free security, a bond, available for investment. The price of the bond, $\{B_t, t \geq 0\}$, evolves according to

$$dB_t = rB_t dt, \tag{2.2}$$

where $r > 0$ is a constant. Thus the stock prices follow a multi-dimensional geometric Brownian motion, as introduced in Merton [22].

An investment policy is a (column) vector control process $\{\boldsymbol{\pi}_t : t \geq 0\}$ with individual components $\pi_i(t), i = 1, \ldots, k$, where $\pi_i(t)$ is the proportion of the investor's wealth invested in the risky asset i at time t, for $i = 1, \ldots, k$, with the remainder invested in the risk-free bond. Thus, under the policy π, the investor's wealth process, X_t^π evolves according to

$$
\begin{aligned}
dX_t^\pi &= X_t^\pi \left(\sum_{i=1}^k \pi_i(t) \frac{dS_t^{(i)}}{S_t^{(i)}} \right) + X_t^\pi \sum_{i=1}^k (1 - \pi_i(t)) \frac{dB_t}{B_t} \\
&= X_t^\pi \left(r + \boldsymbol{\pi}_t'(\boldsymbol{\mu} - r\mathbf{1}) \right) dt + X_t^\pi \sum_{i=1}^k \sum_{j=1}^k \pi_i(t) \sigma_{ij} \, dW_t^{(j)}, \tag{2.3}
\end{aligned}
$$

upon substitution from (2.1) and (2.2), where $\boldsymbol{\pi}_t = (\pi_1(t), \ldots, \pi_k(t))'$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)'$, and $\mathbf{1} = (1, \ldots, 1)'$ (transposition is denoted by the superscript $'$). It is of course assumed that $\boldsymbol{\pi}_t$ is an admissible control process, i.e., it is a nonanticipative adapted vector satisfying $\int_0^T \boldsymbol{\pi}_t' \boldsymbol{\pi}_t dt < \infty$, for all $T < \infty$. We place no further restrictions on $\boldsymbol{\pi}$. For example we allow $\pi_i(t) < 0$, in which case the investor is selling the ith stock short, as well as $\pi_i(t) > 1$, for any and all $i = 1, \ldots, k$.

Let $\boldsymbol{\sigma}$ denote the matrix $\boldsymbol{\sigma} = (\sigma)_{ij}$, and let $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}'$. We will assume for the remainder that the matrix $\boldsymbol{\sigma}$ is of full rank, and hence $\boldsymbol{\sigma}^{-1}$ as well as $\boldsymbol{\Sigma}^{-1}$ exist.

For the sequel, let

$$\tilde{\boldsymbol{\mu}} := \boldsymbol{\mu} - r\mathbf{1} \tag{2.4}$$

denote the vector of the excess returns of the risky stocks over the risk-free return.

2.1 Optimality of constant proportions

Of particular interest to us is the case where $\boldsymbol{\pi}_t$ is a *constant* vector for all $t \geq 0$. Such a policy is called a constant proportions policy and is in fact the *optimal* investment policy for many interesting objective functions. For example, it is well known that if the investor's objective is to choose an admissible investment strategy to maximize expected utility of terminal wealth for a utility function that is either a power function, or logarithmic, then the optimal policy is a constant vector. Furthermore, a constant vector is also the optimal policy for other objective criteria, such as minimizing the expected time to reach a given level of wealth, as well. We summarize these main properties in the following:

3

Proposition 2.1 *For an investor whose wealth evolves according to (2.3), consider the following four problems:*

Problem 1.   Given a utility function $u(x)$ of the form

$$u(x) = \begin{cases} \frac{\delta}{\delta-1} x^{1-\frac{1}{\delta}} & \text{for } x > 0, \ \delta > 0, \delta \neq 1 \\ \\ \ln(x) & \text{for } x > 0, \ \delta = 1 \end{cases} \tag{2.5}$$

choose an admissible investment policy to maximize expected utility of terminal wealth at a fixed terminal time T. Let $\boldsymbol{\pi}^*(\delta) = \{\boldsymbol{\pi}^*(\delta)_t, \ 0 \leq t \leq T\}$ denote this optimal policy, i.e., $\boldsymbol{\pi}^*(\delta) = \arg\sup_{\boldsymbol{\pi}} E_x\left[u(X_T^{\pi})\right]$, where the function $u(\cdot)$ is given by (2.5) and the dynamics of $\{X_t^{\pi}, \ t \geq 0\}$ is given in (2.3).

Problem 2.   For a given discount rate $\lambda > 0$, choose an admissible investment policy to maximize the expected discounted reward of reaching a given target level, say b, of wealth. Specifically, for $X_0 = x < b$, let

$$\tau_b^{\pi} := \inf\{t \geq 0 : X_t^{\pi} \geq b\}, \tag{2.6}$$

denote the first passage time to the wealth level b, and let $\boldsymbol{\pi}^{(b)}(\lambda) = \{\boldsymbol{\pi}^{(b)}(\lambda)_t, t \geq 0\}$ denote the optimal policy, i.e., $\boldsymbol{\pi}^{(b)}(\lambda) = \arg\sup_{\boldsymbol{\pi}} E_x\left[e^{-\lambda\tau_b^{\pi}}\right]$.

Problem 3. Choose an admissible investment policy to minimize the expected time to reach a given level, say b, of wealth. Specifically, for $X_0 = x < b$ and τ_b^{π} as in (2.6), let $\boldsymbol{\pi}^{(b)} = \{\boldsymbol{\pi}_t^{(b)}, t \geq 0\}$ denote the optimal policy, i.e., $\boldsymbol{\pi}^{(b)} = \arg\inf_{\boldsymbol{\pi}} E_x\left[\tau_b^{\pi}\right]$.

Problem 4.   Choose an admissible investment policy to maximize the actual (asymptotic) rate at which wealth compounds, defined as $\liminf_{t\to\infty}(1/t)\ln[X_t^{\pi}]$, and let $\underline{\boldsymbol{\pi}} = \{\underline{\boldsymbol{\pi}}_t, t \geq 0\}$ denote the optimal policy, i.e., $\underline{\boldsymbol{\pi}} = \arg\sup_{\boldsymbol{\pi}}\left\{\liminf_{t\to\infty}(1/t)\ln[X_t^{\pi}]\right\}$.

Let $\boldsymbol{f}^*$ denote the constant vector given by

$$\boldsymbol{f}^* := \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r\mathbf{1}\right) \equiv \boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}. \tag{2.7}$$

Then the optimal policies for the four problems are given respectively by the constant vectors

$$\begin{aligned}
\boldsymbol{\pi}_t^*(\delta) &= \delta\boldsymbol{f}^*, &&\text{for all } t \geq 0, \quad \text{and all } \delta > 0 & (2.8)\\
\boldsymbol{\pi}_t^{(b)}(\lambda) &= \left(\frac{1}{1-\eta}\right)\boldsymbol{f}^*, &&\text{for all } t \geq 0, \quad \text{and all } \lambda \geq 0, \quad \text{and any } b > x, & (2.9)\\
\boldsymbol{\pi}_t^{(b)} &= \boldsymbol{f}^*, &&\text{for all } t \geq 0, \quad \text{and any } b > x, & (2.10)\\
\underline{\boldsymbol{\pi}}_t &= \boldsymbol{f}^*, &&\text{for all } t \geq 0, & (2.11)
\end{aligned}$$

4

where in (2.9), $\eta = \eta(\lambda)$ satisfies $0 < \eta < 1$ and is the (smaller) root to the quadratic equation $\eta^2 r - \eta(\gamma + \lambda + r) + \lambda = 0$, where the parameter $\gamma := (1/2)\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}$.

Problem 1 and its optimal policy (2.8) were first considered in Merton [22]. The identical policy is in effect if the investor maximizes utility obtained from *consumption* as well (see Merton [22, 23] for more details). Problem 2 and its optimal policy (2.9) follows from a somewhat more general result of Browne [6, Theorem 4.3] (see also Orey et al.[24]). Problem 3 and its optimal policy (2.10) was first considered in Heath et al.[15] for the case $k = 1$ (see also Merton[23, Section 6.5]). The multivariate case follows directly from the more general result in Browne [6]. It is interesting to note that the optimal policies for Problems 2 and 3 are *independent of the target level of wealth b*. Problem 4 was considered in Karatzas [17, Section 9.6].

2.2 Logarithmic utility and optimal growth

For logarithmic utility, the optimal policy for Problem 1 is $\boldsymbol{\pi}^*(1) \equiv \boldsymbol{f}^*$. A comparison with the results for Problems 3 and 4 then shows an interesting correspondence between the logarithmic utility function and objective criteria related to growth; specifically, $\boldsymbol{\pi}^*(1) = \boldsymbol{\pi}^{(b)} = \underline{\boldsymbol{\pi}} \equiv \boldsymbol{f}^*$. For this reason the log-optimal policy, $\boldsymbol{f}^*$ of (2.7), is sometimes referred to as the *optimal growth policy* (e.g., Merton [23, Chapter 6]). (While the equivalence between Problem 1 with logarithmic utility and the asymptotic compounding rate of Problem 4 holds for more general price processes than that given by (2.1), it appears that the exact equivalence of this policy to the policy that minimizes the expected time to a target level of wealth for Problem 3 holds *only* for the constant coefficients model treated here.)

