

THE RING OF POLAR PRESERVING ENDOMORPHISMS OF AN ABELIAN LATTICE-ORDERED GROUP

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1. Introduction

Let G be an abelian lattice-ordered group (l -group). We investigate the ring $P(G)$ generated by the semiring $P^+(G)$ of all group endomorphisms α of G such that for $x, y \in G$

$$x \wedge y = 0 \text{ implies } x \wedge y\alpha = 0.$$

$P(G)$ is a po subring of the ring $B(G)$ of all order-bounded endomorphisms of G with $P^+(G)$ as its positive cone. If A is any ring of l -endomorphisms of G that contains the identity automorphism I , then $A \subseteq P(G)$. Thus $P(G)$ is the largest such ring. We show (Theorem 3.4) that the class

$$\{P(G) : G \text{ is an archimedean } l\text{-group}\}$$

is identical with the class of archimedean f -rings with identity. This allows us to derive many useful properties of $P^+(G)$.

For an archimedean l -group G , the largest f -ring of $B(G)$ that contains the identity is $P(G)$. Let G be an archimedean l -group with a weak order unit e . Then there is at most one multiplication on G so that G is an f -ring with identity e , and such a multiplication exists if and only if $\{e\alpha : \alpha \in P^+(G)\} = G^+$.

The elements in $P^+(G)$ preserve minimal prime subgroups. In Section 6 we investigate those group endomorphisms of G which preserve all the prime subgroups. In Section 7 we apply our theory to solve a problem posed by G. Birkhoff.

Notation and terminology. If G is an l -group, then we denote its positive cone by $G^+ = \{g \in G : g \geq 0\}$. An l -subgroup of G is a subgroup K which is also a sublattice. If, in addition, $0 < x < k \in K$ implies $x \in K$, then we say that K is a *convex l -subgroup*. An l -ideal is a normal convex l -subgroup. A *prime subgroup* is a convex l -subgroup M such that $x \wedge y \in M$ implies $x \in M$ or $y \in M$. Various other characterizations of prime subgroups are given in [4] and [9]. An l -endomorphism of G is a group endomorphism that also preserves the lattice operations. Thus an endomorphism α of G is an l -endomorphism if and only if $x \wedge y = 0$ implies $x\alpha \wedge y\alpha = 0$ [7].

If X is a subset of G , then

$$X' = \{g \in G : |g| \wedge |x| = 0 \text{ for all } x \in X\}$$

is called the *polar* of X . X' is a convex l -subgroup of G and the set $p(G)$ of

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all polars of G forms a complete Boolean algebra [11]. Note that $P^+(G)$ consists of those group endomorphisms of G that preserve each polar.

2. Polar preserving endomorphisms of a lattice-ordered group

Let G be a lattice-ordered group (l -group). Then a *polar preserving endomorphism* or *p-endomorphism* of G is a group endomorphism α of G such that for all $x, y \in G$

$$x \wedge y = 0 \text{ implies } x \wedge y\alpha = 0.$$

Note that α is an l -endomorphism and hence $G^+\alpha \subseteq G^+$. For if $x \wedge y = 0$, then $x \wedge y\alpha = 0$ and hence $x\alpha \wedge y\alpha = 0$.

In the first part of this section we give several characterizations of p -endomorphisms. Then we show that the semiring of all p -endomorphisms of an abelian l -group G is the unique maximal subsemiring of the ring of group endomorphisms of G which contains the identity automorphism I and which consists of lattice endomorphisms. Finally we investigate this semiring in the case where G is a subdirect sum of subgroups of the reals.

2.1 PROPOSITION. *Let α be a group endomorphism of the l -group G . Then the following are equivalent.*

- (a) α is a p -endomorphism.
- (b) $G^+\alpha \subseteq G^+$ and $M\alpha \subseteq M$ for each minimal prime subgroup M of G .
- (c) $G^+\alpha \subseteq G^+$ and $P\alpha \subseteq P$ for each polar P of G .
- (d) $G^+\alpha \subseteq G^+$ and $x' \subseteq (x\alpha)'$ for each $x \in G^+$.

Moreover if G is abelian, then (a) is equivalent to

- (e) $G^+\alpha \subseteq G^+$ and $\alpha + I$ is an l -endomorphism.

Proof. (a) implies (b). If $0 < x \in M$, then there is a $y \notin M$ such that $x \wedge y = 0$ (see [6] or [9]). But then $x\alpha \wedge y = 0$, so that $x\alpha \in M$ since $y \notin M$ and M is prime. It follows that $M\alpha \subseteq M$.

(b) implies (c). This follows from the fact that a polar of G is the intersection of minimal prime subgroups of G [4].

