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THE RIGIDITY OF GRAPHS

BY

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ABSTRACT. We regard a graph G as a set $\{1, \ldots, v\}$ together with a nonempty set E of two-element subsets of $\{1, \ldots, v\}$. Let $p = (p_1, \ldots, p_v)$ be an element of $\mathbf{R}^{nv}$ representing v points in $\mathbf{R}^n$. Consider the figure $G(p)$ in $\mathbf{R}^n$ consisting of the line segments $[p_i, p_j]$ in $\mathbf{R}^n$ for $\{i, j\} \in E$. The figure $G(p)$ is said to be rigid in $\mathbf{R}^n$ if every continuous path in $\mathbf{R}^{nv}$, beginning at p and preserving the edge lengths of $G(p)$, terminates at a point $q \in \mathbf{R}^{nv}$ which is the image $(Tp_1, \ldots, Tp_v)$ of p under an isometry T of $\mathbf{R}^n$. Otherwise, $G(p)$ is flexible in $\mathbf{R}^n$. Our main result establishes a formula for determining whether $G(p)$ is rigid in $\mathbf{R}^n$ for almost all locations p of the vertices. Applications of the formula are made to complete graphs, planar graphs, convex polyhedra in $\mathbf{R}^3$, and other related matters.

1. Introduction. Consider a triangle or a square in $\mathbf{R}^2$ for which the edges are rods which are joined but rotate freely at the vertices. The square is said to be flexible in $\mathbf{R}^2$ since the square can move continuously into a family of rhombi. However, the triangle is said to be rigid in $\mathbf{R}^2$ since the three rods determine the relative positions of the three vertices. Similarly, a tetrahedron in $\mathbf{R}^3$ consisting of six rods connected but freely pivoting at the four vertices is rigid while the one-skeleton of a cube in $\mathbf{R}^3$ is flexible. A figure consisting of two triangles with a common edge is rigid in $\mathbf{R}^2$ but flexible in $\mathbf{R}^3$ since one triangle can then rotate relative to the other along the common edge.

Several kinds of physical problems, including the one just described, share the following mathematical description. Consider a finite set V of points in $\mathbf{R}^n$ together with a collection E of pairs of points in V, which is to be thought of as the set of pairs of points that are connected. A continuous time dependent transformation of the points in V is a flexing of the structure if the distances between pairs of points in E remain fixed in time but the final configuration is not congruent (in the Euclidean sense) to the original configuration. If no flexing exists, the structure is said to be rigid.

When the problem is formulated in this way, the usefulness of the language of graph theory becomes apparent. For example, a disconnected graph embedded in $\mathbf{R}^n$ is flexible in $\mathbf{R}^n$ whereas a complete graph embedded in $\mathbf{R}^n$ is rigid in $\mathbf{R}^n$. However, the rigidity or flexibility of a graph embedded in $\mathbf{R}^n$ cannot be determined simply from the abstract structure of the graph, for it is

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not difficult to find a graph which when embedded in $\mathbf{R}^n$ in one way is flexible and in another way rigid. (See Examples 1, 2, 3 of §5.)

In this paper we present criteria for determining whether a particular graph will be rigid or flexible in $\mathbf{R}^n$ for almost all locations of its vertices, where "almost all" has both a topological and a measure theoretic meaning. Furthermore, we find that a graph in $\mathbf{R}^n$ is either rigid for almost all locations of its vertices or flexible for almost all locations of its vertices.

Throughout this paper, our interest focuses on a notion of rigidity that is sometimes referred to as "continuous rigidity". The related concept of "infinitesimal rigidity" which is not discussed in this paper will be dealt with in a sequel. In the literature, the term "rigid" is used in both of these senses as well as several others. In this paper, "rigid" is always meant in the continuous sense that is discussed informally here in the introduction and defined in §2.

Much of the present paper was inspired by Herman Gluck's paper [2] on rigidity. We are also grateful to Branko Grünbaum for providing us with several interesting examples.

2. Preliminaries. For our purposes, an (abstract) graph G is to be thought of as a set $V = \{1, 2, \ldots, v\}$ together with a nonempty set E of two-element subsets of V. Each element of V is referred to as a vertex of G and each element of E is called an edge of G. On the other hand, a graph $G(p)$ in $\mathbf{R}^n$ is a graph $G = (V, E)$ together with a point $p = (p_1, \ldots, p_v) \in \mathbf{R}^n \times \cdots \times \mathbf{R}^n = \mathbf{R}^{nv}$. We refer to the points p_i for $i \in V$ as the vertices and the line segments $[p_i, p_j]$ in $\mathbf{R}^n$ for $\{i, j\} \in E$ as the edges of the graph $G(p)$ in $\mathbf{R}^n$. Note that for a graph $G(p)$ in $\mathbf{R}^n$, we have not required that $p_i \neq p_j$ for $i \neq j$ and thus it is surely inappropriate to speak of $G(p)$ as an embedding of G in $\mathbf{R}^n$.