These optimality results and the connection between logarithmic utility and the objective criteria of growth have their precedents in earlier discrete-time results. In particular, Kelly [20] was the first to observe the relationship between maximizing the logarithm of wealth and the *expected* asymptotic rate at which wealth compounds. (In light of this, maximizing the logarithm of wealth is sometimes referred to as the *Kelly criterion.*) This was amplified to the portfolio setting in the papers of Hakansson [14], Thorp [30] and Markowitz [21] and others. Breiman [4] greatly expanded on the results on [20] and among other fundamental results established that the policy that maximizes the logarithm of wealth is *asymptotically* (as $b \to \infty$) optimal for the objective of minimizing the expected time to b, however it is only in continuous-time that the relationship is *exact.* Thorp [29], Hakansson [14], Finkelstein and Whitely [10] and Ethier and Tavare [9] among others contain penetrating analyses of optimality properties of constant proportional investment policies in discrete-time. A Bayesian version of the optimal growth policy in both discrete and continuous-time is studied in Browne and Whitt [7].

A comparison of (2.9) with (2.5) and (2.8) shows that just as the logarithmic utility function is related to optimal growth, a *power* utility function with $\delta \neq 1$ is also related to certain growth objectives related to that of Problem 2 (in particular for $\delta = (1 - \eta)^{-1}$ in (2.5)). (Relationships between objective criteria other than growth and particular utility functions for some other portfolio problems are discussed in Browne [5].)

Remark 2.1: Risk aversion. The Arrow-Pratt measure of *relative risk aversion* for a utility function $u(\cdot)$, denoted here by $\Lambda(\cdot)$, is defined as $\Lambda(x) := -xu^{(2)}(x)/u^{(1)}(x)$, where $u^{(i)}(x) := \frac{d^i u(x)}{dx^i}$, see [26]. Thus for the power utility function in (2.5) we have $\Lambda(x) = \delta^{-1}$ for all $x > 0$, corresponding to *constant relative risk aversion*. Since the logarithmic case corresponds to a relative risk aversion $\Lambda(x) = 1$, it is seen from (2.8) that for the case where $\boldsymbol{f}^{*'}\boldsymbol{1} > 0$, an investor who is more risk averse than a logarithmic investor (i.e., $\delta < 1$) will in fact under-invest in the risky assets relative to the logarithmic investor while an investor who is less risk averse ($\delta > 1$) will over-invest in the risky assets relative to the logarithmic investor.

Since constant proportion investment policies play such a fundamental role in portfolio theory, and is the focus of study here, we summarize its main properties next. For models where the optimal policy is a constant proportion of the *surplus* of wealth over some level — instead of wealth itself — see Gottlieb [11], Sundaresan[27], Black and Perold[2], Grossman and Zhou[13], Dybvig[8] and Browne[6].

2.3 Wealth under Proportional Investment

Let $\boldsymbol{f} = (f_1, \ldots, f_k)'$ denote a fixed constant vector, where f_i is the percentage of the investor's wealth invested in risky security i, for $i = 1, \ldots, k$. The investor's wealth under this policy, denoted by X_t^f, then evolves as $dX_t^f = X_t^f \left(r + \boldsymbol{f}' \tilde{\boldsymbol{\mu}} \right) dt + X_t^f \sum_{i=1}^{k} \sum_{j=1}^{k} f_i \sigma_{ij} \, dW_t^{(j)}$. Equivalently, X_t^f is the geometric Brownian motion

$$X_t^f = X_0 \exp \left\{ \left(r + \boldsymbol{f}' \tilde{\boldsymbol{\mu}} - \frac{1}{2} \boldsymbol{f}' \boldsymbol{\Sigma} \boldsymbol{f} \right) t + \sum_{i=1}^{k} \sum_{j=1}^{k} f_i \sigma_{ij} W_t^{(j)} \right\}. \tag{2.12}$$

As such, it is then well known (e.g., [18, Pg. 349]) that

$$\inf_{0 \leq t < \infty} X_t^f > 0, \quad \text{and} \quad \lim_{t \to \infty} X_t^f = \infty, \quad \text{a.s.} \tag{2.13}$$

so long as

$$r + \boldsymbol{f}' \tilde{\boldsymbol{\mu}} - \frac{1}{2} \boldsymbol{f}' \boldsymbol{\Sigma} \boldsymbol{f} > 0. \tag{2.14}$$

For the sequel, we will assume that (2.14) holds.

Recognize of course that the resulting wealth process (2.12) is distributionally equivalent to the process $X_0 \exp\left\{\left(r + \boldsymbol{f}'\tilde{\boldsymbol{\mu}} - \frac{1}{2}\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f}\right)t + \sqrt{\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f}}\,W_t\right\}$, where $\{W_t, t \geq 0\}$ is a *one-dimensional* ordinary Brownian motion. In this paper, our focus is on the *rate of return on investment*, which is defined here to be the net gain in wealth divided by the cumulative investment. Since both of these are one-dimensional processes, for expositional purposes we will first treat the case $k = 1$, where there are only two investment opportunities: a single risky stock whose price process follows $dS_t = \mu S_t dt + \sigma S_t dW_t$, and the bond of (2.2), and then in a later section (Section 6) simply outline how to extend the results obtained for single risky stock to the case of multiple stocks, and highlight the differences between them. For the single stock case, let f denote a fixed constant, which is the percentage of the investor's wealth invested in the risky stock. This investor's wealth, denoted by X_t^f then evolves as $dX_t^f = X_t^f\left(r + f\tilde{\mu}\right)dt + f\sigma X_t^f dW_t$, where $\tilde{\mu} := \mu - r$ is the excess return of the risky stock over the return from the risk-free bond. Since f is constant, the wealth process is the geometric Brownian motion $X_t^f = X_0 \exp\left\{\left(r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2\right)t + f\sigma W_t\right\}$. The condition (2.14) becomes

$$r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2 > 0\,, \tag{2.15}$$

equivalently,

$$\frac{\tilde{\mu}}{\sigma^2} - \frac{1}{\sigma}\sqrt{\frac{\tilde{\mu}^2}{\sigma^2} + 2r} \;\; < \; f \; < \;\; \frac{\tilde{\mu}}{\sigma^2} + \frac{1}{\sigma}\sqrt{\frac{\tilde{\mu}^2}{\sigma^2} + 2r}\;. \tag{2.16}$$

For the sequel we will assume that the constant f is such that condition (2.15) is met and hence that (2.13) holds, i.e., $\inf_{0 \leq t < \infty} X_t^f > 0$, and $\lim_{t \to \infty} X_t^f = \infty$, a.s.

Note that we allow both $f > 1$ as well as $f < 0$. In the former holds, the investor is borrowing money at the risk-free rate to invest long in the stock, while if the latter holds, then the investor is selling the stock short and putting the proceeds into the risk-free asset.

Since logarithmic utility and the optimal growth policy play a prominent role in the sequel, for the remainder of this paper we will denote it by f^*, i.e., for the sequel we will take $f^* = (\mu - r)/\sigma^2$. In terms of $\tilde{\mu}$, this gives $f^* := \tilde{\mu}/\sigma^2$.

In the next section we turn our attention to the rate of return on investment.

3   The Rate of Return from Total Investment

Our interest here is the rate of return from *total* investment (RROI), which for the fixed policy f will be denoted by the process $\{\rho_f(t), t \geq 0\}$. This is defined here as the ratio of the net gain to the cumulative investment (see Ethier and Tavare [9]), i.e., since X_s^f is the investor's wealth at time s, and this wealth is completely invested in the risky stock and the riskless bond, the cumulative

investment until time t is then $\int_0^t X_s^f ds$, and the RROI is then defined by

$$\rho_f(t) := \frac{X_t^f - X_0}{\int_0^t X_s^f ds}, \quad \text{for} \quad t > 0. \tag{3.1}$$

Thus $\rho_f(t)$ is a measure of the amount of wealth it takes to finance a gain. If $\rho_f(t)$ is large, then the investor is accumulating gains at a faster rate than if it is small (see Remark 3.1 below for a discussion of other measures of rate of return). Note that if we divide the numerator and denominator of the ratio in (3.1) by t, we may also interpret the measure $\rho_f(t)$ as the *average* net gain over the *average* wealth level.

Substitution of (2.3) into (3.1) shows that for a fixed f,

$$\rho_f(t) = \frac{\exp\left\{ \left(r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2 \right) t + f\sigma W_t \right\} - 1}{\int_0^t \exp\left\{ \left(r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2 \right) s + f\sigma W_s \right\} ds}. \tag{3.2}$$

Placing $f = 0$ in (3.2) shows that for that policy, under which total wealth is always invested in the risk-free asset, we do have

$$\rho_0(t) := \frac{e^{rt} - 1}{\int_0^t e^{rs} ds} = r$$

the risk-free interest rate, as expected. However, for $f \neq 0$, the RROI process $\{\rho_f(t), t > 0\}$ is quite complicated and does not yield to a simple direct analysis. Nevertheless, we will show that for any fixed proportion f such that (2.15) holds, the process $\{\rho_f(t), t > 0\}$ admits a unique limiting distribution, which is a *gamma* distribution. We make this more precise in the next theorem.

Remark on notation: For reference, we note that throughout the remainder of the paper, when we say that a random variable $X \sim \text{gamma}(\alpha, \beta)$, we mean that X is a random variable with density function $\psi(x) = e^{-\beta x} x^{\alpha-1} \beta^\alpha / \Gamma(\alpha)$, and so $E(X) = \alpha/\beta$, and $Var(X) = \alpha/\beta^2$.

Our main result can now be stated as the following:

Theorem 3.1 *For any fixed proportional strategy f which satisfies condition (2.15), the RROI process $\{\rho_f(t), t > 0\}$ converges (as $t \to \infty$) in distribution to a random variable which has a gamma distribution. Specifically, as $t \to \infty$,*

$$\rho_f(t) \xrightarrow{\text{d}} \rho_f \sim \text{gamma}\left(\frac{2(r + f\tilde{\mu})}{\sigma^2 f^2} - 1, \frac{2}{\sigma^2 f^2} \right), \tag{3.3}$$

where $\xrightarrow{\text{d}}$ denotes convergence in distribution. As such

$$E[\rho_f] = r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2 > 0. \tag{3.4}$$

The proof of this theorem is given in the next section. However, since the precise distributional nature of the result allows us a basis upon which to compare different investment strategies, we will investigate some of its ramifications directly.

