The $R_h = ct$ universe

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ABSTRACT

The backbone of standard cosmology is the Friedmann–Robertson–Walker solution to Einstein's equations of general relativity (GR). In recent years, observations have largely confirmed many of the properties of this model, which are based on a partitioning of the universe's energy density into three primary constituents: matter, radiation and a hypothesized dark energy which, in Λ cold dark matter (ΛCDM), is assumed to be a cosmological constant Λ. Yet with this progress, several unpalatable coincidences (perhaps even inconsistencies) have emerged along with the successful confirmation of expected features. One of these is the observed equality of our gravitational horizon $R_h(t_0)$ with the distance ct_0 light has travelled since the big bang, in terms of the current age t_0 of the universe. This equality is very peculiar because it need not have occurred at all and, if it did, should only have happened once (right now) in the context of ΛCDM. In this paper, we propose an explanation for why this equality may actually be required by GR, through the application of Birkhoff's theorem and the Weyl postulate, at least in the case of a flat space–time. If this proposal is correct, $R_h(t)$ should be equal to ct for all cosmic time t, not just its present value t_0. Therefore, models such as ΛCDM would be incomplete because they ascribe the cosmic expansion to variable conditions not consistent with this relativistic constraint. We show that this may be the reason why the observed galaxy correlation function is not consistent with the predictions of the standard model. We suggest that an $R_h = ct$ universe is easily distinguishable from all other models at large redshift (i.e. in the early universe), where the latter all predict a rapid deceleration.

Key words: cosmic background radiation – cosmological parameters – cosmology: observations – cosmology: theory – dark energy – distance scale.

1 INTRODUCTION

The standard model of cosmology, based on cold dark matter and a cosmological constant Lambda, is today confronted with several inconsistencies and unpalatable coincidences, even though it arguably represents the most successful attempt at accounting for the cosmological observations. Many have written extensively on this subject, including, e.g. Spergel et al. (2003) and Tegmark et al. (2004). For example, ΛCDM has been used with measurements of the cosmic microwave background (CMB) radiation to infer that the universe is flat, so its energy density ρ is at (or very near) its 'critical' value

$$\rho_c \equiv 3c^2 H^2/8\pi G, \tag{1}$$

where H is the Hubble constant and the other symbols have their usual meanings. Yet among the many peculiarities of the standard model is the inference that the density ρ_{de} of dark energy must

itself be of order ρ_c. Worse, no reasonable explanation has yet been offered as to why such a fixed, universal density ought to exist at this scale. It is well known that if Λ is associated with the energy of the vacuum in quantum theory, it should have a scale representative of phase transitions in the early universe – 120 orders of magnitude greater than ρ_c.

The most recent – and perhaps most disturbing – coincidence with ΛCDM is the apparent equality of our gravitational horizon $R_h(t_0)$ with the distance ct_0 light has travelled since the big bang (in terms of the presumed current age t_0 of the universe). This equality was first identified as a peculiarity of the standard model in Melia (2003) and has come under greater scrutiny in recent years (Melia 2007, 2009; Melia & Abdelqader 2009; van Oirschot, Kwan & Lewis 2010; see also Lima 2007, for a related, though unpublished, work).

The purpose of this paper is to advance a possible explanation for why the observed equality $R_h(t_0) = ct_0$ may in fact not be a coincidence of any particular model, such as ΛCDM. Rather, we suggest a reason why it may be required for all cosmologies, by an application of Birkhoff's theorem and its corollary, together with

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the Weyl postulate, to the properties of the Friedmann–Robertson–Walker space–time. More importantly, we show that, at least for flat cosmologies, this equality may actually be upheld for all cosmic time t which, however, would not be entirely consistent with ΛCDM, or any other cosmological model we know of. We shall see that if our proposal turns out to be correct, models such as ΛCDM would then be compelled to fit the data subject to the constraint $R_h(t_0) = ct_0$ today, but would therefore incorrectly ascribe the universal expansion to variable conditions inconsistent with this time-independent general relativity–Weyl (GR–Weyl) constraint in the past. We conclude by suggesting that an $R_h = ct$ universe is unmistakably distinguishable from all other models through a comparison with standard candles at redshifts extending beyond the current Type Ia supernova limit at ~ 1.8, therefore providing a reliable test of our proposal when compared to other models.

2 THE FRW EQUATIONS

Standard cosmology is based on the Friedmann–Robertson–Walker (FRW) metric for a spatially homogeneous and isotropic three-dimensional space, in which the coordinates expand or contract as a function of time:

$$ds^2 = c^2 \, dt^2 - a^2(t)[dr^2(1 - kr^2)^{-1} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)]. \quad (2)$$

The coordinates for this metric have been chosen so that t represents the time measured by a comoving observer (and is the same everywhere, so it functions as a 'community' time), $a(t)$ is the expansion factor and r is an appropriately scaled radial coordinate in the comoving frame. The geometric factor k is $+1$ for a closed universe, 0 for a flat, open universe, or -1 for an open universe.

Applying the FRW metric to Einstein's field equations of GR, one obtains the corresponding FRW differential equations of motion. These are the Friedmann equation,

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{kc^2}{a^2}, \quad (3)$$

and the 'acceleration' equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\rho + 3p). \quad (4)$$

An overdot denotes a derivative with respect to cosmic time t, and ρ and p represent the total energy density and total pressure, respectively. A further application of the FRW metric to the energy conservation equation in GR yields the final equation,

$$\dot{\rho} = -3H(\rho + p), \quad (5)$$

which, however, is not independent of equations (3) and (4).