(c) implies (d). Let $y \in x'$ be positive. Then $x \wedge y = 0$ and so $x \in y'$. Thus $x\alpha \in y'\alpha \subseteq y'$. But $x\alpha \geq 0$, so that $x\alpha \wedge y = 0$. Hence $y \in (x\alpha)'$, and it follows that $x' \subseteq (x\alpha)'$.

(d) implies (a). If $x \wedge y = 0$, then $y \in x' \subseteq (x\alpha)'$ and $x\alpha \in G^+$. Thus $x\alpha \wedge y = 0$ and hence α is a p -endomorphism.

Finally suppose that G is abelian and let α be a p -endomorphism of G . Then $x \wedge y = 0$ implies $x\alpha \wedge y = 0$ and hence $(x\alpha + x) \wedge y = 0$. Therefore $\alpha + I$ is a p -endomorphism and hence an l -endomorphism. Conversely if $\alpha + I$ is an l -endomorphism and $G^+\alpha \subseteq G^+$ and $x \wedge y = 0$, then

$$0 = x(\alpha + I) \wedge y(\alpha + I) = (x\alpha + x) \wedge (y\alpha + y) \geq x\alpha \wedge y \geq 0.$$

Thus $0 = x\alpha \wedge y$ and so α is a p -endomorphism.

We now characterize p -endomorphisms in the case where G is a subdirect

sum of totally ordered groups (*o*-groups). In particular, this characterization holds for abelian *l*-groups. Recall that an *l*-group is *representable* if there is an *l*-isomorphism σ of G into a cardinal sum πG_λ of a family $\{G_\lambda : \lambda \in \Gamma\}$ of *o*-groups. In general, the intersection of all minimal prime subgroups of G is zero, and Byrd has shown that G is representable if and only if each minimal prime subgroup is normal [3].

2.2 PROPOSITION. *Let G be a representable *l*-group and let α be a group endomorphism of G such that $G^+\alpha \subseteq G^+$. Then the following are equivalent.*

- (a) α is a *p*-endomorphism.
- (b) There is a set \mathfrak{N} of normal prime subgroups of G such that $\cap \mathfrak{N} = 0$ and $N\alpha \subseteq N$ for each $N \in \mathfrak{N}$.

Proof. (a) implies (b). Let \mathfrak{N} be the set of all minimal prime subgroups of G .

(b) implies (a). Since $N\alpha \subseteq N$, α induces an *o*-endomorphism on the *o*-group G/N and so it induces an *o*-endomorphism $\bar{\alpha}$ of $\pi\{G/N : N \in \mathfrak{N}\}$. Let σ be the natural map of G into $\pi G/N$:

$$g\sigma = (\dots, g + N, \dots).$$

Then it is clear that $\sigma\bar{\alpha} = \alpha\sigma$ and for $x \in \pi G/N$ we have that the support of $x\bar{\alpha}$ is contained in the support of x (since $N\alpha \subseteq N$ for each $N \in \mathfrak{N}$). Thus $\bar{\alpha}$ is a *p*-endomorphism of $\pi G/N$ and so α is a *p*-endomorphism of G .

Let G be an abelian *l*-group and let $E(G)$ denote the endomorphism ring of G . We make $E(G)$ into a po ring by setting

$$E(G)^+ = \{\alpha \in E(G) : G^+\alpha \subseteq G^+\}.$$

Elements of $E(G)^+$ are called *o*-endomorphisms of G . In general $E(G)$ is not directed under this partial order and so we define $B(G) = E(G)^+ - E(G)^+$. Then $B(G)$, the ring of order bounded endomorphisms of G , is a po ring with positive cone $B(G)^+ = E(G)^+$.

2.3. *If G is an abelian *l*-group, then the set $P^+(G)$ of all *p*-endomorphisms of G is a subsemiring of $E(G)^+$ and*

$$P(G) = \{\alpha - \beta : \alpha, \beta \in P^+(G)\}$$

is a po subring of $B(G)$ with positive cone

$$P(G) \cap E(G)^+ = P^+(G).$$

Proof. Consider $\alpha, \beta \in P^+(G)$ and suppose that $x \wedge y = 0$. Then $x\alpha \wedge y = 0$ and hence $x\alpha\beta \wedge y = 0$, so that $\alpha\beta \in P^+(G)$. Also $x\beta \wedge y = 0$ and hence $(x\alpha + x\beta) \wedge y = 0$ so that $\alpha + \beta \in P^+(G)$. Thus $P^+(G)$ is a subsemiring of $E(G)^+$ and so $P(G)$ is a po subring of $B(G)$.