Remark 3.1: Our definition of $\rho_f(t)$ as the rate of return on investment, while based directly on measures studied in Ethier and Tavare [9] (see also Griffin [12]), is not quite the rate of return that is typically used in corporate finance. In continuous-time the rate of return until time t from the portfolio strategy f is more usually defined in the financial literature as the value of $r_f(t)$ such that $X_0 e^{r_f(t)\,t} = X_t^f$. (Recall that in discrete time, if an investment yields a return r_i in period i, then an initial investment of X_0 grows (if all gains are reinvested) after n investment periods to $X_n = X_0 \prod_{i=1}^{n}(1+r_i)$. The rate of return is then typically defined as that value of r_n^* such that $[(1+r_n^*)^n - 1] = [X_n - X_0]/X_0$, from which it follows that $r_n^* = [\prod_{i=1}^{n}(1+r_i)]^{1/n} - 1$. It should be clear that the analog of this for our continuous-time model with continuous reinvestment is indeed $r_f(t)$. The rate of return measure studied by Ethier and Tavare [9] is defined in discrete-time by $R_n := [X_n - X_0]/\sum_{i=0}^{n-1} X_i$, the continuous-time analog of which is clearly our $\rho_f(t)$.)

Using (2.12), it is seen that $r_f(t) = r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2 + \frac{1}{t}f\sigma W_t$, and so for any $t > 0$, $E[r_f(t)] = r + f\tilde{\mu} - f^2\sigma^2/2$, which by (3.4) is equivalent to $E[\rho_f]$. However, the measure $r_f(t)$ has a degenerate stochastic limit, since by the law of large numbers $r_f(t) \xrightarrow{\text{a.s.}} r + f\tilde{\mu} - f^2\sigma^2/2 \equiv E[\rho_f]$, as $t \to \infty$. Thus the more usual measure $r_f(t)$ does not convey any information about the variation of the asymptotic return around the mean. By Theorem 3.1 above, we see that such (more refined) information is indeed provided by the measure ρ_f.

Remark 3.2: The expectation in (3.4) should not be confused with the ratio of the *expected* gain to the *expected* total investment, which for *any* $t > 0$, is equal to

$$\frac{E[\text{total gain}]}{E[\text{total investment}]} \equiv \frac{E\left[X_t^f - X_0\right]}{E\left[\int_0^t X_s^f \, ds\right]} = r + f\tilde{\mu}. \tag{3.5}$$

(This can be established by recognizing that $EX_t^f = X_0 e^{(r+f\tilde{\mu})t}$, and then by using Fubini's theorem to compute the expectation in the denominator.)

The distinction between measures related to (3.4) and (3.5) for simple discrete-time gambling models is discussed in Griffin [12].

3.1 Optimal Growth Policy and Stochastic Dominance

Note that while $r + f\tilde{\mu}$ is unbounded in f, hence implying that the quantity in (3.5) is maximized by a strategy that invests as much as possible in the risky asset, the mean RROI of (3.4), $r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2$, *is* maximized at a finite value. In particular at the value $f^* = \tilde{\mu}/\sigma^2$, which is the same policy that is optimal for maximizing logarithmic utility of wealth at a fixed terminal time and hence for maximizing the asymptotic (exponential) rate of growth, and by the results of [15, 23, 6] is also optimal for minimizing the expected time to a goal, i.e., the mean RROI is maximized by the log-optimal or ordinary optimal-growth strategy of (2.7). Thus, we find that a byproduct of our analysis is that it provides yet another objective justification for the use of logarithmic utility. We make this precise in the following.

Corollary 3.2 *The mean of the limiting distribution of the RROI process is maximized at the value*

$$f^* = \frac{\tilde{\mu}}{\sigma^2} \,, \tag{3.6}$$

with resulting mean $E[\rho_*] = r + \gamma$, where $\gamma := \frac{1}{2}\left(\tilde{\mu}/\sigma\right)^2$. For this strategy the RROI, $\rho_*(t)$, satisfies

$$\rho_*(t) \xrightarrow{\mathrm{d}} \rho_* \sim \mathrm{gamma}\left(\frac{r+\gamma}{\gamma}, \frac{1}{\gamma}\right) \,. \tag{3.7}$$

In fact, the precise distributional characterization of the limiting RROI allows for a somewhat stronger statement, in terms of stochastic orderings. Recall first (see e.g. Stoyan [28]) that for two random variables, X, Y, we say that $X \leq_{\mathrm{icx}} Y$ if $E(X - x)^+ \leq E(Y - x)^+$ for all x. This is equivalent to saying that $E[g(X)] \leq E[g(Y)]$ for all increasing convex functions g, and is referred to as the *increasing convex ordering*. We also say that $X \leq_{\mathrm{icv}} Y$ if $E(x - X)^+ \geq E(x - Y)^+$, for all x (provided the expectations are finite). This is equivalent to saying that $E[h(X)] \leq E[h(Y)]$ for all increasing concave functions h, and is hence referred to as the *increasing concave ordering*. (This last ordering is also referred to as *second degree stochastic monotonic dominance* (see [16, Section 2.9]).)

The following corollary shows that the RROI obtained from using the optimal growth policy dominates in the increasing *convex* stochastic order the RROI from any other proportional strategy that *under-invests* relative to it, and dominates in the increasing *concave* stochastic order the RROI for any proportional strategy that *over-invests* relative to it.

Corollary 3.3 *Let ρ_* denote the RROI obtained from using the policy f^* defined in (3.6), and let ρ_f be the RROI from any other constant proportional strategy $f := cf^*$, where c is an arbitrary constant. Then the following hold:*

$$\rho_f \leq_{\mathrm{icx}} \rho_* \qquad \text{for } c \leq 1, \tag{3.8}$$

10

$$\rho_f \leq_{\text{icv}} \rho_* \qquad for\ c \geq 1. \tag{3.9}$$

Proof: See Appendix A.1.

Comparing Corollary 3.3 with equations (2.5) and (2.8), and the earlier discussion of risk aversion in Remark 2.1, shows clearly that (3.8) is in effect for an investor with greater relative risk aversion than a log investor, while (3.9) is in effect for an investor with less relative risk aversion.

Remark 3.3: It should be noted that while the previous two corollaries highlight some optimal properties of the log-optimal strategy, it is not claimed that the log-optimal strategy, f^*, is in fact the strategy that maximizes the mean RROI over *all* admissible strategies — only among all *constant* proportional ones. It is an open question as to what the actual (global) optimal strategy is for this criterion.

Remark 3.4: It is interesting to observe that we *cannot* extend Corollary 3.3 to get a standard stronger stochastic ordering in general with respect to over or under investing relative to the log-optimal policy. Rather, we can only obtain a (sharp) characterization of the type of constant proportional strategies that yield RROIs that are stochastically dominated by the RROI from the log-optimal or optimal growth policy. However, before we proceed we recall the *likelihood ratio order*, denoted by $\leq_{\text{LR}}$. This stochastic order is stronger than (and hence implies) the stochastic order $\leq_{\text{st}}$, which is also known as "first degree stochastic dominance" (see [16]). (Recall that $X \leq_{\text{st}} Y$ if $P(X \geq z) \leq P(Y \geq z)$, for all z.)

For two continuous nonnegative random variables, X_1, X_2, with densities $\psi_1(x)$ and $\psi_2(x)$, we say that $X_2 \leq_{\text{LR}} X_1$ if

$$\frac{\psi_1(y)}{\psi_1(x)} \geq \frac{\psi_2(y)}{\psi_2(x)} \quad \text{for all} \quad x \leq y\,. \tag{3.10}$$

Corollary 3.4 *For a proportional strategy $f = cf^*$, with $c \neq 1$, where f^* is the optimal policy of (3.6), the relationship $\rho_f \leq_{\text{LR}} \rho_*$ (which implies the weaker relationship $\rho_f \leq_{\text{st}} \rho_*$) holds if and only if c satisfies*

$$1 - \sqrt{\frac{r+\gamma}{\gamma}} < c < -\frac{r}{r+2\gamma}\,. \tag{3.11}$$

Proof: See Appendix A.2.

Corollary 3.4 shows that the only type of constant proportional policy (other than f^*) for which the RROI is stochastically dominated by the optimal growth policy is one that is shorting the stock to the degree required by (3.11).

4 Proof of Theorem 3.1 and a Related Diffusion Process

In this section we provide the proof of Theorem 3.1. While the process $\{\rho_f(t), t \geq 0\}$ does not admit a simple direct analysis, there is a related Markov process amenable to analysis which holds the key for the limiting behavior of $\{\rho_f(t)\}$. Specifically, the process $\{R_f(t), t \geq 0\}$ defined by

$$R_f(t) := \frac{X_t^f}{X_0 + \int_0^t X_s^f \, ds} \, . \tag{4.1}$$

(Note that $R_f(0) = 1$.) We will first show that the limiting behavior of $\{\rho_f(t)\}$ is equivalent to the limiting behavior of the process $\{R_f(t)\}$. Since our main interest here is in fact on the limiting behavior of the RROI process $\{\rho_f(t)\}$, this is quite convenient, since as we will show below $\{R_f(t)\}$ is in fact a diffusion process whose limiting behavior we can analyze precisely.

