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The Riemannian Geometry of the Yang-Mills Moduli Space

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Abstract. The moduli space $\mathscr{M}$ of self-dual connections over a Riemannian 4-manifold has a natural Riemannian metric, inherited from the L^2 metric on the space of connections. We give a formula for the curvature of this metric in terms of the relevant Green operators. We then examine in great detail the moduli space $\mathscr{M}_1$ of $k=1$ instantons on the 4-sphere, and obtain an explicit formula for the metric in this case. In particular, we prove that $\mathscr{M}_1$ is rotationally symmetric and has "finite geometry:" it is an incomplete 5-manifold with finite diameter and finite volume.

Introduction

The moduli spaces of self-dual connections on vector bundles over a Riemannian 4-manifold have been studied from two different viewpoints. Mathematicians have sought to understand the topology of these moduli spaces. Most notable here is the work of S. Donaldson showing that even a rudimentary knowledge of this topology can lead to important results about smooth 4-manifolds. Physicists, on the other hand, study these spaces because the semiclassical – or "instanton" – approximation to the Green functions of (Euclidean) quantum Yang-Mills theory is expressed in terms of integrals over the moduli spaces. The evaluation of such integrals requires a detailed description of the *metric* and the *volume form* of the moduli spaces. In this paper we investigate moduli spaces with the goal of describing them as concrete Riemannian manifolds.

The relevant Riemannian metric on the moduli space $\mathscr{M}$ is the "L^2 metric", defined as follows. First, the space of connections on a principal bundle P is an affine space $\mathscr{A}$ whose tangent space is the space of 1-forms with values in an associated vector bundle $\mathrm{Ad}P$. The L^2 inner product of such forms defines a Riemannian metric on $\mathscr{A}$. This metric is invariant under the action of the gauge group $\mathscr{G}$, and splits the tangent bundle $T\mathscr{A}$ into $\mathscr{G}$-invariant "vertical" and

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"horizontal" subbundles. The metric on the horizontal subbundle passes to $\mathcal{A}/\mathcal{G}$. We then obtain a metric on the moduli space $\mathcal{M}$ – a submanifold of $\mathcal{A}/\mathcal{G}$ – by restriction.

We have tried to make this paper as self-contained as possible. Thus the first section is a review of the construction of the moduli space; it contains many of the definitions, notations, and analytic facts which are used later. In Sect. 2 we derive a formula for the curvature of the moduli space in terms of Green operators. (This has been done independently by M. Itoh [I].) This formula is simple and general, but would seem to be of limited utility unless one can obtain specific information about the Green operators.

The remainder of the paper is devoted to a detailed analysis of the most fundamental example: the moduli space $\mathcal{M}_1$ of self-dual connections on the $SU(2)$-bundle with instanton number 1 over the standard four-sphere S^4. The *topology* of this space is well-known; Atiyah et al. [AHS, Sect. 9] proved that $\mathcal{M}_1$ is diffeomorphic to $\mathbb{R}^5$. More specifically, they showed that the group $SO(5,1)$ of conformal diffeomorphisms of S^4 acts transitively on $\mathcal{M}_1$ with isotropy subgroup $SO(5)$, so $\mathcal{M}_1$ is diffeomorphic to hyperbolic 5-space $SO(5,1)/SO(5)$.

The problem of describing the *metric* on $\mathcal{M}_1$ is more difficult. The first problem is that the approach taken in [AHS, Theorem 9.1] – using the Weitzenböck formula and a vanishing theorem to characterize $\mathcal{M}_1$ – is not constructive. Sections 3 and 4 are devoted to obtaining a *concrete* construction of $\mathcal{M}_1$. We begin in Sect. 3 by giving a very careful description of the action of the conformal group on $\mathcal{M}_1$. Again, our purpose is to introduce the notation and derive the explicit formulas which are required (frequently) in subsequent sections. Using these formulas we define, at the beginning of Sect. 4, a map, $Q:\mathbb{R}^5 \to \mathcal{A}$, into the space of connections. We then show that the image of Q projects onto the moduli space $\mathcal{M}_1$ and gives a diffeomorphism

$$\bar{Q} = \pi \circ Q : \mathbb{R}^5 \to \mathcal{A} \to \mathcal{M}_1 \subset \mathcal{A}/\mathcal{G}\,.$$

This provides the coordinate system that later helps us describe the metric on $\mathcal{M}_1$.

The remaining two sections are independent of one another. In Sect. 5, we use the machinery developed in the first four sections to compute the geometry of $\mathcal{M}_1$ at its "origin," the $SO(5)$-invariant connection A_0. We find that $\mathcal{M}_1$ has positive sectional curvature at A_0; in fact the sectional curvatures at A_0 are the same as those of the 5-sphere of radius $R_0 = 4\pi/\sqrt{5}$. (This can also be derived from the results of Sect. 6.) In particular, this shows that $\mathcal{M}_1$ is *not* isometric to hyperbolic space.

While the methods of Sect. 5 work nicely for the invariant connection A_0, it is difficult to similarly analyze the geometry of $\mathcal{M}_1$ at a non-symmetric connection. Therefore in Sect. 6 we take the more direct approach of explicitly computing the pullback $\bar{Q}^*g$ of the metric g on $\mathcal{M}_1$ to $\mathbb{R}^5$. The computations are complicated, but the result is strikingly simple.

Theorem A. *There exists a coordinate diffeomorphism $\phi:\mathbb{R}^5 \to \mathcal{M}_1$ for which the pullback of the natural metric g on $\mathcal{M}_1$ is given by*

$$(\phi^* g)_{ij} = \psi^2(\varrho)\delta_{ij}$$

for some smooth function ψ of $\varrho = |x|$.

The precise formula for this function ψ is messy; it is given in Sect. 6. Theorem A, together with the equation of ψ, allows us to compute the basic geometric properties of $\mathcal{M}_1$.

Corollary B. *The Riemannian manifold* $(\mathcal{M}_1, g)$ *has the following properties:*

(a) *It is conformally flat.*

(b) *It is radially symmetric. More precisely, the action of* $SO(5)$ *on* S^4 *induces an isometry of* $\mathcal{M}_1$ *whose pullback, via* ϕ, *is the usual* $SO(5)$*-action on* $\mathbb{R}^5$.

(c) *It has finite radius, and hence is incomplete.*

(d) *It has finite volume.*

Properties (b) and (c) suggest that $\mathcal{M}_1$ can be given a boundary consisting of a (round) sphere at infinity of finite radius. The precise statement is:

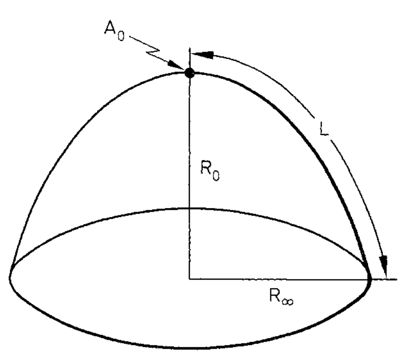
Corollary C. $\mathcal{M}_1$ *can be isometrically included as the interior of a compact Riemannian manifold-with-boundary* $\bar{\mathcal{M}}_1$, *whose boundary* $\partial\bar{\mathcal{M}}_1$ *is isometric to the* 4*-sphere of radius* 2π. *Furthermore, the embedding* $\partial\bar{\mathcal{M}}_1 \hookrightarrow \bar{\mathcal{M}}_1$ *is* totally geodesic.

Note that this sphere at infinity $\partial\bar{\mathcal{M}}_1$ is conformally equivalent to the original manifold (S^4, g) with constant conformal factor $4\pi^2$. Intuitively, points of $\partial\bar{\mathcal{M}}_1$ correspond to instantons which are concentrated at a single point $x \in S^4$. Of course, these idealized instantons cannot be represented as smooth, or even continuous, self-dual connections. Thus while $\mathcal{M}_1$ lies in $\mathcal{A}/\mathcal{G}$, the boundary $\partial\bar{\mathcal{M}}_1$ does not.

Corollary C is related to the "collar theorem" [D, FU] which states that for any 1-connected 4-manifold M with positive-definite intersection form, the moduli space of $k=1$ instantons has an end diffeomorphic to $M \times (1, \infty)$, so can be given a boundary diffeomorphic to M. Corollary C shows that, at least for $M = S^4$, this topological construction is naturally implemented by the L^2 metric.

Further geometric properties of $(\mathcal{M}_1, g)$ are more difficult to obtain because the function $\psi(\varrho)$ is so complicated. However, computer calculations show that the radius of $\mathcal{M}_1$ (L in the diagram below) is approximately 3.37π. Also, a calculation of the scalar curvature $s = -4\psi^{-3}[\psi'' + 8\varrho^{-1}\psi' + \psi^{-1}(\psi')^2]$ shows that $\mathcal{M}_1$ does not have constant curvature.

These data yield a good picture of the moduli space with its L^2-induced geometry, sketched below. It is closely approximated by half the ellipsoid of revolution in $\mathbb{R}^6$ whose semiminor axis is $R_\infty = 2\pi$, and whose semimajor axis (obtained by matching the radius of curvature of $\mathcal{M}_1$ at A_0 with the corresponding



radius of curvature of an ellipsoid) is $R_\infty^2/R_0=\sqrt{5}\pi$, or about 1.12 times the semiminor axis. However, this is only an approximation: computer calculations show that the ratios $L:R_\infty:R_0$ for $\mathcal{M}_1$ cannot be realized by an ellipsoid. (The numerical values of L, R_∞, and R_0 depend on a choice of scale discussed later in this paper, but the ratios of these radii are invariant).

An important feature of the geometry of $\mathcal{M}_1$ is that the diameter and volume are finite. This suggests that the integrals over $\mathcal{M}$ which appear in the semiclassical approximation to quantum Yang-Mills theory may be finite. It also indicates that one can expect other self-dual-moduli spaces to have finite diameter and volume. We will consider these subjects in subsequent papers.

1. The Moduli Space

We begin by briefly reviewing the construction of the Yang-Mills moduli spaces. Most of this material is standard, and detailed expositions can be found in [AHS, FU, L].

Let (M,g) be a compact, oriented, Riemannian 4-manifold and $P\to M$ a principal bundle whose structure group is a compact semi-simple Lie group G with Lie algebra $\mathfrak{g}$. We can then form the adjoint bundle $\mathrm{Ad}\,P=P\times_{\mathrm{Ad}}\mathfrak{g}$ and consider the spaces

$$\Omega^q(\mathrm{Ad}\,P)=\Gamma(\Lambda^q T^*M\otimes \mathrm{Ad}\,P)$$

of $\mathrm{Ad}\,P$-valued q-forms. The bundles $\Lambda^q T^*M\otimes\mathrm{Ad}\,P$ have natural metrics $(\cdot\,,\cdot)$ (induced by the Riemannian metric and a constant negative multiple of the Killing form on $\mathfrak{g}$), and hence $\Omega^q(\mathrm{Ad}\,P)$ has an L^2 inner product

$$\langle\phi,\psi\rangle_{L^2}=\int_M(\phi,\psi).$$

(Integration is with respect to the Riemannian measure determined by g.)

A connection A on P determines a covariant derivative

$$\nabla^A:\Omega^0(\mathrm{Ad}\,P)\to\Omega^1(\mathrm{Ad}\,P)$$

with the property that for any vector field X on M and any $\phi,\psi\in\Omega^0(\mathrm{Ad}\,P)$,

$$X(\phi,\psi)=(\nabla^A_X\phi,\psi)+(\phi,\nabla^A_X\psi).$$

This operator extends to

$$\nabla^A:\Gamma(\Lambda^q T^*M\otimes\mathrm{Ad}\,P)\to\Gamma(T^*M\otimes\Lambda^q T^*M\otimes\mathrm{Ad}\,P)$$

by $\nabla^A=\nabla^g\otimes 1+1\otimes\nabla^A$, where ∇^g is the Levi-Civita connection of the metric on M. By composing ∇^A with exterior multiplication we obtain the covariant exterior derivative

$$d_A:\Omega^q(\mathrm{Ad}\,P)\to\Omega^{q+1}(\mathrm{Ad}\,P),$$

and by composing with contraction (the adjoint of exterior multiplication) we obtain the L^2 adjoint d_A^*. In a local orthonormal frame $\{e_i\}$ of TM with dual coframe $\{\theta^i\}$,

$$d_A\phi=\sum\theta^i\wedge\nabla^A_{e_i}\phi,\qquad d_A^*\phi=-\sum e_i\lrcorner\nabla^A_{e_i}\phi$$

for $\phi \in \Omega^*(\mathrm{Ad}\,P)$. The curvature of the connection A is

$$d_A \circ d_A : \Omega^0(\mathrm{Ad}\,P) \to \Omega^2(\mathrm{Ad}\,P).$$

This is a zeroth-order operator given by bracketing with an element $F_A \in \Omega^2(\mathrm{Ad}\,P)$. Locally,

$$\begin{aligned} d_A d_A \phi &= [F_A, \phi] = 1/2 \sum_{i \neq j} \theta^i \wedge \theta^j [(F_A)_{ij}, \phi] \\ &= 1/2 \sum_{i \neq j} \theta^i \wedge \theta^j (\nabla^A_{e_i} \nabla^A_{e_j} - \nabla^A_{e_j} \nabla^A_{e_i} - \nabla^A_{[e_i, e_j]}) \phi . \end{aligned}$$

