THE RIEMANN PROBLEM FOR
GENERAL 2 × 2 CONSERVATION LAWS

BY

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ABSTRACT. The Riemann Problem for a system of hyperbolic conserva-
tion laws of form

$$u_t + f(u, v)_x = 0,$$

(1)

$$v_t + g(u, v)_x = 0$$

with arbitrary initial constant states

(2)
$$(u_0(x), v_0(x)) = \begin{cases} (u_l, v_l), & x < 0, \\ (u_r, v_r), & x > 0, \end{cases}$$

is considered. We assume that $f_v < 0$, $g_u < 0$. Let l_i (r_i) be the left (right)
eigenvectors of $dF \equiv d(f, g)$ for eigenvalues $\lambda_1 < \lambda_2$. Instead of assuming the
usual convexity condition $d\lambda_i(r_i) \neq 0$, $i = 1, 2$, we assume that $d\lambda_i(r_i) = 0$ on
disjoint union of 1-dim manifolds in the (u, v) plane. Oleinik's condition (E)
for single equation is extended to system (1); again call this new condition (E).
Our condition (E) implies Lax's shock inequalities and, in case $d\lambda_i(r_i) \neq 0$, the
two are equivalent. We then prove that there exists a unique solution to the
Riemann Problem (1) and (2) in the class of shocks, rarefaction waves and con-
tact discontinuities which satisfies condition (E).

Introduction. We consider the system

(0.1)
$$\begin{aligned} u_t + f(u, v)_x &= 0, \\ v_t + g(u, v)_x &= 0, \end{aligned} \quad t \geqslant 0, -\infty < x < \infty,$$

where $u = u(x, t)$, $v = v(x, t)$ and $f, g \in C^3(U)$ for some open set U in R^2.

The problem to be solved is the Riemann Problem $\{(u_l, v_l); (u_r, v_r)\}$ for
arbitrary constants $(u_l, v_l) \in U$, $(u_r, v_r) \in U$; i.e. solve the system (0.1) with initial
data

(0.2)
$$(u(x, 0), v(x, 0)) \equiv (u_0(x), v_0(x)) = \begin{cases} u_l, v_l & \text{for } x < 0, \\ u_r, v_r & \text{for } x > 0. \end{cases}$$

Besides the physical meaning, the significance of the Riemann Problem is that it is commonly served to solve the Cauchy Problem (0.1) with general initial data. In fact, Riemann Problems are the building blocks in the proof of existence theorems in Glimm [1], Smoller and Johnson [2], Nishida [4] and Nishida and Smoller [5].

Since the solution to (0.1) is usually discontinuous, see e.g. [3], we make the following definition.

DEFINITION. The bounded measurable function (u, v) is said to be a weak solution to (0.1), (0.2) if

(0.3)
$$\iint_{t \geqslant 0} [u\phi_t + f(u, v)\phi_x] \, dx \, dt + \int_{t=0} u_0 \phi \, dx = 0,$$

$$\iint_{t \geqslant 0} [v\phi_t + g(u, v)\phi_x] \, dx \, dt + \int_{t=0} v_0 \phi \, dx = 0$$

for all smooth functions $\phi = \phi(t, x)$ with compact support in $t \geqslant 0$.

Hereafter, we assume that, for $(u, v) \in U$,

(0.4) $f_v < 0, \quad g_u < 0,$

(0.5) $f_u \geqslant 0, \quad g_v \leqslant 0.$

Let $F = (f, g)$, dF the Fréchet derivative of F and d^2F the Fréchet derivative of dF. Condition (0.4) implies that system (0.1) is hyperbolic, i.e. dF has real and distinct eigenvalues $\lambda_1 < \lambda_2$. (0.5) implies that, for $(u, v) \in U$,

(0.6) $\lambda_1 < 0 < \lambda_2.$

Let r_i (resp. l_i) be right (resp. left) eigenvectors corresponding to eigenvalues λ_i, $i = 1, 2$. These can be taken in the form

(0.7)
$$r_1 = (1, a_1)^t, \quad r_2 = (1, a_2)^t,$$
$$l_1 = (-a_2, 1), \quad l_2 = (-a_1, 1),$$

where

$$a_i = \frac{g_u}{\lambda_i - g_v} = \frac{\lambda_i - f_u}{f_v}, \qquad i = 1, 2,$$

$$a_2 < 0 < a_1.$$

If $d\lambda_i \cdot r_i \neq 0$, then system (0.1) is said to be *genuinely nonlinear* in the ith class.

Suppose (u, v) is a solution which is discontinuous across curve $x = x(t)$, then (0.3) implies the following *Hugoniot condition* (e.g. [3])

$$s = \frac{f(u_1, v_1) - f(u_0, v_0)}{u_1 - u_0} = \frac{g(u_1, v_1) - g(u_0, v_0)}{v_1 - v_0} \equiv \sigma(u_0, v_0; u_1, v_1)$$

where $s = \dot{x}(t)$, $(u_0, v_0) = (u, v)(x - 0, t)$, $(u_1, v_1) = (u, v)(x + 0, t)$.

DEFINITION. For $(u_0, v_0) \in U$, the *shock curve* $s(u_0, v_0)$ is the set $(u, v) \in U$ which satisfies the Hugoniot condition

$$\frac{f(u, v) - f(u_0, v_0)}{u - u_0} = \frac{g(u, v) - g(u_0, v_0)}{v - v_0}.$$

The *forward shock curve* $S_2(u_0, v_0)$ and *backward shock curve* $S_1(u_0, v_0)$ are:

$$S_1(u_0, v_0) = S(u_0, v_0) \cap \{(u, v) | u \geqslant u_0, v \geqslant v_0 \text{ or } u \leqslant u_0, v \leqslant v_0\},$$

$$S_2(u_0, v_0) = S(u_0, v_0) \cap \{(u, v) | u \geqslant u_0, v \leqslant v_0 \text{ or } u \leqslant u_0, v \geqslant v_0\}.$$
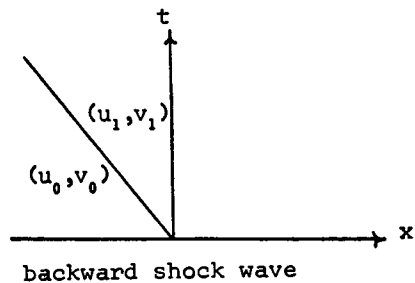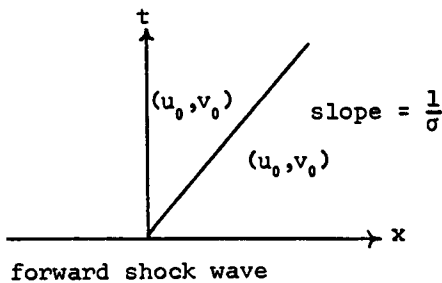
Let $(u_1, v_1) \in S_2(u_0, v_0)$, then we can define a weak solution (u, v) to (0.1) by

$$(u, v)(x, t) = \begin{cases} (u_0, v_0) & \text{for } x < \sigma t, \\ (v_1, v_1) & \text{for } x > \sigma t, \end{cases}$$

where

$$\sigma = \sigma(u_1, v_1; u_0, v_0) = \frac{f(u_1, v_1) - f(u_0, v_0)}{u_1 - u_0} = \frac{g(u_1, u_1) - g(u_0, v_0)}{v_1 - v_0}.$$

We call such solution a forward shock wave. Similarly, for a S_2 curve, we have backward shock wave. We denote them by the following pictures.


forward shock wave   backward shock wave

Let

$$V_{\pm}^2 = \{(u, v) \in U | d\lambda_2(r_2) \gtrless 0\},$$

$$V_{\pm}^1 = \{(u, v) \in U | d\lambda_1(r_1) \gtrless 0\}.$$

Let $(u_1, v_1) \in R_2(u_0, v_0)$, $u_1 > u_0$, be such that every point (u, v) on $R_2(u_0, v_0)$ between (u_0, v_0) and (u_1, v_1) lies in region V_-^2. Then the Riemann Problem $\{(u_0, v_0); (u_1, v_1)\}$ can be solved by (cf. [3])

$$(u, v)(x, t) = (\widetilde{u}, \widetilde{v})(x/t),$$

(1.1) $(\widetilde{u}, \widetilde{v}) \in R_2(u_0, v_0), u_1 \geqslant \widetilde{u} \geqslant u_0.$

$$\lambda_2(\widetilde{u}, \widetilde{v})(x/t) = x/t,$$

We call this solution, which takes values along a rarefaction curve R_2, a *forward rarefaction wave*. We can also have a forward rarefaction wave when $u_1 < u_0$ and the corresponding region is V_+^2. The backward rarefaction wave, which takes values on a R_1 curve can be treated similarly. These can be pictured as:

backward rarefaction wave   forward rarefaction wave
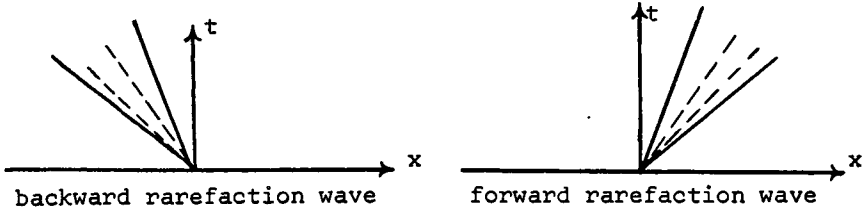
Joel A. Smoller [7] and [8] solved the Riemann Problem in the class of shock and rarefaction waves under the assumption that system (0.1) is genuinely nonlinear and that the shock interaction condition holds. The solution is required to satisfy the following Lax shock inequalities, e.g. [3], across shocks:

$$\lambda_1(u_0, v_0) > \sigma > \lambda_1(u_1, v_1) \quad \text{and} \quad \sigma < \lambda_2(u_1, v_1)$$

(L)   or

$$\lambda_2(u_0, v_0) > \sigma > \lambda_2(u_1, v_1) \quad \text{and} \quad \sigma > \lambda_1(u_0, v_0)$$

where $\sigma = \sigma(u_0, v_0; u_1, v_1)$.