Clearly $P(G) \cap E(G)^+ \supseteq P^+(G)$. Consider $\alpha - \beta \in P(G) \cap E(G)^+$ and

$x \in G^+$. Then $x\alpha \geq x\alpha - x\beta = x(\alpha - \beta) \geq 0$ and so if $x \wedge y = 0$, then

$$0 = x\alpha \wedge y \geq x(\alpha - \beta) \wedge y \geq 0.$$

Therefore $\alpha - \beta \in P^+(G)$.

Now let S be a subsemiring of $E(G)^+$ and let $\tilde{S} = \{\alpha - \beta : \alpha, \beta \in S\}$ be the subring of $B(G)$ that is generated by S . If each element of S is an l -endomorphism, we say that \tilde{S} is a ring of l -endomorphisms of G .

2.4 THEOREM. *Let \tilde{S} be a ring of l -endomorphisms of an abelian l -group G which satisfies (*) for each $0 < g \in G$ there is a $\beta \in \tilde{S}$ such that $g\beta \geq g$. Then $S \subseteq P^+(G)$ and hence $\tilde{S} \subseteq P(G)$.*

Proof. Let $\alpha \in S$ and let $x, y \in G$ be such that $x \wedge y = 0$. Now choose $\beta \in \tilde{S}$ such that $x\beta \geq x$ and write $\beta = \beta_1 - \beta_2$ where $\beta_1, \beta_2 \in S$. Then $\beta + \beta_2 = \beta_1 \in S$ so that $\beta + \beta_2 + \alpha \in S$. But then

$$\begin{aligned} 0 &= x(\beta + \beta_2 + \alpha) \wedge y(\beta + \beta_2 + \alpha) \\ &= (x\beta + x\beta_2 + x\alpha) \wedge (y\beta + y\beta_2 + y\alpha) \\ &\geq x\beta \wedge y\alpha \geq x \wedge y\alpha \geq 0. \end{aligned}$$

Thus $x \wedge y\alpha = 0$, so that $\alpha \in P^+(G)$. The result now follows.

2.5 COROLLARY. *If \tilde{S} is a ring of l -endomorphisms of an abelian l -group G , which contains I , $S \subseteq P^+(G)$.*

Remarks. (i) The proof of 2.4 actually shows that if S is an additive subsemigroup of l -endomorphisms of G , where G is an abelian l -group, and if $I \in S$, then $S \subseteq P^+(G)$.

The examples referred to in the next three remarks can be found in §8.

(ii) A ring of l -endomorphisms of an abelian l -group G need not be contained in $P(G)$ (Example 1).

(iii) If \tilde{S} is a ring of l -endomorphisms of an abelian l -group G and if $I \in \tilde{S}$, then I need not belong to S (Example 4).

(iv) If \tilde{S} is a ring of l -endomorphisms of an abelian l -group G and if $I \in \tilde{S}$, then $(\tilde{S})^+$ need not be contained in S (Example 4).

We now identify the ring $P(G)$ in case G is a subdirect sum of subgroups of the reals. First recall that if G is an o -group, then the set of convex subgroups of G is totally ordered. Moreover, if G has a largest convex subgroup M , then M is normal and G/M is o -isomorphic to a subgroup of the reals \mathbb{R} [7]. Now let π be an o -endomorphism of G , and let $x \in M$ with $x > 0$ be such that $x\pi \notin M$. Then since G is an o -group and $x\pi \notin M$, we have that $x\pi \geq nx$ for all $n \geq 0$. Thus $x\pi^2 \geq nx\pi$ for all $n \geq 0$, so that $x\pi^2 + M \geq n(x\pi + M)$ for all $n \geq 0$. This contradicts the fact that G/M is archimedean and hence $x\pi \in M$, so that $M\pi \subseteq M$. Thus we have shown the following.

2.6. Let G be an o -group with a largest convex subgroup M . Then for each o -endomorphism π of G , $M\pi \subseteq M$.

2.7. LEMMA. Let G be a representable l -group, let $\alpha \in P^+(G)$, and suppose that M is a maximal convex l -subgroup of G . Then $M\alpha \subseteq M$.

Proof. Since M is a maximal convex l -subgroup of G , M is prime. Let P be a minimal prime convex l -subgroup contained in M [4]. Then G/P is an o -group and M/P is the largest convex subgroup of G/P . Now by 2.1, $P\alpha \subseteq P$ so that α induces an o -endomorphism $\bar{\alpha}$ defined by $(x + P)\bar{\alpha} = x\alpha + P$ of G/P . But by 2.6, $(M/P)\bar{\alpha} \subseteq M/P$, so that $M\alpha \subseteq M$.