3 THE BIRKHOFF THEOREM AND THE OBSERVER'S GRAVITATIONAL HORIZON

In comoving coordinates, the proper distance $R(t)$ is measured at constant t and one can easily see from equation (2) that for purely radial paths in a flat cosmology, $R(t) = a(t)r$. It is sometimes useful to recast equation (2) in terms of $R(t)$ (see equation 9) which can reveal e.g. the dependence of the metric coefficients on the observer's gravitational horizon, which we now define.

The Hubble radius is the point at which the universal expansion rate $\dot{R}(t) = \dot{a}(t)r$ equals the speed of light c. But though this radius is well known, it is rarely recognized as just a manifestation of the gravitational radius (see Melia 2007), because every observer experiences zero *net* acceleration from a surrounding

isotropic mass, suggesting that no measure of distance equivalent to the Schwarzschild radius is present in cosmology.

But in fact the *relative* acceleration between an observer and any other space–time point in the cosmos is not zero; it depends on the mass–energy content between himself/herself and that point. This is most easily understood in the context of Birkhoff's theorem and its corollary (Birkhoff 1923) – a relativistic generalization of Newton's theory, that the gravitational field outside a spherically symmetric body is indistinguishable from that of the same mass concentrated at its centre.

What is particularly germane to our discussion here is the corollary to this theorem, describing the field *inside* an empty spherical cavity at the centre of an isotropic distribution. The metric inside such a cavity is equivalent to the flat-space Minkowski metric $\eta_{\alpha\beta}$, a situation not unlike that found in electromagnetism, where the electric field inside a spherical cavity embedded within an otherwise uniform charge distribution is zero. Not surprisingly, the corollary to Birkhoff's theorem is itself analogous to another Newtonian result – that the gravitational field of a spherical shell vanishes inside the shell. So even in the classical limit, one can argue that the medium exterior to a spherical cavity may be thought of as a sequence of ever-increasing spherical shells, each of which produces a net zero effect within the cavity.

To understand the emergence of a gravitational radius in cosmology, imagine placing an observer at the centre of this spherical cavity with proper radius R_{cav}, surrounding him/her by a spherically symmetric mass with a proper surface radius $R_s < R_{cav}$. The metric in the space between the mass and the edge of the cavity is given by the Schwarzschild solution, and the relative acceleration between the observer and R_s is simply due to the mass enclosed within R_s, which we may write in terms of the cosmic energy density $\rho(t)$ as

$$M(R_s) = V_{prop}\frac{\rho(t)}{c^2}, \quad (6)$$

where

$$V_{prop} = \frac{4\pi}{3}R_s^3 \quad (7)$$

is the proper volume.[1]

The criterion we will use to define the gravitational radius R_h is

$$R_h \equiv \frac{2GM(R_h)}{c^2} \quad (8)$$

(see Melia 2007, Melia & Abdelqader 2009). As we shall see below, the FRW equations in principle allow many different kinds of solutions with their own particular form of the expansion factor $a(t)$. When we impose the condition in equation (8), however, only one of these solutions is permitted. This unique solution corresponds to the observed equality $R_h(t_0) = ct_0$, which is most easily inferred from the measurement of H_0 in the project to measure H_0 and the equation of state (SHOES) (Riess et al. 2009), refining the value previously obtained through the Hubble Space Telescope Key Project on the extragalactic distance scale (Mould et al. 2000). The Hubble constant, $H_0 \equiv H(t_0) = 74.2 \pm 3.6\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, is now known with unprecedented accuracy. In the context of ΛCDM, the density ρ is at (or very near) its 'critical' value ρ_c, and with this H_0, $R_h(t_0) \approx 13.7$ billion light-years ($\approx ct_0$).

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[1] To be absolutely clear about this definition, we emphasize the fact that V_{prop} is the volume within which the comoving density of particles remains fixed as the universe expands.

Equation (8) explains why the Hubble radius exists in the first place and is our proposal for a resolution of the $R_h(t_0) = ct_0$ coincidence in the standard model. Ironically, though many may be unaware of the existence of this radius, de Sitter's own solution to Einstein's equations was actually first written in terms of what we now call the proper distance $R(t) = a(t)r$; a limiting radius equivalent to R_h appeared in his form of the metric (see de Sitter 1917).

It is now well known that de Sitter's space–time describes a universe driven by an exponential scale factor $a(t)$. In the more general case, it is not difficult to show, in terms of the proper radii R and R_h, that equation (2) transforms to

$$ds^2 = \Phi\, c^2 dt^2 + 2 \left(\frac{R}{R_h} \right) c\, dt\, dR - dR^2 - R^2\, d\Omega^2 \qquad (9)$$

(Melia & Abdelqader 2009), where the function

$$\Phi \equiv 1 - \left(\frac{R}{R_h} \right)^2 \qquad (10)$$

signals the dependence of the metric on the proximity of the proper radius R to the gravitational radius R_h. We have here assumed a flat universe with $k = 0$, as indicated by the precision measurements of the CMB radiation (Spergel et al. 2003). The reader will also note that, formally, R_h functions as the *static* limit, since the interval ds becomes unphysical at any *fixed* proper distance R beyond R_h. However, there is no such exclusion on the viability of this metric beyond R_h when $\dot{R} \neq 0$, such as we have for sources receding from us with the Hubble expansion (more on this below).

The impact of equation (8) may now be gauged with the use of equation (3), yielding (with $k = 0$)

$$R_h = \frac{c}{H(t)} = c \frac{a}{\dot{a}} \qquad (11)$$

(see Melia & Abdelqader 2009). This is in fact also the definition of the better known Hubble radius, which is therefore simply another manifestation of the gravitational radius R_h. Thus, given what we know about the analogous gravitational radius of a static spherical mass, it is not surprising that the expansion rate $\dot{R}$ should equal c when $R \to R_h$, just as the speed of matter falling towards a black hole reaches c at the event horizon. This may be seen most easily from the definition of R and equation (11), which together give

$$\dot{R} = c \frac{R}{R_h}, \qquad (12)$$

and therefore $\dot{R} = c$ when $R = R_h$. Below we analyse the role of R_h further and see that, even though equation (9) is quite general as written, the definition of the gravitational radius in equation (8) actually selects out only one specific FRW solution, which we are proposing as the correct cosmic space–time.