There are several other important algebraic operators on the spaces $\Omega^*(\mathrm{Ad}\,P)$. First, the Hodge star operator $*$ acts on all such forms, and decomposes $\Omega^2(\mathrm{Ad}\,P)$ into its ± 1-eigenspaces $\Omega^2_\pm(\mathrm{Ad}\,P)$. The orthogonal projections onto these eigenspaces are

$$p_\pm = 1/2(1 \pm *).$$

Also, each $X = \sum \omega_i \otimes B_i \in \Omega^1(\mathrm{Ad}\,P)$ defines an operator

$$P_X : \Omega^q(\mathrm{Ad}\,P) \to \Omega^{q+1}(\mathrm{Ad}\,P)$$

by

$$P_X(\omega' \otimes B') = \sum \omega_i \wedge \omega' \otimes [B_i, B'] .$$

The adjoint of P_X is

$$P^*_X = - * P_X * : \Omega^q(\mathrm{Ad}\,P) \to \Omega^{q-1}(\mathrm{Ad}\,P).$$

If X, Y are in $\Omega^1(\mathrm{Ad}\,P)$ and we use a local orthonormal coframe to write $X = X_i \theta^i$, $Y = Y_i \theta^i$, then

$$P^*_X Y = -P^*_Y X = -\sum [X_i, Y_i] . \tag{1.1}$$

The curvature form decomposes as the sum $F_A = F^+_A + F^-_A$, where $F^\pm_A = p_\pm F_A$. The connection A is called self-dual if $F^-_A \equiv 0$. For a self-dual connection, or "instanton", A, the sequence

$$0 \to \Omega^0(\mathrm{Ad}\,P) \xrightarrow{d_A} \Omega^1(\mathrm{Ad}\,P) \xrightarrow{d^-_A} \Omega^2_-(\mathrm{Ad}\,P) \to 0, \tag{1.2}$$

where $d^-_A = p_- d_A$, is an elliptic complex. We can make it into a complex of Hilbert spaces as follows. Fix a smooth connection A_0 and an integer $\ell \geqq 0$. For $\phi \in \Omega^q(\mathrm{Ad}\,P)$ write

$$(\nabla^{A_0})^\ell \phi = (\nabla^{A_0} \circ \ldots \circ \nabla^{A_0}) \phi \in \Gamma\left(\bigotimes^{\ell} T^*M \otimes \Lambda^q(T^*M) \otimes \mathrm{Ad}\,P \right).$$

The Sobolev space $\Omega^q_\ell(\mathrm{Ad}\,P)$ is defined as the completion of $\Omega^q(\mathrm{Ad}\,P)$ with respect to the norm

$$\|\phi\|_\ell = \left(\sum_{j=0}^{\ell} \int_M |(\nabla^{A_0})^j \phi|^2 \right)^{1/2} .$$

This completion is a Hilbert space under the associated bilinear form.

We can similarly complete the space of connections. Let $\mathscr{A}$ denote the set of smooth connections on P. The covariant derivatives of two connections A_0, $A \in \mathscr{A}$ are related by

$$d_A = d_{A_0} + P_X \tag{1.3}$$

for some $X \in \Omega^1(\operatorname{Ad} P)$. Hence $\mathscr{A}$ is an affine space, and at each $A \in \mathscr{A}$ there is a natural identification of the tangent space $T_A\mathscr{A}$ with $\Omega^1(\operatorname{Ad} P)$. The space $\mathscr{A}_\ell$ of Sobolev connections is defined by fixing $A_0 \in \mathscr{A}$, identifying $\mathscr{A}$ with $\Omega^1(\operatorname{Ad} P)$ via (1.3), and completing in the Sobolev ℓ-norm. The space $\mathscr{A}_\ell$ obtained this way is independent of the choice of A_0. The curvature map $A \mapsto F_A$ extends to a smooth map from $\mathscr{A}_{\ell+1}$ to $\Omega^2_\ell(\operatorname{Ad} P)$ provided $\ell \geqq 1$ [U].

Each $A \in \mathscr{A}_\ell$ determines Laplacians $\Delta^0_A = d^*_A d_A$, $\Delta^1_A = d_A d^*_A + (d^-_A)^* d^-_A$, and $\Delta^2_A = d^-_A (d^-_A)^*$, on $\Omega^q_\ell(\operatorname{Ad} P)$ for $q = 0, 1, 2$ respectively. By the spectral theorem for self-adjoint elliptic operators each space $\Omega^q_\ell(\operatorname{Ad} P)$ decomposes as the direct sum of the finite-dimensional eigenspaces of Δ^q_A, where the eigenvalues $\{\lambda_i\}$ are real, non-negative, and discrete. Thus there are L^2-orthogonal decompositions

$$\Omega^q_\ell(\operatorname{Ad} P) = K^q \oplus B^q_\ell,$$

where $K^q = \ker(\Delta^q_A) \subset C^\infty(\operatorname{Ad} P)$ is the finite-dimensional space of Δ^q_A-harmonic forms. Furthermore $\Delta^q_A : B^q_{\ell+2} \to B^q_\ell$ is a bounded map with a bounded inverse, or Green operator,

$$G^q_A : B^q_\ell \to B^q_{\ell+2}.$$

The connection A is called *irreducible* if $K^0 = \{0\}$, or, equivalently, if $\ker(d_A : \Omega^0 \to \Omega^1) = \{0\}$; Δ^0_A is then invertible on $\Omega^0_\ell(\operatorname{Ad} P)$. It is not hard to show that the set of irreducible connections is an open dense set $\hat{\mathscr{A}}_\ell \subset \mathscr{A}_\ell$.

The set of all smooth automorphisms of P is called the *gauge group* $\mathscr{G}$. This group can be naturally identified with the space of sections of the associated bundle $\operatorname{Aut} P = P \times_{\operatorname{Ad}} G$. For $\ell \geqq 3$ the Sobolev completion $\mathscr{G}_\ell$ is defined by choosing a faithful representation $\varrho : G \to \operatorname{End}(V)$. This gives an inclusion $\mathscr{G} \subset \Gamma(P \times_\varrho \operatorname{End}(V))$, and $\mathscr{G}_\ell$ is the closure of $\mathscr{G}$ in the Sobolev ℓ-completion of $\Gamma(P \times_\varrho \operatorname{End}(V))$. Thus defined, $\mathscr{G}_\ell$ is a smooth Hilbert Lie group whose Lie algebra is $\Omega^0_\ell(\operatorname{Ad} P)$ (cf. [MV]). If we replace ϱ by $\operatorname{Ad} : G \to \operatorname{End}(\mathfrak{g})$ we obtain a similar group $\tilde{\mathscr{G}}_\ell$. Since the kernel of Ad is the center Z of G, we have $\tilde{\mathscr{G}}_\ell = \mathscr{G}_\ell / \mathscr{Z}$, where $\mathscr{Z} = \Gamma(P \times_{\operatorname{Ad}} Z)$, is the center of $\mathscr{G}$. ($\mathscr{Z}$ is isomorphic to the finite group Z.)

A gauge transformation $g \in \mathscr{G}$ acts on connections by $d_A \mapsto g \circ d_A \circ g^{-1}$, and hence sends F_A to $(\operatorname{Ad} g)\, F_A$. This action extends to a smooth action of $\mathscr{G}_{\ell+1}$ on $\mathscr{A}_\ell (\ell \geqq 2)$ whose differential at $A \in \mathscr{A}_\ell$ is

$$-d_A : T_{\operatorname{Id}} \mathscr{G}_{\ell+1} = \Omega^0_{\ell+1}(\operatorname{Ad} P) \to T_A \mathscr{A}_\ell = \Omega^1_\ell(\operatorname{Ad} P).$$

Consequently, the tangent space at an irreducible connection $A \in \hat{\mathscr{A}}_\ell$ is the direct sum of "vertical" and "horizontal" subspaces

$$T_A \hat{\mathscr{A}}_\ell = V_A \oplus H_A, \tag{1.4}$$

where $V_A = \operatorname{im} d_A$ is the tangent space to the gauge-orbit through A and $H_A = \ker d^*_A$ is the L^2-orthogonal complement. ($V_A \cap H_A = \{0\}$ since A is irreducible, and the spectral theorem implies that the splitting is an isomorphism.) For $A \in \hat{\mathscr{A}}_\ell$, the stabilizer of the action of $\mathscr{G}_{\ell+1}$ is precisely the center $\mathscr{Z}$. Moreover, a standard slice theorem asserts that the orbit through A has a tubular neighborhood which is equivariantly diffeomorphic to $\tilde{\mathscr{G}}_{\ell+1} \times U$, where U is an open neighborhood of A in H_A. It follows that the orbit space $\mathcal{O}_\ell$ is a Hausdorff Hilbert manifold, and that

$$\hat{\mathscr{A}}_\ell \xrightarrow{\pi} \mathcal{O}_\ell = \hat{\mathscr{A}}_\ell / \tilde{\mathscr{G}}_{\ell+1}$$

is a principal $\tilde{\mathscr{G}}_{\ell+1}$-bundle. We will denote the gauge-equivalence class of a connection A by $[A] \in \mathcal{O}_\ell$, and will frequently identify $T_{[A]}\mathcal{O}_\ell$ with the horizontal subspace $H_A \subset T_A \mathscr{A}_\ell$ of (1.4).

The L^2-orthogonal splitting (1.4) determines vertical and horizontal projection operators v_A, h_A at each $A \in \hat{\mathscr{A}}_\ell$. To identify these, write any $X \in \Omega^1(\mathrm{Ad}\,P)$ as

$$X = d_A G_A^0 d_A^* X + (X - d_A G_A^0 d_A^* X).$$

Since the first term is in the image of d_A and the second is in the kernel of d_A^*, we have

$$v_A = d_A G_A^0 d_A^*, \qquad h_A = \mathrm{Id} - d_A G_A^0 d_A^*. \tag{1.5}$$

The Yang-Mills action of a connection A,

$$\mathscr{YM}(A) = 1/2 \int_M |F_A|^2 \,\mathrm{vol}_g,$$

is a smooth gauge-invariant function on $\mathscr{A}_\ell$, $\ell \geqq 1$. Of course, the value of $\mathscr{YM}(A)$ depends on which multiple of the Killing form of $\mathfrak{g}$ is used to define the metric on $\mathrm{Ad}\,P$. When $G = SU(N)$ it is common to use minus the trace form of the standard representation of G on $\mathbb{C}^N$; this is equal to $-1/2N$ times the Killing form. We will adhere to this convention when we consider $SU(2)$-bundles in Sects. 3–6. In general, if we take the metric on $\mathrm{Ad}P$ induced by $-\lambda$ times the Killing form, then we have

$$\mathscr{YM}(A) = \frac{\lambda}{2} \int_M -\mathrm{tr}(\mathrm{ad}\,F_A \wedge \mathrm{ad}\,F_A) + \int_M |F_A^-|^2 \mathrm{vol}_g.$$

The first integral is a (positive) multiple of the characteristic number $p_1(\mathrm{Ad}\,P)[M]$, where $p_1(\mathrm{Ad}P)$ is the first pontryagin class of the real orthogonal bundle $\mathrm{Ad}\,P$ [1]. This integral depends only on P, not on A. Thus self-dual connections can exist only if $p_1(\mathrm{Ad}P)[M] \geqq 0$, and, when they exist, they absolutely minimize the Yang-Mills functional.

Remark. When $G = SU(N)$ there is a vector bundle E associated to the standard representation, and we have $2N \cdot c_2(E) = c_2(\mathrm{Ad}\,P \otimes \mathbb{C}) = -p_1(\mathrm{Ad}\,P)$. It is then traditional to express the characteristic number of P in terms of the "instanton number" $k = -c_2(E)[M]$.

The existence theorems of Taubes [T] show that self-dual connections exist on all bundles with $p_1(\mathrm{Ad}\,P)[M] \geqq 0$ over many 4-manifolds, including those with positive-definite intersection form.

Let $\mathscr{B}_\ell \subset \mathscr{A}_\ell$ be the (gauge-invariant) set of all self-dual $\underline{SU}(2)$ connections. The image of $\mathscr{B}_\ell$ in the orbit space is called the *self-dual moduli space*

$$\mathscr{M} = \{[A] / A \text{ is self-dual}\} \subset \mathcal{O}_\ell.$$

[1] In [AHS], $2p_1(\mathrm{Ad}\,P) = p_1(\mathrm{Ad}\,P \otimes \mathbb{C})$ is denoted by $p_1(\mathfrak{g})$

It contains, as an open dense subset, the moduli space

$$\hat{\mathscr{M}} = \{[A]/A \text{ is self-dual and } \ker \Delta_A^0 = \{0\}\}$$

of irreducible self-dual connections.