The purpose of this paper is to prove the existence and uniqueness theorem for the Riemann Problem in the class of shocks, rarefaction waves and contact discontinuities, when we relax the genuine-nonlinearity condition and let $d\lambda_i(r_i) = 0$, $i = 1, 2$, on a disjoint union of 1-manifolds in the (u, v) plane. The solution is required to satisfy, instead of condition (L), the following extended entropy condition

(E) $\sigma(u_1, v_1; u_0, v_0) \leqslant \sigma(u, v; u_0, v_0)$

for every (u, v) on $S(u_0; v_0)$ between (u_0, v_0) and (u_1, v_1).

Condition (E) extends Oleinik's celebrated condition (E) (cf. [6]) to systems and reduces to condition (L) when the system (0.1) is genuinely nonlinear.

1. **Preliminary results.** In this section, we shall make some basic observations about the shock and rarefaction curves.

Let $d/d\mu$ (resp. $d/d\nu_i$) be the derivative along curve S_2 (resp. R_i). Thus,

$$\frac{d}{d\mu} = \frac{d}{du} + h_2\frac{d}{dv},$$

(1.2) $$h_2 = h_2(u_0, v_0; u, v) = \frac{(u - u_0)g_u + (\sigma - f_u)(v - v_0)}{(v - v_0)f_v + (\sigma - g_v)(u - u_0)},$$

$$\sigma = \sigma(u_0, v_0; u, v) = \frac{f(u, v) - f(u_0, v_0)}{u - u_0} = \frac{g(u, v) - g(u_0, v_0)}{v - v_0};$$

(1.3) $$\frac{d}{dv_i} = \frac{d}{du} + a_i\frac{d}{dv}, \qquad i = 1, 2,$$

where g_u, g_v, f_u, f_v and a_i, $i = 1, 2$, are evaluated at (u, v).

LEMMA 1.1. *For* $(u_1, v_1) \in S_2(u_0, v_0)$ *(resp.* $S_1(u_0, v_0)$), *we have* $\sigma(u_1, v_1; u_0, v_0) \geqslant 0$ *(resp.* $\leqslant 0$).

LEMMA 1.2. *The set* $S_2(u_0, v_0)$ *(resp.* $S_1(u_0, v_0)$) *is a smooth curve defined for all* u *(resp.* v).

Lemmas 1.1 and 1.2 can be proved by using (0.4) and (0.5); we omit the proofs.

LEMMA 1.3. *Given* $(u_0, v_0) \in U$, *and supposing that for any* (u, v) *on* $S_2(u_0, v_0)$, $u > u_0$, $|u - u_0|$ *small, we have* $(u, v) \in V_+^2$. *Then, for such* (u, v),

$$\lambda_2(u, v) < \sigma(u, v; u_0, v_0) < \lambda_2(u_0, v_0);$$

$$\sigma(u, v; u_0, v_0) = \lambda_2(u_0, v_0) + O|u - u_0|;$$

$$h_2(u, v; u_0, v_0) = a_2(u_0, v_0) + O|u - u_0|;$$

and σ *is decreasing along* $S_2(u_0, v_0)$.

PROOF. See Lax [3].

LEMMA 1.4. *For any* $(u, v) \in S_2(u_0, v_0)$, $\sigma = \sigma(u_0, v_0; u, v)$, $u > u_0$, *the following are equivalent*:
   (i) $d\sigma/d\mu > 0$ *(resp.* $d\sigma/d\mu < 0$).
   (ii) $\sigma < \lambda_2$ *(resp.* $\sigma > \lambda_2$).
If $u < u_0$, *then the following are equivalent*:
   (iii) $d\sigma/d\mu > 0$ *(resp.* $d\sigma/d\mu < 0$).
   (iv) $\sigma > \lambda_2$ *(resp.* $\sigma < \lambda_2$).
Thus $d\sigma/d\mu = 0$ *if and only if* $\sigma = \lambda_2$.

PROOF. We only consider the case $u > u_0$. Thus, by Lemma 1.2, $v < v_0$.
   Case 1: $h_2 \geqslant a_2$ at (u, v). Assume (i), $f_u + h_2f_v - \sigma > 0$; thus $\lambda_2 = f_u + a_2f_v \geqslant f_u + h_2f_v > \sigma$ which is (ii).

Assume (ii), then

$$\frac{d\sigma}{d\mu} = \frac{g_u + h_2 g_v - \sigma h_2}{v - v_0} \geqslant \frac{g_u + a_2 g_v - \sigma a_2}{v - v_0} > \frac{g_u + a_2 g_v - \lambda_2 a_2}{v - v_0} = 0$$

which is (i).

Case 2: $h_2 < a_2$ at (u, v). Assume (i), $g_u + h_2 g_v - \sigma h_2 < 0$, thus

$$0 < g_u + h_2 g_v - \sigma h_2 < g_u + a_2 g_v - a_2 \sigma = a_2 \lambda_2 - a_2 \sigma = a_2 (\lambda_2 - \sigma).$$

So $\lambda_2 - \sigma > 0$ which is (ii).

Assume (ii), then

$$\frac{d\sigma}{d\mu} = \frac{f_u + h_2 f_v - \sigma}{u - u_0} > \frac{f_u + h_2 f_v - \lambda_2}{u - u_0} = \frac{f_u + h_2 f_v - (f_u + a_2 f_v)}{u - u_0}$$

$$= \frac{(h_2 - a_2) f_v}{u - u_0} > 0$$

which is (i).

The fact that $d\sigma/d\mu < 0$ and $\sigma > \lambda_2$ are equivalent can be proved similarly.   Q.E.D.

Using Lemmas 1.3 and 1.4 we can easily prove the following:

LEMMA 1.5.   If $\sigma = \lambda_2$ at $(u, v) \in S_2(u_0, v_0)$, then $h_2 = a_2$ at (u, v).

LEMMA 1.6.   Let $(u, v) S_2(u_0, v_0) \in V_+^2$ and $\lambda_2 = \sigma$ at (u, v). Then $d(\sigma - \lambda_2)/d\mu = d\lambda_2/d\mu > 0$ and σ has local maximum at (u, v).

LEMMA 1.7.   Suppose $(u_1, v_1) \in S_2(u_0, v_0)$ and condition (E) is satisfied for $\{(u_0, v_0); (u_1, v_1)\}$. Then, for $\sigma = \sigma(u_0, v_0; u_1, v_1)$,

$$\lambda_1(u_0, v_0) \leqslant \sigma \quad and \quad \lambda_2(u_0, v_0) \geqslant \sigma \geqslant \lambda_2(u_1, v_1).$$

We have analogous lemmas as above for $S_1(u_0, v_0)$; in this case, we use $d/d\mu = k \, d/du + d/dv$,

$$k = k(u_0, v_0; u, v) = \frac{(\sigma - g_v)(u - u_0) + f_v(v - v_0)}{(\sigma - f_u)(v - v_0) + g_v(u - u_0)}.$$

THEOREM 1.1.   Condition (E) is equivalent to Lax's shock inequalities when system (0.1) is genuinely nonlinear.

PROOF. We only prove the necessity part of the theorem for forward shocks. The other cases can be treated similarly. Thus we assume that $(u_1, v_1) \in S_2(u_0, v_0)$ and $\{(u_0, v_0); (u_1, v_1)\}$ satisfies condition (E). Without loss of generality, assume $d\lambda_2(r_2) < 0$.

If $u_1 > u_0$, then, by Lemma 1.3, $\lambda_2 < \sigma$ at all points on $S_2(u_0, v_0)$ between (u_0, v_0) and (u_0, v_0) and close to (u_0, v_0). For such points, by Lemma 1.4, we have $d\sigma/d\mu < 0$. We claim that, for all points on $S_2(u_0, v_0)$ between (u_0, v_0) and (u_1, v_1), we actually have $d\sigma/d\mu < 0$. Indeed if (u, v) is the first point on $S_2(u_0, v_0)$, $u > u_0$, such that $d\sigma/d\mu = 0$, then, by Lemma 1.6, $\sigma < \lambda_2$ at some point $(\tilde{u}, \tilde{v})$ on $S_2(u_0, v_0)$ between (u_0, v_0) and (u, v). Thus, by Lemma 1.4, $d\sigma/d\mu > 0$ at $(\tilde{u}, \tilde{v})$. But since $d\sigma/d\mu < 0$ at points close to (u_0, v_0), we then have $d\sigma/d\mu = 0$ at some point between (u_0, v_0) and $(\tilde{u}, \tilde{v})$. This contradicts the fact that (u, v) is the first point with $d\sigma/d\mu = 0$. So we have $d\sigma/d\mu < 0$ at all points on $S_2(u_0, v_0)$ between (u_0, v_0) and (u_1, v_1). In particular, $\sigma(u_1, v_1; u_0, v_0) < \sigma(u_0, v_0; u_0, v_0) = \lambda_2(u_0, v_0)$ and $\sigma(u_1, v_1; u_0, v_0) > \lambda_2(u_1, v_1)$ by Lemma 1.4. Since $\sigma > 0$, $\lambda_1 < 0 < \lambda_2$, we have condition (L).

If $u_1 < u_0$, a similar argument gives that σ is increasing as (u, v) moves from (u_0, v_0) to (u_1, v_1) along $S_2(u_0, v_0)$. This would contradict condition (E). Hence $u_1 > u_0$ and we are done. Q.E.D.

2. Existence. In this section, we want to solve the Riemann Problem in the class of shocks, rarefaction waves and contact discontinuities.

We make the following assumption:

(2.1)   If $d\lambda_i(r_i) = 0$ at point (u, v), then $d(d\lambda_i(r_i))/dv_i \neq 0$, $i = 1, 2$, where $d/dv_i = d/du + a_i d/dv$.

