Discrete Comput Geom 10:411–420 (1993)

Discrete & Computational
Geometry

The Reverse Isoperimetric Problem for Gaussian Measure*

Keith Ball

Department of Mathematics, University College London,
Gower Street, London, England, and
Department of Mathematics, Texas A&M University,
College Station, TX 77843, USA

Abstract. It is shown that if g is the standard Gaussian density on $\mathbb{R}^n$ and C is a convex body in $\mathbb{R}^n$,

$$\int_{\partial C} g \leq 4n^{1/4}.$$

The arguments presented raise several questions in integral geometry.

1. Introduction

Borell [B] and Sudakov and Tsirel'son [ST] proved that if γ is the standard Gaussian measure on $\mathbb{R}^n$, $t \in (0, 1)$ and $\varepsilon > 0$, then among all measurable $A \subset \mathbb{R}^n$ with $\gamma(A) = t$, the sets for which A^ε (the ε-neighborhood of A) has smallest Gaussian measure, are half-spaces. Thus, half-spaces solve the isoperimetric problem for Gaussian measure on $\mathbb{R}^n$.

Motivated by several probabilistic problems, Mushtari and Kwapien asked about reverse inequalities: in particular, if g is the density of γ, how large can the integral

$$\int_{\partial C} g \tag{1}$$

be if C is a convex body in $\mathbb{R}^n$? The integral is taken with respect to the

* This research was supported in part by NSF Grant DMS-8807243.

$(n-1)$-dimensional Hausdorff measure on ∂C. Let

$$v_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$$

be the volume of the Euclidean unit ball in $\mathbb{R}^n$. Then, if C is a ball of radius r, centered at 0, expression (1) is

$$n v_n r^{n-1} \frac{1}{(\sqrt{2\pi})^n} e^{-r^2/2},$$

whose maximum value (attained when $r = \sqrt{n-1}$) is

$$\frac{n}{2^{n/2}\Gamma(n/2+1)} (n-1)^{(n-1)/2} e^{-(n-1)/2} \sim \frac{1}{\sqrt{\pi}}.$$

If C is a convex body which is very flat, and so approximately a "two-sided" one-codimensional subspace, then

$$\int_{\partial C} g \approx 2 \int_{\mathbb{R}^{n-1}} \frac{1}{(\sqrt{2\pi})^n} e^{-|x|^2/2}\, dx$$
$$= \sqrt{\frac{2}{\pi}}.$$

However, neither of these examples can be extremal for the problem, since if Q is a centrally symmetric cube of appropriate size, in $\mathbb{R}^n$,

$$\int_{\partial Q} g \geq \frac{\sqrt{\log n}}{e}.$$

This observation is proved in Proposition 2 below.

It is very easy to obtain an upper estimate of $\sqrt{n}$ for expression (1) for general bodies. The purpose of this paper is to improve this trivial estimate. The argument used is a rather natural approach to the problem, suggested by Cauchy's formula for the surface area of convex bodies. (See p. 89 of [E] for a statement and proof of Cauchy's formula.) Unfortunately, this approach cannot give an estimate better than $Kn^{1/4}$ for some constant K, whereas vague analogies between this problem and others suggest that the correct estimate should be a multiple of $\log n$. It was noticed by Milman that, as a consequence of Pisier's estimate for the K-convexity constant of a finite-dimensional normed space [P], every n-dimensional normed space has a representation $(\mathbb{R}^n, \|\cdot\|)$ with unit ball C of volume 1 and

$$\int_{\mathbb{R}^n} \|x\| g(x)\, dx \leq (\text{constant}) \log n.$$

On the other hand, the measure constructed in Theorem 4 answers a question which is itself quite natural, and it is the appearance of the "unusual" growth rate, $n^{1/4}$, in this context, which is perhaps the most interesting feature of this argument. The method used here raises several further geometric questions which are described in Section 4. Section 2 contains the simple upper and lower estimates mentioned above and Section 3 gives the main result.