4 CONSISTENCY WITH THE WEYL POSTULATE

As a prelude to our further consideration of R_h, we re-affirm the fact that the Universe appears to be homogeneous and isotropic on large scales, meaning that observations made from our vantage point are representative of the cosmos as viewed from anywhere else. Known as the cosmological principle, the assumption of homogeneity and isotropy is essential to any attempt at using what we see here from Earth as a basis for testing cosmological models.

On large scales, at least, the universe appears to be expanding in an orderly manner, with galaxies moving apart from one another (except for the odd collision or two due to some peculiar motion on top of the 'Hubble flow'). Galactic trajectories on a space–time diagram would therefore show world lines forming a funnel-like structure in which the separation between any two paths is steadily increasing with time t.

Homogeneity and isotropy are consistent with this type of regularity and together suggest that the evolution of the universe may be represented as a time-ordered sequence of three-dimensional space-like hypersurfaces, each of which satisfies the cosmological principle. This intuitive picture of regularity is often expressed formally as the *Weyl postulate*, after the mathematician Hermann Weyl, who did much of the early work on this subject in the 1920s (see e.g. Weyl 1923).

The most general line element satisfying the Weyl postulate and the cosmological principle is given by equation (2), in which the spatial coordinates (r, θ, ϕ) are constant from hypersurface to hypersurface in the expanding flow, while the temporal behaviour of the scale factor $a(t)$ reflects the dynamics of the expanding cosmos. This metric was first rigorously derived in the 1930s by Robertson (1935) and (independently) Walker (1936), using the ideas espoused earlier by Weyl.

It is therefore clear that any proper distance in this space–time is measured on a space-like hypersurface in the foliated sequence orthogonal to the non-intersecting geodesics. We have shown in Section 3 that the Hubble radius is itself the distance R_h. But according to the definition of R_h in terms of V_{prop} in equation (8), R_h must itself be a proper distance

$$R_h = a(t)r_h, \qquad (13)$$

with the property that r_h is a constant comoving coordinate; otherwise, V_{prop} would not represent the volume within which the particle density is constant in the comoving frame. Comparing equations (11) and (13), we therefore see that

$$r_h \equiv \frac{c}{\dot{a}}, \qquad (14)$$

which means that $\dot{a}$ itself must be constant for consistency with the Weyl postulate. This is the most important consequence of our definition of R_h in equation (8).

From equation (4), we infer that the acceleration $\ddot{a}$ is zero either for an empty universe (in which $\rho = p = 0$) or one characterized by an equation of state $w = -1/3$ (refer to Appendix A for some additional insight into why these two conditions are actually related). And it is trivial to see from equation (11) that

$$R_h = ct \qquad (15)$$

for all cosmic times t, not just the current value t_0.

Within the framework of our proposal, one may then understand why today we 'measure' $R_h(t_0)$ to be equal to ct_0 (within the observational errors), because in a flat universe ($k = 0$) consistent with the Weyl postulate and the cosmological principle, these two quantities must always be equal.

5 COSMOLOGICAL MODELS

Let us now see how this result impacts the standard model of cosmology. We suppose that

$$\rho = \rho_m + \rho_r + \rho_{de}, \qquad (16)$$

where, following convention, we designate the matter, radiation and dark energy densities, respectively, as ρ_m, ρ_r and ρ_{de}. We will also assume that these energy densities scale according to $\rho_m \propto a^{-3}$, $\rho_r \propto a^{-4}$ and $\rho_{de} \propto f(a)$. (If dark energy is indeed a cosmological constant Λ, then $f(a) = $ constant.) Thus, defining

$$\Omega_m \equiv \frac{\rho_m(t_0)}{\rho_c}, \qquad (17)$$

$$\Omega_r \equiv \frac{\rho_r(t_0)}{\rho_c} \tag{18}$$

and

$$\Omega_{de} \equiv \frac{\rho_{de}(t_0)}{\rho_c}, \tag{19}$$

with the (flatness) constraint

$$\Omega_m + \Omega_r + \Omega_{de} = 1, \tag{20}$$

we may rewrite the Friedmann equation as

$$\left(\frac{da}{dt}\right)^2 = H_0^2 \left\{ 1 + \Omega_m \left(\frac{1}{a} - 1\right) + \Omega_{de}\left(a^2 f - 1\right) \right\}. \tag{21}$$

We have here normalized the expansion factor so that $a(t_0) = 1$, which we assume throughout this paper.

Introducing the cosmological redshift z, where

$$1 + z = \frac{1}{a(t)}, \tag{22}$$

we can re-arrange this equation to read

$$\frac{1}{(1+z)^2}\frac{dz}{dt} = -H_0 \left\{ 1 + \Omega_m\left(\frac{1}{a}-1\right) + \Omega_{de}\left(a^2 f - 1\right)\right\}^{1/2}, \tag{23}$$

so that

$$H_0 \int_{t_e}^{t_0} dt = \int_0^{z(t_e)} \frac{dz}{(1+z)^2[1+\Omega_m z - g(z)\Omega_{de}]^{1/2}}. \tag{24}$$

That is

$$c(t_0 - t_e) = R_h(t_0)\int_0^{z(t_e)} \frac{dz}{(1+z)^2[1+\Omega_m z - g(z)\Omega_{de}]^{1/2}}, \tag{25}$$

where we have also defined the function $g(z) \equiv f/(1+z)^2 - 1$, and $z(t_e)$ is the redshift of light reaching us at t_0 but emitted at cosmic time t_e. In this expression, we have used the equality $R_h = c/H$, which is valid in a flat ($k = 0$) cosmology. Other than this flat condition, equation (25) is identical to that obtained in the concordance model, subject to the density in equation (16).