Consider a graph $G = (V, E)$ with v vertices and e edges, that is, $V = \{1, \ldots, v\}$ and E has e elements. Order the e edges of G in some way and define f_G: $\mathbf{R}^{nv} \to \mathbf{R}^e$ by

$$f_G(t_1, \ldots, t_v) = \left(\ldots, \|t_i - t_j\|^2, \ldots \right)$$

where $\{i, j\} \in E$, $t_k \in \mathbf{R}^n$ for $1 \leqslant k \leqslant v$, and $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^n$. Note that if $G(p)$ is a graph in $\mathbf{R}^n$, then $f_G(p) \in \mathbf{R}^e$ consists of the squares of the lengths of the e edges of $G(p)$ and thus we refer to f_G as the *edge function* of the graph G. If $f_G(p) = f_G(q)$ for $p, q \in \mathbf{R}^{nv}$, then the corresponding edges of the graphs $G(p)$ and $G(q)$ in $\mathbf{R}^n$ have the same length. Consequently, the structure near p of the real algebraic variety $f_G^{-1}(f_G(p))$ is pertinent to the determination of the rigidity or flexibility of the graph $G(p)$ in $\mathbf{R}^n$.

Let K_v (or simply K) denote the complete graph with v vertices, which means that every two-element subset of $V = \{1, \ldots, v\}$ is an edge of K_v.

Note that $f_K(p) = f_K(q)$ for p, $q \in \mathbf{R}^{nv}$ if and only if the map $p_i \to q_i$, $1 \leqslant i \leqslant v$, extends to an isometry of $\mathbf{R}^n$. Recall that an isometry T of $\mathbf{R}^n$ is a map $T\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\|Tx - Ty\| = \|x - y\|$ for all $x, y \in \mathbf{R}^n$. Since the set $\mathcal{I}(n)$ of isometries of $\mathbf{R}^n$ is a smooth manifold, $f_K^{-1}(f_K(p))$ is also a smooth manifold. (See §3 for details.) If G is a graph with v vertices and K the complete graph with v vertices, then clearly $f_K^{-1}(f_K(p)) \subset f_G^{-1}(f_G(p))$ and it is the nature near p of this inclusion that determines the flexibility or rigidity of the graph $G(p)$ in $\mathbf{R}^n$.

Next, we provide definitions of rigidity and flexibility.

DEFINITION. Let G be a graph with v vertices, K the complete graph with v vertices, and $p \in \mathbf{R}^{nv}$. The graph $G(p)$ is *rigid in* $\mathbf{R}^n$ if there exists a neighborhood U of p in $\mathbf{R}^{nv}$ such that

$$f_K^{-1}(f_K(p)) \cap U = f_G^{-1}(f_G(p)) \cap U.$$

The graph $G(p)$ is *flexible in* $\mathbf{R}^n$ if there exists a continuous path $x : [0, 1] \to \mathbf{R}^{nv}$ such that $x(0) = p$ and $x(t) \in f_G^{-1}(f_G(p)) - f_K^{-1}(f_K(p))$ for all $t \in (0, 1]$.

Thus $G(p)$ is rigid in $\mathbf{R}^n$ if and only if for every q sufficiently close to p with $f_G(q) = f_G(p)$, there exists an isometry of $\mathbf{R}^n$ taking p_i to q_i for $1 \leqslant i \leqslant v$. On the other hand, $G(p)$ is flexible in $\mathbf{R}^n$ if and only if it is possible to continuously move the vertices of $G(p)$ to noncongruent positions while preserving the edge lengths of the figure.

In addition to establishing the equivalence of nonrigidity and flexibility, the following proposition demonstrates the equivalence of another reasonable notion of flexibility.

PROPOSITION 1. *Let G be a graph with v vertices, K the complete graph with v vertices, and $p \in \mathbf{R}^{nv}$. The following are equivalent*:

(a) *$G(p)$ is not rigid in* $\mathbf{R}^n$;

(b) *$G(p)$ is flexible in* $\mathbf{R}^n$;

(c) *there exists a continuous path y in $f_G^{-1}(f_G(p))$ with $y(0) = p$ and $y(t) \notin f_K^{-1}(f_K(p))$ for some $t \in (0, 1]$.*

PROOF. If $G(p)$ is not rigid in $\mathbf{R}^n$, then every neighborhood of p contains points of the algebraic variety $f_G^{-1}(f_G(p))$ not belonging to the subvariety $f_K^{-1}(f_K(p))$ and thus the existence of an analytic path x with $x(0) = p$ and $x(t) \in f_G^{-1}(f_G(p)) - f_K^{-1}(f_K(p))$, $0 < t \leqslant 1$, follows from [**5**, Lemma 18.3]. (See also [**4**].) Thus (a) implies (b). Clearly (b) implies (c).