Theorem 4.1 *Suppose that for some random variable R_f, we have $R_f(t) \xrightarrow{\mathrm{d}} R_f$ as $t \to \infty$. Then for any f that satisfies (2.15) we have, as $t \to \infty$,*

$$\rho_f(t) \xrightarrow{\mathrm{d}} R_f \, . \tag{4.2}$$

Proof: Observe that

$$\rho_f(t) = R_f(t) \left[\frac{X_t^f - X_0}{X_t^f} \right] \left[\frac{X_0 + \int_0^t X_s^f \, ds}{\int_0^t X_s^f \, ds} \right] \equiv R_f(t) \left[1 - e^{-B_t} \right] \left[1 + \frac{1}{\int_0^t e^{B_s} \, ds} \right] \tag{4.3}$$

where $\{B_s, s \geq 0\}$ is the linear Brownian motion defined by

$$B_s := \left(r + f\tilde{\mu} - \frac{f^2 \sigma^2}{2} \right) s + f\sigma W_s \, , \quad \text{with } B_0 = 0.$$

If (2.15) holds, then $\{B_s\}$ has *positive* drift, which then implies that $\lim_{t\to\infty} e^{-B_t} = 0$ a.s., as well as $\lim_{t\to\infty} \left(\int_0^t e^{B_s} \, ds \right)^{-1} = 0$ a.s., which in turn implies the distributional limit (4.2) as a consequence of Theorem 4.4 of Billingsley [1]. ∎

As such, Theorem 3.1 will be completely established if we can now prove that $R_f(t) \xrightarrow{\mathrm{d}} R_f$, for some random variable R_f with $R_f = \rho_f$, i.e., we must establish that $\{R_f(t)\}$ has the limiting gamma distribution posited earlier. To that end, we will first show that the process $\{R_f(t)\}$ is in fact a temporally homogeneous diffusion. This enables us to use standard techniques from the theory of diffusion processes to establish its limiting stationary distribution exactly. We will also prove that $\{R_f(t)\}$ is uniformly integrable. We begin by first noting the following.

Proposition 4.2 *For a fixed proportional investment policy f, the process $\{R_f(t)\}$ follows the stochastic differential equation*

$$dR_f(t) = \left[(r + f\tilde{\mu})R_f(t) - R_f^2(t) \right] dt + f\sigma R_f(t) dW_t \, , \tag{4.4}$$

12

i.e., $\{R_f(t)\}$ *is a temporally homogeneous diffusion process with drift function* $b(x) = (r + f\tilde{\mu})x - x^2$ *and diffusion function* $v^2(x) = f^2\sigma^2 x^2$.

Proof: Defining $A_t := \int_0^t X_s^f \, ds$ as the cumulative wealth (investment) process, we may write $R_f(t) = X_t^f/(X_0 + A_t)$. Since A_t is a process of bounded first variation, its quadratic variation is zero, and so Ito's rule shows that $dA_t = X_t^f \, dt$. Applying Ito's rule to $X_t^f/(X_0 + A_t)$ gives

$$d\frac{X_t^f}{X_0 + A_t} = \frac{1}{X_0 + A_t} dX_t^f - \frac{X_t^f}{(X_0 + A_t)^2} dA_t$$

which upon substitution gives

$$d\frac{X_t^f}{X_0 + A_t} = \frac{X_t^f}{X_0 + A_t}(r + f\tilde{\mu}) dt + \frac{X_t^f}{X_0 + A_t} f\sigma dW_t - \left(\frac{X_t^f}{X_0 + A_t}\right)^2 dt$$

which is equivalent to (4.4). ∎

The fact that the $\{R_f(t)\}$ follows the stochastic differential equation (4.4) allows us to conclude that it is in fact a strongly ergodic Markov process with a gamma stationary distribution. We state this as a theorem next. The proof will follow as an application of the more general lemma that follows directly.

Theorem 4.3

$$R_f(t) \xrightarrow{d} R_f \sim \text{gamma}\left(\frac{2(r + f\tilde{\mu})}{\sigma^2 f^2} - 1, \frac{2}{\sigma^2 f^2}\right). \tag{4.5}$$

Moreover the the process $R_f(t)$ *is uniformly integrable, and hence*

$$\lim_{t \to \infty} E(R_f(t)) = E[R_f] = r + f\tilde{\mu} - \frac{1}{2}f^2\sigma^2. \tag{4.6}$$

Proof: Since we will have other occasions to examine stochastic differential equations similar to (4.4), we will establish the result for the more general stochastic differential equation $dZ_t = \left[aZ_t - bZ_t^2\right] dt + cZ_t dW_t$, of which (4.4) is just a special case with $a = r + f\tilde{\mu}, b = 1, c = f\sigma$. Therefore, Theorem 4.3 (and then in light of Theorem 4.1, by implication Theorem 3.1) is just a consequence of the following lemma:

Lemma 4.4 *Suppose* Z_t *evolves as the stochastic differential equation*

$$dZ_t = \left[aZ_t - bZ_t^2\right] dt + cZ_t dW_t \tag{4.7}$$

with $a > 0, b > 0$. *Then*

(i)

$$Z_t = \frac{Z_0 \exp\left\{\left(a - \frac{1}{2}c^2\right)t + cW_t\right\}}{1 + Z_0 b \int_0^t \exp\left\{\left(a - \frac{1}{2}c^2\right)s + cW_s\right\} ds}. \tag{4.8}$$

13

(ii) *Furthermore, for $a > c^2/2, b > 0$, the process Z_t is strongly ergodic with*

$$Z_t \xrightarrow{\text{d}} Z_\infty \sim \text{gamma}\left(\frac{2a}{c^2} - 1, \frac{2b}{c^2}\right). \tag{4.9}$$

(iii) *Moreover, Z_t is uniformly integrable, so that in particular we have*

$$\lim_{t\to\infty} E(Z_t) = E(Z_\infty) = \frac{1}{b}\left(a - \frac{1}{2}c^2\right). \tag{4.10}$$

Proof:

(i) Let $Y_t := Z_t^{-1}$. Then an application of Ito's formula shows that Y_t evolves according to the stochastic differential equation $dY_t = \left[b + (c^2 - a)Y_t\right]dt - cY_t dW_t$, which is a linear stochastic differential equation, and so can be solved by standard methods (see e.g.[18, Section 5.6.C]) to yield

$$Y_t = \exp\left\{-\left(a - \frac{1}{2}c^2\right)t - cW_t\right\}\left[Y_0 + b\int_0^t \exp\left\{\left(a - \frac{1}{2}c^2\right)s + cW_s\right\}ds\right].$$

Using $Z_t = 1/Y_t$ then gives (4.8).

(ii) The stationary distribution for Z can be recovered by recognizing that Z is a temporally homogeneous diffusion with drift function $\mu(z) = az - bz^2$ and diffusion function $\sigma^2(z) = c^2 z^2$. As such, its *scale density*, $s(z)$ is given by ([19, Chapter 15])

$$s(z) \equiv \exp\left\{-\int^z \frac{2(ax - bx^2)}{c^2 x^2}dx\right\} = z^{-2a/c^2}e^{z2b/c^2}.$$

Letting $S(z) = \int^z s(x)dx$, we see that $S(0+) = -\infty$ while $S(z) \to \infty$ as $z \to \infty$. As such, it is well known (see e.g. [19, Section 15.6]) that Z is a strongly ergodic process with unique stationary distribution given by $m(z)/\int_0^\infty m(x)dx$, where $m(z)$ is the *speed density* defined by

$$m(z) := \left[\sigma^2(z)s(z)\right]^{-1} \equiv c^{-2}z^{2a/c^2-2}e^{-z2b/c^2},$$

which is the kernel of the gamma$(2a/c^2 - 1, 2b/c^2)$ density, which establishes (4.9).

(iii) It remains to prove that Z_t is in fact *uniformly integrable*. A sufficient condition for this is that $\sup_t E\left[Z_t^2\right] < \infty$. We can establish that this holds by letting $H_t = Z_t^2$, and noting that by Ito's rule, H_t satisfies

$$dH_t = \left[(2a + c^2)H_t - 2bH_t^{3/2}\right]dt + 2cH_t dW_t, \quad \text{with} \quad H_0 = Z_0^2.$$

14

Since $b > 0$, we may now choose constants $\alpha > 0$ and $\beta > 0$ so that

$$\alpha - \beta x > (2a + c^2)x - 2bx^{3/2}, \quad \text{for} \quad x \geq 0,$$

and construct a new process $\tilde{H}$ where

$$d\tilde{H}_t = (\alpha - \beta\tilde{H}_t)dt + 2c\tilde{H}_t dW_t, \quad \text{with} \quad \tilde{H}_0 \equiv Z_0^2. \tag{4.11}$$

Note that $\tilde{H}_t$ is always nonnegative.

By the comparison theorem for diffusions (e.g.[18, Proposition 5.2.18]) it follows that $\tilde{H}_t \geq H_t$, a.s., and hence $P_z(H_t > x) \leq P_z(\tilde{H}_t > x)$, from which it follows that $E(H_t) \leq E(\tilde{H}_t)$. Note now that since (4.11) is a linear equation, we can solve it exactly to find

$$\tilde{H}_t = e^{-(\beta + 2c^2)t + 2cW_t}\left[H_0 + \alpha\int_0^t e^{(\beta + 2c^2)s - 2cW_s}ds\right]$$

from which it is follows that

$$E(\tilde{H}_t) = H_0 e^{-\beta t} + \frac{\alpha}{\beta}(1 - e^{-\beta t}),$$

and for which it is therefore evident (since $\beta > 0$) that $\sup_t E(\tilde{H}_t) < \infty$. Since $E(Z_t^2) = E(H_t) \leq E(\tilde{H}_t)$ for every $t \geq 0$, it follows that Z_t is uniformly integrable. $\blacksquare$

5 Rate of return from investment in the risky asset alone.

In the previous sections we concentrated on the return from *total* investment, which included both investment in the risky stock as well as investment in the risk-free asset. Here we focus on the *excess return above the risk-free rate*, i.e., the gain and return from the investment in the *risky stock* alone. To that end, recognize that the excess gain in wealth above what could have been obtained by investing in the risk-free asset is the discounted, or present value, of the gain, $e^{-rt}X_t^f - X_0$. Note that if $f = 0$, whereby all the wealth is always invested in the risk-free asset, then this quantity is simply 0. (This follows since if all the money is invested in the risk-free asset, then $X_t^0 = e^{rt}X_0$.) Since f is the proportion of wealth invested in the risky stock, the total amount invested in the risky stock at time t is fX_t^f, and so the *cumulative* amount of money invested in the risky stock until t is $f\int_0^t X_s^f ds$. The discounted, or present value, of the cumulative amount of money invested in the risky stock until time t is therefore $f\int_0^t e^{-rs}X_s^f ds$.