LEMMA 2.1. *The set $V_0^i = \{(u, v) \mid d\lambda_i(r_i) = 0\}$ is union of disjoint 1-manifolds and transversal to integral curves of $dv/du = a_i$, $i = 1, 2$.*

LEMMA 2.2. *Given $(u, v) \in S_2(u_0, v_0) \cap V_0^2$, $\sigma = \sigma(u_0, v_0; u, v) = \lambda_2(u, v) = \lambda_2$, $u > u_0$, we have, at (u, v),*

$$\frac{d\sigma}{d\mu} = \frac{d\lambda_2}{d\mu} = \frac{d^2\sigma}{d\mu^2} = 0 \quad and \quad \frac{d^2\lambda_2}{d\mu^2} > 0$$

and σ is increasing at (u, v) if immediately to the right of (u, v) along $R_2(u, v)$ is region V_-^2, $d^2\lambda_2/d\mu^2 < 0$ and σ is decreasing at (u, v) if immediately to the left of (u, v) along $R_2(u, v)$ is region V_+^2.

Lemmas 2.1 and 2.2 are easy consequences of (2.1) and Lemma 1.5; we omit the proofs.

Before we state and prove our rather long existence theorem, we sketch the construction of the solution.

Given a fixed point (u_0, v_0) in U, we first construct a curve $\gamma(u_0, v_0) = \gamma$ so that points (u, v) on γ, $u > u_0$, $v < v_0$, can be connected to (u_0, v_0) on the right by forward waves.

Suppose $(u_0, v_0) \in V_+^2$; we then let the first segment of γ be $S_2(u_0, v_0)$ and so the solution can be pictured as
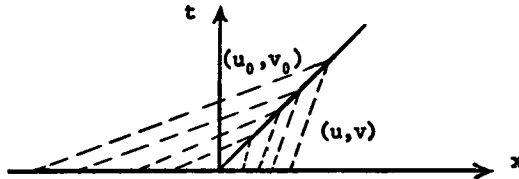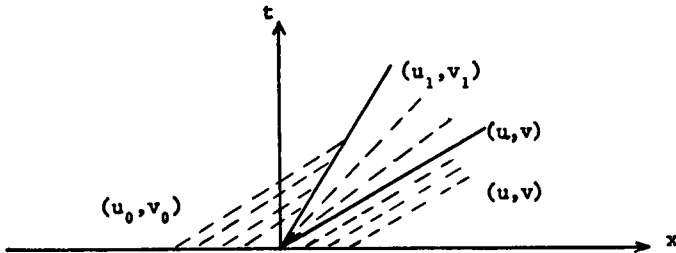

which is a forward shock, (where light lines denote characteristic lines).

As (u, v) moves further to the right along $S_2(u_0, v_0)$, the picture becomes at some point (u_1, v_1),
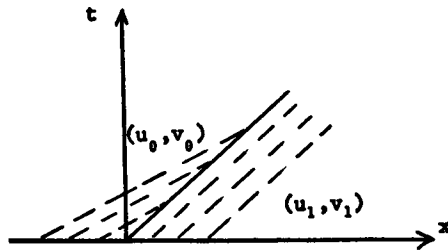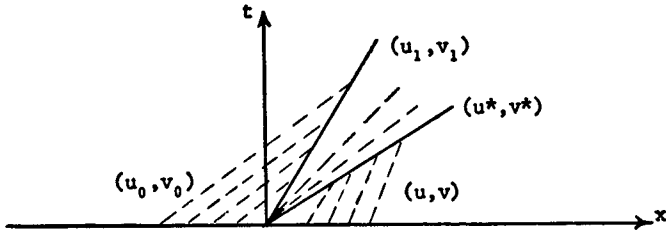

i.e. $\sigma(u_0, v_0; u_1, v_1) = \lambda_2(u_1, v_1)$. We then continue γ by $R_2(u_1, v_1)$, so that the solution is a shock connecting (u_0, v_0) and (u_1, v_1) followed by a rarefaction wave connecting (u_1, v_1) and (u, v) on $R_2(u_1, v_1)$. The diagram is
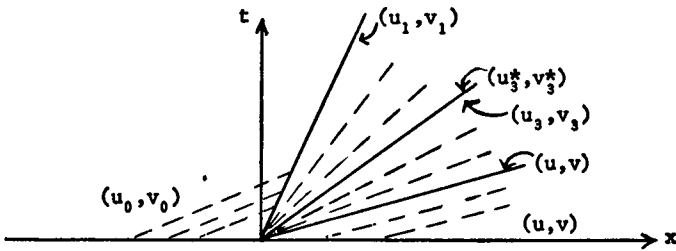

When $R_2(u_1, v_1)$ leaves region V_-^2 at (u_2, v_2), we continue γ so that

the point (u, v) can be connected by shock to some (u^*, v^*) on $R_2(u_1, v_1)$ between (u_1, v_1) and (u_2, v_2) and $\lambda_2(u^*, v^*) = \sigma(u^*, v^*; u, v)$. This can be pictured as

Continue γ until some point (u_3, v_3) so that $\sigma(u_3^*, v_3^*; u_3, v_3) = \lambda_2(u_3, v_3)$. We then continue γ by $R_2(u_3, v_3)$ and point (u, v) on $R_2(u_3, v_3)$ is connected to (u_3, v_3) by rarefaction wave:

Continue these processes so that γ is defined for all $u > u_0$. During the process we must always make sure that condition (E) holds for any discontinuity.

To be precise, we construct the curve γ as follows:

Step 1. By Lemma 2.1, we know that any curve in V_0^2 is transversal to R_2 curves, so we have two cases:

(i) Immediately to the right of (u_0, v_0) along $R_2(u_0, v_0)$ is region V_+^2:

The curve γ_1, the first segment of γ, is $S_2(u_0, v_0)$ starting at (u_0, v_0) and extended until there exists a point (u_1, v_1) which is the first point on $S_2(u_0, v_0)$ such that $\lambda_2(u_1, v_1) = \sigma(u_0, v_0; u_1, v_1)$ and immediately to the right of (u_1, v_1) along $R_2(u_1, v_1)$ is region V_-^2. The point (u, v) on γ_1 is connected to (u_0, v_0) by a forward shock. Analogous to the proof of Theorem 1.1, with the aid of Lemma 2.2, we can show that σ is decreasing between (u_0, v_0) and (u_1, v_1). Thus condition (E) is clearly satisfied for $\{(u_0, v_0),$ $(u, v)\}, (u, v) \in \gamma_1$.

(ii) Immediately to the right of (u_0, v_0) along $R_2(u_0, v_0)$ is region V_-^2:

The curve γ_1 is $R_2(u_0, v_0)$ starting at (u_0, v_0) and extended to (u_1, v_1), the first point at which $R_2(u_0, v_0)$ leaves V_-^2. Point (u, v) on γ_1 is connect-

ed to (u_0, v_0) by a forward rarefaction wave.

Step 2. (i) For case (i) of Step 1:

Since immediately to the right of (u_1, v_1) is region V_-^2, we proceed as in case (ii) of Step 1, i.e. continue the curve from (u_1, v_1) by $R_2(u_1, v_1)$ until there exists a point (u_2, v_2) at which $R_2(u_1, v_1)$ first leaves region V_-^2. The point (u, v) on $R_2(u_1, v_1)$ between (u_1, v_1) and (u_2, v_2) is connected to (u_1, v_1) by a forward rarefaction wave. The diagrams look like:
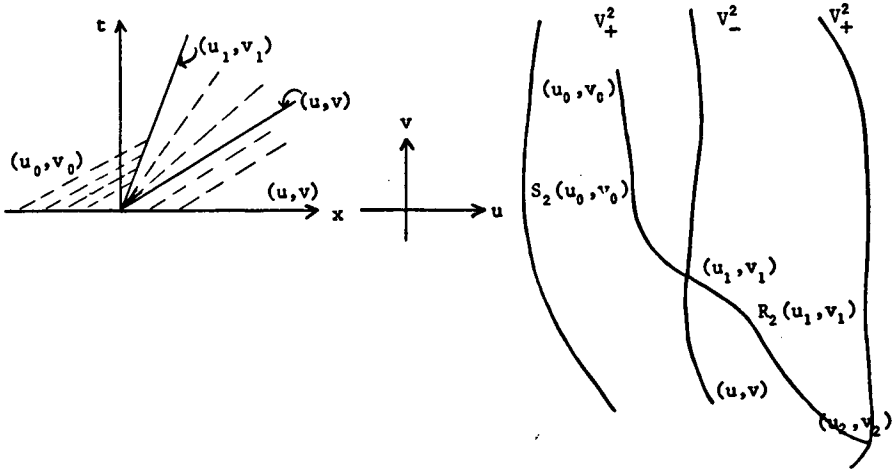

(ii) For case (ii) of Step 1:

We continue the curve from (u_1, v_1) by a *mixed curve* γ_1^* defined as follows: $(u, v) \in \gamma_1^*$ if there is a point $(u^*, v^*) \in \gamma_1$ such that (u, v) is the first point on $S_2(u^*, v^*)$, $u > u^*$, at which $\sigma(u^*, v^*, u, v) = \lambda_2(u^*, v^*)$. Such a γ_1^* exists at least when $|u^* - u_1|$ is small. In fact, since immediately to the right of (u_1, v_1) is region V_+^2, we know that $\sigma(u_1, v_1; u, v)$ is decreasing. By continuity of σ, we then have, for $(\tilde{u}^*, \tilde{v}^*) \in \gamma_1$ and near (u_1, v_1), the following diagram:
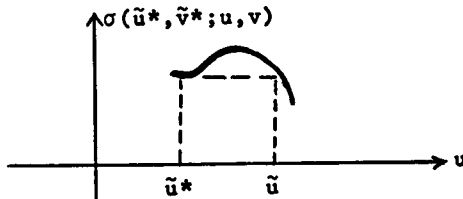

The reason $d\sigma/d\mu > 0$ near $(\tilde{u}^*, \tilde{v}^*)$ is that $(\tilde{u}^*, \tilde{v}^*)$ lies in region V_-^2.