If we now put $t_e \to 0$ and $z(t_e) \to \infty$, then clearly

$$ct_0 = R_h(t_0)\int_0^\infty \frac{dz}{(1+z)^2[1+\Omega_m z - g(z)\Omega_{de}]^{1/2}}. \tag{26}$$

Our proposed form of the gravitational (i.e. Hubble) radius in equation (8) leads to the equality $R_h(t_0) = ct_0$. Therefore, any cosmological model consistent with the Weyl Postulate and the cosmological principle must satisfy the condition

$$\int_0^\infty \frac{dz}{(1+z)^2[1+\Omega_m z - g(z)\Omega_{de}]^{1/2}} = 1. \tag{27}$$

Although not immediately obvious, this constraint implies that no matter what period of deceleration or acceleration the universe may have experienced in its past, its overall acceleration averaged over the time t_0 must be zero (Melia 2009). We can best see this directly from the FRW equations, which indicate that

$$\dot{R}_h \equiv \frac{dR_h}{dt} = \frac{3}{2}(1+w)c, \tag{28}$$

where the parameter

$$w \equiv \frac{p}{\rho} \tag{29}$$

characterizes the total pressure p in terms of the total energy density ρ. Under the assumption that R_h was much smaller in the distant past than it is today, we can easily integrate this equation to get

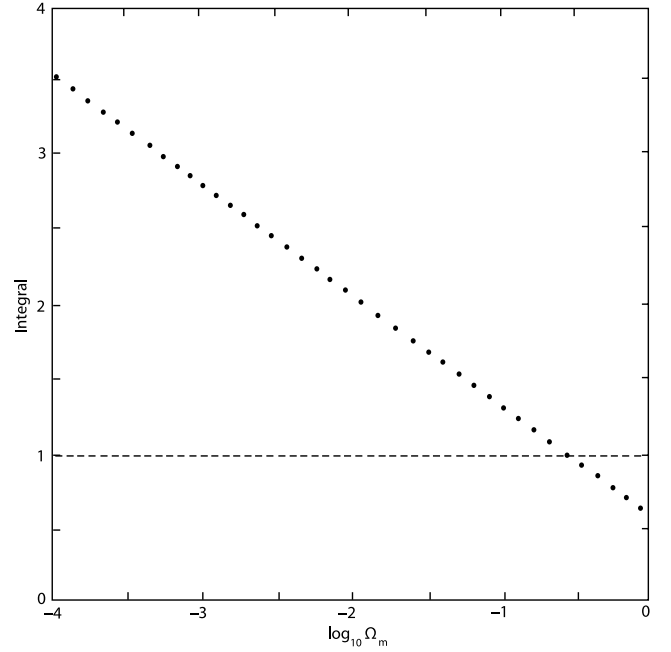$$R_h(t_0) = \frac{3}{2}(1 + \langle w \rangle)ct_0, \tag{30}$$

Figure 1. The integral in equation (27) as a function of Ω_m, assuming a flat cosmology, for the standard model (i.e. ΛCDM). The integral equals 1 when $\Omega_m \approx 0.27$ (and $\Omega_{de} \equiv \Omega_\Lambda \approx 0.73$). It is important to emphasize that this inferred value of Ω_m comes, not from fits to the cosmological data using the ΛCDM decomposition in equation (16) but, rather, from the imposition of the Weyl postulate expressed through equation (27).

where

$$\langle w \rangle \equiv \frac{1}{t_0}\int_0^{t_0} w(t)\, dt. \tag{31}$$

Thus, in order for $R_h(t_0)$ to be equal to ct_0 (which in turn leads to equation 27), we must have $\langle w \rangle = -1/3$, corresponding to an average acceleration $\langle \ddot{a} \rangle = 0$ in equation (4).

Any cosmological model that purports to correctly trace the universal expansion must simultaneously satisfy equation (27) and the condition $\langle w \rangle = -1/3$. In ΛCDM, for example, dark energy is considered to be a cosmological constant, so $g(z) = z(2+z)/(1+z)^2$. In Fig. 1, we plot the value of the integral in equation (27) as a function of Ω_m for a flat ΛCDM cosmology. Not surprisingly, the integral is 1 when $\Omega_m \approx 0.27$, consistent with the optimized parameters of the concordance model (see e.g. Spergel et al. 2003).