If (c) holds, then there exists $t_0 \in [0, 1)$ such that $y(t_0)$ is the last point in $f_K^{-1}(f_K(p))$ as t increases. Let $y(t_0) = q = (q_1, \ldots, q_v)$ and $p = (p_1, \ldots, p_v)$. Then there is an isometry T of $\mathbf{R}^n$ with $Tq_i = p_i$, $1 \leqslant i \leqslant v$, and thus $(T \circ y_1, \ldots, T \circ y_v)$ maps $(t_0, 1]$ into $f_G^{-1}(f_G(p)) - f_K^{-1}(f_K(p))$. Since $(T \circ y_1, \ldots, T \circ y_v)(t_0) = p$, every neighborhood of p intersects $f_G^{-1}(f_G(p)) - f_K^{-1}(f_K(p))$ and therefore $G(p)$ is not rigid in $\mathbf{R}^n$.

For a smooth map f: $X \to Y$ where X and Y are smooth manifolds, we denote the derivative of f at $x \in X$ by $df(x)$. Let $k = \max\{\text{rank } df(x)$: $x \in X\}$. We say that $x \in X$ is a regular point of f if rank $df(x) = k$ and a singular point otherwise.

PROPOSITION 2. *Let f: $\mathbf{R}^n \to \mathbf{R}^m$ be a smooth map and $k = \max\{\text{rank } df(x) : x \in \mathbf{R}^n\}$. If $x_0 \in \mathbf{R}^n$ is a regular point of f, then the image under f of some neighborhood of x_0 is a k-dimensional manifold.*

PROOF. Let $f = (f_1, f_2)$ where f_1 consists of the first k coordinate functions of f and assume that rank $df_1(x_0) = k$. Since rank $df_1 = k$ in a neighborhood of x_0, the Inverse Function Theorem yields local coordinates at x_0 such that $f_1(x_1, x_2) = x_1$. Thus in local coordinates

$$df = \begin{bmatrix} I & 0 \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}.$$

Since rank $df = k$ near x_0, $\partial f_2 / \partial x_2 = 0$ near x_0. Hence $f_2(x_1, x_2) = g(x_1)$ and therefore $f(x_1, x_2) = (x_1, g(x_1))$ near x_0. Thus f maps some neighborhood of x_0 onto the graph of g which is a k-dimensional manifold.

It follows that if p is a regular point of f_G, then $f_G^{-1}(f_G(p))$ is a manifold of co-dimension k near p. In this case, the construction of a smooth path in (a) implies (b) of Proposition 1 is straightforward since near p we then have that $f_K^{-1}(f_K(p))$ is a proper submanifold of $f_G^{-1}(f_G(p))$.

Finally, a subset M of $\mathbf{R}^n$ is said to be an affine set if M contains the entire line through each pair of distinct points in M. The dimension of an affine set M in $\mathbf{R}^n$ is defined to be the dimension of the subspace $M - M = \{x - y : x, y \in M\}$ parallel to M and the affine hull of a set $S \subset \mathbf{R}^n$ is the smallest affine set containing S. For $p = (p_1, \ldots, p_v) \in \mathbf{R}^{nv}$, let $\dim p$ be the dimension of the affine hull of $\{p_1, \ldots, p_v\}$.

3. The main result.

THEOREM. *Let G be a graph with v vertices, e edges, and edge function f_G: $\mathbf{R}^{nv} \to \mathbf{R}^e$. Suppose that $p \in \mathbf{R}^{nv}$ is a regular point of f_G and let $m = \dim p$. Then the graph $G(p)$ is rigid in $\mathbf{R}^n$ if and only if*

$$\text{rank } df_G(p) = nv - (m + 1)(2n - m)/2,$$

and $G(p)$ is flexible in $\mathbf{R}^n$ if and only if

$$\text{rank } df_G(p) < nv - (m + 1)(2n - m)/2.$$

PROOF. Let $k = \max\{\text{rank } df_G(x) : x \in \mathbf{R}^{nv}\}$. Then $k = \text{rank } df_G(p)$. By Proposition 2 of §2, there exists a neighborhood V of p in $\mathbf{R}^{nv}$ such that $f_G^{-1}(f_G(p)) \cap V$ is an $(nv - k)$-dimensional manifold.