We then connect (u^*, v^*) on γ_1 to (u, v) on γ_1^* by shock wave. Since $(u^*, v^*) \in \gamma_1 \subset V_-^2$, by definition of γ_1^*, condition (E) is satisfied for $\{(u^*, v^*); (u, v)\}$. Indeed, the diagrams look like the following:
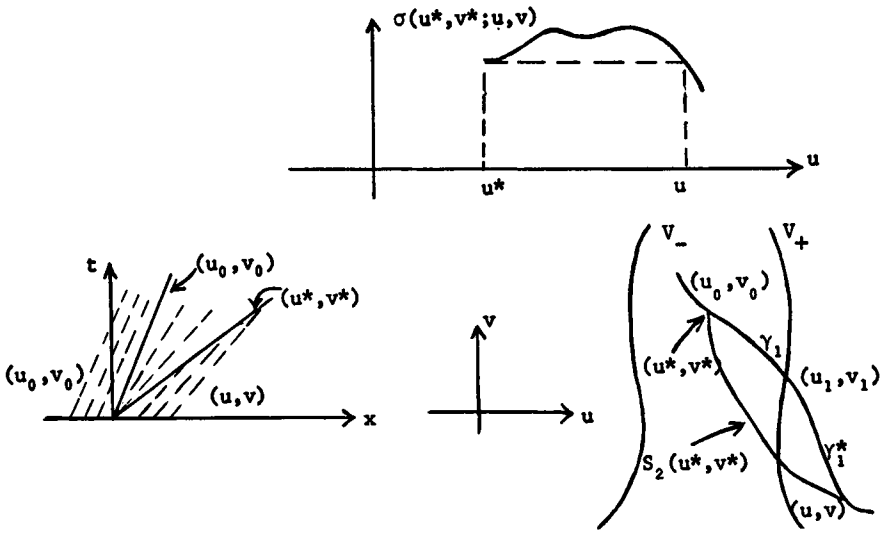
We continue the curve by γ_1^* starting at (u_1, v_1) until point (u_2, v_2) defined as follows:

(ii) (a) There exists point (u_2, v_2) on γ_1^* such that the corresponding point $(u_2^*, v_2^*) = (u_0, v_0)$. In this case we continue the curve from (u_2, v_2) by $S_2(u_0, v_0)$ and point (u, v) on $S_2(u_0, v_0)$ is connected to (u_0, v_0) by a forward shock. Condition (E) is clearly satisfied for those $(u, v) \in S_2(u_0, v_0)$ and close to (u_2, v_2). We have diagrams like the following:
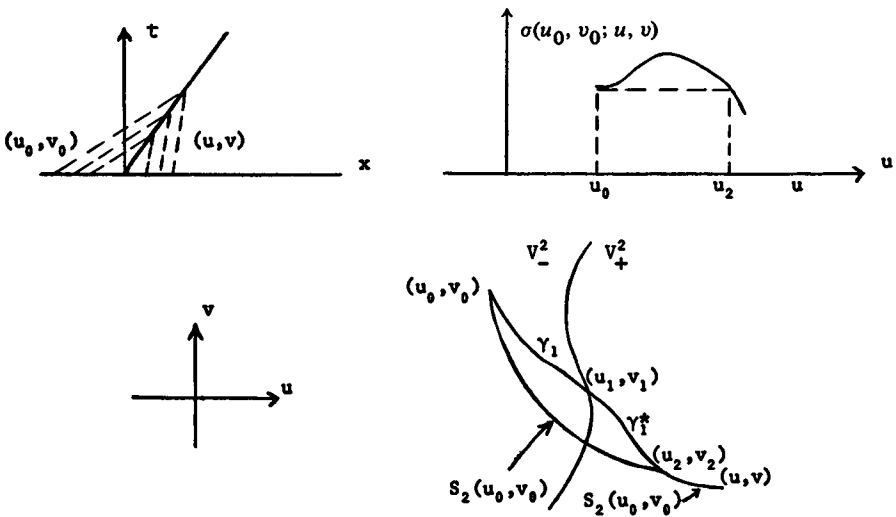
(ii) (b) There exists a point (u_2, v_2) on γ_1^* and a corresponding point (u_2^*, v_2^*) on γ_1 such that $\sigma(u_2^*, v_2^*, u, v)$ attains local minimum at (u_2, v_2).

For this case, we know, by Lemmas 1.4, 1.6 and 2.1 that $\sigma(u_2^*, v_2^*; u_2, v_2) = \lambda_2(u_2, v_2)$ and $(u_2, v_2) \in V_-^2$. We then continue the curve from (u_2, v_2) by $R_2(u_2, v_2)$ and the point $(u, v) \in R_2(u_2, v_2)$ is connected to (u_2, v_2) by a forward wave. The diagrams look like the following:
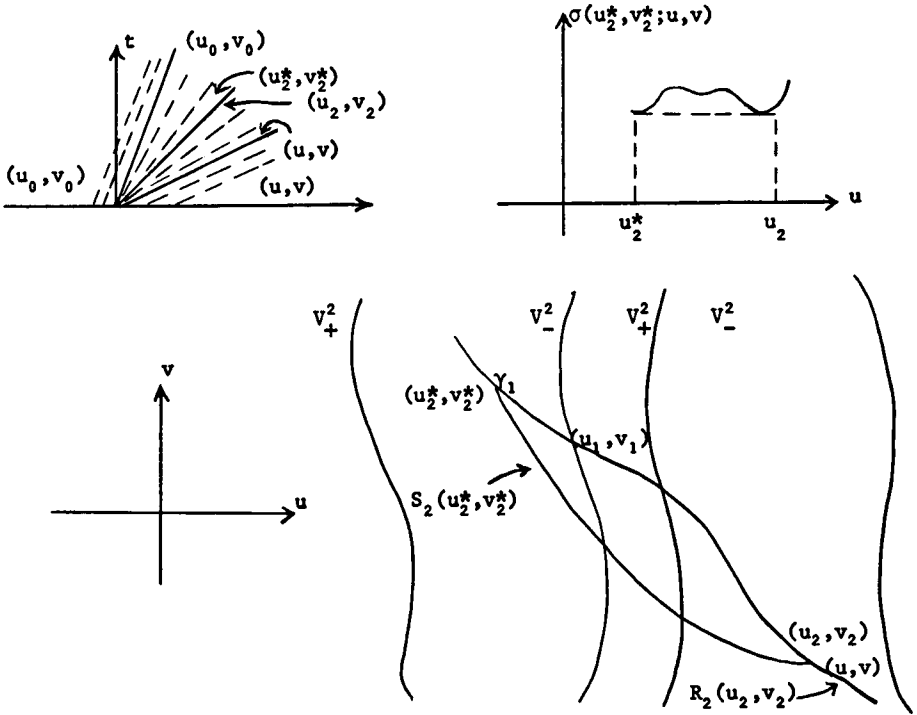
The discontinuous part of the solution (u_2^*, v_2^*), (u_2, v_2) has the property that the shock speed σ coincides with eigenvalue λ_2 on either side. We call such discontinuity a *contact discontinuity*.

REMARK. Lemma 2.3–Lemma 2.5 to be presented later will show that a mixed curve is continuous and decreasing as a function of u.

Step 3. (i) For case (i) of Step 2:

We continue the curve by the mixed curve γ_2^* starting at (u_2, v_2) where γ_2 is the rarefaction curve $R_2(u_1, v_1)$ between (u_1, v_1) and (u_2, v_2). The process is exactly the same as in case (ii) of Step 2.

If there is a point (u_3, v_3) on γ_2^*, (u_3^*, v_3^*) on γ_2, such that $\sigma(u_3^*, v_3^*, u, v)$ attain a local minimum at (u_3, v_3), we then continue the curve starting at (u_3, v_3) by $R_2(u_3, v_3)$.

If there is a point (u_3, v_3) on γ_2^* such that $(u_3^*, v_3^*) = (u_1, v_1)$, we then continue the curve starting at (u_3, v_3) by $S_2(u_0, v_0)$. In fact, in this case we have $\sigma(u_0, v_0; u_1, v_1) = \lambda_1(u_1, v_1) = \sigma(u_1, v_1; u_3, v_3)$ which then implies that $(u_3, v_3) \in S_2(u_0, v_0)$ and $\sigma(u_0, v_0; u_3, v_3) = \lambda_2(u_1, v_1)$. We have to check