Using the same optimized parameters to evaluate the integral in equation (31), we obtain the time-averaged value of w plotted as a function of cosmic time in Fig. 2. We see that $\langle w \rangle \approx -1/3$ at $t \approx 1/H_0$, consistent with the fit shown in Fig. 1. Clearly, the simplest way to satisfy both equation (27) and the constraint $\langle w \rangle = -1/3$ would be to have $w = -1/3$ for all cosmic time t. But this is not what happens in ΛCDM, as one can trivially see from Fig. 2. Instead, one must adjust the values of Ω_m and Ω_{de} in order to make the integral in equation (27) come out to be 1, which ensures that $\langle w \rangle = -1/3$ today, but neither w nor $\langle w \rangle$ is equal to $-1/3$ at any other time. This is far from satisfactory, however, because (as noted previously by Melia 2009) the time-averaged value of w could then be equal to $-1/3$ only once in the entire history of the universe, and that would have to happen right now.
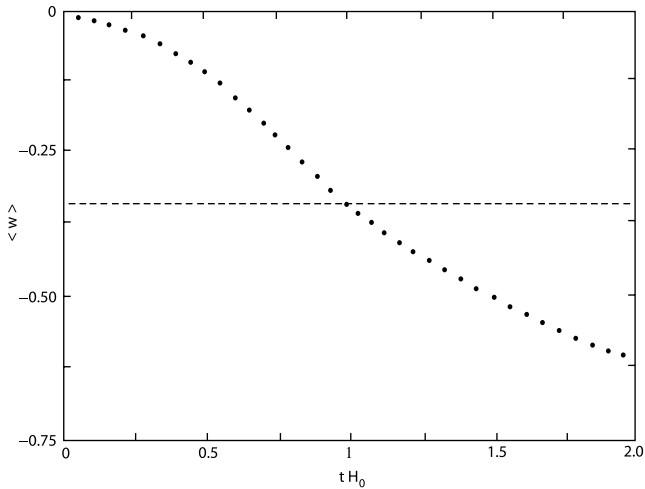
Figure 2. Average value of w (equation 29) as a function of t_0, in units of $1/H_0$, for the standard model (i.e. ΛCDM). The dashed line corresponds to $\langle w \rangle = -1/3$.

6 THE LUMINOSITY DISTANCE

The distinction between our proposed cosmology with $R_h = ct$ (for all t, not just t_0) and other FRW models with past epochs of deceleration is quite pronounced at redshifts larger than the current limits (~ 1.5–2) of study. This happens because the application of Birkhoff's theorem, together with the Weyl postulate and the cosmological principle, suggests that $w = -1/3$ for all t, whereas $\langle w \rangle$ in ΛCDM changes with cosmic time (see Fig. 2).

Based on current Type Ia supernova measurements, the use of ΛCDM as the standard evolutionary model seems to provide an adequate fit to the data. This could present a problem for our proposal because our explanation for the observed equality $R_h(t_0) = ct_0$ would suggest that the ΛCDM version of the luminosity distance d_L used to fit the Type Ia supernova data (e.g. the 'gold sample' in Riess et al. 2004) is not correct in a flat space–time (see also Riess et al. 1998, Perlmutter et al. 1999). However, the disparity between this version of d_L and that required by a flat cosmology with $w = -1/3$ increases with redshift, so in principle we should be able to distinguish between the two by observing events at sufficiently early times.

In ΛCDM, the luminosity distance is given as

$$d_L = (1 + z) R_h(t_0) \int_0^z \frac{du}{[\Omega_m(1+u)^3 + \Omega_{de}(1+u)^\alpha]^{1/2}}, \quad (32)$$

where, strictly speaking, dark energy is a cosmological constant, so that $\Omega_{de} \equiv \Omega_\Lambda$ and $\alpha \equiv 3(1 + w_{de})$ is zero, since $w_{de} \equiv w_\Lambda = -1$. Using this distance measure, Riess et al. (2004) find that the 'gold sample' of 157 Type Ia supernovae (SNe Ia) is consistent with an $\Omega_m = 0.27$, $\Omega_\Lambda = 0.73$ cosmology, yielding $\chi^2_{d.o.f.} = 1.13$. Adding several free parameters, specifically an acceleration parameter $q_0 \equiv -\ddot{a}(t_0)a(t_0)/\dot{a}(t_0)^2$ and dq/dz evaluated at $z = 0$, Riess et al. (2004) find an even better fit with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$, yielding $\chi^2_{d.o.f.} = 1.06$.

At face value, this is a reasonable fit. The caveat, of course, is that one must use many free parameters with this model. One should also question the validity of introducing two new parameters (q_0 and dq/dz) independent of Ω_m and Ω_{de}, given that the expansion history of the universe in ΛCDM is completely specified once the latter two are selected. As it turns out, the additional free parameters improve the fit because the current acceleration needs to be counterbalanced

by an earlier deceleration that together yield an overall expansion consistent with a coasting universe (i.e. $\langle q \rangle = 0$, equivalent to $\langle w \rangle = -1/3$).

In contrast, the luminosity distance in a universe with $R_h = ct$ is given by the expression

$$d_L = (1 + z) R_h(t_0) \ln(1 + z) \quad (33)$$

(see also Melia 2009). Here, the only parameter is the Hubble constant H_0, which enters through our gravitational radius $R_h(t_0)$. This is the proper form of the luminosity distance to use in the analysis of Type Ia supernova data if our understanding of the relativistic constraint $R_h = ct$ is correct. However, this form of the luminosity distance, without the luxury of extra free parameters, does not fit the current sample of Type Ia supernova as well as equation (32).

Interestingly, equation (33) *does* fit the data adequately at low and high redshifts, but not in between, as may be seen, e.g. in fig. 6 of Riess et al. (2004). This could be an important clue, because the difficulty with interpreting the data at intermediate redshifts is made more evident through a comparison of the gold sample with other, newer compilations. Though all of the currently available SNe Ia catalogues yield a consistent and robust value of Ω_m (i.e. ≈ 0.27), they vary significantly when it comes to the inferred redshift z_{acc} at which deceleration is meant to have switched over to acceleration in the present epoch. For example, the gold sample gives a value $z_{acc} = 0.46 \pm 0.13$ (Riess et al. 2004). The so-called Union2 sample contains 557 events in the redshift range $0.015 < z < 1.4$ (Amanullah et al. 2010). The analysis of these data alone yields $z_{acc} \approx 0.75$, though with a fairly large uncertainty (± 0.35), and a combination of the Union2 sample with the CMB measurements yields $z_{acc} = 1.2 \pm 0.10$. The data gathered by the project known as 'Equation of State: SupErNovae trace Cosmic Expansion (ESSENCE)' span the redshift range $z = 0.2$–0.8 (Wu et al. 2008). Their analysis yields a transition redshift $z_{acc} \approx 0.632$, roughly in the range of the others, but not as tightly consistent with them as the value of Ω_m, which ESSENCE finds to be ≈ 0.278, quite close to the value calculated from both the gold and Union2 samples.