Let $\mathcal{I}(n)$ be the $n(n+1)/2$-dimensional manifold of isometries of $\mathbf{R}^n$ and define $F : \mathcal{I}(n) \to \mathbf{R}^{nv}$ by $F(T) = (Tp_1, \ldots, Tp_v)$ for $T \in \mathcal{I}(n)$. Note that F is smooth and $\operatorname{im}(F) = f_K^{-1}(f_K(p))$ where K is the complete graph with v vertices. Let M be the affine hull of $\{p_1, \ldots, p_v\}$. Thus $\dim M = \dim p = m$. Clearly $F^{-1}(p)$ is the subgroup of $\mathcal{I}(n)$ consisting of isometries which equal the identity on M and $F^{-1}(p)$ can be identified with the $(n-m)(n-m-1)/2$-dimensional manifold $\mathcal{O}(N)$ of orthogonal linear transformations of N where N is the $(n-m)$-dimensional subspace orthogonal to the m-dimensional subspace $M - M$. Let

$$\pi: \mathcal{I}(n) \to \mathcal{I}(n)/F^{-1}(p)$$

be the natural projection and define

$$\bar{F} : \mathcal{I}(n)/F^{-1}(p) \to \mathbf{R}^{nv}$$

so that $F = \bar{F} \circ \pi$. Then $\bar{F}$ is smooth since F is smooth and, moreover, $\bar{F}$: $\mathcal{I}(n)/F^{-1}(p) \to \operatorname{im}(\bar{F})$ is a diffeomorphism. Since $\mathcal{I}(n)/F^{-1}(p)$ is a manifold of dimension

$$n(n+1)/2 - (n-m)(n-m-1)/2 = (m+1)(2n-m)/2,$$

we conclude that $\operatorname{im}(\bar{F}) = \operatorname{im}(F) = f_K^{-1}(f_K(p))$ is an $(m+1)(2n-m)/2$-dimensional manifold.

The $(m+1)(2n-m)/2$-dimensional manifold $f_K^{-1}(f_K(p)) \cap V$ is a submanifold of the $(nv-k)$-dimensional manifold $f_G^{-1}(f_G(p)) \cap V$. Therefore we have

$$k \leqslant nv - (m+1)(2n-m)/2.$$

Clearly $k = nv - (m+1)(2n-m)/2$ if and only if there exists a neighborhood U of p in $\mathbf{R}^{nv}$ such that $f_K^{-1}(f_K(p)) \cap U = f_G^{-1}(f_G(p)) \cap U$, that is, if and only if $G(p)$ is rigid in $\mathbf{R}^n$. Since $k \leqslant nv - (m+1)(2n-m)/2$, $G(p)$ is flexible in $\mathbf{R}^n$ if and only if $k < nv - (m+1)(2n-m)/2$, which concludes the proof.

Let $P(x)$ be the sum of the squares of the determinants of all the $k \times k$ submatrices of $df_G(x)$ for $x \in \mathbf{R}^{nv}$. Then P is a nontrivial polynomial in nv variables and thus the set R of regular points of f_G is a dense open subset of $\mathbf{R}^{nv}$ since $R = \{x \in \mathbf{R}^{nv} : P(x) \neq 0\}$. Moreover, Fubini's Theorem enables one to conclude that the set $\mathbf{R}^{nv} - R$ of singular points of f_G has Lebesgue measure zero in $\mathbf{R}^{nv}$. Therefore the above theorem determines the rigidity or flexibility in $\mathbf{R}^n$ of a graph $G(p)$ for almost all $p \in \mathbf{R}^{nv}$, where "almost all" can be interpreted both topologically and measure theoretically. Furthermore, in §4 we show that this determination is constant for regular points of f_G, that is, either $G(p)$ is rigid in $\mathbf{R}^n$ for all $p \in R$ or $G(p)$ is flexible for all $p \in R$.

4. Corollaries. In the following lemma and our first two corollaries, the dimension n of the Euclidean space in which we consider a graph varies, so we temporarily denote the edge function of a graph G by $f_{G,n}$. Thus $f_{G,n}$: $\mathbf{R}^{nv} \to \mathbf{R}^e$ and

$$f_{G,n}(t_1, \ldots, t_v) = (\ldots, \|t_i - t_j\|^2, \ldots)$$

where $\{i, j\}$ is an edge of G and each $t_k \in \mathbf{R}^n$.

LEMMA. *Let G be a graph with v vertices. Suppose $p \in \mathbf{R}^{nv}$ is a regular point of $f_{G,n}$ and let $m = \dim p$. Then there exists $q \in \mathbf{R}^{mv}$ such that q is a regular point of $f_{G,m}$, $\dim q = m$, and $\operatorname{rank} df_{G,m}(q) = \operatorname{rank} df_{G,n}(p)$. Moreover, if $G(p)$ is rigid in $\mathbf{R}^n$, then $G(q)$ is rigid in $\mathbf{R}^m$.*