2. Simple Bounds

Proposition 1. *If g is the standard Gaussian density on $\mathbb{R}^n$ and C is a convex body in $\mathbb{R}^n$,*

$$\int_{\partial C} g < \sqrt{n}.$$

Proof. For $x \in \mathbb{R}^n$,

$$e^{-|x|^2/2} = \int_0^\infty t e^{-t^2/2} \chi_{B(t)}(x)\, dt,$$

where $\chi_{B(t)}$ is the characteristic function of the n-dimensional Euclidean ball of radius t, centered at 0. Hence

$$\begin{aligned}
\frac{1}{(\sqrt{2\pi})^n} \int_{\partial C} e^{-|x|^2/2} &= \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty t e^{-t^2/2} |\partial C \cap B(t)|\, dt \\
&\le \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty t e^{-t^2/2} |\partial (C \cap B(t))|\, dt \\
&< \frac{1}{(\sqrt{2\pi})^n} \int_0^\infty t e^{-t^2/2} |\partial B(t)|\, dt \\
&= \frac{n v_n}{(\sqrt{2\pi})^n} \int_0^\infty t^n e^{-t^2/2}\, dt < \sqrt{n}.
\end{aligned}$$

□

Proposition 2. *If $Q \subset \mathbb{R}^n$ is a centrally symmetric cube of appropriate size, then*

$$\int_{\partial Q} g \ge \frac{\sqrt{\log n}}{e}.$$

Proof. Let Q be a symmetric cube of side $2r$. Then

$$\int_{\partial Q} g = \frac{2n}{(\sqrt{2\pi})^n} e^{-r^2/2} \left(\int_{-r}^{r} e^{-t^2/2}\, dt \right)^{n-1}.$$

Since

$$\frac{2}{\sqrt{2\pi}}\int_r^\infty e^{-t^2/2}\,dt \le \frac{2}{\sqrt{2\pi}}\frac{1}{r}\int_r^\infty te^{-t^2/2}\,dt = \sqrt{\frac{2}{\pi}}\frac{e^{-r^2/2}}{r},$$

$$\frac{1}{\sqrt{2\pi}}\int_{-r}^r e^{-t^2/2}\,dt \ge 1 - \sqrt{\frac{2}{\pi}}\frac{e^{-r^2/2}}{r}.$$

Hence

$$\int_{\partial Q} g \ge \frac{2n}{\sqrt{2\pi}} e^{-r^2/2}\left(1 - \sqrt{\frac{2}{\pi}}\frac{e^{-r^2/2}}{r}\right)^{n-1}.$$

Choose r so that

$$\sqrt{\frac{2}{\pi}}\frac{e^{-r^2/2}}{r} = \frac{1}{n}.$$

Then $r \ge \sqrt{\log n}$ and so

$$e^{-r^2/2} \ge \sqrt{\frac{\pi}{2}}\frac{\sqrt{\log n}}{n}.$$

So, for this r,

$$\int_{\partial Q} g \ge \sqrt{\log n}\left(1 - \frac{1}{n}\right)^{n-1} > \frac{\sqrt{\log n}}{e}. \qquad \square$$

3. The Main Result

The proof of Theorem 4, below, makes use of the following simple rearrangement inequality. Henceforth, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ and σ_{n-1} denotes the rotation invariant probability on S^{n-1}

Lemma 3. *Let f and g be nondecreasing, real-valued functions on* $[0, 1]$. *Then the integral*

$$\int_{S^{n-1}} f(|\langle\theta, \varphi\rangle|)g(|\langle\theta, \psi\rangle|)\,d\sigma_{n-1}(\theta)$$

is minimized over $\varphi, \psi \in S^{n-1}$ when $\langle\varphi, \psi\rangle = 0$.

Remark. The integral is obviously maximized when $\varphi = \pm\psi$ since in this case the functions $f(|\langle\cdot, \varphi\rangle|)$ and $g(|\langle\cdot, \psi\rangle|)$ are similarly arranged on S^{n-1}: i.e.,

$$f(|\langle\theta_1, \varphi\rangle|) > f(|\langle\theta_2, \varphi\rangle|)$$

if and only if

$$g(|\langle\theta_1, \psi\rangle|) > g(|\langle\theta_2, \psi\rangle|).$$

Proof of Lemma 3. Assume $\varphi \neq \pm\psi$ and let H be the two-dimensional subspace of $\mathbb{R}^n$ spanned by $\{\varphi, \psi\}$. For $u \in H^\perp$, $|u| < 1$, consider the circle

$$C_u = S^{n-1} \cap (H + u).$$

The restrictions of $f(|\langle\cdot, \varphi\rangle|)$ and $g(|\langle\cdot, \psi\rangle|)$ to C_u, are oppositely arranged if $\langle\varphi, \psi\rangle = 0$, and the distribution of each of these functions on C_u depends only upon $|u|$. So, if λ is the obvious measure on C_u,

$$\int_{C_u} f(|\langle\theta, \varphi\rangle|)g(|\langle\theta, \psi\rangle|)\, d\lambda(\theta)$$

is minimized when $\langle\varphi, \psi\rangle = 0$. Integration over all $u \in H^\perp$ of norm at most 1, with respect to an appropriate measure, gives the desired inequality. □

Remark. Presumably, analogous inequalities hold for n functions "on" S^{n-1}. For $n \geq 2$, the trivial argument used above does not seem to be of much use.