7 DISCUSSION

We draw several conclusions from this comparison. It is possible, though we believe unlikely, that ΛCDM is correct after all and that equation (27) is simply a coincidence, as improbable as that may be. It would then be incumbent upon us to understand where our argument for the constraint $R_h = ct$ has gone wrong. We stress, however, that we have examined the need for this equality only for a flat cosmology (i.e. $k = 0$). The disparity between this condition and the Type Ia supernova data may be telling us that the universe is not flat after all – if it turns out that the constraint $R_h = ct$ does not apply when $k \neq 0$. We will examine this situation next and report the results elsewhere.

On the other hand, it could very well be that ΛCDM is currently providing a reasonable fit to the Type Ia supernova data only because (i) it has several free parameters, some of them (q_0 and dq/dz) possibly inconsistent with the others (e.g. Ω_m and Ω_{de}); and (ii) other factors, perhaps astrophysical in origin, are biasing the observed supernova luminosities at intermediate redshifts. Certainly, the fact that z_{acc} varies widely from sample to sample could be an indication that this might be happening.

Of course, there are many other consequences of the $R_h = ct$ constraint, e.g. with regard to baryogenesis, nucleosynthesis and structure formation, all of which would have been affected in terms

of when they could have occurred, if not the physical conditions prevalent at those times. Although it is beyond the scope of the present work to fully explore all of these processes, a detailed account is necessary before the viability of our proposal can be fully assessed.

This extended analysis is necessary because the current situation with the standard model is far from adequate. For example, ΛCDM does not provide a compelling explanation for the galaxy correlation function. Over the past four decades, the successively larger galaxy redshift surveys have mapped the distribution of galaxies with ever-increasing precision, confirming correlation functions consistent with a single power law on all scales (e.g. Marzke et al. 1995; Zehavi et al. 2002), from large regions ($r > 10$ Mpc) exhibiting slight density fluctuations, to collapsed, virialized galaxy groups and clusters ($r < 1$ Mpc). The lack of any observational feature signalling the transition from one physical domain to the next is surprising when viewed within the context of the standard model (see e.g. Li & White 2009), because the matter correlation function in the concordance model differs significantly from a power law.

The most recent attempts at accounting for the unexpected galaxy correlation function have relied on several new, fine-tuning additions in order to get the correct profile (see e.g. Watson, Berlind & Zentner 2011). But the various contributing effects are intertwined and no simple, universal rule exists for which a power-law correlation function emerges. The evolving competition between accretion and destruction rates of subhaloes over time is *required* to have struck just the right balance at $z \approx 0$, leading Watson et al. (2011) to conclude that the power-law galaxy correlation function is a cosmic coincidence.

Part of the difficulty with this type of analysis is that, besides gravity and pressure, other physical processes can play an important role in the formation of structure and these are not easy to handle. For example, in baryonic models, the most important physical phenomenon is the interaction between baryons and photons during the pre-recombination era and the consequent dissipation due to viscosity and heat conduction.

In so far as the $R_{\rm h} = ct$ universe is concerned, we can leave these elements aside for the moment and at least suggest how the fundamental equation describing the dynamical growth of density fluctuations would appear in this cosmology. Defining the density contrast $\delta \equiv \delta\rho/\rho$ in terms of the density fluctuation $\delta\rho$ and unperturbed density ρ, we can form the wave-like decomposition

$$\delta = \sum_{\kappa} \delta_{\kappa}(t) e^{i\kappa \cdot r}, \tag{34}$$

where the Fourier component δ_{κ} depends only on cosmic time t, and κ and r are the comoving wavevector and radius, respectively. In the linear regime, the κth perturbative mode satisfies the equation

$$\ddot{\delta}_{\kappa} + 2\frac{\dot{a}}{a}\dot{\delta}_{\kappa} = \left(\frac{4\pi G}{c^2}\rho - \frac{v_{\rm s}^2 \kappa^2}{a^2}\right)\delta_{\kappa}, \tag{35}$$

where a dot signifies differentiation with respect to t, $a = a(t)$ is the cosmic expansion factor we defined earlier and $v_{\rm s}^2 \equiv {\rm d}p/{\rm d}\rho$ is the adiabatic sound speed squared in terms of the pressure p and energy density ρ (see e.g. Tsagas 2002).

The second term on the left-hand side is due to the cosmic expansion and always suppresses the growth of δ_{κ}. The combined term on the right-hand side reflects the conflict between gravity ($4\pi G\rho/c^2$) and pressure support ($-v_{\rm s}^2\kappa^2/a^2$). Defining the proper wavelength of the perturbation $\lambda \equiv 2\pi a/\kappa$, one sees immediately that whether gravity or pressure support dominates depends on whether λ is greater or smaller than the so-called Jeans length

$$\lambda_{\rm J} \equiv v_{\rm s}\sqrt{\frac{\pi c^2}{G\rho}}. \tag{36}$$

In the standard model, one solves equation (35) by first choosing the constituents of the universe (e.g. baryonic matter, cold dark matter and radiation) contributing to ρ, adopting an equation of state to calculate p and therefore $v_{\rm s}$, and then integrating δ_{κ} over time from an assumed set of initial conditions.