condition (E) for $\{(u_0, v_0); (u_3, v_3)\}$. Suppose condition (E) fails, then since $\{(u_0, v_0); (u_1, v_1)\}$ satisfies (E), we would have a picture like the following:
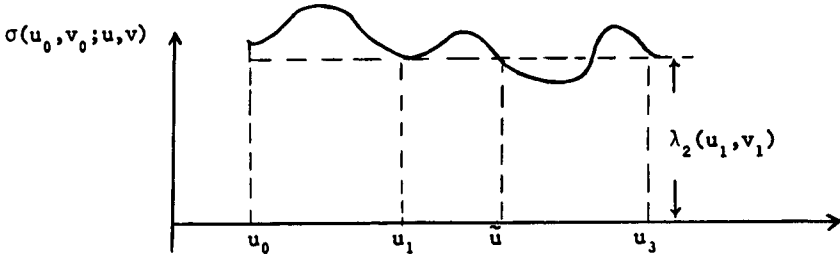
i.e. there is a point $(\widetilde{u}, \widetilde{v})$ on $S_2(u_0, v_0)$, $u_1 < \widetilde{u} < u_3$, and $\sigma(u_0, v_0; \widetilde{u}, \widetilde{v}) = \lambda_2(u_1, v_1) = \sigma(u_0, v_0; u_1, v_1)$. The last equalities imply that $(\widetilde{u}, \widetilde{v}) \in S_2(u_1, v_1)$ and $\sigma(u_1, v_1; u, v) = \lambda_2(u_1, v_1)$. But we have $(u_1, v_1) = (u_3^*, v_3^*)$; that is, (u_3, v_3) is the first point on $S_2(u_1, v_1)$ with $\sigma(u_1, v_1; u_3, v_3) = \lambda_2(u_1, v_1)$. This is a contradiction because $u_1 < \widetilde{u} < u_3$. So condition (E) is satisfied for $\{(u_0, v_0); (u_3, v_3)\}$. We have diagrams like the following:
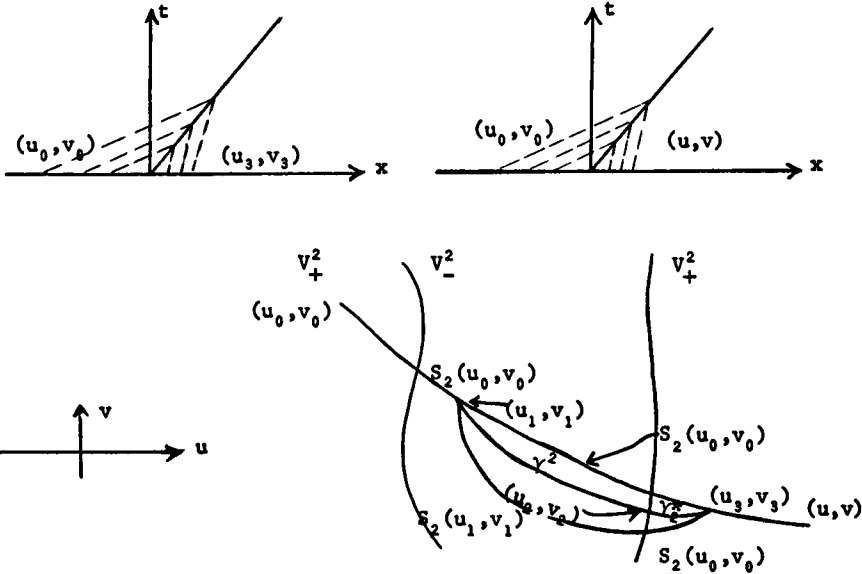
(ii) (a) For case (ii) (a) of Step 2:

We extend $S_2(u_0, v_0)$ until there exists a point (u_3, v_3) which is the first point on $S_2(u_0, v_0)$, $u_3 > u_2$ such that $\sigma(u_0, v_0; u_3, v_3) = \lambda_2(u_3, v_3)$ and that immediately to the right of (u_3, v_3) along $R_2(u_3, v_3)$ is region V_-^2. Then as in case (i) of Step 2, we continue the curve from (u_3, v_3) by $R_2(u_3, v_3)$ until it leaves region V_-^2.

(ii) (b) For case (ii) (b) of Step 2:

We continue $R_2(u_2, v_2)$ until there exists a point (u_3, v_3) at which $R_2(u_2, v_2)$ first leaves region V_-^2. Then we continue the curve by γ_2^*, where γ_2 is actually the curve $R_2(u_2, v_2)$ between (u_2, v_2) and (u_3, v_3). The diagrams look like the following:
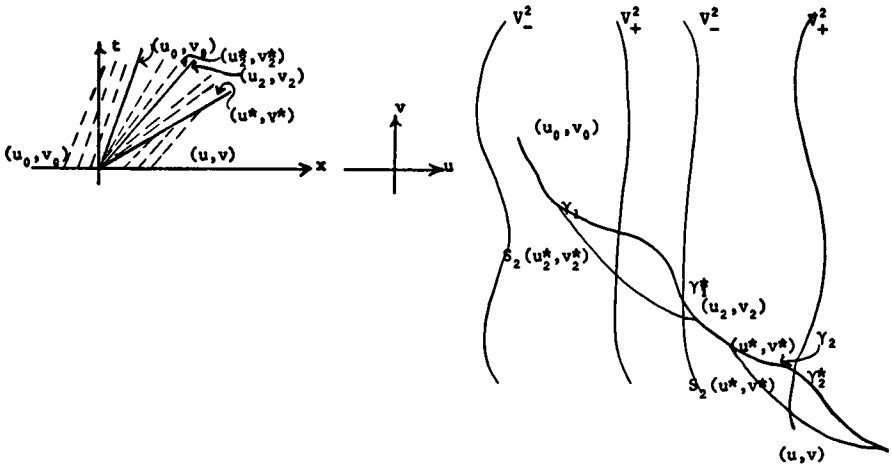
We need not describe how to continue the curve γ; it is analogous to the above steps. Thus the solution to the Riemann Problem $\{(u_0, v_0); (u, v)\}$ for $(u, v) \in \gamma$ takes the form
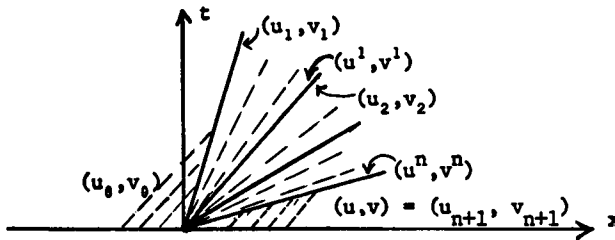
i.e. $\{(u_0, v_0); (u_1, v_1)\}$ and $\{(u^n, v^n); (u, v)\}$ are connected by one-sided contact discontinuities; $\{(u_i, v_i); (u^i, v^i)\}$, $1 \leqslant i \leqslant n$, are connected by rarefaction waves and $\{(u^i, v^i); (u_{i+1}, v_{i+1})\}$, $1 \leqslant i \leqslant n - 1$, are connected by two-sided contact discontinuities. Of course, we might have $(u_1, v_1) = (u^n, v^n)$ and $\{(u_0, v_0); (u, v)\}$ are connected by a single shock.

To prove that γ_i^* is continuous and defined for u and that condition (E) is satisfied for all discontinuities, we need some lemmas.

LEMMA 2.3. *Let γ be a curve with corresponding mixed curve γ^*, that is, γ is a segment of an R_2 curve in the region V_-^2, and for every point (u, v) on γ^*, there is a point (u^*, v^*) on γ such that $(u, v) \in S_2(u^*, v^*), u > u^*$,*

and (u, v) is the first point on $S_2(u^*, v^*)$ such that $\sigma(u^*, v^*; u, v) = \lambda_2(u, v)$. Then, along the curve γ^*,

$$\frac{dv}{du} = \frac{(u - u^*)g_u + (\lambda^* - f_u)(v - v^*)}{(v - v^*)f_v + (\lambda^* - g_v)(u - u^*)}$$

where g_v, f_v, f_u, g_u and dv/du are evaluated at (u, v) and $\lambda^* = \lambda_2(u^*, v^*)$. Since $\lambda^* = \sigma$, we thus have $dv/du = h_2(u^*, v^*; u, v)$.
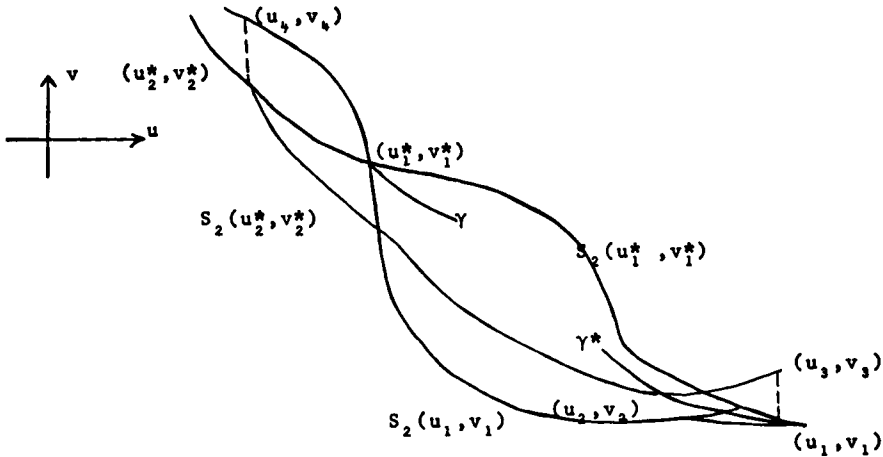
PROOF. Use the Hugoniot condition for $\{(u^*, v^*); (u, v)\}$ along with equalities:

$$\frac{dv^*}{du^*} = a^* \equiv a_2(u^*, v^*), \qquad \frac{dv}{du} = \frac{v_1 + v_2 dv^*/du^*}{u_1 + u_2 dv^*/du^*},$$

$$u_1 = \frac{\partial u}{\partial u^*}\bigg|_{(u,v)}, \qquad u_2 = \frac{\partial u}{\partial v^*}\bigg|_{(u,v)}, \quad \text{etc.}$$

LEMMA 2.4. Let $(u_1^*, v_1^*) \in \gamma$, $(u_2^*, v_2^*) \in \gamma$, $(u_1, v_1) \in \gamma^*$, $(u_2, v_2) \in \gamma^*$, and $u_1^* > u_2^*$, then $u_2 > u_1$.

PROOF. We have only to prove the lemma when $|u_1^* - u_2^*|$ is small. Suppose, otherwise, we have $u_2 < u_1$, and the picture looks like:

By continuity of σ, we have $|u_1 - u_2|$ small. Pick points $(u_3, v_3) \in S_2(u_2^*, v_2^*)$ and $(u_4, v_4) \in S_2(u_1, v_1)$, $u_3 = u_1, u_4 = u_2^*$. By Lemma 2.3, we know $S_2(u_2^*, v_2^*)$ is tangent to γ^* at (u_2, v_2); thus $|v_3 - v_1| = O_2|u_1 - u_2|$. By Lemma 2.3 and continuity of σ, we have $O_2|u_1 - u_2| = O_2|u_1^* - u_2^*|$. So $|v_3 - v_1| = O_2|u_1^* - u_2^*|$. The definition of γ^* gives $\sigma(u_1, v_1; u_1^*, v_1^*) = \lambda_2(u_1^*, v_1^*)$, and thus, by Lemma 1.5, $h_2(u_1, v_1; u_1^*, v_1^*) = a_2(u_1^*, v_1^*)$, so $|v_4 - v_2^*| = O_2|u_1^* - u_2^*|$.