PROOF. Define C: $\mathbf{R}^m \to \mathbf{R}^n$ by $C(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$. There exists an isometry T of $\mathbf{R}^n$ taking the m-dimensional subspace $\operatorname{im}(C)$ onto the affine hull M of $\{p_1, \ldots, p_v\}$. Then $T \circ C$ maps $\mathbf{R}^m$ onto M and it is not difficult to show that for all $(t_1, \ldots, t_v) \in \mathbf{R}^{mv}$,

$$\operatorname{rank} df_{G,m}(t_1, \ldots, t_v) = \operatorname{rank} df_{G,n}(TCt_1, \ldots, TCt_v).$$

Choose $q = (q_1, \ldots, q_v) \in \mathbf{R}^{mv}$ such that $TCq_i = p_i$, $1 \leqslant i \leqslant v$. Clearly $\dim q = \dim p$ and $\max\{\operatorname{rank} df_{G,m}\} \leqslant \max\{\operatorname{rank} df_{G,n}\} = \operatorname{rank} df_{G,n}(p) = \operatorname{rank} df_{G,m}(q) \leqslant \max\{\operatorname{rank} df_{G,m}\}$. Therefore q is a regular point of $f_{G,m}$.

If $G(q)$ is flexible in $\mathbf{R}^m$, then the flexing of $G(q)$ guaranteed by the definition together with $T \circ C$ provides a flexing of $G(p)$ in $\mathbf{R}^n$.

Consider a graph G with v vertices. Then $\mathbf{R}^{nv}$ is partitioned in three ways. There is the set where $G(p)$ is rigid in $\mathbf{R}^n$ and the set where $G(p)$ is flexible in $\mathbf{R}^n$; there are the sets of regular and singular points of the edge function f_G; and there are the sets where $\dim p = \min\{v - 1, n\}$ and $\dim p < \min\{v - 1, n\}$. How are these three partitions related? Our first corollary establishes that it is never the case that $G(p)$ is rigid in $\mathbf{R}^n$ where p is a regular point of f_G with $\dim p < \min\{v - 1, n\}$. It is not difficult to find graphs G and points $p \in \mathbf{R}^{2v}$ showing the seven remaining possibilities can all occur.

COROLLARY 1. *Let G be a graph with v vertices. If $G(p)$ is rigid in $\mathbf{R}^n$ where $p \in \mathbf{R}^{nv}$ is a regular point of $f_{G,n}$, then $\dim p = \min\{v - 1, n\}$.*

PROOF. Let $m = \dim p$. By the lemma, there exists $q \in \mathbf{R}^{mv}$ with q a regular point of $f_{G,m}$, $\dim q = m$, $\operatorname{rank} df_{G,m}(q) = \operatorname{rank} df_{G,n}(p)$, and $G(q)$ rigid in $\mathbf{R}^m$. Applying the theorem of §3 to $G(p)$ in $\mathbf{R}^n$ and $G(q)$ in $\mathbf{R}^m$, we obtain

$$\begin{aligned} mv - (m + 1)(2m - m)/2 &= \operatorname{rank} df_{G,m}(q) = \operatorname{rank} df_{G,n}(p) \\ &= nv - (m + 1)(2n - m)/2. \end{aligned}$$

Since the function $g(x) = vx - (m + 1)(2x - m)/2$ is affine and $g(m) = g(n)$, either $m = n$ or the coefficient $v - (m + 1)$ of x in g equals zero. Therefore $m = \min\{v - 1, n\}$.

In connection with Corollary 1, we note that if $\dim p = v - 1$, then for each j the set $\{p_i - p_j : i \neq j\}$ is linearly independent. It follows that p is a regular point since rank $df_G(p) = e$.

Let G be the graph consisting of two triangles sharing a common edge and let $f_G : \mathbf{R}^{12} \to \mathbf{R}^5$ be the edge function of G in $\mathbf{R}^3$. Then it is a simple matter to find regular points $p,q \in \mathbf{R}^{12}$ of f_G where $\dim p = 3$ and $\dim q = 2$. In this case, $G(p)$ and $G(q)$ are both flexible in $\mathbf{R}^3$ and our next corollary states that regular points of different dimensions cannot occur if one of the graphs is rigid.

COROLLARY 2. *Let G be a graph with v vertices and edge function $f_{G,n}$: $\mathbf{R}^{nv} \to \mathbf{R}^e$. If $p, q \in \mathbf{R}^{nv}$ are regular points of $f_{G,n}$ and $G(p)$ is rigid in $\mathbf{R}^n$, then $G(q)$ is also rigid in $\mathbf{R}^n$ and* $\dim p = \dim q$.