Define now the *return on the risky investment* (RRORI) to be the discounted gain divided by the discounted cumulative investment in the risky stock, i.e., let $\tilde{\rho}_f(t)$ denote the RRORI, which is

defined by

$$\tilde{\rho}_f(t) = \frac{e^{-rt}X_t^f - X_0}{f \int_0^t e^{-rs}X_s^f \, ds}. \tag{5.1}$$

As will be seen presently, if $\tilde{\mu} > 0$ we require $f > 0$ for $\{\tilde{\rho}_f(t)\}$ to have a nondegenerate limit, i.e., shortselling the stock is prohibited, as then the discounted gain tends to 0. The RRORI is thus the present value of the gain from risky investment divided by the present value of the total amount of wealth invested in the risky stock that is needed to obtain the gain. As such, it is again a measure of the effectiveness of an investment strategy, with larger values signifying a better strategy.

We will show that the limiting distribution of $\tilde{\rho}_f(t)$ is again a gamma distribution, specifically,

Theorem 5.1 *For any constant f such that*

$$0 < f < \frac{2\tilde{\mu}}{\sigma^2}, \tag{5.2}$$

the RRORI process $\tilde{\rho}_f(t)$ converges as $t \to \infty$ to a gamma distribution, i.e.,

$$\tilde{\rho}_f(t) \xrightarrow{\mathrm{d}} \tilde{\rho}_f \sim gamma\left(\frac{2\tilde{\mu}}{f\sigma^2} - 1, \frac{2}{f\sigma^2}\right), \tag{5.3}$$

and hence

$$E[\tilde{\rho}_f] = \tilde{\mu} - \frac{1}{2}f\sigma^2. \tag{5.4}$$

Remark 5.1: Note that the mean RROI in (5.4) is a *strictly decreasing* function of f, the proportion invested in the risky stock (for $f > 0$). However, it is interesting to note that if we look at the *ratio of the expected values* of the discounted gain to the discounted cumulative investment, we get a value that is *independent of the proportion invested*, for any $t > 0$. Specifically, note that $E(e^{-rt}X_t^f) = X_0 e^{f\tilde{\mu}t}$, and so by Fubini we have $E \int_0^t e^{-rs}X_s^f \, ds = X_0(f\tilde{\mu})^{-1}\left(e^{f\tilde{\mu}t} - 1\right)$. Hence, it follows that

$$\frac{E(\text{discounted gain})}{E(\text{ discounted cumulative } \textit{risky} \text{ investment})} \equiv \frac{E\left(e^{-rt}X_t^f - X_0\right)}{fE \int_0^t e^{-rs}X_s^f \, ds} = \tilde{\mu}, \tag{5.5}$$

for any value of $f > 0$, and $t > 0$. (See Griffin [12] for a discussion of the different interpretation of measures related to (5.4) and (5.5) for simple gambling models.)

5.1   Proof of Theorem 5.1

To study the limiting distribution of $\tilde{\rho}_f(t)$, we first define the process

$$\tilde{R}_f(t) := \frac{e^{-rt}X_t^f}{f\left(X_0 + \int_0^t e^{-rs}X_s^f \, ds\right)}. \tag{5.6}$$

In the next theorem we show that the limiting behavior of the process $\{\tilde{\rho}_f(t)\}$ is determined by the limiting behavior of the process $\{\tilde{R}_f(t)\}$.

Theorem 5.2 *If for some random variable $\tilde{R}_f$ we have $\tilde{R}_f(t) \xrightarrow{\text{d}} \tilde{R}_f$, then for any constant f that satisfies (5.2), we have, as $t \to \infty$,*

$$\tilde{\rho}_f(t) \xrightarrow{\text{d}} \tilde{R}_f . \tag{5.7}$$

Proof: It is easiest to first define the discounted wealth process $Y_t^f := e^{-rt}X_t^f$, since then Ito's formula applied to Y shows that $dY_t^f = f\tilde{\mu}Y_t^f dt + \sigma f Y_t^f dW_t$, and so

$$Y_t^f = X_0 \exp\left\{\left(f\tilde{\mu} - \frac{1}{2}f^2\sigma^2\right)t + f\sigma W_t\right\} .$$

Noting now that $\tilde{\rho}_f(t) = \left(Y_t^f - Y_0\right) / \left(f\int_0^t Y_s^f ds\right)$, it is clear that we can write

$$\tilde{\rho}_f(t) := \tilde{R}_f(t) \left[\frac{Y_t^f - Y_0}{Y_t^f}\right] \left[\frac{Y_0 + \int_0^t Y_s^f ds}{\int_0^t Y_s^f ds}\right] . \tag{5.8}$$

From classical results on geometric Brownian motion we know that so long as (5.2) holds, $\inf_{t \geq 0} Y_t^f > 0$ and $Y_t^f \xrightarrow{\text{a.s.}} \infty$, which also implies that $\int_0^t Y_s^f ds \xrightarrow{\text{a.s.}} \infty$. Therefore it follows from (5.8) that if (5.2) holds we have

$$\lim_{t \to \infty}\left[\frac{Y_t^f - Y_0}{Y_t^f}\right] = 1 \quad \text{a.s}, \quad \text{and} \quad \lim_{t \to \infty}\left[\frac{Y_0 + \int_0^t Y_s^f ds}{\int_0^t Y_s^f ds}\right] = 1 \quad \text{a.s}, \tag{5.9}$$

which then implies (5.7) as a consequence of Theorem 4.4 of Billingsley[1]. ∎

Since (5.7) shows that the limiting behavior of the RRORI process $\{\tilde{\rho}_f(t)\}$ is determined by the limiting behavior of $\{\tilde{R}_f(t)\}$, we now turn to the study of the process $\tilde{R}_f(t)$. We will show that $\tilde{R}_f(t)$ is an ergodic diffusion process, which will allow us to identify its limiting distribution from previous results, and extract its limiting moments accordingly.

To begin, we note the following, which should be compared with Proposition 4.2.

Proposition 5.3 *The process $\tilde{R}_f(t)$ satisfies the stochastic differential equation*

$$d\tilde{R}_f(t) = \left(f\tilde{\mu}\tilde{R}_f(t) - f\tilde{R}_f(t)^2\right)dt + f\sigma\tilde{R}_f(t)dW_t , \tag{5.10}$$

i.e., $\tilde{R}_f(t)$ is a temporally homogeneous diffusion process with drift function $b(x) = f\tilde{\mu}x - fx^2$, and diffusion function $v^2(x) = f^2\sigma^2 x^2$.

Proof: Note that $\tilde{R}_f(t) = Y_t^f/D_t$, where $D_t := f\left(Y_0 + \int_0^t Y_s^f ds\right)$. Recognize that $dD_t = fY_t^f dt$, and apply Ito's formula to (Y_t^f/D_t) to get

$$d\left(\frac{Y_t^f}{D_t}\right) = D_t^{-1}dY_t^f - \left(\frac{Y_t^f}{D_t^2}\right)dD_t \equiv \left(\frac{Y_t^f}{D_t}\right)(f\tilde{\mu}dt + f\sigma dW_t) - f\left(\frac{Y_t^f}{D_t}\right)^2 dt\,,$$

which is equivalent to (5.10). ∎

Since for $f > 0$ the process $\tilde{R}_f(t)$ follows the stochastic differential equation (5.10), we may now identify $a = f\tilde{\mu}$, $b = f$ and $c = f\sigma$ in (4.7), (4.10) and therefore appeal to Lemma 1 to conclude the following:

Theorem 5.4 *For $0 < f < 2\tilde{\mu}/\sigma^2$, the process $\tilde{R}_f(t)$ is strongly ergodic and has a limiting gamma distribution. Specifically, as $t \to \infty$*

$$\tilde{R}_f(t) \xrightarrow{\mathrm{d}} \tilde{R}_f \sim gamma\left(\frac{2\tilde{\mu}}{f\sigma^2} - 1, \frac{2}{f\sigma^2}\right) \tag{5.11}$$

Furthermore, we have uniform integrability, and so

$$\lim_{t\to\infty} E\left[\tilde{R}_f(t)\right] = E[\tilde{R}_f] = \tilde{\mu} - \frac{1}{2}f\sigma^2\,. \tag{5.12}$$

This in conjunction with Theorem 5.2, and the fact that $\tilde{R}_f \stackrel{\mathrm{d}}{=} \tilde{\rho}_f$, proves Theorem 5.1. ∎

We next move on to study some consequences of Theorem 5.1.

5.2 Optimal Growth Policy

It is clear from (5.4) that here, where our interest is in the return from the *risky* investment, logarithmic utility and the optimal growth policy, f^*, no longer gives the maximal mean RRORI. In fact, $E(\tilde{\rho}_f) := \tilde{\mu} - f\sigma^2/2$ is a strictly decreasing function of the (fixed) proportion f and so has no maximizer for $f > 0$. This would seem to preclude any optimality properties for the optimal growth policy under this performance measure. Furthermore it turns out that the RRORI under the optimal growth policy has a limiting *exponential* distribution, which agrees with the asymptotic results obtained in Ethier and Tavare [9] for the discrete-time gambling model studied there.