The last equality along with $|v_3 - v_1| = O_2|u_1^* - u_2^*|$ gives $\sigma(u_1, v_1; u_4, v_4) = \sigma(u_3, v_3; u_2^*, v_2^*) + O_2|u_1^* - u_2^*|$.

Since condition (E) is satisfied for the pair $\{(u_2^*, v_2^*); (u_2, v_2)\}$, we know that $d\sigma/d\mu \leqslant 0$ along $S_2(u_2^*, v_2^*)$ at the point (u_2, v_2). This implies that $\sigma(u_2^*, v_2^*; u_2, v_2) \geqslant \sigma(u_2^*, v_2^*; u_3, v_3) + O_2|u_1 - u_2|$.

Since $\sigma(u_1, v_1; u_1^*, v_1^*) = \lambda_2(u_1^*, v_1^*)$, we have, by Lemma 1.4, $d\sigma/d\mu = 0$ along $S_2(u_1, v_1)$ at (u_1^*, v_1^*); thus $\sigma(u_1, v_1; u_1^*, v_1^*) = \sigma(u_1, v_1; u_4, v_4) + O_2|u_1 - u_2|$.

Since $\gamma \subset V_-^2$, we have $\lambda_2(u_1^*, v_1^*) \geqslant \lambda_2(u_2^*, v_2^*) + k|u_1 - u_2|$ for some $k > 0$.

Using the inequalities just derived, we have

$$\lambda_2(u_2^*, v_2^*) = \sigma(u_2^*, v_2^*; u_2, v_2) \geqslant \sigma(u_2^*, v_2^*; u_3, v_3) + O_2|u_1^* - u_2^*|$$

$$= \sigma(u_1, v_1; u_4, v_4) + O_2|u_1^* - u_2^*|$$

$$= \sigma(u_1, v_1; u_1^*, v_1^*) + O_2|u_1^* - u_2^*|$$

$$= \lambda_2(u_1^*, v_1^*) + O_2|u_1^* - u_2^*|$$

$$\geqslant \lambda_2(u_2^*, v_2^*) + k|u_1^* - u_2^*| + O_2|u_1^* - u_2^*|, \quad k > 0,$$

which is a contradiction. The lemma is proved.   Q.E.D.

Similarly, we can prove

LEMMA 2.5. *Let* $(u_1^*, v_1^*) \in \gamma$, $(u_1, v_1) \in \gamma^*$, *and suppose that* $\sigma(u_1^*, v_1^*; u, v)$ *attains a local minimum at* (u_1, v_1). *Then, for* $(u^*, v^*) \in \gamma$, $|u^* - u_1^*|$ *small, we have*

(i) *There is no point* $(u, v) \in S_2(u^*, v^*)$, $|u - u_1|$ *small or* $u^* < u \leqslant u_1$, *so that condition* (E) *is satisfied for* $\{(u^*, v^*); (u, v)\}$, *provided* $u^* < u_1^*$.

(ii) *There is point* $(u, v) \in S_2(u^*, v^*)$, $|u - u_1|$ *small,* $u^* < u \leqslant u_1$, *so that condition* (E) *is satisfied for* $\{(u^*, v^*); (u, v)\}$, *provided* $u^* > u_1^*$.

Using Lemmas 2.4 and 2.5, we can prove

LEMMA 2.6. *Let* $(u_3, v_3) \in S_2(u_1, v_1)$, $u_2 > u_1$, $\sigma(u_1, v_1; u_3, v_3) = \lambda_2(u_1, v_1)$ *and suppose that* $\{(u_1, v_1); (u_3, v_3)\}$ *satisfies* (E). *Take any* $(u^*, v^*) \in R_2(u_1, v_1)$, $u^* < u_1$, *such that any point on* $R_2(u_1, v_1)$ *between* (u^*, v^*) *and* (u_1, v_1) *lies in region* V_-^2. *Then there is no point* (u, v) *on* $S_2(u^*, v^*)$, $u^* < u \leqslant u_3$, *such that condition* (E) *is satisfied for* $\{(u^*, v^*); (u, v)\}$.

REMARK. Part (i) of Lemma 2.5 implies that the curve γ^* cannot be extended beyond (u_1, v_1); and part (ii) implies that there are points on γ^* left of (u_1, v_1). Lemma 2.6 extends the results globally.

Using Lemmas 2.4, 2.5 and 2.6, we finally have

LEMMA 2.7. *Any mixed curve is a smooth and decreasing function of* u.

We make the following assumption:

$$(2.2) \begin{cases} \text{Either (i) } g_u \text{ is finite for finite } u, \\ \text{or (ii) the integral curve of } dv/du = g_u \text{ does not escape along any} \\ \text{vertical line and every finite-width vertical zone contains only finite} \\ \text{many curves in } V_0^2. \end{cases}$$

THEOREM 2.1. *Under the assumptions* (0.4), (0.5), (2.1) *and* (2.2), *the curve* $\gamma(u_0, v_0)$ *constructed is defined for all* u *and is a decreasing function of* u. *Here* $\gamma(u_0, v_0)$ *is such that points* (u, v) *on* $\gamma(u_0, v_0)$ *can be connected to* (u_0, v_0) *on the right by forward waves.*

PROOF. The fact that the curve γ is a decreasing function of u is obvious by now. We have only to show that γ does not have vertical asymptotes. With Lemma 1.1 and Lemma 2.3, we need only to treat the cases when γ is composed of a single rarefaction curve $R_2(\widetilde{u}, \widetilde{v})$ or when γ is composed of infinitely many mixed and rarefaction curves eventually. In the first case, λ_2 is increasing along $R_2(\widetilde{u}, \widetilde{v})$, and thus $dv/du = a_2 = g_u/\lambda - g_v < g_u/\lambda_2(u, v) < g_u/\lambda_2(\widetilde{u}, \widetilde{v})$. By (2.2), we know $R(\widetilde{u}, \widetilde{v})$ does not have vertical asymptote. In the case γ is composed of infinitely many rarefaction and mixed curves, and γ has vertical asymptote, we show this is impossible by using (2.2). From (ii) of (2.2), this cannot happen, since the intersection of a rarefaction curve with the adjacent mixed curve is a point in V_0^2. We treat the case when (i) of (2.2) is assumed. With Lemma 1.1, we may assume that $S_2(u_0, v_0)$ does not appear in γ eventually, thus the solution to the Riemann Problem $\{(u_0, v_0); (u, v)\}$, u close to $\bar{u}$ and $(u, v) \in \gamma$ is of the form that (u_0, v_0) is connected to a fixed (u_1, v_1) on γ by a contact discontinuity and (u_1, v_1) is connected to (u, v) by contact discontinuities and rarefaction waves. With the preceding argument, in order to have γ escape along $u = \bar{u}$, there must be sequences $\{(u_n, u_n)\}$ and $\{(u^n, v^n)\}$ on γ, such that (u_n, v_n) is connected to (u^n, v^n) by a two-sided contact discontinuity and $\lim (v_n - v^n)/(u_n - u^n) = -\infty$. However, since $g_v \leqslant 0, g_u < 0$, the last equality implies

$$\sigma(u^n, v^n; u_n, v_n) = \frac{g(u^n, v^n) - g(u_n, v_n)}{v^n - v_n} < \frac{g(u^n, v_n) - g(u_n, v_n)}{v^n - v_n}$$

$$= \frac{(u^n - u_n)g_u(\widetilde{u}, v_n)}{v^n - v_n} \longrightarrow 0 \quad \text{as } u_n \longrightarrow \bar{u}.$$

But since

$$\sigma(u^n, v^n; u_n, v_n) > \lambda_2(u_1, v_1) > 0,$$

this is a contradiction. We have finished the proof of the theorem.　　Q.E.D.
Similarly, under the assumptions (0.4), (0.5), (2.1) and the following:

(2.3)　The integral curve of $dv/du = a$ does not have horizontal asymptote and any finite-width horizontal zone contains only finite many curves in V_0^1,

we have

THEOREM 2.2. *Given any* $(u_0, v_0) \in U$, *there is a curve* $\beta(u_0, v_0) = \beta$ *defined for all* v *and contained in the quadrants* $\{(u, v) | u \geqslant u_0, v \geqslant v_0\}$ *and* $\{(u, v) | u \leqslant u_0, v \leqslant v_0\}$ *such that point* (u, v) *on* β *can be connected to* (u_0, v_0) *on the right by forward shock, rarefaction waves and contact discontinuities.*

From Theorems 2.1 and 2.2, we finally have the following theorem.