A theorem of Atiyah et al. [AHS] shows that, when nonempty, the subset

$$\hat{\mathscr{M}}' = \{[A] \in \hat{\mathscr{M}} | \operatorname{Ker} \Delta_A^2 = \{0\}\}$$

is a manifold of dimension $2p_1(\operatorname{Ad}P)[M] - (\dim G)(1 - b_1 + b_2^-)$, where $b_1 = \dim(H^1(M;\mathbb{R}))$ and b_2^- is the dimension of the space of anti-self-dual harmonic 2-forms on M. The difficulties caused by the presence of connections A with $\ker \Delta_A^2 \neq \{0\}$ can be avoided in at least two ways. First, a vanishing theorem based on the Weitzenböck formula (5.2) shows that, under a certain curvature condition on M, the kernel of Δ_A^2 is zero for all $A \in \hat{\mathscr{M}}$. The specific condition is that $-2\mathscr{W}_- + \frac{s}{3} \cdot \operatorname{Id} \in \operatorname{End}(\Lambda_-^2 T^*M)$ be positive-definite, where s is the scalar curvature and $\mathscr{W}_-$ is the Weyl curvature endomorphism defined by $2\mathscr{W}_-(\theta^i \wedge \theta^j)_- = W_{ijk\ell}(\theta^k \wedge \theta^\ell)_-$. Second, Freed and Uhlenbeck [FU] have shown that when $G = \underline{SU}(2)$ it is always possible to perturb the metric on M to ensure that $\hat{\mathscr{M}}' = \hat{\mathscr{M}}$. We will henceforth assume that we are in one of these situations, so $\hat{\mathscr{M}}$ is a manifold.

There is a simple description of the tangent space of $\hat{\mathscr{M}}$. If $A_t = A_0 + tB + O(t^2)$ is a one-parameter family of connections, then $F_{A_t} = F_{A_0} + t d_{A_0} B + O(t^2)$, and if A_t is self-dual (so $F_{A_t}^- \equiv 0$) for all t, then $d_{A_0}^- B = 0$. Thus when we identify $T_{[A]}\mathscr{O}_\ell$ with the horizontal subspace H_A in (1.4) we have

$$T_{[A]}\hat{\mathscr{M}} = \{B \in \Omega_\ell^1(\operatorname{Ad}P)/d_A^* B = 0, d_A^- B = 0\}. \tag{1.6}$$

Elliptic regularity arguments show that any $A' \in B_\ell$ is $\mathscr{G}_{\ell+1}$-equivalent to a smooth connection A, and that for such A the right-hand side of (1.6) consists of smooth forms B.

The simplest examples of moduli spaces occur when M is 1-connected with positive-definite intersection form, and P is the principal $SU(2)$-bundle with instanton number $k=1$. The moduli space $\hat{\mathscr{M}}$ is then a 5-dimensional manifold whose topological structure has been described by Donaldson [D]. The space $\hat{\mathscr{M}}$ is orientable, $\mathscr{M} - \hat{\mathscr{M}}$ consists of finitely many reducible connections $\{A_i\}$ [2] around each of which $\mathscr{M}$ is locally diffeomorphic to an open cone on $\mathbb{C}P^2$, and there is a compact set in $\mathscr{M}$ whose complement is a "collar" diffeomorphic to $M \times (1, \infty)$.

In the next section we begin to study the Riemannian geometry of a general moduli space $\mathscr{M}$.

2. Curvature of the Moduli Space

In this and subsequent sections we fix a Sobolev index $\ell \geqq 2$ and denote $\mathscr{A}_\ell$, $\mathscr{G}_{\ell+1}$, and $\mathscr{O}_\ell$ by $\mathscr{A}$, $\mathscr{G}$, and $\mathscr{O}$ respectively.

The affine space $\mathscr{A}$ inherits a weak Riemannian metric, via the identification $T_A\mathscr{A} = \Omega_\ell^1(\operatorname{Ad}P)$, from the L^2 inner product on $\Omega^*(\operatorname{Ad}P)$. This metric is

[2] For simplicity we sometimes refer to a connection A as lying in $\mathscr{M}$ when we mean $[A] \in \mathscr{M}$

translation-invariant and flat. The group $\mathscr{G}$ acts isometrically on $\mathscr{A}$ and, at each $A \in \hat{\mathscr{A}}$, preserves the splitting (1.4) of $T_A\mathscr{A}$ into horizontal and vertical subspaces. We can therefore give the orbit space $\mathcal{O} = \hat{\mathscr{A}}/\mathscr{G}$ a weak Riemannian metric by identifying $T_{[A]}\mathcal{O}$ with the horizontal subspace $H_A \subset T_A\hat{\mathscr{A}}$. (This definition is independent of the representative A of $[A]$.) Finally, the moduli space $\hat{\mathscr{M}}$ is a smoothly embedded submanifold of $\mathcal{O}$, so there is an induced smooth Riemannian metric on $\hat{\mathscr{M}}$.

Equivalently, we can observe that $T_{[A]}\mathscr{M}$ is naturally isomorphic to the cohomology $\ker d_A^- / \operatorname{im} d_A$ of the elliptic complex (1.2), and that this cohomology embeds in $\Omega^1_\ell(\mathrm{Ad}\,\mathbf{P})$ as the Δ^1_A-harmonic forms [as in (1.6)]. The metric on $T_{[A]}\mathscr{M}$ is then simply the restriction of the L^2 metric on $\Omega^1_\ell(\mathrm{Ad}\,\mathbf{P})$.

In this section we will compute the curvature of this L^2-induced metric on $\hat{\mathscr{M}}$. This is done in two steps, using the diagram

$$\begin{array}{ccc} & & \hat{\mathscr{A}} \\ & & \big\downarrow \pi \\ \hat{\mathscr{M}} & \longrightarrow & \mathcal{O} = \hat{\mathscr{A}}/\mathscr{G}\,. \end{array}$$

We first observe that π is a Riemannian submersion (cf. [CE]). After describing the Levi-Civita connection of the metrics on $\hat{\mathscr{A}}$ and $\mathcal{O}$, we apply O'Neill's formula to compute the curvature of $\mathcal{O}$ (this has previously been computed by other methods [BV, S]). We then calculate the second fundamental form of the embedding of $\hat{\mathscr{M}}$ into $\mathcal{O}$. The curvature $\hat{\mathscr{M}}$ is then obtained from the Gauss equation.

Before beginning the curvature calculations we must describe the Levi-Civita connections on the infinite-dimensional manifolds $\hat{\mathscr{A}}$ and $\mathcal{O}$. On a finite-dimensional Riemannian manifold M, the Levi-Civita connection ∇ is the unique connection on TM that is torsion-free and compatible with the metric. These conditions are equivalent to the formula

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle \\ - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle \end{aligned} \tag{2.1}$$

for all smooth vector fields X, Y, Z. The right-hand side of (2.1) is $C^\infty(M)$-linear in Z, so the metric isomorphism $TM \simeq T^*M$ yields the existence and uniqueness of the connection. This argument is valid in infinite dimensions provided each tangent space is a Hilbert space with respect to the Riemannian metric. However, in our case $T_A\mathscr{A}$ and $T_A\mathcal{O}$ are not complete under the L^2 metric. For such weak Riemannian metrics (2.1) still guarantees the uniqueness of the Levi-Civita connection, but not its existence. One generally handles the existence problem by exhibiting an explicit formula for ∇, as we do below.

The connection on $\mathscr{A}$ is easily described because the tangent bundle $T\mathscr{A}$ is canonically trivial. Using the natural identification $T_A\mathscr{A} = \Omega^1_\ell(\mathrm{Ad}\,P)$, we regard a vector field on an open set $U \subset \mathscr{A}$ as a map $Y: U \to \Omega^1_\ell(\mathrm{Ad}\,P)$ and set

$$(\nabla^{\mathscr{A}}_X Y)_A = \frac{d}{dt}\, Y(A + tX(A))|_{t=0} \in \Omega^1_\ell(\mathrm{Ad}\,P)\,. \tag{2.2}$$

This defines a connection that is clearly torsion-free and compatible with the (translation-invariant) L^2 metric.

We now apply the standard formula giving the covariant derivative in the image of a Riemannian submersion. Given $X \in T_{[A]}\mathcal{O}$ and a vector field Y defined in a neighborhood of $[A]$, we choose $\bar{X} \in H_A$ and a horizontal vector field $\bar{Y}$ with $\pi_* \bar{X} = X$ and $\pi_* \bar{Y} = Y$, and set

$$(\nabla_X^{\mathcal{O}} Y)_{[A]} = \pi_* (\nabla_{\bar{X}}^{\mathscr{A}} \bar{Y})_A . \tag{2.3}$$

It is not hard to verify (2.1) for this connection $\nabla^{\mathcal{O}}$.

The Curvature of $\mathcal{O}$. For any Riemannian submersion, the curvatures upstairs and downstairs are related by O'Neill's formula (cf. [CE]),

$$\langle R_{\text{down}}(X_p, Y_p) Y_p, X_p \rangle = \langle R_{\text{up}}(\bar{X}_q, \bar{Y}_q) \bar{Y}_q, \bar{X}_q \rangle + 3/4 \, \| \text{vert}[\bar{X}, \bar{Y}]_q \|^2 , \tag{2.4}$$

where X_p, Y_p are tangent to the base at p, $\bar{X}_q$, $\bar{Y}_q$ are their horizontal lifts to an arbitrary point q over p, $[\bar{X}, \bar{Y}]_q$ is the Lie bracket of arbitrary horizontal extensions $\bar{X}$, $\bar{Y}$ of $\bar{X}_q$, $\bar{Y}_q$ at q, and "vert" denotes projection onto the vertical subspace. In our situation $R_{\text{up}} \equiv 0$, so we need only compute the final term.

Fix $A_0 \in \hat{\mathscr{A}}$ and vectors $X_0, Y_0 \in H_{A_0}$. To apply (2.4) we must choose horizontal extensions $\bar{X}$, $\bar{Y}$ of X_0 and Y_0. This is conveniently done by regarding X_0, Y_0 as constant vector fields on $\mathscr{A}$ and taking their horizontal projections:

$$\bar{X}_A = h_A X_0 , \qquad \bar{Y}_A = h_A Y_0 . \tag{2.5}$$

By (2.2) and the formula (1.5) for h_A we have

$$(\nabla_{X_0}^{\mathscr{A}} Y)_A = \frac{d}{dt} [- d_{A_0 + tX_0} G^0_{A_0 + tX_0} d^*_{A_0 + tX_0} Y_0]|_{t=0} .$$

But $d^*_{A_0} Y_0 = 0$ and, using (1.3), $d^*_{A_0 + tX_0} = d^*_{A_0} + P^*_X$, so

$$(\nabla_{X_0}^{\mathscr{A}} \bar{Y})_A = - d_{A_0} G^0_{A_0} P^*_{X_0} Y_0 . \tag{2.6}$$

Applying (1.1) we find

$$[\bar{X}, \bar{Y}]_{A_0} = (\nabla_{X_0}^{\mathscr{A}} \bar{Y} - \nabla_{Y_0}^{\mathscr{A}} \bar{X})_{A_0} = - 2 d_{A_0} G^0_{A_0} P^*_{X_0} Y_0 .$$

Observe that this is already vertical. Hence

$$\begin{aligned} \| v_{A_0} [\bar{X}, \bar{Y}] \|^2 = 4 \, \| d_{A_0} G^0_{A_0} P^*_{X_0} Y_0 \|^2 &= 4 \langle d^*_{A_0} d_{A_0} G^0_{A_0} P^*_{X_0} Y_0, G^0_{A_0} P^*_{X_0} Y_0 \rangle \\ &= 4 \langle P^*_{X_0} Y_0, G^0_A \, P^*_{X_0} Y_0 \rangle . \end{aligned}$$

Putting this into (2.4) yields

$$\langle R_{\mathcal{O}}(X, Y) Y, X \rangle = 3 \langle P^*_{\bar{X}} \bar{Y}, G^0_A P^*_{\bar{X}} \bar{Y} \rangle , \tag{2.7}$$

where $X, Y \in T_{[A]}\mathcal{O}$ and $\bar{X}, \bar{Y} \in H_A$ project to X, Y. The full curvature tensor is then obtained by polarization.

Proposition 2.1. *The curvature of the L^2 metric on $\mathcal{O} = \hat{\mathscr{A}}/\mathscr{G}$ at $[A]$ is given by*

$$\langle R_{\mathcal{O}}(X, Y) Z, W \rangle = \langle P^*_{\bar{X}} \bar{W}, G^0_A P^*_{\bar{Y}} \bar{Z} \rangle - \langle P^*_{\bar{Y}} \bar{W}, G^0_A P^*_{\bar{X}} \bar{Z} \rangle + 2 \langle P^*_{\bar{W}} \bar{Z}, G^0_A P^*_{\bar{X}} \bar{Y} \rangle \tag{2.8}$$

where $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{W}$ are the horizontal lifts to $A \in \hat{\mathscr{A}}$ of X, Y, Z, $W \in T_{[A]}\mathcal{O}$. □

As pointed out by Singer [S], (2.7) shows that $\mathcal{O}$ has non-negative sectional curvature (for each A, the Laplacian $d^*_A d_A$ and its inverse G^0_A are non-negative operators).