Theorem 4. *Let g be the standard Gaussian density on $\mathbb{R}^n$, $n \geq 2$, and let C be a convex body in $\mathbb{R}^n$. Then*

$$\int_{\partial C} g \leq 4n^{1/4}.$$

Proof. For a unit vector θ in $\mathbb{R}^n$, let P_θ be the orthogonal projection onto the one-codimensional subspace of $\mathbb{R}^n$, $\langle\theta\rangle^\perp$. The aim is to construct a measure μ on $\mathbb{R}^{n-1}$ which is rotation invariant, absolutely continuous with respect to Lebesgue measure and which satisfies

$$\int_S g \leq \int_{S^{n-1}} \mu(P_\theta S)\, d\sigma_{n-1}(\theta) \tag{2}$$

whenever S is a Borel subset of a hyperplane in $\mathbb{R}^n$. (The quantity $\mu(P_\theta S)$ is interpreted via some isometry between $P_\theta(\mathbb{R}^n)$ and $\mathbb{R}^{n-1}$.)

Now, if C is a convex body, then "μ almost every" line in a given direction hits ∂C at most twice: i.e., almost all of $P_\theta(\mathbb{R}^n)$ is covered at most twice by $P_\theta(\partial C)$. Hence

$$\int_{\partial C} g \le 2\mu(\mathbb{R}^{n-1}).$$

If μ is a measure on $\mathbb{R}^n$ with density F where $F(x) = f(|x|)$ for some

$$f\colon [0, \infty) \to [0, \infty)$$

then, in the limit for small pieces of hyperplane centered at $r\varphi$ ($r \ge 0$, $\varphi \in S^{n-1}$) with unit normal ψ, inequality (2) becomes

$$\tilde{g}(r) \le \int_{S^{n-1}} f(r(1 - \langle \theta, \varphi\rangle^2)^{1/2}) \cdot |\langle \theta, \psi\rangle| \, d\sigma_{n-1}(\theta), \tag{3}$$

where $\tilde{g}(r) = (1/(\sqrt{2\pi})^n)e^{-r^2/2}$.

A function f will be constructed that is (nonnegative and) nonincreasing on $[0, \infty)$ and that satisfies (3) whenever φ and ψ are perpendicular. Since the function $t \mapsto f(r(1 - t^2)^{1/2})$ will then be increasing on $[0, 1]$, Lemma 3 will imply that the right-hand side of (3) is minimized when φ and ψ are perpendicular, and hence that (3) holds for all φ and ψ.

Fix $\varphi \in S^{n-1}$: the right-hand side of (3) is the same for all ψ perpendicular to φ. The average of $|\langle \theta, \psi\rangle|$ over all such ψ, as a function of θ, is proportional to the length of the projection $P_\varphi(\theta)$; i.e., to $(1 - \langle \theta, \varphi\rangle^2)^{1/2}$. The constant of proportionality is

$$\int_{S^{n-2}} |\psi_1| \, d\sigma_{n-2}(\psi),$$

where ψ_1 is the first coordinate of ψ. Hence if $\psi \perp \varphi$, the right-hand side of (3) is

$$\begin{aligned}
&\int_{S^{n-2}} |\psi_1| \, d\sigma_{n-2}(\psi) \int_{S^{n-2}} f(r(1 - \langle \theta, \varphi\rangle^2)^{1/2})(1 - \langle \theta, \varphi\rangle^2)^{1/2} \, d\sigma_{n-1}(\theta) \\
&\quad = \frac{\int_0^{\pi/2} \cos\theta \sin^{n-3}\theta \, d\theta}{\int_0^{\pi/2} \sin^{n-3}\theta \, d\theta} \frac{\int_0^{\pi/2} f(r\sin\theta)\sin^{n-1}\theta \, d\theta}{\int_0^{\pi/2} \sin^{n-2}\theta \, d\theta} \\
&\quad = \frac{2}{\pi}\int_0^{\pi/2} f(r\sin\theta)\sin^{n-1}\theta \, d\theta.
\end{aligned}$$

So the aim is to find a nonincreasing $f\colon [0, \infty) \to [0, \infty)$ satisfying

$$\tilde{g}(r) \le \frac{2}{\pi} \int_0^{\pi/2} f(r \sin \theta) \sin^{n-1} \theta \, d\theta \qquad \text{for all} \quad r \ge 0 \tag{4}$$

for which $\mu(\mathbb{R}^{n-1})$ is as small as possible.