PROOF. Let $m = \dim p$ and $l = \dim q$. By Corollary 1, $m = \min\{v - 1, n\}$ and, as always, $l \leq \min\{v - 1, n\}$. Since p and q are both regular points of $f_{G,n}$, the lemma gives $p' \in \mathbf{R}^{mv}$ and $q' \in \mathbf{R}^{lv}$ with p' a regular point of $f_{G,m}$, q' a regular point of $f_{G,l}$, $\dim p' = m$, $\dim q' = l$, and rank $df_{G,m}(p') =$ rank $df_{G,n}(p) =$ rank $df_{G,n}(q) =$ rank $df_{G,l}(q')$. By the theorem of §3, rank $df_{G,l}(q') \leq lv - l(l + 1)/2$. By the lemma, $G(p')$ is rigid in $\mathbf{R}^m$ and thus rank $df_{G,m}(p') = mv - m(m + 1)/2$ by the theorem. But the function $g(x) = vx - x(x + 1)/2$ is strictly increasing for $x \leq v - 1$ and thus $l = m$ since $g(l) = g(m)$. Therefore $\dim p = \dim q$ which implies that $G(q)$ is also rigid in $\mathbf{R}^n$.

Since Corollary 2 guarantees that $G(p)$ is rigid in $\mathbf{R}^n$ for all regular points p whenever it is rigid for a single regular point, we can conclude that either $G(p)$ is rigid in $\mathbf{R}^n$ for all regular points or $G(p)$ is flexible in $\mathbf{R}^n$ for all regular points.

Our next corollary makes precise the appealing idea that a graph with too few edges can almost never be rigid. (See §5 for examples of rigid graphs having too few edges.)

COROLLARY 3. *Let G be a graph with v vertices and e edges. If $e < nv - n(n + 1)/2$, then $G(p)$ is flexible in $\mathbf{R}^n$ for all regular points p of f_G.*

PROOF. Let $p \in \mathbf{R}^{nv}$ be a regular point of f_G and $m = \dim p$. Then

$$\operatorname{rank} df_G(p) \leq e < nv - n(n + 1)/2 \leq nv - (m + 1)(2n - m)/2$$

and thus $G(p)$ is flexible in $\mathbf{R}^n$ by the theorem of §3.

Corollary 3 represents a first step in the direction of a purely combinatorial method for determining whether a graph is almost always rigid or flexible in $\mathbf{R}^n$. A graph theoretic characterization in the case $n = 2$ is given by Laman [3, Theorem 6.5].

Let K be a complete graph. By the definition of rigidity, it is clear that $K(p)$ is rigid in $\mathbf{R}^n$ for all n and all $p \in \mathbf{R}^{nv}$. One consequence of our next corollary is that an incomplete graph is almost always flexible in $\mathbf{R}^n$ for all $n \geqslant v - 1$.

COROLLARY 4. *Let G be a graph with v vertices. The following are equivalent*:
(a) *G is a complete graph*;
(b) *for all n, $G(p)$ is rigid in $\mathbf{R}^n$ for all $p \in \mathbf{R}^{nv}$*;
(c) *for some $n \geqslant v - 1$, $G(p)$ is rigid in $\mathbf{R}^n$ for all regular points p.*

PROOF. The fact that (a) implies (b) is a consequence of the definition of rigidity and (b) obviously implies (c). Suppose there exists $n \geqslant v - 1$ such that $G(p)$ is rigid in $\mathbf{R}^n$ where $p \in \mathbf{R}^{nv}$ is a regular point. By Corollary 1, $\dim p = v - 1$ and thus rank $df_G(p) = v(v-1)/2$ by the theorem of §3. But $e \leqslant v(v-1)/2 = \text{rank } df_G(p) \leqslant e$ and therefore $e = v(v-1)/2$, that is, G is a complete graph.

Our last two corollaries concern planar graphs, which are graphs that can be embedded in the plane $\mathbf{R}^2$, that is, drawn in the plane in such a way that edges intersect only at the appropriate vertices. For a connected planar graph G, Euler's formula implies that the number f of faces of G is well-defined since $v - e + f = 2$ for any embedding of G in the plane. For a connected planar graph, we define the average number A of edges per face by $A = 2e/f$. A version of our next corollary was first suggested and proved by Sherman Stein.

COROLLARY 5. *Let G be a planar graph such that $G(p)$ is rigid in $\mathbf{R}^2$ for all regular points p of f_G. Then the average number A of edges per face of G is less than four and if G has more than two vertices, then G contains a triangle.*

PROOF. Since $G(p)$ is rigid in $\mathbf{R}^2$ for all regular points $p \in \mathbf{R}^{2v}$, G is connected and Corollary 3 implies that $e \geqslant 2v - 3$. Therefore

$$A = 2e/f = 2e/(2 - v + e) \leqslant 4e/(e+1) < 4.$$

Now suppose $v \geqslant 3$ and consider any embedding of the connected planar graph G in $\mathbf{R}^2$. Every face of the embedded graph is bounded by at least three edges, where an edge is counted twice for a face if the face lies on both sides of the edge. If no face has exactly three boundary edges, then $2e \geqslant 4f$ which contradicts $A = 2e/f < 4$. Therefore some face of the embedded graph has exactly three boundary edges which implies that G contains a subgraph isomorphic to the graph K_3, that is, G contains a triangle.