The Curvature of $\hat{\mathcal{M}}$. The second fundamental form of the embedding $i:\hat{\mathcal{M}}\to\mathcal{O}$ is a section b of $\mathrm{Sym}^2(T^*\mathcal{M})\otimes v$, where v is the normal bundle. The form b is defined, as in the finite-dimensional setting, by

$$b(X,Y)=(\nabla^{\mathcal{O}}_{i_*X}i_*Y)^{\perp},\tag{2.9}$$

where $X, Y\in T\mathcal{M}$ and $\perp$ denotes the L^2-orthogonal projection from $i^*T\mathcal{O}$ onto v; this projection is well-defined since each $T_{[A]}\hat{\mathcal{M}}$ is finite-dimensional. [In (2.9), an extension of i_*Y must be chosen along $\hat{\mathcal{M}}$, but $b(X,Y)$ is independent of this choice.]

Fix $[A_0]\in\hat{\mathcal{M}}$ and consider the "slice" $S_0=A_0+\ker d^*_{A_0}\subset\mathcal{A}$. The manifold $N=\pi^{-1}\hat{\mathcal{M}}\cap S_0$ passes through A, and $\pi:N\to\hat{\mathcal{M}}$ is a local diffeomorphism. Furthermore, $T_{A_0}N$ is the subspace $T_0=\mathrm{Ker}\,d^*_{A_0}\cap\mathrm{Ker}\,d^-_{A_0}$ of $\Omega^1_\ell(\mathrm{Ad}\,P)$ [cf. (1.6)], and the normal bundle v pulls back to a bundle over N whose fiber at A_0 is

$$v_0=\{Z\in\ker d^*_{A_0}\,|\,Z\perp T_0\}\subset T_{A_0}\mathcal{A}\,.$$

Now given $X_0, Y_0\in T_0$, we can construct their horizontal extensions as in (2.5). If we project further, setting

$$X^{\tan}_A=(1-(d^-_A)^*G^2_Ad^-_A)X_A\,,$$

we obtain horizontal vector fields $X^{\tan}$, $Y^{\tan}$ whose projections $\pi_*X^{\tan}$, $\pi_*Y^{\tan}$ are tangent to $\hat{\mathcal{M}}$ along $\hat{\mathcal{M}}$. Computing as we did to arrive at (2.6), now noting that $d^-_{A_0}Y_0=0$ and that $d^-_{A+tX}=d^-_A+P^-_X$ (where $P^-_X=p_-\circ P_X$), we find

$$(\nabla^{\mathcal{A}}_{X_0}Y^{\tan})_{A_0}=-(d^-_{A_0})^*G^2_{A_0}P^-_{X_0}Y_0-d_{A_0}G^0_{A_0}P^*_{X_0}Y_0\in T_{A_0}\mathcal{A}\,.$$

The last term is vertical, so it drops out upon projection to $\mathcal{O}$. The remaining pieces are already in the normal space $v_{A_0}\supset\mathrm{im}(d_{A_0})^*$, so by (2.9) and (2.3) we have

$$b(X,Y)_A=-(d^-_A)^*G^2_AP^-_{\bar X}\bar Y,\tag{2.10}$$

where $\bar X$, $\bar Y$ are horizontal lifts of the vector fields X, Y on $\mathcal{M}$.

The curvature of $\hat{\mathcal{M}}$ is now given by the Gauss equation:

$$\langle R_{\hat{\mathcal{M}}}(X,Y)Z,W\rangle=\langle R_{\mathcal{O}}(X,Y)Z,W\rangle+\langle b(Y,Z),b(X,W)\rangle-\langle b(X,Z),b(Y,W)\rangle,\tag{2.11}$$

where X, Y, Z, W are vector fields on $\hat{\mathcal{M}}$. By (2.10),

$$\langle b(Y,Z),b(X,W)\rangle=\langle(d^-_A)^*G^2_AP^-_{\bar Y}\bar Z,(d^-_A)^*G^2_AP^-_{\bar X}\bar W\rangle=\langle P^-_{\bar Y}\bar Z,G^2_AP^-_{\bar X}\bar W\rangle\,.\tag{2.12}$$

From (2.8), (2.11), and (2.12), we obtain

Theorem 2.2. *The curvature of the self-dual-moduli space* $\hat{\mathcal{M}}$ *at* $[A]$ *is given by*

$$\begin{aligned}\langle R_{\hat{\mathcal{M}}}(X,Y)Z,W\rangle=&\langle P^*_{\bar X}\bar W,G^0_AP^*_{\bar Y}\bar Z\rangle-\langle P^*_{\bar Y}\bar W,G^0_AP^*_{\bar X}\bar Z\rangle+2\langle P^*_{\bar W}\bar Z,G^0_AP^*_{\bar X}\bar Y\rangle\\&+\langle P^-_{\bar Y}\bar Z,G^2_AP^-_{\bar X}\bar W\rangle-\langle P^-_{\bar X}\bar Z,G^2_AP^-_{\bar Y}\bar W\rangle,\end{aligned}$$

where $\bar X$, $\bar Y$, $\bar Z$, $\bar W$ *are the horizontal lifts to* $A\in\hat{\mathcal{A}}$ *of* X, Y, Z, $W\in T_{[A]}\hat{\mathcal{M}}$. *In particular, the sectional curvatures are given by*

$$\langle R_{\hat{\mathcal{M}}}(X,Y)Y,X\rangle=3\langle P^*_{\bar X}\bar Y,G^0_AP^*_{\bar Y}\bar Y\rangle+\langle P^-_{\bar Y}\bar Y,G^2_AP^-_{\bar X}\bar X\rangle-\langle P^-_{\bar X}\bar Y,G^2_AP^-_{\bar X}\bar Y\rangle.\quad\square\tag{2.13}$$

In contrast to the formula of Proposition 2.1, this formula does not immediately yield information on the sign of the sectional curvatures of $\hat{\mathcal{M}}$. In fact, any application of (2.13) would seem to require some concrete information about the Green operators beyond crude spectral estimates. Rather than try to analyze the general moduli space from this perspective we will, in subsequent sections, focus our attention on the model case of the $k=1$ instantons on S^4.

3. The Conformal Group Action on $k=1$ Instantons over S^*

The best-known Yang-Mills moduli space is the space $\mathcal{M}_1$ of 't Hooft instantons. This is the moduli space of self-dual connections on the principal $SU(2)$-bundle P with $k=-c_2(P)=1$ over the 4-sphere with the standard metric. Since $H^2(S^4;\mathbb{Z})=0$, $\mathcal{M}_1$ contains no reducible connections [D], and the vanishing theorem mentioned in Sect. 1 shows that $\ker(\Delta_A^2)=\{0\}$ for any $A\in\mathcal{M}_1$. Thus $\mathcal{M}_1=\hat{\mathcal{M}}_1=\hat{\mathcal{M}}'_1$ is a 5-dimensional manifold with a natural Riemannian metric.

The conformal diffeomorphisms of the 4-sphere form a Lie group isomorphic to $SO(5,1)$. Atiyah et al. [AHS, Sect. 9] have shown that there is a natural, transitive action of the group on the moduli space $\mathcal{M}_1$. The isotropy subgroup of this action at each instanton is isomorphic to $SO(5)$, so $\mathcal{M}_1$ is diffeomorphic to hyperbolic 5-space:

$$\mathcal{M}_1\cong SO(5,1)/SO(5)\cong\mathbb{R}^5. \tag{3.1}$$

In the next several sections we will use this description to help compute the geometry of $\mathcal{M}_1$. We will need a much more explicit description of the conformal group action than that given by Atiyah, Hitchin, and Singer. This section provides these details.

We begin with some linear algebra. Let V be a vector space with inner product $(\cdot,\cdot)$. There is an induced inner product on $\Lambda^2 V$ defined by

$$(v\wedge w, v'\wedge w')=(v,v')(w,w')-(v,w')(v',w); \tag{3.2}$$

in particular,

$$|v\wedge w|^2=|v|^2|w|^2-(v,w)^2. \tag{3.3}$$

There is also an isomorphism of $\Lambda^2 V$ with the Lie algebra $\mathbf{so}(V)$ of skew-adjoint endomorphisms of V given by letting $v\wedge w=v\otimes w-w\otimes v$ act according to the rule

$$(v\wedge w)\cdot u=(w,u)v-(v,u)w. \tag{3.4}$$

This makes $\Lambda^2 V$ a Lie algebra with brackets given by

$$\begin{aligned}[v\wedge w, v'\wedge w'] &= -(v,v')w\wedge w'+(v,w')w\wedge v'+(w,v')v\wedge w'-(w,w')v\wedge v'\\ &= ((v\wedge w)\cdot v')\wedge w'+v'\wedge((v\wedge w)\cdot w').\end{aligned} \tag{3.5}$$

When $V=\mathbb{R}^4$, the star operator determines projection operators $p_\pm=1/2(1\pm *)$ from $\Lambda^2 V$ onto its self-dual subspace $\Lambda^2_+ V$ and anti-self-dual subspace $\Lambda^2_- V$. We will usually write $p_\pm(v\wedge w)$ as $(v\wedge w)_\pm$. With the above bracket operation, the decomposition $\Lambda^2 V\simeq\Lambda^2_+ V\oplus\Lambda^2_- V$ corresponds to the Lie algebra isomorphism $\mathfrak{so}(4)=\mathfrak{so}(3)\oplus\mathfrak{so}(3)$ and the metric (3.2) corresponds to $-1/4$ times the Killing form on each factor $\Lambda^2_\pm V=\mathfrak{so}(3)$. Since $p_\pm$ are Lie algebra projections, we have

$$[(v\wedge w)_+,(v'\wedge w')_-]=0, \tag{3.6}$$

$$[v\wedge w,(v'\wedge w')_\pm]=[(v\wedge w)_\pm,(v'\wedge w')_\pm]=[v\wedge w, v'\wedge w']_\pm. \tag{3.7}$$

On a Riemannian 4-manifold M, the metric provides isomorphisms $TM \cong T^*M$ and $\Lambda^2 TM \cong \Lambda^2 T^*M$, and induces a metric and Lie algebra structure on $\Lambda^2 T^*M$ by (3.2) and (3.5). When M is oriented, the star operator determines subbundles $\Lambda^2_\pm(TM)$. Let P' denote the $SO(3)$-bundle of orthonormal frames of $\Lambda^2_+ TM$, and let P be the unique lift of P' to a (connected) principal $SU(2)$-bundle. Since $\mathfrak{su}(2)=\mathfrak{so}(3)=\Lambda^2_+\mathbb{R}^4$, there is a natural identification $\Lambda^2_+ TM=\mathrm{Ad}\,P$ of bundles of Lie algebras with metric.

When M is the 4-sphere $S=S^4$, this bundle P is the $k=1$ $SU(2)$-bundle on S (see [FU]). We can then explicitly describe the action of the conformal group on $\mathscr{A}$ as follows. Any diffeomorphism Φ of S induces bundle maps $\Phi_*: TS \to TS$ and $\Lambda^2\Phi_*: \Lambda^2 TS \to \Lambda^2 TS$. When Φ is conformal and orientation-preserving $\Lambda^2\Phi_*$ commutes with the star operator, and hence preserves the subbundle $\Lambda^2_+ TS$. If $\Phi^* g=\gamma^2 g$, then $\Lambda^2\Phi_*$ is not norm-preserving (unless $\gamma\equiv 1$). However, $\gamma^{-2}\Lambda^2\Phi_*$ does preserve norms and therefore induces a bundle automorphism of P' [taking the frame $\{(e_i\wedge e_j)_+(x)\}$ to $\{\gamma^{-2}(x)(\Phi_* e_i\wedge\Phi_* e_j)_+\}$].

In this way, $SO(5,1)$ acts on P', and hence on the space of connections on P'. But connections on P' are in $1-1$ correspondence with those on the covering bundle P, so this determines an $SO(5,1)$ action on $\mathscr{A}$.

Remark. Alternatively, one can lift the $SO(5,1)$ action to an action of $\mathrm{Spin}(5,1)$ on P, and hence on $\mathscr{A}$. But then the center $Z\simeq\mathbb{Z}_2$ of $\mathrm{Spin}(5,1)$ acts trivially on $\mathscr{A}$, so we recover the action of $\mathrm{Spin}(5,1)/Z=SO(5,1)$ on $\mathscr{A}$ described above. We will henceforth avoid taking this lift by considering $\mathscr{A}$ as the space of connections on P'.

We next assemble some facts about conformal vector fields on S. We will use the following notation in our computations. Throughout, x will represent a point of S (sometimes viewed as a unit vector in $\mathbb{R}^5$), $\{e_i\}$ a local orthonormal frame of TS, and $\{\theta^i\}$ the dual coframe. A frame is *special at* x if the covariant derivatives of the $\{e_i\}$ vanish at x (such frames always exist). Repeated indices are summed over. The Levi-Civita connection ∇ on TS induces connections on all associated vector bundles (e.g. $\Lambda^2_+ TS$); we will also denote these connections by ∇ or will more explicitly indicate the associating representation (e.g. $\Lambda^2_+\nabla$). The L^2 adjoint of ∇ is denoted by ∇^*.