It is easy to check (for example by considering monomial functions) that if $h\colon [0, \infty) \to \mathbb{R}$ is defined by

$$\int_0^t h(r) r^{n-2} \, dr = t^{n-1} \int_0^{\pi/2} \tilde{g}(t \sin \theta) \sin^{n-2} \theta \, d\theta, \tag{5}$$

then

$$\frac{2}{\pi} \int_0^{\pi/2} h(r \sin \theta) \sin^{n-1} \theta \, d\theta = \tilde{g}(r), \qquad 0 \le r < \infty.$$

So the function $f = h^+$, the positive part of h, satisfies (4) for all r. It remains to show that this f is nonincreasing and that the corresponding measure μ satisfies $\mu(\mathbb{R}^{n-1}) \le 2n^{1/4}$. Now, from (4)

$$\begin{aligned} h(t) &= \frac{1}{t^{n-2}} \frac{d}{dt} \left(t^{n-1} \int_0^{\pi/2} \tilde{g}(t \sin \theta) \sin^{n-2} \theta \, d\theta \right) \\ &= \frac{1}{(\sqrt{2\pi})^n} \int_0^{\pi/2} (n - 1 - t^2 \sin^2 \theta) \sin^{n-2} \theta \cdot e^{-(t^2/2)\sin^2\theta} \, d\theta \\ &= \frac{e^{-t^2/2}}{(\sqrt{2\pi})^n} \int_0^{\pi/2} (n - 1 - t^2 \sin^2 \theta) \sin^{n-2} \theta \cdot e^{(t^2/2)\cos^2\theta} \, d\theta. \end{aligned} \tag{6}$$

This expression is obviously positive if

$$0 \le t \le \sqrt{n-1}. \tag{7}$$

Also, it is evident that the integral has a power series expansion in t^2, valid for all t; for $k \ge 1$, the coefficient of t^{2k} is

$$\begin{aligned} \frac{n-1}{2^k k!} \int_0^{\pi/2} \cos^{2k} \theta \sin^{n-2} \theta \, d\theta - \frac{1}{2^{k-1}(k-1)!} \int_0^{\pi/2} \cos^{2k-2} \theta \sin^n \theta \, d\theta \\ = -\frac{1}{2^k k!} \int_0^{\pi/2} \cos^{2k-2} \theta \sin^n \theta \, d\theta < 0. \end{aligned}$$

This shows that the function $t \mapsto e^{t^2/2} h(t)$ is decreasing on $[0, \infty)$ so that h is decreasing where it is nonnegative and $f = h^+$ is nonincreasing.

Now, the coefficient of t^0 in the integral in (6) is

$$(n-1)\int_0^{\pi/2} \sin^{n-2}\theta\, d\theta = n\int_0^{\pi/2} \sin^n \theta\, d\theta$$

and that of t^2 is $-\frac{1}{2}\int_0^{\pi/2} \sin^n \theta\, d\theta$. So

$$h(t) = \frac{e^{-t^2/2}}{(\sqrt{2\pi})^n}\, n \int_0^{\pi/2} \sin^n \theta\, d\theta \left(1 - \frac{t^2}{2n} - \cdots\right)$$

so the unique zero of h in $[0, \infty)$ occurs at some number $s < \sqrt{2n}$. Remark (7) shows that $s > \sqrt{n-1} \geq 1$. Observe that

$$\begin{aligned}
\mu(\mathbb{R}^{n-1}) &= \int_{\mathbb{R}^{n-1}} f(|x|)\, dx \\
&= (n-1)v_{n-1} \int_0^\infty f(r) r^{n-2}\, dr \\
&= \frac{(n-1)\pi^{(n-1)/2}}{\Gamma((n+1)/2)} \int_0^s h(r) r^{n-2}\, dr \\
&= \frac{(n-1)}{2^{n/2}\Gamma((n+1)/2)\sqrt{\pi}}\, s^{n-1} \int_0^{\pi/2} e^{-(s^2/2)\sin^2\theta} \sin^{n-2}\theta\, d\theta \\
&= \frac{1}{2^{n/2-1}\Gamma((n-1)/2)\sqrt{\pi}} \int_0^s e^{-t^2/2} \frac{t^{n-2}}{\sqrt{1-(t/s)^2}}\, dt.
\end{aligned}$$