It can be shown that four is the best possible bound in Corollary 5.

If G is a connected planar graph with $v \geqslant 3$, then $3f \leqslant 2e$ and $3f = 2e$ if and only if every face (including the unbounded one) of some (or equivalently every) embedding of G in the plane has exactly three boundary edges, where again an edge is counted twice for a face if the edge lies inside the face. Thus we refer to connected planar graphs with $3f = 2e$ as *triangular*.

We say that a graph is *polyhedral* if the graph can be embedded in $\mathbf{R}^3$ in such a way that its vertices and edges are the vertices and edges of a convex polyhedron in $\mathbf{R}^3$. Thus G is polyhedral if and only if there exists $p = (p_1, \ldots, p_v) \in \mathbf{R}^{3v}$ such that $p_i \neq p_j$ for $i \neq j$ and the edges $[p_i, p_j]$ of $G(p)$ are the edges of a convex polyhedron in $\mathbf{R}^3$. Clearly every polyhedral graph is 3-connected and planar.

If G is polyhedral and $p \in \mathbf{R}^{3v}$ is chosen so that the vertices and edges of $G(p)$ are the vertices and edges of a convex polyhedron in $\mathbf{R}^3$, then p is a regular point of f_G since rank $df_G(p) = e$. This fact stems from some approaches to Cauchy's Theorem on the rigidity of polyhedra in $\mathbf{R}^3$. (For example, see [**2**, Lemma 5.2, Lemma 5.3, and the proof of Theorem 5.1] where we interpret "strictly convex" to mean edges as well as vertices are extremal.) The following corollary is essentially a reformulation of results of Gluck [**2**] obtained in a somewhat different setting.

COROLLARY 6. *Let G be a connected planar graph with $v \geqslant 4$. The following are equivalent*:

(a) *$G(p)$ is rigid in $\mathbf{R}^3$ for all regular points $p \in \mathbf{R}^{3v}$*;

(b) *G is triangular*;

(c) *G is triangular and polyhedral.*

PROOF. If (a) holds, then $e \geqslant 3v - 6$ by Corollary 3. Thus

$$e \geqslant 3v - 6 = 3(v - 2) = 3(e - f) = e + (2e - 3f) \geqslant e$$

and therefore $2e = 3f$ which means that G is triangular.

The fact that (b) implies (c) follows from Steinitz's Theorem [**1**], which provides necessary and sufficient conditions for a collection of vertices, edges, and faces to be realized as the vertices, edges, and faces of a convex polyhedron. It is here that we need $v \geqslant 4$.

Suppose (c) holds. Then there exists $q \in \mathbf{R}^{3v}$ such that the vertices and edges of $G(q)$ are the vertices and edges of a convex polyhedron in $\mathbf{R}^3$ and $2e = 3f$. Thus rank $df_G(q) = e = 3(e - f) = 3(v - 2) = 3v - 6$ which implies that $G(q)$ is rigid in $\mathbf{R}^3$ by the theorem of §3. Therefore $G(p)$ is rigid in $\mathbf{R}^3$ for all regular points p.

In particular, the one-skeleton of a convex polyhedron in $\mathbf{R}^3$ is rigid in $\mathbf{R}^3$ if and only if every face of the polyhedron is a triangle. Moreover, the use of the result on rank arising from Cauchy's Theorem in conjunction with Corollary 3 shows that for a convex polyhedron in $\mathbf{R}^3$ with all triangular faces the removal of any edge from its one-skeleton leads to flexibility.

Another condition equivalent to (a), (b), and (c) in Corollary 6 is "G is triangular and 3-connected". Therefore if G is a connected planar graph with $v \geqslant 4$ such that $G(p)$ is almost always rigid in $\mathbf{R}^3$, then G is 3-connected. More generally, the referee has observed that if G is a graph with $v \geqslant n + 1$

such that $G(p)$ is rigid in $\mathbf{R}^n$ for all regular points p, then G is n-connected.

5. Examples. It is quite easy to induce pathological behavior by allowing some of the vertices of a graph $G(p)$ in $\mathbf{R}^n$ to coincide; we have chosen examples in which this does not occur.

EXAMPLE 1. Let G be the graph shown in Figure 1 for which $v = 6$ and $e = 8$. Since $e < 2v - 3$, Corollary 3 implies that $G(p)$ is flexible in $\mathbf{R}^2$ for all regular points $p \in \mathbf{R}^{12}$. However, if the location q of the vertices is as shown in Figure 1, then a simple geometrical argument shows that $G(q)$ is rigid in $\mathbf{R}^2$. By perturbing the four collinear vertices of $G(q)$, one obtains graphs isomorphic to $G(q)$ but flexible in $\mathbf{R}^2$. We note also that the average number of edges per face of the rigid graph $G(q)$ equals four.