Each vector $v\in\mathbb{R}^5$ determines a linear function $f_v=(v,\cdot)$ on S. The negative gradient $V(x)=-\mathrm{grad}\,f_v(x)=(v,x)x-v$ has covariant derivative

$$(\nabla_Y V)(x)=(\partial_Y V)^T(x)=f_v(x)\,Y, \tag{3.8}$$

where ∂ is the usual directional derivative in $\mathbb{R}^5$ and $T:\mathbb{R}^5\to T_x S$ denotes orthogonal projection. Hence for tangent vectors Y, Z,

$$(\mathscr{L}_V g)(Y,Z)=g(\nabla_Y V,Z)+g(Y,\nabla_Z V)=2f_v g(Y,Z);$$

i.e. $\mathscr{L}_V g=2f_v g$. Therefore each V is a conformal vector field on S.

Lemma 3.1. *The following equations are true pointwise:*

(a) $|V|^2=|\mathrm{grad}\,f_v|^2=|v|^2-f_v^2$,

(b) $\nabla df_v=-f_v g$,

(c) $\nabla^*\nabla f_v=4f_v$,

(d) $\nabla^*\nabla V=V$.

Furthermore, under the flow Φ_t^v *of* V, *the metric pulls back to*

$$(\Phi_t^v)^* g = \gamma_v(t)^2 g, \tag{3.9}$$

where

$$\gamma_v(t) = \left[\cosh(|v|t) - \left(\frac{f_v}{|v|}\right)\sinh(|v|t)\right]^{-1}. \tag{3.10}$$

Proof. (a) Immediate from the definition of V.

(b) For $Y, Z \in TS$, we have

$$(\nabla df_v)(Y,Z) = \langle \nabla_Y df_v, Z\rangle = g(-\nabla_Y V, Z) = -f_v g(Y,Z).$$

(c) $\nabla^*\nabla f_v = -(\nabla df_v)(e_i, e_i) = f_v g(e_i, e_i) = 4 f_v$.

(d) Computing in a special frame at an arbitrary x,

$$\begin{aligned}(\nabla^*\nabla V)(x) &= -\nabla_{e_i}\nabla_{e_i} V|_x = -\nabla_{e_i}(f_v, e_i)|_x \\ &= -\langle df_v, e_i\rangle e_i|_x = (V, e_i) e_i|_x = V(x).\end{aligned}$$

Finally, set $\gamma(t) = \exp\left(\int_0^t (\Phi_s^v)^* f_v ds\right)$ and $h(t) = \gamma^{-2}(t)(\Phi_t^v)^* g$. Then

$$\begin{aligned} dh/dt &= \gamma^{-2}(-2(\Phi_t^v)^* f_v)(\Phi_t^v)^* g + \gamma^{-2}(\Phi_t^v)^* \mathcal{L}_V g \\ &= \gamma^{-2}(\Phi_t^v)^* g[-2(\Phi_t^v)^* f_v + 2(\Phi_t^v)^* f_v] = 0.\end{aligned}$$

Hence h is constant, and since $h(0) = g$ we obtain (3.9). Since V commutes with its own flow Φ_t^v, the function $p(t) = (\Phi_t^v)^* f_v$ satisfies

$$p'(t) = (\Phi_t^v)^*(V(f_v)) = (\Phi_t^v)^*(\langle df_v, V\rangle).$$

But $\langle df_v, V\rangle = -|V|^2 = f_v^2 - |v|^2$ by part (a), so $p' = p^2 - |v|^2$. Consequently $\gamma^{-1}(t)$ satisfies

$$(\gamma^{-1})''(t) = [-p\gamma^{-1}]' = \gamma^{-1}[p^2 - p'] = \gamma^{-1}|v|^2,$$

with initial conditions $\gamma^{-1}(0) = 1$ and $(\gamma^{-1})'(0) = -p(0) = -f_v$. The unique solution of this ODE gives (3.9). □

Notation. Henceforth we write

$$\begin{aligned} r &= |v| \\ a_r &= \cosh(r) \\ b_r &= r^{-1}\sinh(r) \qquad (\text{and } b_0 = 1) \\ \gamma_r &= (a_r - b_r f_r)^{-1} \qquad [\text{this is (3.9) with } t = 1]. \end{aligned} \tag{3.11}$$

Next we observe that for any oriented Riemannian 4-manifold there is a natural conformally invariant injection

$$\alpha : TM \to T^*M \otimes \Lambda_+^2 TM$$

given by

$$\alpha(Y) = \theta^i \otimes (e_i \wedge Y)_+ .$$

Because the projection p_+ is covariant constant, $\nabla_Y(\alpha(Z)) = \alpha(\nabla_Y Z)$, where ∇ is the Levi-Civita connection.

The Levi-Civita connection $\nabla=\nabla^0$ on S^4 is the unique $SO(5)$-invariant connection on TS. The corresponding homogeneous connection $A_0=\Lambda^2\nabla^0$ on $\Lambda_+^2 TS$ is self-dual; we refer to it as the "standard instanton" and view it as the base point or origin of $\mathcal{M}_1$. We can obtain additional self-dual connections using conformal diffeomorphisms. For each $v\in\mathbb{R}^5$ let ∇^v be the connection on TS given by

$$\nabla^v Y=\gamma_v(\Phi_v^*\nabla)\gamma_v^{-1}Y, \qquad Y\in\Gamma(TS),$$

where

$$(\Phi_t^*\nabla)_X Y=\Phi_{v*}^{-1}(\nabla_{\Phi_{v*}X}\Phi_{v*}Y).$$

∇^v is the pullback of ∇^0 under the norm-preserving automorphism $\gamma^{-1}\Phi_{v*}$ of TS, and hence defines a connection on P'. The corresponding connection on $\Lambda_v^2 TS$ is

$$\Lambda^2\nabla^v=\gamma_v^2(\Lambda_+^2\Phi_v^*\nabla)\gamma_v^{-2}.$$

Lemma 3.2. *For each $v\in\mathbb{R}^5$, the connection $\Lambda^2\nabla^v$ on $\Lambda_+^2(TS)$ is compatible with the standard metric on $\Lambda_+^2(TS)$, is self-dual, and satisfies*

(a) $\Lambda^2\nabla^v=\Lambda^2\nabla^0-b_v\gamma_v\,\mathrm{ad}(\alpha(V))$,

(b) $F^v=\Phi_v^*F^0=\gamma_v^2F^0$, *where F^v is the curvature $\Lambda^2\nabla^v$.*

Proof. These connections are associated to P', so are compatible with the standard metric. Conjugating a covariant derivative by a function does not affect curvature, so $F^v=\Phi_v^*F^0$. But $F^0=\Lambda_+^2R$ is obtained from the Riemann curvature tensor R of S^4, which is given by

$$R(X,Y)Z=g(Y,Z)X-g(X,Z)Y. \tag{3.12}$$

It follows that $\Phi_v^*F^0=\gamma_v^2F^0$. This implies self-duality, and establishes (b).

To prove (a), observe that $\Phi_v^*\nabla$ is compatible with $\Phi_v^*g=\gamma_v^2g$, while $\Lambda^2\Phi_v^*\nabla$ is compatible with the standard metric on TS. An easy calculation shows that

$$(\Phi^*\nabla)_Y Z=\nabla_Y Z+Y(\psi)Z+Z(\psi)Y-g(Y,Z)\,\mathrm{grad}\,\psi,$$

where $\psi=\log\gamma_v$. Hence

$$\nabla_Y^v Z=\nabla_Y Z+Z(\psi)Y-g(Y,Z)\,\mathrm{grad}\,\psi=\nabla_Y Z+(Y\wedge\mathrm{grad}\,\psi)\cdot Z; \tag{3.13}$$

equivalently, $\nabla^v=\nabla+\theta^i\otimes(e_i\wedge\mathrm{grad}\,\psi)$. Part (a) follows by applying the representation Λ_+^2, using (3.7), and noting that

$$\mathrm{grad}\,\psi=-\gamma_v\,\mathrm{grad}(a_v-b_vf_v)=-\gamma_v b_v V. \quad \square$$

It will be useful later to have an explicit formula for F^0.

Lemma 3.3. *The curvature F^0 satisfies*

(a) $F^0(X,Y)=\mathrm{ad}(X\wedge Y)_+$,

(b) $F^0=1/2(\theta^i\wedge\theta^j)\otimes(e_i\wedge e_j)_+=1/2(\theta^i\wedge\theta^j)_+\otimes(e_i\wedge e_j)_+$.

Proof. From (3.12) and (3.5) we have

$$(\Lambda^2R)(X,Y)(Z\wedge W)=((X\wedge Y)\cdot Z)\wedge W+Z\wedge((X\wedge Y)\cdot W)=[X\wedge Y,Z\wedge W],$$

so $(\Lambda^2 R)(X, Y) = \mathrm{ad}(X \wedge Y)$. Restricting to $\Lambda^2_+ TS$ then gives part (a) [in view of (3.7)]. Writing (a) in terms of a local frame gives the first part of (b). Finally, observe that

$$(\theta^i \wedge \theta^j)_- \otimes (e_i \wedge e_j)_+ = 0; \tag{3.14}$$

this is true because the left-hand side is a scalar in the irreducible representation $\Lambda^2_- \otimes \Lambda^2_+$ of $SO(4)$. The last part of (b) follows. □

4. A Coordinate System on $\mathcal{M}_1$

Each $v \in \mathbb{R}^5$ determines a conformal vector field V on S and an $\mathrm{Ad}\,P$-valued 1-form X_v defined by

$$X_v = \alpha(V) \in \Omega^1(\mathrm{Ad}\,P)$$

[α was defined after (3.11).] It also determines a self-dual connection on P' by the formula of Lemma 3.2a. To simplify notation we will henceforth omit the Λ^2 and the ad from that formula, and will write ∇^0 as ∇. Thus the lemma defines an affine map

$$Q : \mathbb{R}^5 \to \mathcal{A},$$

by

$$Q(v) = \nabla^v = \nabla - b_v \gamma_v X_v, \tag{4.1}$$

into the subspace of self-dual connections which arise by applying conformal transformations of the form $\Phi^v = \exp(V)$ to the standard instanton. Since every conformal diffeomorphism of S can be written as $g \cdot \exp(V)$ for some V and some $g \in SO(5)$, the Atiyah-Hitchin-Singer result [AHS, Sect. 9] shows that the image of Q projects *onto* the moduli space $\mathcal{M}_1$ in $\mathcal{A}/\mathcal{G}$. Furthermore, since $|F^v|^2 = \gamma_v^2 |F^0|$ (see Lemma 3.2b) is gauge-invariant and the functions γ_v^2, γ_w^2 are distinct for $v \neq w$, the composition

$$\mathbb{R}^5 \xrightarrow{Q} \mathcal{A} \xrightarrow{\pi} \mathcal{M}_1 \subset \mathcal{A}/\mathcal{G}$$

is injective, and hence a homeomorphism. This section will be devoted primarily to the proof of the following sharper statement.

Proposition 4.1. *Q induces a diffeomorphism $\bar{Q} = \pi \circ Q : \mathbb{R}^5 \to \mathcal{M}_1$, and therefore provides coordinates on $\mathcal{M}_1$.*

Remark. We will later derive a formula [Eq. (4.11) evaluated at A_0] that shows that

$$d^*_{A_0}(b_v \gamma_v X_v) = 0.$$

Thus the image of Q lies in the horizontal slice of the gauge group action at the standard instanton A_0.

The first step in the proof of Proposition 4.1 is to calculate the derivative of Q. As a preliminary, we derive an alternative expression for X_v. Lemmas 3.2b and 3.3b show that the contraction $i_Y F^v$ of $Y \in TS$ with F^v [defined by $(i_Y F^v)(Z) = F(Y, Z)$] is

$$\begin{aligned} i_Y F^v &= 1/2\gamma_v^2 (Y^i \theta^j - Y^j \theta^i) \otimes (e_i \wedge e_j)_+ \qquad (\text{where } Y^i = \langle \theta^i, Y \rangle) \\ &= -\gamma_v^2 \theta^i \otimes (e_i \wedge Y)_+ = -\gamma_v^2 \alpha(Y). \end{aligned}$$

Taking $v=0$ and specializing to a conformal vector field V,

$$X_v=\alpha(V)=-i_V F^0. \tag{4.3}$$

Lemma 4.2. *The derivative of Q at $v\in\mathbb{R}^5$*

$$Y_v^w=(DQ)_v(w)=\frac{d}{dt}Q(v+tw)|_{t=0}$$

is given by

$$Y_v^w=\begin{cases}-\gamma_v^2X_v=-\gamma^2\alpha(V)=i_vF^v & \text{if } w=v\\ -b_v^2\gamma_v^2f_wX_v-b_v\gamma_vX_w=\alpha(b_v^2\gamma_v^2f_wV+b_v\gamma_vW) & \text{if } w\perp v.\end{cases} \tag{4.4}$$

Proof. Set $v_t=v+tw$, $a_t=a_{v_t}$, etc. (see the definitions in Sect. 3) and let prime denote t-differentiation at $t=0$. Simple computations show that $a'=(v,w)b_v$, $b'=(a_v-b_v)|v|^{-2}(v,w)$, $f'=f_w$, and hence

$$Y_v^w=-(b_t\gamma_t)'X_v-b_v\gamma_vX_w=-\alpha[(b_t\gamma_t)'V+b_v\gamma_vW]. \tag{4.5}$$

Using the identity $a_v^2-b_v^2|v|^2=1$, we obtain

$$(b_t\gamma_t)'=\gamma_v^2[(1-a_vb_v)|v|^{-2}(v,w)+b_v^2f_w]. \tag{4.6}$$

When $v\perp w$ this simplifies to $(b\gamma)'=\gamma^2b^2f$. When $v=w$ the identity $bf=a-\gamma^{-1}$ shows that $(b\gamma)'+b\gamma=\gamma^2$. The lemma follows by combining these formulas and using (4.3). □

We now give an explicit description of $T_{V^v}\mathcal{M}_1$ and show that $\bar{Q}$ is a local diffeomorphism. Since we have already observed that $\bar{Q}$ is a homeomorphism, this will complete the proof of Proposition 4.1.