Recall first that the optimal growth proportional investment strategy is $f^* := \tilde{\mu}/\sigma^2$. Therefore by taking $f = cf^*$ in (5.3), where f^* is the optimal growth policy and $0 < c < 2$, we find that

$$\tilde{\rho}_f \sim \text{gamma}\left(\frac{2}{c} - 1, \frac{2}{c\tilde{\mu}}\right) \tag{5.13}$$

which is independent of σ.

Remark 5.2: This now allows us to recover the asymptotic results for the discrete-time gambling model in Ethier and Tavare [9], since (5.13) implies that

$$\frac{\tilde{\rho}_f}{\tilde{\mu}} \sim \text{gamma}\left(\frac{2}{c} - 1, \frac{2}{c}\right) ,$$

which should be compared to (1.9) of Ethier and Tavare [9].

Taking $c = 1$ in (5.13), and recognizing that a $\text{gamma}(1, \beta) \equiv \text{Exp}(\beta)$, where $\text{Exp}(\lambda)$ denotes the exponential density function with mean $1/\lambda$, then allows us to deduce directly that the RRORI for the optimal growth policy tends to an exponential distribution.

Corollary 5.5 *For the case where $f = f^*$, the optimal growth policy, the associated RRORI, say $\tilde{\rho}_*(t)$, has a limiting* exponential *distribution. Specifically, $\tilde{\rho}_*(t) \xrightarrow{\text{d}} \tilde{\rho}_* \sim \text{Exp}\left(2/\tilde{\mu}\right)$, and so*

$$E[\tilde{\rho}_*] = \frac{\tilde{\mu}}{2} . \tag{5.14}$$

Checking the conditions given in the appendices for stochastic dominance of gamma random variables, we find that no orderings are possible in general, except for the following, which shows that the RRORI from the optimal growth policy dominates in the *increasing concave* stochastic order the RRORI from any other constant proportional strategy which *over invests* in the risky stock relative to the optimal growth policy, i.e., for an investor who is less risk averse than an investor with a logarithmic utility function. (This is quite limited compared to what we obtained earlier in Section 3 for the rate of return on *total* investment, the RROI.)

Corollary 5.6 *For $f > f^*$, where f^* is the optimal growth policy, we have $\tilde{\rho}_f \leq_{\text{icv}} \tilde{\rho}_*$.*

Proof: The conditions needed for the stochastic order relation $X_2 \leq_{\text{icv}} X_1$, for $X_i \sim \text{gamma}(\alpha_i, \beta_i)$, are $\beta_1 > \beta_2$ and $\alpha_1/\beta_1 \geq \alpha_2/\beta_2$ (see Appendix A.1). Taking $X_2 = \tilde{\rho}_f$ and $X_1 = \tilde{\rho}_*$, (5.13) and the fact that $\tilde{\rho}_* \sim \text{gamma}(1, 2/\tilde{\mu})$, shows that these two conditions are equivalent to the requirement $c > 1$, i.e., $f > f^*$. ∎

It is interesting to note that the conditions for the other forms of stochastic dominance (see Appendix A.1 and A.2), such as $\leq_{\text{icx}}$, $\leq_{\text{LR}}$ and $\leq_{\text{st}}$ here give contradictory conditions on c, so that no domination can be established. It is easily checked that the previous result generalizes to arbitrary constant proportional strategies as

Corollary 5.7 *Consider two proportional strategies, with $f_i = c_i f^*$ for $i = 1, 2$, and let $\tilde{\rho}_i$ denote their respective RRORIs. Then $c_2 > c_1$ implies $\tilde{\rho}_2 \leq_{\text{icv}} \tilde{\rho}_1$.*

19

5.3 Constant investment

For $\tilde{\mu} > 0$, the mean RRORI, $E(\tilde{\rho}_f)$ is a decreasing function of f, and we see that the maximal RRORI is taken at $f = 0$ with corresponding value $\tilde{\mu}$. As noted earlier, this of course does not correspond to an investor who invests all of his wealth in the risk-free asset, since for such an investor $\tilde{\rho}_0(t) \equiv 0$, for all $t > 0$. Instead, as we now show, this maximal return is instead achieved for an investor who invests a fixed *constant amount* (as opposed to a constant proportion) in the risky asset. Such investment policies are optimal when the investor has an *exponential* utility function (see e.g., Merton[22] or Browne[5]).

Interestingly enough, the RRORI for such an investment policy is independent of the particular constant amount invested.

Proposition 5.8 *Consider an investment strategy that always keeps a fixed* constant total amount *of money, say $\$\kappa$, invested in the risky stock, regardless of the wealth level with the remainder invested in the risk-free bond. Then the RRORI for this strategy, $\{\tilde{\rho}_\kappa(t), t \geq 0\}$, is an ergodic Gaussian process with constant mean function $E[\tilde{\rho}_\kappa(t)] = \tilde{\mu}$, and covariance function, for $s \leq t$*

$$E\left[(\tilde{\rho}_\kappa(t) - \tilde{\mu})(\tilde{\rho}_\kappa(s) - \tilde{\mu})\right] = \frac{r\sigma^2}{2}\left(\frac{1 - e^{-2rs}}{(1 - e^{-rt})(1 - e^{-rs})}\right). \tag{5.15}$$

Therefore, as $t \to \infty$

$$\tilde{\rho}_\kappa(t) \xrightarrow{\mathrm{d}} \tilde{\rho}_\kappa \sim N\left(\tilde{\mu}, \frac{r\sigma^2}{2}\right), \tag{5.16}$$

where $N(\alpha, \beta^2)$ denotes a normal distribution with mean α and variance β^2.

Proof:

If the investor always keeps $\$\kappa$ invested in the risky stock, regardless of his wealth level, with the remainder invested in the bond, then this investor's wealth, say X_t^κ, evolves as

$$dX_t^\kappa = \kappa\frac{dS_t}{S_t} + (X_t^\kappa - \kappa)\frac{dB_t}{B_t} = (rX_t^\kappa + \kappa\tilde{\mu}) + \kappa\sigma dW_t, \tag{5.17}$$

which is a simple linear stochastic differential equation, sometimes called "compounding Brownian motion". It follows that (see e.g.[18, Section 5.6])

$$e^{-rt}X_t^\kappa = X_0 + \frac{\kappa\tilde{\mu}}{r}(1 - e^{-rt}) + \kappa\sigma\int_0^t e^{-rs}dW_s \tag{5.18}$$

and so the discounted gain for a constant investor, $e^{-rt}X_t^\kappa - X_0$ is a Gaussian process with mean function $\kappa\tilde{\mu}(1 - e^{-rt})/r$ and variance function $\kappa^2\sigma^2(1 - e^{-2rt})/(2r)$.

The cumulative amount invested in the risky stock for a constant investor is $\int_0^t \kappa ds = \kappa t$, and the discounted cumulative amount invested is $\int_0^t \kappa e^{-rs} ds = \kappa(1 - e^{-rt})/r$. Therefore, the RRORI for a constant investor, $\tilde{\rho}_\kappa(t)$ is given by

$$\tilde{\rho}_\kappa(t) := \frac{e^{-rt} X_t^\kappa - X_0}{\frac{\kappa}{r}(1 - e^{-rt})} = \tilde{\mu} + \frac{r\sigma}{1 - e^{-rt}} \int_0^t e^{-rs} dW_s \qquad (5.19)$$

where the second equality follows from (5.18). It is clear from (5.19) that *for any constant* κ, $\tilde{\rho}_\kappa(t)$ is a Gaussian process with mean $\tilde{\mu}$ and covariance function given by (5.15). It is also clear from (5.19) that $\tilde{\rho}_\kappa(t) \xrightarrow{d} \tilde{\rho}_\kappa$, where $\tilde{\rho}_\kappa \sim N\left(\tilde{\mu}, \frac{r\sigma^2}{2}\right)$. ∎

Remark 5.3: Comparing the RRORI for the optimal growth policy with the RRORI for a constant investment policy shows that the mean RRORI under the optimal growth policy is half the mean RRORI for a constant amount investment policy, i.e., (5.14) and (5.16) shows

$$E(\tilde{\rho}_*) = \frac{1}{2} E(\tilde{\rho}_\kappa).$$

This rather disturbing result can best be understood in the context of the fact that the constant investor can go bankrupt under his investment policy, while a proportional investor will never go bankrupt. Thus the halved return can be considered the insurance premium for never going bankrupt. This is discussed in Ethier and Tavare [9] as well.

It turns out that this comparison is specific to the case of a single risky stock, and does *not generalize to the multiple stock case*, as we will see in the next section.

The distributional results in Theorem 5.1 and Proposition 5.8 allow for the explicit computation of various probabilities. The probability that $\tilde{\rho}_\kappa$ is positive is directly seen to be $P(\tilde{\rho}_\kappa > 0) = \Phi\left(\sqrt{\frac{2}{r}} \frac{\tilde{\mu}}{\sigma}\right)$, where $\Phi(\cdot)$ is the CDF of the standard normal variate. This quantity is of course *not* the probability that the constant investor never goes bankrupt. This latter quantity is in fact

$$P\left(\inf_{0 \le t \le \infty} X_t^\kappa > 0\right) = \frac{\Phi\left(\frac{\sqrt{2r}}{\kappa\sigma}(X_0 + \kappa\tilde{\mu})\right) - \Phi\left(\frac{\sqrt{2r}}{\sigma}\tilde{\mu}\right)}{1 - \Phi\left(\frac{\sqrt{2r}}{\sigma}\tilde{\mu}\right)}. \qquad (5.20)$$

This last result can be established from the fact that (5.17) shows that X_t^κ is a diffusion process with drift $b(x) = rx + \kappa\tilde{\mu}$ and diffusion function $v^2(x) = \kappa^2\sigma^2$. Hence, it follows that its *scale* function is given by

$$s_\kappa(x) = e^{r\tilde{\mu}/\sigma^2} \exp\left\{-\frac{r}{\kappa^2\sigma^2}(x + \kappa\tilde{\mu})^2\right\},$$

and that therefore $P\left(\inf_{0 \le t \le \infty} X_t^\kappa > 0\right) = \int_0^{X_0} s_\kappa(x)dx / \int_0^\infty s_\kappa(x)dx$, which reduces to (5.20).