THEOREM 2.3. *Suppose that* (0.4), (0.5), (2.1), (2.2) *and* (2.3) *hold. Then the Riemann Problem* $\{(u_l, v_l); (u_r, v_r)\}$ *for general data can be solved in the class of shocks, rarefaction waves and contact discontinuities, and condition* (E) *is satisfied across discontinuities.*

PROOF. Divide the region U into four quadrants as shown in the diagram that follows
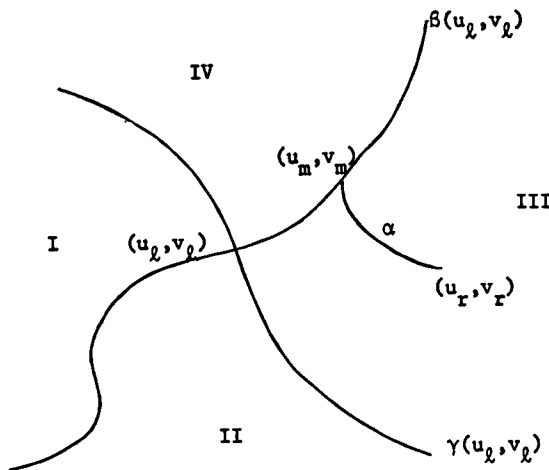
Suppose, say, $(u_r, v_r) \in \text{III}$. Construct a curve $\alpha(u_r, v_r)$ from (u_r, v_r) such that a point (u, v) on α can be connected to (u_r, v_r) on the left by forward waves. An analogy of Theorems 2.1 and 2.2 yields that α is defined for all $u < u_r$. Since β is defined for all v, α intercepts β at some point (u_m, v_m). We then solve the Riemann Problem $\{(u_l, v_l); (u_r, v_r)\}$ by connecting (u_l, v_l) to (u_m, v_m) by backward waves and (u_m, v_m) to (u_r, v_r) by forward waves, as shown in the picture below
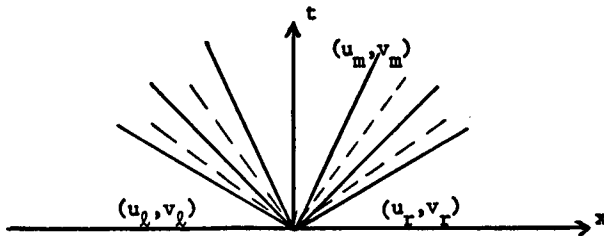

This completes the proof of the theorem.   Q.E.D.

3. **Uniqueness.** In this section we shall prove that the solution to the Riemann Problem is unique, provided condition (E) is satisfied.

THEOREM 3.1.   *Under the assumptions* (0.4), (0.5) *and* (2.1), *if* (u_l, v_l) *is connected to* (u_r, v_r) *on the right by finite number forward shocks, rarefaction waves and contact discontinuities and condition* (E) *is satisfied across discontinuities, then* $(u_r, v_r) \in \gamma(u_l, v_l)$ *and the solution is the one constructed in* §2.

PROOF.   By simple geometric consideration, using Lemma 1.7, we know that the solution must be of the form
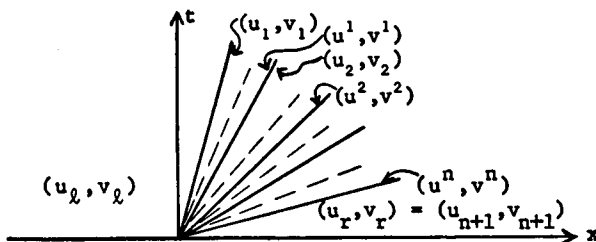
where $\{(u^i, v^i); (u_{i+1}, v_{i+1})\}, 1 \leq i \leq n - 1$, are connected by contact discontinuities, $\{(u_i, v_i); (u^i, v^i)\}, 1 \leq i \leq n - 1$, are connected by rarefaction waves, and $\{(u_l, v_l); (u_1, v_1)\}$ and $\{(u^n, v^n); (u_r, v_r)\}$ are connected by shocks or contact discontinuities.

We assume that $u^i \neq u_{i+1}$, that is, there is a real discontinuity between (u^i, v^i) and (u_{i+1}, v_{i+1}). Suppose that $u_1 > u_l$. The case $u_1 \leqslant u_l$ can be treated similarly. The proof is based on several claims.

CLAIM 1. $(u_i, v_i) \in V_-^2$ and $u_i < u^i, u^i < u_{i+1}, i = 1, 2, \cdots, n$.

PROOF. We shall show that $(u_1, v_1) \in V_+^2 \cup V_0^2$ is impossible.

If $(u_1, v_1) \in V_+^2$, then since $\sigma(u_l, v_l; u_1, v_1) = \lambda_2(u_1, v_1)$, we have, by Lemma 1.6, that $\sigma(u_l, v_l; u, v)$ has a local maximum at (u_1, v_1). This contradicts the condition (E) for $\{(u_l, v_l); (u_1, v_1)\}$.

If $(u_1, v_1) \in V_0^2$, and immediately to the left of (u_1, v_1) is region V_+^2, then, by Lemma 2.2, $\sigma(u_l, v_l; u, v)$ is increasing at (u_1, v_1). This again contradicts condition (E) for $\{(u_l, v_l); (u_1, v_1)\}$.

If $(u_1, v_1) \in V_0^2$, and immediately to the left of (u_1, v_1) is region V_-^2, then, by Lemma 2.1, immediately to the right of (u_1, v_1) is region V_+^2. However, we have $(u^1, v^1) \in R_2(u_1, v_1)$ and $\{(u_1, v_1); (u^1, v^1)\}$ are connected by forward rarefaction wave. This implies that, if $u^1 > u_1$, then the R_2 curve between (u_1, v_1) and (u^1, v^1) lies in V_-^2, and, if $u^1 < u_1$, then the R_2 curve between (u_1, v_1) and (u^1, v^1) lies in V_+^2. In our present case, this is impossible.

In any case, we have $(u_1, v_1) \in V_-^2$. Thus, by Lemma 1.3, since $\{(u^1, v^1); (u_2, v_2)\}$ satisfies condition (E), we know that $u_2 > u^1$. It is clear that Claim 1 can be proved by induction.

Claim 2. (i) *Let* $\hat{u}^{i-1} = \max \{u | (u, v) \in S_2(u^{i-1}, v^{i-1}), u^{i-1} \leqslant u \leqslant u_r, \{(u^{i-1}, v^{i-1}); (u, v)\}$ *satisfies* (E)$\}$, $1 \leqslant i \leqslant n + 1$; *then* $u_i = \hat{u}^{i-1}$.

(ii) *Let* $(u, v) \in R_2(u_i, v_i); (\tilde{u}, \tilde{v}) \in S_2(u, v), u_i \leqslant u \leqslant u^i, u < \tilde{u} \leqslant u_r$; *then* $\{(u, v); (\tilde{u}, \tilde{v})\}$ *does not satisfy* (E), $1 \leqslant i \leqslant n$.

PROOF (by induction). Part (i) is clearly true for $i = n + 1$. By Lemma 2.6, (ii) is true for $i = n$. Suppose that (i) fails for $i = n$. Pick $(\bar{u}, \bar{v}) \in S_2(u^{n-1}, v^{n-1}), u_n < \bar{u} \leqslant u_r$, such that $\sigma(u^{n-1}, v^{n-1}; \bar{u}, \bar{v}) = \lambda_2(u^{n-1}, v^{n-1})$. Since, by Claim 1, $(u_n, v_n) \in V_-^2$, we know that $\sigma(u^{n-1}, v^{n-1}; u, v)$ attains a local minimum at (u_n, v_n) and $\sigma(u^{n-1}, v^{n-1}; u_n, v_n) = \lambda_2(u^{n-1}, v^{n-1}) = \lambda_2(u_n, v_n)$. The last equalities together with the fact that $\sigma(u^{n-1}, v^{n-1}; \bar{u}, \bar{v}) = \lambda_2(u^{n-1}, v^{n-1})$ implies that $(\bar{u}, \bar{v}) \in S_2(u_n, v_n)$ and $\sigma(u_n, v_n; \bar{u}, \bar{v}) = \lambda_2(u_n, v_n)$.

Since $(u_n, v_n) \in V_-^2$, we have $d\sigma/d\mu > 0$ along $S_2(u_n, v_n)$ at points close to (u_n, v_n). Thus there exists $(u, v) \in S_2(u_n, v_n), u_n < u \leqslant \bar{u}$, such that $\{(u_n, v_n), (u, v)\}$ satisfies (E).
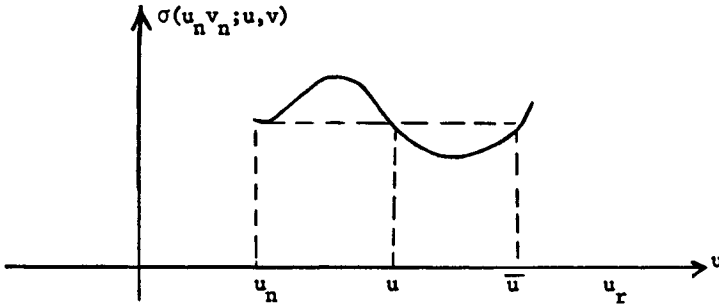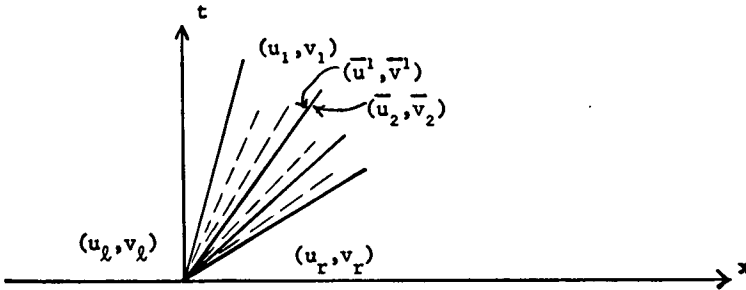
This contradicts (ii) of this claim for $i = n$. Thus (i) holds for $i = n$. We then prove (ii) for $i = n - 1$ by using Lemmas 2.5 and 2.6. This completes the proof of Claim 2 by induction.

We now return to the proof of Theorem 3.1.

Claim 2, (i) implies that $u_1 = \hat{u}_l$. Since $(u_1, v_1) \in S_2(u_l, v_l)$, we know that (u_1, v_1) is uniquely determined by (u_l, v_l) and u_r.