Proposition 4.3. *At $A=V^v\in\mathcal{A}$,*

$$T_A\mathcal{M}_1=\{i_WF^v/W=-Vf_w \text{ for some } w\in\mathbb{R}^5\}.$$

Moreover, $\bar{Q}=\pi Q$ is a local diffeomorphism.

Proof. Each such conformal vector field W determines a flow $\Phi_t=\Phi_t^w$ on M. As described in Sect. 3, Φ_t lifts to a flow $\tilde{\Phi}_t$ on the principal bundle P'. Let $\tilde{W}$ be the infinitesimal generator of $\tilde{\Phi}_t$, let $G=SU(2)$ denote the structure group of P and let $\mathfrak{g}$ denote its Lie algebra. Since $\tilde{\Phi}_t$ is a bundle map covering Φ_t, $\tilde{W}$ is a G-invariant vector field covering W.

Now let $\omega_A\in\Omega^1(P;\mathfrak{g})$ be the connection form of A. Then

$$\begin{aligned}\frac{d}{dt}(\Phi_t^*\omega_A)|_{t=0}&=\mathcal{L}_{\tilde{W}}\omega_A=i_{\tilde{W}}d\omega_A+di_{\tilde{W}}\omega_A\\ &=i_{\tilde{W}}(d\omega_A+1/2[\omega_A,\omega_A])+[\omega_A,i_{\tilde{W}}\omega_A]+d(i_{\tilde{W}}\omega_A)\\ &=i_{\tilde{W}}\tilde{F}_A+d\tilde{u}+[\omega_A,\tilde{u}],\end{aligned} \tag{4.7}$$

where $\tilde{u}=i_{\tilde{W}}\omega_A$, and $\tilde{F}_A$ is the curvature, regarded as a $\mathfrak{g}$-valued 2-form on P. Note that u satisfies $R_h^*\tilde{u}=(\mathrm{Ad}\,h^{-1})(\tilde{u})$ for $h\in G$, and hence represents a section u of $\mathrm{Ad}\,P$. Therefore, as a statement about $\mathrm{Ad}\,P$-valued 1-forms on S^4, (4.7) reads

$$\frac{d}{dt}(\Phi_t^*A)|_{t=0}=i_WF_A+d_Au. \tag{4.8}$$

Note also that $\Phi_t^* A = (\Phi_t^w)^* A = (\Phi_t^w)^* (\Phi_1^v)^* A^0 = (\Phi_1^v \Phi_1^{tw})^* \nabla^0$.

Now $SO(5)$ is the maximal compact subgroup of $SO(5,1)$, and there is a (smooth) decomposition $SO(5,1) = SO(5) \cdot \mathbb{R}^5$, where $\mathbb{R}^5$ is the submanifold $\{\Phi_t^v | v \in \mathbb{R}^5\}$ of $SO(5,1)$. Therefore there exists elements $g_t \in SO(5)$ and $z_t \in \mathbb{R}^5$, depending smoothly on t, such that $\Phi_1^v \Phi_1^{tw} = g_t \Phi_1^{z_t}$. Since $g_t^* \nabla^0 = \nabla^0$ and $(\Phi_1^z)^* \nabla_0 = Q(z)$, it follows that $\Phi_t^* A = Q(z_t)$. Thus if we set $z' = \frac{d}{dt} z_t|_{t=0}$, Eq. (4.8) gives

$$i_W F_A = Q_*(z') - d_A u. \tag{4.9}$$

But $d_A^- \circ d_A = 0$ and, as remarked earlier, $d_A^- \circ Q_* = 0$, so we have

$$d_A^-(i_w F_A) = 0. \tag{4.10}$$

Next fix $x \in S^4$ and a frame $\{e_i\}$ special at x. At x,

$$\begin{aligned} d_A^*(i_W F^v) &= d_A^*(-\gamma^2 \alpha(W)) = \nabla_{e_i}^v(\gamma^2(e_i \wedge W)_+) \\ &= (\operatorname{grad}(\gamma^2) \wedge W + \gamma^2 \nabla^v e_i \wedge W + \gamma^2 e_i \wedge \nabla^v W)_+ . \end{aligned} \tag{4.11}$$

But $\operatorname{grad}(\gamma^2) = -2b\gamma^3 V$, and by (3.13) and (3.4) we have

$$\nabla_i^v W = \nabla_i^0 W - b\gamma(e_i \wedge V) \cdot W = f_w e_i - b\gamma \langle V, W \rangle e_i + b\gamma \langle e_i, W \rangle V, \tag{4.12}$$

and similarly $\nabla_i^v e_i = 3b\gamma V$ at x. Substituting into (4.11) shows that

$$d_A^*(i_W F_A) = 0 \tag{4.13}$$

at x and hence everywhere.

We have now established that $i_W F_A \in \operatorname{Ker}(d_A^*) \cap \ker(d_A^-)$ for each W. Since $\{i_W F_A\} = \{\gamma^2 \alpha(W)\}$ is a 5-dimensional subspace of $\Omega^1(\operatorname{Ad} P)$, it follows that the $i_W F_A$ span $T_A \mathscr{M}_1$. But (4.9) implies that $h_A(Q_*(z')) = i_W F_A$, so $h: \{Y_v^w\} \to T_A \mathscr{M}_1$ is surjective. Moreover, it is clear from (4.4) that $\dim\{Y_v^w\} = 5$. Therefore $h|_{\operatorname{Im} Q}$ is an isomorphism, and hence so is $(\pi \circ Q)$. We conclude that $\pi \circ Q$ is a local diffeomorphism, as claimed. □

5. Curvature of $\mathscr{M}_1$ at the Standard Instanton

Proposition 4.3 and Eq. (4.3) show that the tangent space to $\mathscr{M}_1$ at the base connection A_0 is spanned by $\{\alpha(V) | V = -\nabla f_v\} = \{X_v\}$. We will next compute the sectional curvature

$$\sigma(X_v, X_w) = \frac{\langle R(X_v, X_w) X_w, X_v \rangle}{\|X_v\|^2 \, \|X_w\|^2 - \langle X_v, X_w \rangle^2} \tag{5.1}$$

of the 2-plane spanned by $X_v, X_w \in T_{A_0} \mathscr{M}_1$.

Theorem 5.1. *The sectional curvatures of $\mathscr{M}_1$ at A_0 are all equal to $5/(16\pi^2)$.*

Thus $\mathscr{M}_1$ has constant *positive* sectional curvature at A_0. In particular, this means that the Atiyah-Hitchin-Singer diffeomorphism (3.1) between $\mathscr{M}_1$ and hyperbolic 5-space is *not* an isometry.

Remark. The number $5/16\pi^2$ in Theorem 5.1 depends on a choice of scale for the metric on $\operatorname{Ad} P$. We could have chosen the metric to be induced by any G-invariant inner product on $\mathfrak{g}$. Any such inner product is a multiple of the Killing form; in Sect. 3, we fixed the multiple to be $-1/4$. Had we used $-\mu^2/4$ instead, we would have obtained $5/16\mu^2\pi^2$ in the theorem.

We will prove this theorem by applying the curvature formula (2.13). (In Sect. 6 we will obtain an explicit formula for the metric on $\mathcal{M}_1$, which can also be used to compute the sectional curvature.) For this we need explicit information about $P^*_{X_v}X_w$ and $P^-_{X_v}X_w$.

Lemma 5.2. *Writing D for d_{A_0} we have, for all $v, w \in \mathbb{R}^5$,*

(a) $P^*_{X_v}X_w = 2(V \wedge W)_+$,

(b) $P^-_{X_v}X_w = 2(\theta^i \wedge V^*)_- \otimes (e_i \wedge W)_+$,

(c) $D^*D(P^*_{X_v}X_w) = \nabla^*\nabla(P^*_{X_v}X_w) = 2P^*_{X_v}X_w$,

(d) $D^-(D^-)^*(P^-_{X_v}X_w) = 3P^-_{X_v}X_w$.

Proof. (a) $P^*_{X_v}X_w = -[(e_i \wedge V)_+, (e_i \wedge W)_+] = -[e_i \wedge V, e_i \wedge W]_+ = 2[V \wedge W]_+$ by (3.7) and (3.5).

(b)
$$\begin{aligned} P^-_{X_v}X_w &= p_-[\theta^i \otimes (e_i \wedge V)_+, \theta^j \otimes (e_j \wedge W)_+] \\ &= (\theta^i \wedge \theta^j)_- \otimes [e_i \wedge V, e_j \wedge W]_+ \\ &= (\theta^i \wedge V^*)_- \otimes (e_i \wedge W)_+ + (\theta^j \wedge W^*)_- \otimes (e_j \wedge V)_+, \end{aligned}$$

where in the last step we have used (3.5) and (3.14) and have written V^* for the metric dual of V. It remains to show that these last two terms are equal, and it suffices to verify this when $V = e_1$ and $W = e_2$. But then

$$\begin{aligned} &(V^* \wedge \theta^i)_- \otimes (W \wedge e_i)_+ \\ &= (\theta^1 \wedge \theta^3)_- \otimes (e_2 \wedge e_3)_+ + (\theta^1 \wedge \theta^4)_- \otimes (e_2 \wedge e_4)_+ \\ &= (\theta^2 \wedge \theta^4)_- \otimes (e_1 \wedge e_4)_+ + (\theta^2 \wedge \theta^3)_- \otimes (e_1 \wedge e_3)_+ \\ &= (W^* \wedge \theta^i)_- \otimes (V \wedge e_i)_+, \end{aligned}$$

as desired.

(c) By (a), it suffices to show that $\nabla^*\nabla[(V \wedge W)_+] = 2(V \wedge W)_+$ for conformal vector fields V, W ($D^*D = \nabla^*\nabla$ on 0-forms). But

$$\nabla^*\nabla[(V \wedge W)_+] = (\nabla^*\nabla(V \wedge W))_+ = ((\nabla^*\nabla V) \wedge W - 2\nabla_{e_i}V \wedge \nabla_{e_i}W + V \wedge (\nabla^*\nabla W))_+$$

The middle term vanishes by (3.8) and the other two terms simplify by Lemma 3.1d. We are left with $\nabla^*\nabla(V \wedge W)_+ = 2(V \wedge W)_+$.

(d) On $\Omega^2_-(\operatorname{Ad}P)$ we have $D^-(D^-)^* = 1/2p_- \circ (D^*D + DD^*)$. The Weitzenböck formula on $\Omega^2_-(E)$, where $E \to M$ is any vector bundle with connection A, states that

$$d^*_A d_A + d_A d^*_A = \nabla^*_A \nabla_A + (s/3) - 2\mathcal{W}_- - \mathcal{F}_-. \tag{5.2}$$

Here s is the scalar curvature of M, $\mathcal{W}_-$ is the Weyl endomorphism defined in Sect. 1, and $\mathcal{F}_-$ is proportional to $(F_A)_-$. In our case, $d_A = D$, $\nabla_A = \nabla$, $(F_A)_- = 0$, and, since $M = S^4$, $s = 12$ and $\mathcal{W}_- = 0$. Hence $D^-(D^-)^* = 1/2\nabla^*\nabla + 2$. Using part (b),

$$\begin{aligned} 1/2\nabla^*\nabla(P^-_{X_v}X_w) = &(\theta^i \wedge \nabla^*\nabla V^*)_- \otimes (e_i \wedge W)_+ \\ &- 2(\theta^i \wedge \nabla_j V^*)_- \otimes (e_i \wedge \nabla_j W)_+ \\ &+ (\theta^i \wedge V^*)_- \otimes (e_i \wedge \nabla^*\nabla W)_+. \end{aligned}$$

Again, $\nabla^*\nabla W = W$ and $\nabla^*\nabla(V^*) = (\nabla^*\nabla V)^* = V^*$. The cross-term is proportional to $(\theta^i \wedge \theta^j)_- \otimes (e_i \wedge e_j)_+ = 0$ [see (3.14)], so by (b) the right-hand side above is simply $P^-_{X_v} X_w$. Combining these formulas yields (d). □

Proof of Theorem 5.1. From parts (c) and (d) of Lemma 5.2 we conclude that

$$G^0_{A_0}(P^*_{X_v} X_w) = \tfrac{1}{2} P^*_{X_v} X_w, \qquad G^0_{A_0}(P^-_{X_v} X_w) = \tfrac{1}{3} P^-_{X_v} X_w, \tag{5.3}$$

so the basic curvature formula (2.13) becomes

$$\langle R(X_v, X_w) X_w, X_v \rangle = 3 \cdot \tfrac{1}{2} \| P^*_{X_v} X_w \|^2 - \tfrac{1}{3} [\| P^-_{X_v} X_w \|^2 - \langle P^-_{X_v} X_v, P^-_{X_w} X_w \rangle)] . \tag{5.4}$$

Hence we need only compute the L^2 inner products of objects of the form X_v, $P^*_{X_v} X_w$, and $P^-_{X_v} X_w$. First we make some pointwise calculations.