Another interesting quantity is the *probability that the (optimal) proportional return $\tilde{\rho}_*$ exceeds the return on constant investment $\tilde{\rho}_\kappa$, $P(\tilde{\rho}_* > \tilde{\rho}_\kappa)$*. This can be computed by recognizing that

$$P(\tilde{\rho}_* > \tilde{\rho}_\kappa | \tilde{\rho}_\kappa = x) = e^{-2x/\tilde{\mu}},$$

which then implies

$$
\begin{aligned}
P(\tilde{\rho}_* > \tilde{\rho}_\kappa) &= P(\tilde{\rho}_\kappa < 0) + \frac{1}{\sqrt{\pi r \sigma^2}} \int_0^\infty \exp\left\{-\left[\frac{2x}{\tilde{\mu}} + \frac{1}{r\sigma^2}(x - \tilde{\mu})^2\right]\right\} dx \\
&= 1 - \Phi\left(\sqrt{\frac{2}{r}}\left(\frac{\tilde{\mu}}{\sigma}\right)\right) + \exp\left\{\frac{|\tilde{\mu}^2 - r\sigma^2| - \tilde{\mu}^3}{r\sigma^2\tilde{\mu}}\right\} \Phi\left(\sqrt{\frac{2|\tilde{\mu}^2 - r\sigma^2|}{\tilde{\mu}r\sigma^2}}\right).
\end{aligned}
$$

6 Multiple Risky Stocks

As promised earlier, in this section we return to consider the case with k risky stocks. Since the wealth process, X_t^f of (2.12), is still one dimensional it is not hard to extend the previous analysis to this case. As such, we merely highlight the results, leaving the details for the reader. It does turn out that the results for the RRORI measure differ somewhat from the case with a single risky stock.

6.1 RROI

For any fixed proportional strategy $\boldsymbol{f}$ which satisfies condition (2.14), the RROI process $\rho_f(t)$ converges in distribution to a random variable which has a gamma distribution. I.e., as $t \to \infty$,

$$\rho_f(t) \xrightarrow{\text{d}} \rho_f \sim \text{gamma}\left(\frac{2\left(r + \boldsymbol{f}'\tilde{\boldsymbol{\mu}}\right)}{\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f}} - 1, \frac{2}{\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f}}\right).$$

It is clear once again that the mean of this distribution, $E[\rho_f] = r + \boldsymbol{f}'\tilde{\boldsymbol{\mu}} - \frac{1}{2}\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f}$, is again maximized at the log-optimal, or optimal growth policy, which is now $\boldsymbol{f}^* = \boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}$. For this strategy the RROI, $\rho_*(t)$, still satisfies $\rho_*(t) \xrightarrow{\text{d}} \rho_*$ where $\rho_* \sim \text{gamma}\left(\frac{r+\gamma}{\gamma}, \frac{1}{\gamma}\right)$, and hence with mean $E[\rho_*] = r + \gamma$, but now we have $\gamma := (1/2)\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}$. The comparisons in Corollaries 3.3 and 3.4 still hold in terms of $\boldsymbol{f} = c\boldsymbol{f}^*$.

All of this can be proved in essentially the identical manner as in Section 4, with the only proviso being that now $\{R_f(t)\}$ is a temporally homogeneous diffusion process with drift function $b(x) = (r + \boldsymbol{f}'\tilde{\boldsymbol{\mu}})x - x^2$ and diffusion function $v^2(x) = \boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f} x^2$. As such, no new difficulties are encountered.

6.2 RRORI

Here the situation is slightly different. The amount invested in the risky stock at time t is $(\boldsymbol{f}'\boldsymbol{1})X_t^f$, and so the RRORI, $\tilde{\rho}_f(t)$, is defined by

$$\tilde{\rho}_f(t) := \frac{e^{-rt}X_t^f - X_0}{(\boldsymbol{f}'\boldsymbol{1})\int_0^t e^{-rs}X_s^f\,ds}\,.$$

The analog of Theorem 5.1, which holds so long as $\boldsymbol{f}'\tilde{\boldsymbol{\mu}} - (1/2)\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f} > 0$, is

$$\tilde{\rho}_f(t) \xrightarrow{\text{d}} \tilde{\rho}_f \sim \text{gamma}\left(\frac{2\boldsymbol{f}'\tilde{\boldsymbol{\mu}}}{\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f}} - 1\,,\, \frac{2}{\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f}}\right),$$

and hence $E[\rho_f] = \left(\boldsymbol{f}'\tilde{\boldsymbol{\mu}} - \frac{1}{2}\boldsymbol{f}'\boldsymbol{\Sigma}\boldsymbol{f}\right)/\boldsymbol{f}'\boldsymbol{1}$. For this case we also note that the ratio of the expected values is no longer independent of the policy (see Remark 5.1), and is in fact

$$\frac{E\left(e^{-rt}X_t^f - X_0\right)}{(\boldsymbol{f}'\boldsymbol{1})E\int_0^t X_s^f\,ds} = \frac{\boldsymbol{f}'\tilde{\boldsymbol{\mu}}}{\boldsymbol{f}'\boldsymbol{1}}\,.$$

The RRORI under the optimal growth policy, $\boldsymbol{f}^*$ is still *exponential*, but its mean now depends on the covariance matrix as well: specifically,

$$E\tilde{\rho}_* = \frac{\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\tilde{\boldsymbol{\mu}}}{2\tilde{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\boldsymbol{1}}\,. \tag{6.1}$$

Comparisons with a *constant amount* investment strategy no longer seem relevant. For a constant amount policy, say $\boldsymbol{\kappa} := (\kappa_1,\ldots,\kappa_k)'$, where κ_i is the amount of money invested in stock i, the wealth process is distributionally equivalent to the process X_t^κ which evolves as

$$dX_t^\kappa = (rX_t^\kappa + \boldsymbol{\kappa}'\tilde{\boldsymbol{\mu}}) + \sqrt{\boldsymbol{\kappa}'\boldsymbol{\Sigma}\boldsymbol{\kappa}}\,\,dW_t\,.$$

Since the amount invested at time t is just $\boldsymbol{\kappa}'\boldsymbol{1}$, the cumulative discounted investment is just $(\boldsymbol{\kappa}'\boldsymbol{1}/r)(1 - e^{-rt})$, and hence the RRORI for this policy, $\tilde{\rho}_\kappa(t)$ is the Gaussian process

$$\tilde{\rho}_\kappa(t) := \frac{e^{-rt}X_t^\kappa - X_0}{(\boldsymbol{\kappa}'\boldsymbol{1}/r)(1 - e^{-rt})} \equiv \frac{\boldsymbol{\kappa}'\tilde{\boldsymbol{\mu}}}{\boldsymbol{\kappa}'\boldsymbol{1}} + \frac{\sqrt{\boldsymbol{\kappa}'\boldsymbol{\Sigma}\boldsymbol{\kappa}}}{\boldsymbol{\kappa}'\boldsymbol{1}}\left(\frac{r}{1 - e^{-rt}}\right)\int_0^t e^{-rt}dW_s\,,$$

which depends on the specific value of $\boldsymbol{\kappa}$. It follows that $\tilde{\rho}_\kappa(t) \xrightarrow{\text{d}} \tilde{\rho}_\kappa$, where

$$\tilde{\rho}_\kappa \sim N\left(\frac{\boldsymbol{\kappa}'\tilde{\boldsymbol{\mu}}}{\boldsymbol{\kappa}'\boldsymbol{1}}\,,\, \frac{2\boldsymbol{\kappa}'\boldsymbol{\Sigma}\boldsymbol{\kappa}}{2(\boldsymbol{\kappa}'\boldsymbol{1})^2}\right)\,. \tag{6.2}$$

What is apparent from (6.1) and (6.2) is that *if* the constant amount investor chooses $\boldsymbol{\kappa} \equiv \boldsymbol{f}^*$, then the mean RRORI for the constant amount investor is twice the mean RRORI for a log-optimal, or optimal growth policy investor.

A   Appendix

A.1   Proof of Corollary 3.3

Let $f = cf^*$, where c is an arbitrary constant. Note that it then follows from Theorem 3.1 that under this parameterization we have

$$\rho_f \sim \text{gamma}\left(\frac{r + 2c\gamma - c^2\gamma}{c^2\gamma}, \frac{1}{c^2\gamma}\right). \tag{A.1}$$

Recall first (Stoyan[28]) that if $X_i \sim \text{gamma}(\alpha_i, \beta_i)$ for $i = 1, 2$, then

$$\alpha_1 < \alpha_2 \quad \text{and} \quad \frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2} \quad \text{implies} \quad X_2 \leq_{\text{icx}} X_1. \tag{A.2}$$

Taking $X_1 = \rho_*$ and $X_2 = \rho_f$, we see from the parameterization in (3.7) and (A.1) that the two conditions for the dominance $\rho_f \leq_{\text{icx}} \rho_*$ in (A.2) are equivalent to

$$\text{(i)} \quad \frac{r + \gamma}{\gamma} < \frac{r + 2c\gamma - c^2\gamma}{c^2\gamma}, \quad \text{and} \quad \text{(ii)} \quad r + \gamma \geq r + 2c\gamma - c^2\gamma.$$

Condition (ii) holds trivially by the fact that f^* was chosen to maximize the mean RROI. Condition (i) is equivalent to the requirement that $Q(c) < 0$ where $Q(c)$ is the quadratic defined by

$$Q(c) := c^2(r + 2\gamma) - c2\gamma - r. \tag{A.3}$$

It is easily checked that the two roots to the equation $Q(c) = 0$ are $c^* = 1$ and $c_* = \frac{-r}{r + 2\gamma}$, with $Q(c) \geq 0$ only for $c \geq c^*$ or $c \leq c_*$, and correspondingly with $Q(c) < 0$ for $c_* < c < c^*$. Hence for any $f < f^*$ we have $c < 1$, for which $\rho_f \leq_{\text{icx}} \rho_*$.