The origin of the initial seed perturbations is uncertain, one possible explanation being that they are quantum fluctuations boosted to macroscopic scales by inflation. The primordial power spectrum is usually assumed to have a power-law dependence on scale,

$$P(\kappa) = A\kappa^n, \tag{37}$$

with a scale-invariant spectral index $n = 1$ and an unknown normalization factor A that must be determined observationally. The initial conditions for the solution to equation (35) follow from this because at any redshift z, the power spectrum may also be written as

$$P(\kappa, z) = \langle|\delta_{\kappa}(z)|^2\rangle, \tag{38}$$

so the starting size of the fluctuation is

$$\delta_{\kappa} \propto \kappa^{1/2}. \tag{39}$$

Equation (35) is adequate for most applications, but not in situations where the pressure is a significant fraction of ρ. In general relativity, both ρ and p contribute to the 'active' mass inducing curvature, as evidenced by the appearance of both ρ and p in equations (4) and (5). Thus, to analyse the growth of perturbations in an $R_{\rm h} = ct$ universe, we must resort to the relativistic version of equation (35). Fortunately, this transition is greatly simplified by the very simple equation of state implied by the condition $R_{\rm h} = ct$ given by

$$p = w\rho \tag{40}$$

with $w = -1/3$, as we discussed earlier.

For a universe with density ρ and pressure $p = w\rho$, the linear relativistic version of equation (35) is

$$\ddot{\delta}_{\kappa} + \left(2 - 6w + 3v_{\rm s}^2\right)\frac{\dot{a}}{a}\dot{\delta}_{\kappa} - 3/2\left(1 + 8w - 3w^2 - 6v_{\rm s}^2\right)\left(\frac{\dot{a}}{a}\right)^2 \delta_{\kappa}$$
$$= -\frac{\kappa^2 v_{\rm s}^2}{a^2}\delta_{\kappa}. \tag{41}$$

Therefore, for an $R_{\rm h} = ct$ universe, the dynamical equation for δ_{κ} is

$$\ddot{\delta}_{\kappa} + \frac{3}{t}\dot{\delta}_{\kappa} = \frac{1}{3}c^2\left(\frac{\kappa}{a}\right)^2 \delta_{\kappa}. \tag{42}$$

We need to emphasize several important features of this equation. First of all, the active mass in this universe is proportional to $\rho + 3p = 0$, and therefore the gravitational term normally appearing in the standard model is absent (see equation 35). But this does not mean that δ_{κ} cannot grow. Instead, because $p < 0$, the (usually dissipative) pressure term in equation (35) here becomes an agent of growth. Moreover, there is no Jeans length scale. In its place is the gravitational radius, which we can see most easily by writing equation (42) in the form

$$\ddot{\delta}_{\kappa} + \frac{3}{t}\dot{\delta}_{\kappa} - \frac{1}{3}\frac{\Delta_{\kappa}^2}{t^2}\delta_{\kappa} = 0, \tag{43}$$

where

$$\Delta_{\kappa} \equiv \frac{2\pi R_{\rm h}}{\lambda}. \tag{44}$$

Note, in particular, that both the gravitational radius R_h and the fluctuation scale λ vary with t in exactly the same way, so Δ_κ is therefore a constant in time. But the growth rate of δ_κ depends critically on whether λ is less than or greater than R_h.

A simple solution to equation (43) is the power law

$$\delta_\kappa(t) = \delta_\kappa(0) t^\alpha, \qquad (45)$$

where evidently

$$\alpha^2 + 2\alpha - \frac{1}{3}\Delta_\kappa = 0, \qquad (46)$$

so that

$$\alpha = -1 \pm \sqrt{1 + \Delta_\kappa^2/3}. \qquad (47)$$

Thus, for small fluctuations ($\lambda \ll R_h$),

$$\delta_\kappa \sim C_1 \kappa^{1/2} t^{\Delta_\kappa/\sqrt{3}} + C_2 \kappa^{1/2} t^{-\Delta_\kappa/\sqrt{3}}, \qquad (48)$$

whereas for large fluctuations ($\lambda \gg R_h$),

$$\delta_\kappa \sim C_3 \kappa^{1/2} + C_4 \kappa^{1/2} t^{-2}, \qquad (49)$$

where the C_i constants depend on the initial conditions.

Beyond this point there are too many unknowns to pin down the final galaxy correlation function resulting from these growth functions. For example, we do not know how to set the values of C_1, C_2, C_3 and C_4 in a model-independent way, nor does any of these analyses take into account the non-linear growth that follows. But already we can point to a decided advantage of the $R_h = ct$ universe over ΛCDM. Whereas the concordance model predicts different distributions at different scales, in part because of the influence of the Jeans length, no such transition region exists for the $R_h = ct$ universe. Instead, the fluctuation growth is driven by the pressure term, which looks the same no matter the perturbation length $\lambda \ll R_h$. At least in this regard, the $R_h = ct$ universe appears to be a better match to the observations.

8 CONCLUSION

Fortunately, a resolution to the ΛCDM versus $R_h = ct$ universe dilemma will surely come with the observation of standard candles at redshifts even greater than 1.8 (roughly the current upper limit to the Type Ia samples). A cosmology with the time-independent constraint $R_h = ct$ predicts a luminosity distance unmistakably distinguishable from that of all other models. And the differences will manifest themselves most prominently early in the universe's expansion (i.e. at large redshift z), where all other models (including ΛCDM) predict a rapid deceleration.