Observe that

$$\begin{aligned}((e_i \wedge V)_\pm, (e_j \wedge W)_\pm) &= 1/2(e_i \wedge V, e_j \wedge W \pm *(e_j \wedge W)) \\ &= 1/2((V, W)\delta_{ij} - V^j W^i \pm (e_j \wedge V, *(e_j \wedge W))) \\ &= 1/2((V, W)\delta_{ij} - V^j W^i \mp (V \wedge W, *(e_i \wedge e_j))) .\end{aligned} \tag{5.5}$$

(The last step is verified by checking it for $V = e_k$, $W = e_\ell$.) From this and the definition of α, it follows that $(\alpha(V), \alpha(V)) = 3/2(V, V)$, and, polarizing,

$$(\alpha(V), \alpha(W)) = (X_v, X_w) = 3/2(V, W) . \tag{5.6}$$

Next, Lemma 5.2a implies that

$$|P^*_{X_v} X_w|^2 = 4|(V \wedge W)_+|^2 = 2|V \wedge W|^2 . \tag{5.7}$$

Finally, using Lemma 5.2b and (5.5), we have

$$(P^-_{X_v} X_w, P^-_{X_v} X_w) = 4((\theta^i \wedge V^*)_-, (\theta^j \wedge V^*)_-)((e_i \wedge W)_+, (e_j \wedge W)_+) = 2|V|^2|W|^2 + V, W)^2 .$$

Polarizing (in this case computing the st term in $|P_{X_{v+tw}} X_{v+sw}|^2$, one finds that

$$(P^-_{X_v} X_w, P^-_{X_v} X_w) = 4(V, W)^2 - |V|^2|W|^2 .$$

Combining the last two equations we obtain

$$(P^-_{X_v} X_w, P^-_{X_v} X_w) - (P^-_{X_v} X_v, P^-_{X_w} X_w) = 3(|V|^2|W|^2 - (V, W)^2) = 3|V \wedge W|^2 . \tag{5.8}$$

The L^2 inner products corresponding to these pointwise expressions are given by the following lemma, which is proved in the appendix.

Lemma 5.3. (a) $\langle V, W \rangle_{L^2} = \frac{32}{15}\pi^2(v, w)$,

(b) $\| V \wedge W \|^2_{L^2} = \frac{8}{5}\pi^2 |v \wedge w|^2$.

By (5.7), (5.8) and Lemma 5.3b, the right-hand side of the curvature formula (5.4) reduces to $16\pi^2/5|V \wedge W|^2$. On the other hand, (5.6) and Lemma 5.2a imply that

$$\|X_v\|^2 \|X_w\|^2 - \langle X_v, X_w \rangle^2 = (\tfrac{16}{5}\pi^2)^2 |v \wedge w|^2 .$$

The sectional curvature (5.1) is therefore $\sigma(X_v, X_w) = \frac{5}{16}\pi^2$, as claimed. □

6. The Shape of $\mathcal{M}_1$

In this section we exhibit an explicit formula for the metric $\mathcal{g}$ on $\mathcal{M}_1$ and show how it leads to the results stated in the introduction.

Our method is to directly calculate the expression $\bar{Q}^*\mathcal{g}$ for the metric in the coordinate system $\bar{Q}=\pi Q:\mathbb{R}^5\to\mathcal{M}_1$ which was introduced in Sect. 4. At $v\in\mathbb{R}^5$, this metric is given by

$$(\bar{Q}_*\mathcal{g})_v(w,w)=\mathcal{g}(\pi_* Y_v^w, \pi_* Y_v^w)=\|hY_v^w\|^2,$$

where the $Y_v^w=(DQ)_v(w)$ are given by (4.4). Thus we must determine the horizontal projections hY_v^w at $Q(v)$. Our first step is to check whether the Y_v^w are already horizontal.

Lemma 6.1. $d^*_{\nabla^v} Y_v^w = 2b_v^2\gamma_v^2(V\wedge W)_+$.

Proof. From the proof of Lemma 4.1 we have $Y_v^w=-\alpha(Z)$, where $Z=(b\gamma)'V+b\gamma W$. Fix $x\in S$ and a frame $\{e_k\}$ special at x. Then at x,

$$\begin{aligned} d^*_{\nabla^v} Y^w &= e_k \lrcorner \nabla^v_k[\theta^i\otimes(e_i\wedge Z)_+]=(\nabla^v_i e_i\wedge Z+e_i\wedge\nabla^v_i Z)_+\\ &=(\nabla^v_i e_i\wedge Z+\operatorname{grad}(b\gamma)'\wedge V+\operatorname{grad}(b\gamma)\wedge W\\ &\quad +(b\gamma)'e_i\wedge\nabla^v_i V+b\gamma e_i\wedge \nabla^v_i W)_+ . \end{aligned} \tag{6.1}$$

But $\operatorname{grad}(b\gamma)=-b^2\gamma^2 V$ and, from (4.6), $\operatorname{grad}(b\gamma)'=-b^2\gamma^2 W$. Furthermore, in (4.12) we calculated $\nabla^v_i W$, and also saw that $\nabla^v_i e_i=3b\gamma V$ at x. The lemma then follows from (6.1). □

Lemma 6.1 shows that Y_v^v is horizontal, so we need only compute hY_v^w for $w\perp v$. This is accomplished by projecting Y_v^w onto the subspace spanned by $i_Z F^v$, where Z is conformal (cf. Proposition 4.2). For this, we must calculate the lengths of the $i_Z F^v$ and the inner products of Y_v^w with the $i_Z F^v$; this is done in the next three lemmas.

Notation. We introduce two functions which will arise naturally in the calculations below. For $v\in\mathbb{R}^5$, let $r=|v|$ and define [using the notation (3.11)]

$$\begin{aligned} A(r)&=b_v^{-2}|v|^{-2}+3(b_v-a_v)b_v^{-5}|v|^{-4}\\ &=\sinh^{-2}(r)+3\sinh^{-4}(r)-3r\cosh(r)\sinh^{-5}(r) \end{aligned}$$

and

$$\begin{aligned} B(r)&=(2-A(r))^{-1}[3b_v^{-2}|v|^{-2}(a_v b_v-1)-a_v b_v A(r)]^2\\ &=r^{-2}(2-A(r))^{-1}[3\coth(r)-3r\sinh^{-2}(r)-\cosh(r)\sinh(r)A(r)]^2 . \end{aligned}$$

We will compute the L^2 norms of the Y_v^w and $i_Z F$ by expressing them as integrals of the functions γ_v, f_v on S^4. The specific integrals we will need are given in the following lemma, which is proved in the appendix.

Lemma 6.2. *Let γ, a, b, A denote γ_v, a_v, b_v, $A(|v|)$ respectively. Then*

(a) $\int\gamma^4=\frac{8}{3}\pi^2$ *(independently of v)*,

(b) $\int\gamma^4(|v|^2-f_v^2)=\frac{16}{3}\pi^2|v|^2A$,

(c) $\int\gamma^3=4\pi^2 b^{-2}|v|^{-2}(a-b^{-1})$ *(if $v\neq 0$)*, *and*

(d) *For* $w \perp v$, $\int \gamma^4 f_w^2 = \frac{4}{3}\pi^2 |w|^2 A$,

where all integrals are over S^4.

Lemma 6.3. (a) $\| Y_v^v \|^2 = \| i_v F^v \|^2 = 8\pi^2 |v|^2 A(|v|)$.

(b) *If* $w \perp v$, *then* $\| i_W F^v \|^2 = 2\pi^2 |w|^2 (2 - A(|v|))$.

Proof. Computing the norm of $i_W F^v = -\gamma_v^2 \alpha(W)$ by (5.6) and Lemma 3.1a, we have

$$\| i_W F^v \|^2 = \tfrac{3}{2} \int \gamma^4 (|w|^2 - f_w^2).$$

Part (a) now follows by taking $w = v$ and applying Lemma 6.2b, and (b) follows by taking $w \perp v$ and applying Lemma 6.2a, d. □

Lemma 6.4. *Suppose* $w \perp v$. *Then*

(a) $\langle Y_v^w, i_W F^v \rangle = 2\pi^2 |w|^2 (3b^{-2} |v|^{-2} (ab - 1) - abA)$

(b) *For each* $z \perp w$, $\langle Y_v^w, i_Z F^v \rangle = 0$.

Proof. Using Lemma 4.2 and Eq. (5.6),

$$\begin{aligned} \langle Y_v^w, i_Z F^v \rangle &= \tfrac{3}{2} \langle b^2 \gamma^2 f_w V + b\gamma W, \gamma^2 Z \rangle \\ &= \tfrac{3}{2} \int_S \{ \gamma^4 b f_w [(v, z) - f_v f_z] + \gamma^3 [(w, z) - f_w f_z] \}. \end{aligned} \tag{6.2}$$

Taking $z = w$ and using the identity $b\gamma f = a\gamma - 1$, the integrand simplifies to $|w|^2 \gamma^3 - a\gamma^4 f_w^2$. We then obtain part (a) by integrating this using Lemma 6.2c, d. On the other hand, when $z \perp w$, (6.2) becomes

$$\langle Y_v^w, i_Z F^v \rangle = \tfrac{3}{2} \int_S \{ b\gamma^4 [(v, z) - f_v f_w] - \gamma^3 f_z \} f_w.$$

Choose coordinates $\{x^i\}$ on $\mathbb{R}^5$ with w along the x^5-axis. Then the part of the integrand in brackets is independent of x^5 (since f_v and f_z are). Since f_w – and therefore the entire integrand – is an odd function of x^5 the integral must vanish. □

Proposition 6.5. (a) $\| h Y_v^v \|^2 = \| Y_v^v \|^2 = 8\pi^2 |v|^2 A(|v|)$.

(b) *For* $w \perp v$, $\| h Y_v^w \|^2 = 2\pi^2 |w|^2 B(|v|)$ *and* $\langle h Y_v^w, h Y_v^v \rangle = 0$.

Proof. (a) Lemma 6.1 shows that Y_v^v is horizontal, so this follows from Lemma 6.3a.

(b) The horizontal projection of Y_v^w is obtained by projecting onto the space spanned by $\{i_Z F\}$. In light of Lemma 6.4b this is simply

$$h Y_v^w = \| i_W F^v \|^{-2} \langle Y_v^w, i_W F^v \rangle i_W F^v. \tag{6.3}$$

We then obtain $\| h Y_v^w \|^2 = 2\pi^2 |w|^2 B(|v|)$ using Lemma 6.3b, Lemma 6.4a, and the definition of B. Finally, (6.3) also shows that $\langle h Y_v^w, h Y_v^v \rangle$ is a multiple of

$$\langle i_w F^v, i_v F^v \rangle = \langle \gamma_v^2 \alpha(W), \gamma_v^2 \alpha(V) \rangle = -\tfrac{3}{2} \int \gamma_v^4 f_v f_w$$

[using (5.6) and Lemma 3.1a]. This integral vanishes as in the proof of Lemma 6.4b, so $\langle h Y_v^w, h Y_v^v \rangle = 0$. □

Remark. The formulas in Proposition 6.5 show that $A(r)$ and $B(r)$ are non-negative. In fact they are positive, since $h|_{\mathrm{Im}\,Q_*}$ is an isomorphism and the metric on $\mathcal{M}_1$ cannot be degenerate.

Proposition 6.5 enables us to write the pullback metric $\bar{Q}^*g$ in terms of the standard coordinates $\{x^i\}$ on $\mathbb{R}^5$. Let $u_i = \partial/\partial x_i$ be the unit basis vectors and write $r = |x|$. The vector fields

$$w_i(x) = u_i - (x^i/r^2)x$$

satisfy $w_i \perp x\ \forall i$. Hence

$$\bar{Q}_* u_i = (x^i/r^2)\bar{Q}_* x + \bar{Q}_* w_i = (x^i/r^2)X_x^x + hX_x^{w_i},$$

so by Proposition 6.5

$$\langle \bar{Q}_* u_i, \bar{Q}_* u_j \rangle = 2\pi^2(4x^i x^j r^{-2} A(r) + \langle w_i, w_j \rangle B(r)).$$

This leads immediately to the following formula for the metric.