It is simplest to estimate this integral in two parts. For $0 \leq t \leq s-1$,

$$\frac{1}{\sqrt{1-(t/s)^2}} \leq \frac{1}{\sqrt{2/s - 1/s^2}} \leq \sqrt{s} \qquad \text{since} \quad s \geq 1.$$

Hence

$$\begin{aligned}
&\frac{1}{2^{n/2-1}\Gamma((n-1)/2)\sqrt{\pi}} \int_0^{s-1} e^{-t^2/2} \frac{t^{n-2}}{\sqrt{1-(t/s)^2}}\, dt \\
&= \sqrt{s}\, \frac{1}{2^{n/2-1}\Gamma((n-1)/2)\sqrt{\pi}} \int_0^\infty e^{-t^2/2} t^{n-2}\, dt \\
&= \sqrt{\frac{s}{2\pi}} < n^{1/4} \qquad \text{since} \quad s < \sqrt{2n}.
\end{aligned}$$

On the other hand,

$$\max_t e^{-t^2/2} t^{n-2} = e^{-(n-2)/2}(n-2)^{(n-2)/2}$$

and Stirling's formula shows that

$$\frac{1}{2^{n/2-1}\Gamma((n-1)/2)\sqrt{\pi}} e^{-(n-2)/2}(n-2)^{(n-2)/2} < \frac{1}{\pi}.$$

Hence

$$\begin{aligned}\frac{1}{2^{n/2-1}\Gamma((n-1)/2)\sqrt{\pi}} \int_{s-1}^{s} e^{-t^2/2} \frac{t^{n-2}}{\sqrt{1-(t/s)^2}}\, dt &\le \frac{1}{\pi}\int_{s-1}^{s} \frac{dt}{\sqrt{1-(t/s)^2}} \\ &= \frac{s}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\left(1 - \frac{1}{s}\right)\right) \\ &< \sqrt{\frac{s}{\pi}} < n^{1/4}. \qquad \square\end{aligned}$$

4. Some Further Remarks

The proof of Theorem 4 shows something more general. Let $f: [0, \infty) \to [0, \infty)$ be nondecreasing and set

$$R = (n-1)v_{n-1} \int_0^\infty f(r) r^{n-2}\, dr.$$

Define $g: \mathbb{R}^n \to [0, \infty)$ by $g(x) = \tilde{g}(|x|)$ where

$$\tilde{g}(r) = \frac{2}{\pi} \int_0^{\pi/2} f(r \sin\theta) \sin^{n-1}\theta\, d\theta. \tag{8}$$

Then, for any rectifiable hypersurface $S \subset \mathbb{R}^n$, there is a line which meets S at least

$$\frac{1}{R}\int_S g \text{ times.} \tag{9}$$

The fact that $g(x) = e^{-|x|^2/2}$ cannot arise from a suitable f via (8), suggests that it is the wrong function to consider. It makes sense to ask whether there is a function

g satisfying (9) with $R = 1$ (say), and such that any rectifiable surface S, with small curvature, is hit at most about

$$\sup_{\tilde{S}} \int_{\tilde{S}} g$$

times, by any line (the sup being taken over translates $\tilde{S}$ of S).

References

[B] C. Borell, The Brunn–Minkowski inequality in Gauss spaces, *Invent. Math.* **30** (1975), 207–216.

[E] H. G. Eggleston, *Convexity*, Cambridge University Press, Cambridge (1958).

[P] G. Pisier, Un théorème sur les opérateurs linéaires entre espaces de Banach qui se factorisent par un espace de Hilbert, *Ann. Sci. École Norm. Sup.* **13** (1980), 23–43.

[ST] V. N. Sudakov and B. S. Tsirel'son, Extremal properties of half-spaces for spherically invariant measures. Problems in the theory of probability distributions, II. *Zap. Nauč. Leningrad Otdel. Mat. Inst. Steklov* **41** (1974) 14–24 (in Russian).

Received February 10, 1992, and in revised form July 24, 1992.