Suppose that there exists $(\bar{u}^i, \bar{v}^i)$, and $(\bar{u}_i, \bar{v}_i)$, $i = 1, 2, \cdots , \bar{u}^1 \neq u^1$, and that

solves the Riemann Problem $\{(u_l, v_l); (u_r, v_r)\}$. Without loss of generality, assume $\bar{u}^1 < u^1$. By Claim 1, $\bar{u}^1 < \bar{u}_2 \leqslant u_r$ and $\{(\bar{u}^1, \bar{v}^1); (\bar{u}_2, \bar{v}_2)\}$ satisfies condition (E). This contradicts part (ii) of Claim 2 for $i = 1$. Thus $\bar{u}^1 = u^1$. Since $(u^1, v^1) \in R_2(u_1, v_1)$, it follows that (u^1, v^1) is uniquely determined by (u_l, v_l) and u_r. The proof of Theorem 3.1 is complete by induction. Q.E.D.

THEOREM 3.2. *Assume* (0.4), (0.5), (2.1), (2.2) *and* (2.3). *Take* $(u_0, v_0) \in U$, (u_1, v_1) *and* (u_2, v_2) *on* $\beta(u_0, v_0)$, $v_1 \neq v_2$. *Then* $\gamma(u_1, v_1) \cap \gamma(u_2, v_2) \neq \emptyset$. *Here* β *and* γ *are defined in Theorems* 2.1 *and* 2.2.

PROOF. Suppose, otherwise, there is $(u_3, v_3) \in \gamma(u_1, v_1) \cap \gamma(u_2, v_2)$. Assume that $u_3 > u_1, u_3 > u_2$. The case $u_3 < u_1, u_3 < u_2$ can be treated similarly.

Choose $(u^1, v^1) \in \beta(u_0, v_0)$, (u^1, v^1) lying between (u_1, v_1) and (u_2, v_2). Then (u^1, v^1) intercepts $\gamma(u_i, v_i)$, $i = 1$ or 2, at, say $(\bar{u}^1, \bar{v}^1)$, $u^1 < \bar{u}^1 \leqslant u_3$.

Next choose (u^2, v^2) on $\beta(u_0, v_0)$ between (u^1, v^1) and (u_i, v_i). Then $\gamma(u^2, v^2)$ intercepts $\gamma(u_i, v_i)$ or $\gamma(u^1, v^1)$ at say $(\bar{u}^2, \bar{v}^2)$, $u^2 < \bar{u}^2 \leqslant u_3$. Continuing the process, we then get a sequence $\{(u^i, v^i)\}$. Without introducing a new notation, by passing to a subsequence, we may let (u_1, v_1) be the limit point of the sequence $\{(u^i, v^i)\}$. Thus we can assume that (u_2, u_2) is arbitrarily close to (u_1, v_1). We consider only the case $v_1 \geqslant v_0$, $v_2 \geqslant v_0$. The other cases can be treated similarly.

Let $\alpha(u_3, v_3)$ be a curve such that any point (u, v) on $\alpha(u_3, v_3)$ can be connected to (u_3, v_3) on the left by forward waves. An analogue of Theorem 3.1 says that (u_2, v_2) and (u_1, v_1) are both on $\alpha(u_3, v_3)$ and that $\alpha(u_3, v_3)$ is composed of shock, rarefaction and mixed curves.

By choosing (u_2, v_2) close enough to (u_1, v_1), we have

(3.1)

$$\text{either} \quad \frac{du}{dv}\bigg|_{\alpha(u_3, u_3)} \geqslant \frac{dv}{du}\bigg|_{\beta(u_0, v_0)} \geqslant 0 \quad \text{at } (u_2, v_2);$$

$$\text{or} \quad \frac{dv}{du}\bigg|_{\alpha(u_3, v_3)} \leqslant \frac{dv}{du}\bigg|_{\beta(u_0, v_0)} \leqslant 0 \quad \text{at } (u_2, v_2).$$

We only prove the theorem for the second case; i.e. we assume (3.1). The first case can be treated similarly.

Since $a_1 > 0$, by (3.1), we have $dv/du|_{\beta(u_0, v_0)} \neq a_1$ at (u_2, v_2). Hence, by the construction of $\beta(u_0, v_0)$, there exists $(u^0, v^0) \in \beta(u_0, v_0)$ such that $\{(u^0, v^0); (u_2, v_2)\}$ satisfies (E) and $dv/du|_{\beta(u_0, v_0)} = h_2(u^0, v^0; u_2, v_2)$ at (u_2, v_2).

If $du/dv|_{\alpha(u_3, v_3)} = a_2$ at (u_2, v_2), then by (3.1), $0 > h(u^0, v^0; u_2, v_2) \geqslant a_2(u_2, v_2)$. This would then lead to $\sigma - \lambda_2 > 0$, which is a contradiction, since $\sigma < 0$ and $\lambda_2 > 0$. So there exists (u^3, v^3) on $\alpha(u_3, v_3)$ such that $\{(u_2, v_2); (u^3, v^3)\}$ satisfies (E) and $dv/du|_{\alpha(u_3, v_3)} = h_2(u^3, v^3; u_2, v_2)$ at (u_2, v_2). (3.1) then gives
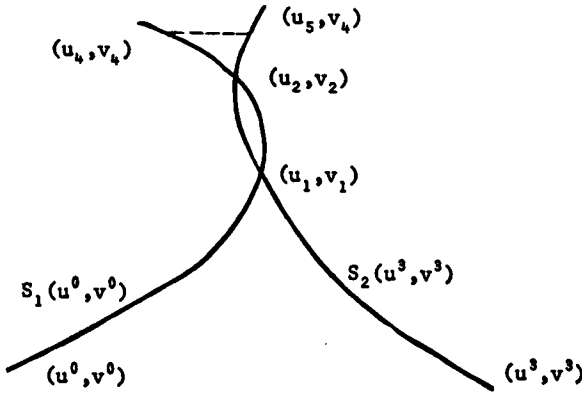
(3.2) $$h_2(u^3, v^3; u_2, v_2) \leqslant h_2(u^0, v^0; u_2, v_2) \leqslant 0.$$

By condition (E) for $\{(u_2, v_2); (u^3, v^3)\}$ and $\{(u^0, v^0); (u_2, v_2)\}$, using Lemmas 1.4, 1.5 and 1.7, we have

(3.3) $$\frac{d\sigma}{d\mu} > 0 \quad \text{along } S_2(u^3, v^3) \text{ at } (u_2, v_2),$$

(3.4) $$\frac{d\sigma}{d\mu} < 0 \quad \text{along } S_1(u^0, v^0) \text{ at } (u_2, v_2).$$

Pick points (u_4, v_4) on $S_1(u^0, v^0)$ and (u_5, v_4) on $S_2(u^3, v^3)$, $|v_4 - v_2|$ small, $v_4 > v_2$. We have the following picture:

By (3.2), we know

(3.5)
$$u_5 \geqslant u_4 + O_2 |v_4 - v_2|.$$

Since $g_u < 0$, (3.5) implies

(3.6)
$$g(u_5, v_4) \leqslant g(u_4, v_4) + O_2 |v_4 - v_2|.$$

Since $\sigma = \Delta g / \Delta u$, (3.3) implies that

$$\frac{g(u_5, v_4) - g(u^3, v^3)}{v_4 - v^3} > \frac{g(u_2, v_2) - g(u^3, v^3)}{v_2 - v^3} + k_1 |v_2 - v_4|, \quad k_1 > 0.$$

Since $v_4 > v_2 > v^3$ and $\sigma|_{\alpha(u^3, v^3)} > 0$, the above inequality gives

$$g(u_5, v_4) - g(u^3, v^3) > g(u_2, v_2) - g(u^3, v^3) + k_2 |v_2 - v_4|, \quad k_2 > 0,$$

(3.7)
$$g(u_5, v_4) > g(u_2, v_2) + k_2 |v_2 - v_4|, \quad k_2 > 0.$$

Similarly, (3.4) gives

(3.8)
$$g(u_2, v_2) > g(u_4, v_4) + k_3 |v_2 - v_4|, \quad k_3 > 0.$$

Now (3.7) and (3.8) imply that $g(u_5, v_4) > g(u_4, v_4) + k|v_2 - v_4|$, $k = k_2 + k_3 > 0$; this contradicts (3.6).

This completes the proof of Theorem 3.2.   Q.E.D.

Finally we have the main result of this section.

THEOREM 3.3. *Assume* (0.4), (0.5), (2.1), (2.2) *and* (2.3). *There exists at most one solution to any Riemann Problem* $\{(u_l, v_l); (u_r, v_r)\}$ *in the class of finite number shocks, rarefaction waves and contact discontinuities; such that condition* (E) *is satisfied across discontinuities.*

PROOF. Suppose the Riemann Problem $\{(u_l, v_l); (u_r, v_r)\}$ can be solved by connecting (u_l, v_l) to (u_m, v_m) by backward waves and (u_m, v_m) to (u_r, v_r) by forward waves; and can also be solved by connecting (u_l, v_l) to $(\bar{u}_m, \bar{v}_m)$ by backward waves and $(\bar{u}_m, \bar{v}_m)$ to (u_r, v_r) by forward waves. Then, by Theorem 3.1, we have $\gamma(u_m, v_m) \cap \gamma(\bar{u}_m, \bar{v}_m) \supset \{(u_r, v_r)\}$ and $(u_m, v_m) \in \beta(u_l, v_l); (\bar{u}_m, \bar{v}_m) \in \beta(u_l, v_l)$. Thus by Theorem 3.2, $(u_m, v_m) = (\bar{u}_m, \bar{v}_m)$. Also Theorem 3.1 says that both $\{(u_l, v_l); (u_m, v_m)\}$ and $\{(u_m, v_m); (u_r, v_r)\}$ have unique solution. This finishes the proof of Theorem 3.3.   Q.E.D.

Combining Theorems 2.3 and 3.3, we finally have the following main theorem.

THEOREM. *Assume* (0.4), (0.5), (2.1), (2.2) *and* (2.3). *There exists a unique solution to any Riemann Problem in the class of shocks, rarefaction waves and contact discontinuities, such that condition* (E) *is satisfied across discontinuities.*

The author wishes to express his gratitude to Professor Joel A. Smoller for his encouragement and also for the inspiration he received from their many stimulating conversations, all of which contributed to the completion of this work.

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