Proposition 6.6. *Let g be the metric on $\mathcal{M}_1$. Under the diffeomorphism $\bar{Q}: \mathbb{R}^5 \to \mathcal{M}_1$, g pulls back to the metric $\bar{Q}^*g = h_{ij} dx^i \otimes dx^j$, where*

$$h_{ij} = 2\pi^2 B(r)\,[\delta_{ij} + (4A(r)B^{-1}(r) - 1)(x^i x^j/r^2)].$$

This formula can be simplified by changing coordinates. First we have

$$\bar{Q}^*g = 2\pi^2 B(r)\,\{\textstyle\sum (dx^i)^2 + C(r)(dr)^2\},$$

where $C(r) = 4A(r)B^{-1}(r) - 1$. Define new coordinates by

$$y^i = E(r)x^i,$$

where

$$E(r) = \exp\left[\int_0^r s^{-1}(\sqrt{1 + C(s)} - 1)\,ds\right]$$

[the integral converges since $C(s) = O(s^2)$ as $s \to 0$]. The function $\varrho = rE(r)$ then satisfies $\varrho^2 = \sum (y^i)^2$ and

$$\varrho'(r) = E(r)\sqrt{1 + C(r)} > 0,$$

so ϱ is a monotonically increasing function of r. It follows that $x \mapsto y = E(r)x$ is a diffeomorphism and that the equation $\varrho = rE(r)$ defines r implicitly as a smooth function of ϱ. In particular, we can define a smooth positive function $\psi(\varrho)$ by

$$[\psi(\varrho)]^2 = 2\pi^2 B(r)\,[E(r)]^{-2}.$$

Now, rewriting the metric (6.4) in these new coordinates $\{y^i\}$, we find that

$$\bar{Q}^*g = [\psi(\varrho)]^2 \textstyle\sum (dy^i)^2. \tag{6.5}$$

This is the formula stated as Theorem A in the introduction. Using it, it is now easy to verify the four basic geometric properties of $\mathcal{M}_1$ listed in Corollary B.

(a) $\mathcal{M}_1$ is conformally flat.

This is immediate from (6.5).

(b) $\mathcal{M}_1$ is radially symmetric.

The metric (6.5) is clearly invariant under the usual action of $SO(5)$ on $\mathbb{R}^5$. By the construction of $\bar{Q}$, this corresponds to the $SO(5)$ action on $\mathcal{M}_1$ induced by the rotations of S^4.

(c) $\mathcal{M}_1$ is incomplete.

Given a metric of the form (6.5), it is easy to check that the radial lines through the origin are geodesics (not necessarily parametrized by arclength); we will show that these geodesics have finite length. Fix $v \in \mathbb{R}^5$ with $|v|=1$ and consider the ray $\{tv\}$, $0 \leqq t < \infty$. The tangent vector T to this ray at tv satisfies $\bar{Q}^* T = t^{-1} Y_v^v$, so by Lemma 6.3a

$$\|\bar{Q}_* T\|^2 = t^{-2} \|Y_v^v\|^2 = 8\pi^2 A(t).$$

The length of this ray is therefore

$$L = 2\pi\sqrt{2} \int_0^\infty A(r)^{1/2} dr. \tag{6.6}$$

But as $r \to \infty$, $A^{1/2}(r) \sim 2e^{-r}$, so the integral is finite (since $A(0)=\frac{2}{5}$, there are no problems at $r=0$). Hence $\mathcal{M}_1$ is incomplete.

(d) $\mathcal{M}_1$ has finite volume.

The volume of $\mathcal{M}_1$ is

$$\operatorname{Vol} \mathcal{M}_1 = \int_{\mathbb{R}^5} J(v) dx^1 \wedge \ldots \wedge dx^5, \tag{6.7}$$

where $J(v)$ is the Jacobian determinant $J(v)$ of $\bar{Q} : \mathbb{R}^5 \to \mathcal{M}_1$, which can be calculated at any non-zero $v \in \mathbb{R}^5$ as follows. Choose an orthonormal basis $\{w_1 = v/|v|, w_2, w_3, w_4, w_5\}$. By Proposition 6.5, $\{\bar{Q}_* w_i\}$ is an orthogonal basis of $T_{\bar{Q}(v)} \mathcal{M}_1$, and hence

$$J(v) = \prod_{i=1}^{5} \|\bar{Q}_* w_i\|.$$

Using Proposition 6.5 and the fact that $\bar{Q}_* w_1 = |v|^{-1} Y_v^v$, we obtain

$$J(v) = 8\sqrt{2}\pi^5 A^{1/2}(r) B^2(r),$$

where $r=|v|$. This is continuous at $r=0$, and for large r we have $A^{1/2}(r) \sim 2e^{-r}$, $B(r) \sim 2r^{-2}$, and so $J(v) \sim 64\sqrt{2}\pi^5 r^{-4} e^{-r}$. It follows that the integral (6.7) converges. Thus $\mathcal{M}_1$ has finite volume.

Next, we examine the hypersurfaces in $\mathcal{M}_1$ of constant distance from A_0. These are the spheres $\{r=\text{const}\}$, or, equivalently, $\{\varrho=\text{const}\}$. Formula (6.5) shows that the metric on the sphere S_ϱ of fixed ϱ is $\psi(\varrho)^2$ times the metric on the standard sphere of radius ϱ. Hence S_ϱ is isometric to the standard sphere of radius

$$R(\varrho) = \varrho\psi(\varrho) = \pi r \sqrt{2B(r)}. \tag{6.8}$$

Now the geodesic rays from A_0 have finite length L, given by (6.6). As the arclength parameter ranges from 0 to L, r ranges from 0 to ∞, and ϱ, one checks, ranges from 0 to some limiting value ϱ_∞. Then as $r \to \infty$, $B(r) \sim 2r^{-2}$, and hence by (6.8) $R(\varrho) \to R(\varrho_\infty) = 2\pi$. Thus the closure of $\mathcal{M}_1$ in the metric topology is compact, and its boundary is isometric to the sphere of radius 2π.

Finally, we compute the second fundamental form of these spheres S_r as $r \to \infty$. Using formula (6.5) for the metric, one sees that the unit normal to S is $N = \psi^{-1}(\varrho(r))\partial/\partial\varrho$. A straightforward calculation shows that, for any two vectors X, Y tangent to S_ϱ, the second fundamental form $b(X, Y) = \langle \nabla_X N, Y \rangle$ is

$$b(X, Y) = \psi^{-1} \frac{\partial}{\partial\varrho} [\log(\varrho\psi)] \langle X, Y \rangle.$$

We can then use the definitions of ϱ and $\psi(\varrho)$ to express this in terms of r:

$$b(X,Y)=(8\pi^2r^2AB)^{-1/2}\frac{d}{dr}(rB^{1/2})\langle X,Y\rangle.$$

Computing the asymptotics of this using the definitions of $A(r)$ and $B(r)$, we find that

$$b(X,Y)\sim\frac{3}{2\sqrt{2\pi}}e^{-r}\langle X,Y\rangle\sim\frac{3d}{16\pi^2}\langle X,Y\rangle,$$

where $d=2\pi\sqrt{2}\int_r^\infty A^{1/2}\sim 2\pi\sqrt{2}\int_r^\infty 2e^{-r}\sim 4\pi\sqrt{2}\,e^{-r}$ is the distance to the boundary of $\mathcal{M}_1$. Thus $b\equiv 0$ on $\partial\bar{\mathcal{M}}_1$, and hence the embedding $\partial\bar{\mathcal{M}}_1\hookrightarrow\bar{\mathcal{M}}_1$ is totally geodesic. This proves Corollary C of the introduction.

Remark. These results depend on two scale choices, as follows. For general M, if we replace the metric g by a constant multiple c^2g and the fiber metric on AdP by the one induced by a constant $-\mu^2/4$ times the Killing form (instead of just $-1/4$), then the differentiable manifold $\hat{\mathcal{M}}$ does not change, but its metric $\mathcal{g}$ changes to $c^2\mu^2\mathcal{g}$. (The metric on $\mathcal{A}$ scales by $c^2\mu^2$ and the horizontal distribution on $\mathcal{A}$ does not change.) This has the effect of multiplying distances in $\hat{\mathcal{M}}$ by $c\mu$ and sectional curvatures by $(c\mu)^{-2}$. In particular, when $M=S^4$ the numbers R_0, R_∞, and L scale according to these rules. However, the ratio $R_0:R_\infty:L$ and the ratio $4\pi^2$ of the metric on $\partial\mathcal{M}_1$ to that on S^4 remain unchanged.

Appendix

In this appendix we prove several of the computational lemmas used in the paper. We will repeatedly use the facts that $\mathrm{Vol}(S^3)=2\pi^2$ and $\mathrm{Vol}(S^4)=8\pi^2/3$. Throughout, S denotes S^4.

Lemma A.1. $\int_S f_v f_w=\frac{8}{15}\pi^2(v,w)$.

Proof. We have $f_v=(v,x)=|v|\cos\theta$, where θ is the angle between v and x. Hence

$$\int_S f_v^2=|v|^2\int_0^\pi\cos^2\theta\cdot\mathrm{Vol}(S^3)\sin^3\theta d\theta=\tfrac{8}{15}\pi^2|v|^2.$$

The desired formula follows by polarization. □

Proof of Lemma 5.3. (a) We have $(V,W)=(v,w)-f_vf_w$ pointwise, so

$$\langle V,W\rangle=(v,w)\mathrm{Vol}(S)-\int_S f_vf_w=\tfrac{32}{15}\pi^2(v,w),$$

using Lemma A.1.

(b) Pointwise, $|V\wedge W|^2=|V|^2|W|^2-(V,W)^2$ is

$$|V\wedge W|^2-(|V|^2f_w^2+|w|^2f_v^2-2(v,w)f_vf_w).$$

The expression in parentheses integrates to $\frac{16}{15}\pi^2|v\wedge w|^2$ by Lemma A.1, so

$$\|V\wedge W\|_{L^2}^2=[\mathrm{Vol}(S^4)-\tfrac{16}{15}\pi^2]\,|v\wedge w|^2=\tfrac{8}{5}\pi^2|v\wedge w|^2.\quad\square$$

Lemma A.2. *Suppose $\alpha, \beta > 0$ and $\alpha^2 - \beta^2 = 1$. Then*

(a) $\int_0^\pi (\alpha - \beta\cos\theta)^{-4}\sin^5\theta\, d\theta = \frac{8}{3}\beta^{-2} + 8\beta^{-4} + 4\alpha\beta^{-5}\log\left[\frac{\alpha-\beta}{\alpha+\beta}\right]$,

(b) $\int_0^\pi (\alpha - \beta\cos\theta)^{-3}\sin^3\theta\, d\theta = 2\beta^{-2} + \beta^{-3} + \log\left[\frac{\alpha-\beta}{\alpha+\beta}\right]$.

Proof. In each case let $u = \cos\theta$ and integrate by parts. We omit the details. □

Proof of Lemma 6.2. (a) The flow $\{\Phi_t^v\}$ of the conformal vector V satisfies Eq. (3.9), so the volume element of $(\Phi_t^v)^* g$ is γ_v^4 times the volume element dv_g of g. Since integrals are invariant under oriented diffeomorphisms,

$$\int_S \gamma^4 dv_g = \int_S (\Phi_t^u)^* dv_g = \int_S dv_g = \mathrm{Vol}(S) = \tfrac{8}{3}\pi^2 .$$

(b) For each non-zero vector $v \in \mathbb{R}^5$, set $\alpha = \cosh|v|$, $\beta = \sinh|v|$, and write $f_v = |v|\cos\theta$ as in the proof of Lemma A.1. Then $\gamma_v = (\alpha - \beta\cos\theta)^{-1}$ and

$$\int_S \gamma_v^4(|v|^2 - f_v^2) = |v|^2 \int_0^\pi (\alpha - \beta\cos\theta)^{-4}\,\mathrm{Vol}(S^3)\sin^5\theta\, d\theta .$$

By Lemma A.2a this is equal to $|v|^2 \cdot 2\pi^2 \cdot \frac{8}{3}A(|v|)$.

(c) Similarly, Lemma A.2b shows that

$$\int_S \gamma_v^3 = \mathrm{Vol}(S^3)\int_0^\pi (\alpha - \beta\cos\theta)^{-3}\sin^3\theta\, d\theta$$

is $4\pi^2 b^{-2}(\alpha - \beta^{-1}|v|)$, where $\alpha = a$ and $\beta = b|v|$.

(d) Again, let θ be the angle between v and $x \in S$, and let ϕ be the angle between w and the orthogonal projection of x onto the subspace $H \subset \mathbb{R}^5$ of vectors perpendicular to v. Then $w \in H$ and $f_w = \langle x, w\rangle = |w|\cos\phi\sin\theta$. Hence

$$\int_S \gamma_v^4 f_w^2 = \int_0^\pi (\alpha - \beta\cos\theta)^{-4}|w|^2\left[\int_{\Sigma_\theta} \sin^2\theta\cos^2\phi\right] d\theta ,$$

where Σ_θ is the 3-sphere of radius $\sin\theta$ in H. The inner integral is $\sin^5\theta \cdot I$, where I is the integral

$$\int_{S^3} \cos^2\phi = \int_{S^3} x^2 = 1/4\int_{S^3} x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1/4\,\mathrm{Vol}(S^3) = \pi^2/2 .$$

The result follows from the last two equations, Lemma A.2a, and the definition of $A(r)$. □

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