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REFERENCES

Amanullah R. et al., 2010, ApJ, 716, 712
Birkhoff G., 1923, Relativity and Modern Physics. Harvard Univ. Press, Cambridge
de Sitter W., 1917, Proc. Akad. Wetensch. Amsterdam, 19, 1217
Li C., White S. D. M., 2009, MNRAS, 398, 2177
Lima J. A. S., 2007, preprint (arXiv:0708.3414)

Marzke R. O., Geller M. J., da Costa L. N., Huchra J. P., 1995, AJ, 110, 477
Melia F., 2003, The Edge of Infinity: Supermassive Black Holes in the Universe. Cambridge Univ. Press, Cambridge, p. 119
Melia F., 2007, MNRAS, 382, 1917
Melia F., 2009, Int. J. Modern Phys. D, 18, 1113
Melia F., Abdelqader M., 2009, Int. J. Modern Phys. D, 18, 1889
Milne E. A., 1940, ApJ, 91, 129
Mould J. R. et al., 2000, ApJ, 529, 786
Perlmutter S. et al., 1999, ApJ, 517, 565
Riess A. G. et al., 1998, AJ, 116, 1009
Riess A. G. et al., 2004, ApJ, 607, 665
Riess A. G. et al., 2009, ApJ, 699, 539
Robertson H. P., 1935, ApJ, 82, 248
Spergel D. N. et al., 2003, ApJS, 148, 175
Tegmark M. et al., 2004, Phys. Rev. D, 69, 103501
Tsagas C. G., 2002, in Cotsakis S., Papantonopoulos E., eds, Lecture Notes in Physics, Vol. 592, Cosmological Crossroads. Springer-Verlag, Berlin, p. 223
van Oirschot P., Kwan J., Lewis G. F., 2010, MNRAS, 404, 1633
Walker A. G., 1936, Proc. Lond. Math. Soc., 42, 90
Watson D. F., Berlind A. A., Zentner A. R., 2011, ApJ, 738, 22
Weinberg S., 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. Wiley, New York
Weyl H., 1923, Z. Phys. 24, 230
Wu Q., Gong Y. G., Wang A. Z., Alcanizd J. S., 2008, Phys. Lett. B, 659, 34
Zehavi I. et al., 2002, ApJ, 608, 16

APPENDIX A

The fact that $R_h = ct$ in *both* an empty universe (Milne 1940) and a flat ($k = 0$) universe is not a coincidence, as one may appreciate from a simple heuristic argument justified by the corollary to Birkhoff's theorem. As noted by Weinberg (1972), the fact that the gravitational influence of any isotropic, external mass–energy is zero within a spherical cavity permits the limited use of Newtonian mechanics to some cosmological problems, which we can use here to gain some insight into the dynamics implied by $k = 0$.

Consider a sphere 'cut out' of a homogeneous and isotropic universal medium with (proper) radius $R_s(t) = a(t) r_s$. Adopting the cosmological principle, we assume that the density within this region is a function of time t only and that every point within and without the sphere expands away from every other point in proportion to the time-dependent scale factor $a(t)$, which itself is the same everywhere. According to Birkhoff's theorem and its corollary, we only need to consider contributions to the energy from the contents enclosed within R_s to determine the local dynamics of this region extending out to R_s.

Relative to an observer at the centre of this sphere, the kinetic energy of a shell with thickness dR at radius R is therefore

$$dK = 4\pi R^2 \, dR \, \frac{1}{2} \frac{\rho(t)}{c^2} \dot{R}^2, \qquad (A1)$$

and integrating this out from $r = 0$ to r_s, one easily gets the total kinetic energy of this sphere relative to the observer at the origin:

$$K = \frac{2\pi}{5} \frac{\rho(t)}{c^2} a^3 \dot{a}^2 r_s^5. \qquad (A2)$$

Let us now calculate the corresponding gravitational potential energy of this spherical distribution (remember that this is a classical approach). The potential energy of the shell at R is

$$dV = -4\pi R^2 \, dR \, \frac{\rho(t)}{c^2} \frac{GM(R)}{R}, \qquad (A3)$$

where

$$M(R) = \frac{4\pi}{3} R^3 \frac{\rho(t)}{c^2} \qquad (A4)$$

is the total mass enclosed inside radius R. And integrating this out from $r = 0$ to r_s, we see that the total potential energy of this sphere (as measured by the observer at the origin) is

$$V = \frac{16\pi^2 G}{15} \frac{\rho(t)^2}{c^4} a^5 r_s^5. \tag{A5}$$

Classically, then, the observer measures a total energy of this sphere given by

$$E = \frac{2\pi}{5} \frac{\rho(t)}{c^2} a^3 \dot{a}^2 r_s^5 - \frac{16\pi^2 G}{15} \frac{\rho(t)^2}{c^4} a^5 r_s^5, \tag{A6}$$

which may be re-arranged to cast it into a more recognizable form:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \rho(t) + \frac{5c^2 E}{2\pi \rho(t) a^5 r_s^5}. \tag{A7}$$

Evidently, the local conservation of energy relative to the observer at the origin is actually the Friedmann equation (3), when we identify the spatial curvature constant as

$$k \equiv -\frac{10}{3 r_s^2} \left(\frac{\epsilon}{\rho}\right), \tag{A8}$$

where

$$\epsilon \equiv \frac{3E}{4\pi R_s^3} \tag{A9}$$

is the *total* local energy density. A universe with positive curvature therefore corresponds to a net negative energy, which means the system is bound, whereas a negative curvature is associated with a positive total energy density ($\epsilon > 0$), characterizing an unbound universe.

A universe with net zero energy is therefore flat ($k = 0$), and the latest cosmological measurements (see e.g. Spergel et al. 2003) are apparently telling us that this is the state we are in. Let us remember that general relativity is a local theory; it tells us only about the gradient of the space–time curvature locally due to the presence of a source at that point. As far as general relativity is concerned, the local dynamics of a universe with net zero energy density ($\epsilon = 0$) is indistinguishable from an empty (or Milne) universe. This is the reason why $\ddot{a} = 0$ in both cases, and why $R_h = ct$.

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