Remark. Recall (2.1) that p -endomorphisms leave invariant all minimal prime subgroups. Moreover, by 2.7, p -endomorphisms leave invariant maximal convex l -subgroups in representable l -groups. However, even in an archimedean l -group a prime subgroup need not be left invariant by a p -endomorphism (See Example 9 of §8). In Section 6 we examine those o -endomorphisms which leave fixed all the prime subgroups of G .

Now suppose that G is an l -group which is a subdirect sum $\pi_S\{R_\lambda : \lambda \in \Gamma\}$ of a family $\{R_\lambda : \lambda \in \Gamma\}$ of subgroups of the reals. Then $\pi_S\{R_\lambda : \lambda \in \Gamma\}$ is an l -subgroup of the l -group R^Γ of all functions from Γ into R where addition and the partial order are defined pointwise. Moreover, pointwise multiplication on R^Γ makes the l -group R^Γ into a lattice-ordered ring (l -ring) which is an f -ring.² We are now ready to prove

2.8 THEOREM. Let G be an l -group which is a subdirect sum $\pi_S\{R_\lambda : \lambda \in \Gamma\}$ of a family $\{R_\lambda : \lambda \in \Gamma\}$ of subgroups of the reals R and let α be an o -endomorphism of G . Then

- (i) $\alpha \in P^+(G)$ if and only if $M\alpha \subseteq M$ for all maximal l -ideals of G ;
- (ii) if $\alpha \in P^+(G)$, then there is a unique extension of α to a p -endomorphism $\bar{\alpha}$ of R^Γ ; and
- (iii) $P(G)$ is isomorphic to the subring $\{f \in R^\Gamma : fg \in G \text{ for all } g \in G\}$ of R^Γ .

Proof. Let $\mathfrak{M}(G)$ denote the set of maximal l -ideals of G . Then since G is a subdirect sum of subgroups of the reals, we have that $\cap \mathfrak{M}(G) = 0$. Thus if $M\alpha \subseteq M$ for all $M \in \mathfrak{M}(G)$, then $\alpha \in P^+(G)$ by 2.2. The converse implication in (i) is immediate from 2.7.

To see (ii) let $x \in G$ and let x_λ denote the value of x , considered as function on Γ , at $\lambda \in \Gamma$. Let $G_\lambda = \{x \in G : x_\lambda = 0\}$. Then $G_\lambda \in \mathfrak{M}(G)$ and hence by (i), $G_\lambda\alpha \subseteq G_\lambda$. Thus α induces an o -endomorphism α^* on G/G_λ . Since G/G_λ is a subgroup of the reals R , there is a real number α_λ such that

$$(g + G_\lambda)\alpha^* = \alpha_\lambda(g + G_\lambda).$$

It follows that there is a unique extension of α to $\bar{\alpha} : R^\Gamma \rightarrow R^\Gamma$. Moreover, it is now clear how to prove (iii).

² An f -ring is an l -ring in which $a \wedge b = 0$ and $c \geq 0$ imply $ca \wedge b = ac \wedge b = 0$.

Remark. In the next section we extend the above result to show that $P(G)$ is an archimedean f -ring whenever G is an archimedean l -group.

2.9 COROLLARY. *Let X be a topological space and let $C(X)$ denote the l -group of continuous real-valued functions on X . Then $P(C(X))$ is isomorphic to $C(X)$.*

To close this section we remark that almost all the results in the sequel are valid only for archimedean l -groups. Consequently, we shall restrict our attention to the class of archimedean l -groups. The final result of this section shows that our results will apply to archimedean vector lattices.

2.10 PROPOSITION. *Suppose that G is a vector lattice. Then*

- (i) *each l -ideal of the l -group G is a subspace and*
- (ii) *If $\alpha : G \rightarrow G$ is an o -endomorphism of the l -group G , then α is linear provided that G is archimedean.*

Proof. (i) Let C be an l -ideal of G , let $c \in C$, and let $r \in R^+$. Then there is a positive integer $n > r$ such that $0 \leq rc^+ < nc^+ \in C$ and $0 \leq rc^- < nc^- \in C$. Thus since C is convex, both rc^+ and rc^- are in C . Hence $rc = rc^+ - rc^- \in C$.