Similarly, it is well known (ibid) that for $X_i \sim \text{gamma}(\alpha_i, \beta_i)$ for $i = 1, 2$,

$$\beta_1 > \beta_2 \quad \text{and} \quad \frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2} \quad \text{implies} \quad X_2 \leq_{\text{icv}} X_1. \tag{A.4}$$

Under the parameterization of (A.1) the two conditions for the dominance $\rho_f \leq_{\text{icv}} \rho_*$ in (A.4) then become (ii), which holds by construction of f^*, and the new condition $\frac{1}{\gamma} > \frac{1}{c^2\gamma}$. But this trivially holds for $c > 1$ which is the same as $f > f^*$. ∎

A.2   Proof of Corollary 3.4

Taking $X_i \sim \text{gamma}(\alpha_i, \beta_i)$, it is clear that the inequality in (3.10) is equivalent to

$$\left(\frac{y}{x}\right)^{\alpha_1 - \alpha_2} \geq e^{(\beta_1 - \beta_2)(y - x)}, \quad \text{for all} \quad x \leq y,$$

for which a necessary and sufficient condition is obviously $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$. These two conditions are in fact the conditions given in Stoyan[28] for the (weaker) relationship $X_2 \leq_{\text{st}} X_1$.

Hence, a sufficient condition for the likelihood ratio order to apply to two gamma random variables is that they be stochastically ordered. (This is obviously not the case in general.)

Using (3.7) and (A.1) we see that these two conditions (i.e., $\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$) for the dominating relationship $\rho_f \leq_{\mathrm{LR}} \rho_*$ (which implies $\rho_f \leq_{\mathrm{st}} \rho_*$) are equivalent to the requirement that c simultaneously satisfy the inequalities

$$\text{(iii)} \quad \frac{r+\gamma}{\gamma} \geq \frac{r + 2c\gamma - c^2\gamma}{c^2\gamma}, \quad \text{and} \quad \text{(iv)} \quad \frac{1}{\gamma} \leq \frac{1}{c^2\gamma}.$$

Condition (iv) is obviously equivalent to $c^2 \leq 1$, while (iii) reduces to the requirement that $Q(c) \geq 0$, where $Q(c)$ is the quadratic of (A.3). However, as noted above, the two roots to the equation $Q(c) = 0$ are $c^* = 1$ and $c_* = \frac{-r}{r+2\gamma}$, with $Q(c) \geq 0$ only for $c \geq c^*$ or $c \leq c_*$. Hence $Q(c) \geq 0$ implies that either $c \geq 1$ or $c \leq -r/(r+2\gamma)$. The former is impossible (except at equality $c = 1$) by (iv). The requirement that $r + 2c\gamma - c^2\gamma > 0$, which is just (2.15), is equivalent to the requirement (recall (2.16))

$$1 - \sqrt{\frac{r+\gamma}{\gamma}} < c < 1 + \sqrt{\frac{r+\gamma}{\gamma}}.$$

Since $-r/(r+2\gamma) > 1 - \sqrt{\frac{r+\gamma}{\gamma}}$, we obtain the result. ∎

It should be noted that Corollary 3.4 extends to arbitrary constant proportional policies as in the following.

Proposition A.1 *Consider two proportional strategies, with $f_i = c_i f^*$, for $i = 1, 2$, where f^* is the optimal growth policy of (3.6), and let ρ_i, $i = 1, 2$ denote their corresponding RROIs. Then a necessary and sufficient condition for $\rho_2 \leq_{\mathrm{LR}} \rho_1$ (and by implication, also $\rho_2 \leq_{\mathrm{st}} \rho_1$) is*

$$1 \leq \frac{c_1^2}{c_2^2} \leq \frac{r + 2c_1\gamma - c_1^2\gamma}{r + 2c_2\gamma - c_2^2\gamma}. \tag{A.5}$$

Proof: Using (A.1) the two conditions ($\alpha_1 \geq \alpha_2$ and $\beta_1 \leq \beta_2$) for the relation $\rho_2 \leq_{\mathrm{LR}} \rho_1$ become

$$\frac{r + 2c_1\gamma - c_1^2\gamma}{c_1^2\gamma} \geq \frac{r + 2c_2\gamma - c_2^2\gamma}{c_2^2\gamma}, \quad \text{and} \quad \frac{1}{c_1^2\gamma} \leq \frac{1}{c_2^2\gamma}, \tag{A.6}$$

which together imply the simultaneous inequality (A.5). ∎

References

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, NY.

[2] BLACK, F., AND PEROLD, A.F. (1992). Theory of Constant Proportion Portfolio Insurance. *Jour. Econ. Dyn. and Cntrl.* **16**, 403-426.

[3] BLACK, F. AND SCHOLES, M. (1973). The Pricing of Options and Corporate Liabilities. *J. Polit. Econom.* **81**, 637-659.

[4] BREIMAN, L. (1961). Optimal Gambling Systems for Favorable Games. *Fourth Berkeley Symp. Math. Stat. and Prob.* **1**, 65 - 78.

[5] BROWNE, S. (1995). Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Math. of Oper. Res.* **20**, 937-958.

[6] BROWNE, S. (1997). Survival and Growth with a Fixed Liability: Optimal Portfolios in Continuous Time. *Math. of Oper. Res.* **22**, 468-493.

[7] BROWNE, S. AND WHITT, W. (1996). The Bayesian Kelly Criterion. *Adv. Applied Prob.* **28**, 1145-1176.

[8] DYBVIG, P.H. (1995). Duesenberry's Ratcheting of Consumption: Optimal Dynamic Consumption and Investment given Intolerance for any Decline in Standard of Living. *Review of Economic Studies*, **62**, 287-313.

[9] ETHIER, S.N. AND TAVARE, S. (1983). The Proportional Bettor's Return on Investment. *Jour. Applied Prob.* **20**, 563 - 573.

[10] FINKELSTEIN, M. AND WHITELY, R. (1981). Optimal Strategies for Repeated Games. *Adv. Applied Prob.* **13**, 415 - 428.

[11] GOTTLIEB, G. (1985). An Optimal Betting Strategy for Repeated Games. *Jour. Applied Prob.* **22**, 787 - 795.

[12] GRIFFIN, P.A. (1984). Different Measures of Win Rate for Optimal Proportional Betting. *Management Science* **30**, 1540-1547.

[13] GROSSMAN, S.J., AND ZHOU, Z. (1993). Optimal Investment Strategies for Controlling Drawdowns. *Mathematical Finance* **3**, 241-276.

[14] HAKANSSON, N.H. (1970). Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions. *Econometrica* **38**, 5, 587-607.

[15] HEATH, D., OREY, S., PESTIEN, V. AND SUDDERTH, W. (1987). Minimizing or Maximizing the Expected Time to Reach Zero. *SIAM J. Contr. and Opt.* **25**, 1, 195-205.

[16] HUANG, C. AND LITZENBERGER, R.H. (1988). *Foundations for Financial Economics.* Prentice Hall.

[17] KARATZAS, I. (1989). Optimization Problems in the Theory of Continuous Trading. *SIAM J. Contr. and Opt.* **7**, 6, 1221-1259.

[18] KARATZAS, I. AND SHREVE, S. (1988). *Brownian Motion and Stochastic Calculus.* Springer-Verlag, NY.

[19] KARLIN, S. AND TAYLOR, H.M. (1981). *A Second Course on Stochastic Processes.* Academic, NY.

[20] KELLY, J. (1956). A New Interpretation of Information Rate. *Bell Sys. Tech. J.* **35**, 917 - 926.

[21] MARKOWITZ, H.M. (1976). Investment for the Long Run: New Evidence for an Old Rule. *The Journal of Finance* **16**, 1273-1286.

[22] MERTON, R. (1971). Optimum Consumption and Portfolio Rules in a Continuous Time Model. *Jour. Econ. Theory* **3**, 373-413.

[23] MERTON, R. (1990). *Continuous Time Finance.* Blackwell, Ma.

[24] OREY, S., PESTIEN, V. AND SUDDERTH, W. (1987). Reaching Zero Rapidly. *SIAM J. Cont. and Opt.* **25**, 5, 1253-1265.

[25] PEROLD, A.F. AND SHARPE, W.F. (1988). Dynamic Strategies for Asset Allocation. *Financial Analyst Journal*, Jan/Feb: 16-27.

[26] PRATT, J.W. (1964). Risk Aversion in the Small and in the Large. *Econometrica* **32**, 122-136.

[27] SUNDARESAN, S.M. (1989). Intertemporally Dependent Preferences and the Volatility of Consumption and Wealth. *Rev. Fin. Studies* **2**, 73-89.

[28] STOYAN, D. (1983). *Comparison Methods for Queues and other Stochastic Models.* Wiley.

[29] THORP, E.O. (1969). Optimal Gambling Systems for Favorable Games. *Rev. Int. Stat. Inst.* **37**, 273 - 293.

[30] THORP, E.O. (1971). Portfolio Choice and the Kelly Criterion. *Bus. and Econ. Stat. Sec., Proc. Amer. Stat. Assoc.*, 215-224.