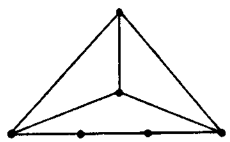

FIGURE 1

Our next example is almost always rigid in $\mathbf{R}^2$ and only occasionally flexible, in contrast to Example 1.

EXAMPLE 2. Let G be the graph shown in Figure 2 for which $v = 6$ and $e = 9$. For the location q of the vertices shown in Figure 2, a simple geometrical argument shows that $G(q)$ is flexible in $\mathbf{R}^2$. However, it is not difficult to find $p \in \mathbf{R}^{12}$ with rank $df_G(p) = 9$ and $\dim p = 2$. Therefore $G(p)$ is rigid in $\mathbf{R}^2$ by the theorem of §3 and thus rigid at all regular points by Corollary 2.

FIGURE 2

EXAMPLE 3. Let G be the graph shown in Figure 3 which consists of a tetrahedron with a triangle inside its base. Since G is planar but not triangular, $G(p)$ is flexible in $\mathbf{R}^3$ for all regular points $p \in \mathbf{R}^{21}$ by Corollary 6. However, if the location q of the vertices is as shown in Figure 3 with the triangle lying in the plane of the base of the tetrahedron, then a simple geometrical argument shows that $G(q)$ is rigid in $\mathbf{R}^3$ even though $G(q)$ has three nontriangular faces.

One consequence of Cauchy's Theorem is that if $G(p)$ forms the one-skeleton of a convex polyhedron in $\mathbf{R}^3$, then rank $df_G(p) = e$. Our last

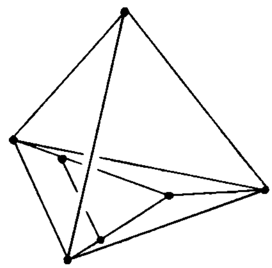

FIGURE 3

example shows that this fact does not generalize to higher dimensional polytopes.

EXAMPLE 4. Let C be a convex polytope in $\mathbf{R}^n$ with nonempty interior. We construct the convex polytope $r(C)$ in $\mathbf{R}^{n+1}$ as follows. Embed C in a hyperplane H in $\mathbf{R}^{n+1}$ and choose new vertices $x, y \in \mathbf{R}^{n+1} - H$ such that the line segment $[x, y]$ intersects the relative interior of C. Then $r(C)$ is the convex hull of C with $[x, y]$ and it can be verified that the vertices of $r(C)$ are x and y together with the vertices of C. Also the edges of $r(C)$ are the edges of C together with all $[x, z]$ and $[y, z]$ where z is a vertex of C. Thus, if v' and e' are the number of vertices and edges of $r(C)$, we have $v' = v + 2$ and $e' = e + 2v$ where v and e are the number of vertices and edges of C.

Now let C be a convex polygon in $\mathbf{R}^2$ with v vertices and v edges. Then for $r(C)$ in $\mathbf{R}^3$, we have $v' = v + 2$ and $e' = 3v$, while for $r^2(C)$ in $\mathbf{R}^4$, we have $v'' = v + 4$ and $e'' = 5v + 4$. Suppose G is the graph of vertices and edges of $r^2(C)$. Then for $v \geqslant 3$,

$$\max\{\operatorname{rank} df_G\} \leqslant 4v'' - 10 = 4v + 6 < 5v + 4 = e''.$$

Finally, we note that for $v = 3$, if C, $r(C)$ and $r^2(C)$ are suitably chosen and $G(p)$ forms the one-skeleton of $r^2(C)$, then rank $df_G(p) = 18$. Therefore p is a regular point of f_G and $G(p)$ is rigid in $\mathbf{R}^4$.

REFERENCES

1. D. Barnette and B. Grünbaum, *On Steinitz's theorem concerning convex 3-polytopes and on some properties of planar graphs*, The Many Facets of Graph Theory, Lecture Notes in Math., vol. 110, Springer-Verlag, Berlin, 1969, pp. 27–40.

2. H. Gluck, *Almost all simply connected closed surfaces are rigid*, Geometric Topology, Lecture Notes in Math., no. 438, Springer-Verlag, Berlin, 1975, pp. 225–239.

3. G. Laman, *On graphs and rigidity of plane skeletal structures*, J. Engrg. Math. **4** (1970), 331–340.

4. J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies, no. 61, Princeton Univ. Press, Princeton, N.J., 1968.

5. A. Wallace, *Algebraic approximation of curves*, Canad. J. Math. **10** (1958), 242–278.

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