(ii) For $g \in G$ and k a rational number, we have that $(kg)\alpha = k(g\alpha)$ for any o -endomorphism α of the l -group G . Now suppose that $g \in G^+$ and $r \in R$ is strictly positive. Then for rationals r_1 and r_2 such that $0 < r_1 \leq r \leq r_2$, we have, since α is an o -endomorphism, that $0 \leq r_1(g\alpha) = (r_1g)\alpha \leq (rg)\alpha \leq (r_2g)\alpha = r_2(g\alpha)$. Moreover, we have that $r_1(g\alpha) \leq r(g\alpha) \leq r_2(g\alpha)$. Thus

$$(rg)\alpha - r(g\alpha) \leq r_2(g\alpha) - r_1(g\alpha) = (r_2 - r_1)(g\alpha)$$

and

$$r(g\alpha) - (rg)\alpha \leq r_2(g\alpha) - r_1(g\alpha) = (r_2 - r_1)(g\alpha),$$

so that $|(rg)\alpha - r(g\alpha)| \leq (r_2 - r_1)(g\alpha)$. It follows that

$$|(rg)\alpha - r(g\alpha)| \leq s(g\alpha)$$

for all positive rationals s . Thus since G is archimedean, $|(rg)\alpha - r(g\alpha)| = 0$ and so $(rg)\alpha = r(g\alpha)$.

Remark. Example 10 of §8 shows that (ii) of 2.10 need not hold in the non-archimedean case.

3. p -endomorphisms of archimedean l -groups

The main result of this section provides a generalization of 2.8 in case the l -group G is archimedean but not necessarily a subdirect sum of subgroups of the reals. Our techniques require a functional representation of the archimedean l -group. We use Bernau's representation [1] of an archimedean l -group as an l -subgroup of a vector lattice $D(X)$ of almost finite continuous functions on a Stone space X . We now collect the pertinent facts about this representation.

Let G be an archimedean l -group, and let $\mathcal{P}(G)$ denote the set of polars of G . Then $\mathcal{P}(G)$ is a complete Boolean algebra [11], and so the associated Stone space X is extremely disconnected, Hausdorff and compact. Let $D(X)$ be the collection of almost finite continuous functions from X into $R \cup \{\pm \infty\}$ (i.e., $D(X) = \{f : X \rightarrow R \cup \{\pm \infty\} : f \text{ is continuous and } \{x \in X : f(x) \in R\} \text{ is dense}\}$). Then $D(X)$ is a complete vector lattice and Bernau established the following result.

THEOREM. *Let G be an archimedean l -group. Then there is an l -isomorphism σ of G in $D(X)$ (where X is the Stone space of the complete boolean algebra of polars of G) which preserves all existing infima and suprema and such that if $d \in D(X)$ is strictly positive, then there is a $g \in G$ and an integer n such that $0 < g < nd$. Moreover, if $\{e_\lambda : \lambda \in \Gamma\}$ is a maximal disjoint subset of G^+ , then σ can be chosen so that each $e_\lambda \sigma$ is the characteristic function of a subset X_λ of X where the family $\{X_\lambda : \lambda \in \Gamma\}$ is a collection of compact open subsets of X whose union is dense in X .*

Now let G be an archimedean l -group and let $\{e_\lambda : \lambda \in \Gamma\}$ be maximal disjoint subset of G . Let σ be an l -isomorphism of G into $D(X)$ with the properties of Bernau's Theorem. We shall identify G with $G\sigma$. Thus G is an l -subgroup of $D(X)$ and all infima and suprema in G agree with those in $D(X)$.

For each $x \in X$, let $G_x = \{g \in G : g(x) = 0\}$. Then for each $h \in G$ with $h(x) > 0$ and real, we have that G_x is a l -ideal maximal with respect to not containing h . Then if G^x is the intersection of all l -ideals of G that properly contain G_x , then $h \in G^x \setminus G_x$ and G^x/G_x is (isomorphic to) a subgroup of the reals R . Conversely, if $g \in G^x \setminus G_x$, then $g(x)$ is real. We are now ready to prove

3.1 LEMMA. *Let α be a p -endomorphism of G , let $e_\lambda \in \{e_\lambda : \lambda \in \Gamma\}$ a maximal disjoint subset of G^+ , let X_λ be the compact open subset of X associated with e_λ , and let $x \in X_\lambda$. Then $e_\lambda \alpha(x) = +\infty$ if either $G^x \alpha \not\subseteq G^x$ or $G_x \alpha \not\subseteq G_x$.*

Proof. First suppose that $G^x \alpha \not\subseteq G^x$. Then there is a strictly positive $g \in G^x$ such that $g\alpha \notin G^x$. Since $e_\lambda(x) = 1$, we have that $e_\lambda \in G^x \setminus G_x$ and hence there is an $n > 0$ such that $G_x + g < G_x + ne_\lambda$ since G^x/G_x is archimedean. Since G_x is a prime l -ideal, there is a minimal prime l -ideal N contained in G_x . Thus, by 2.1, $N\alpha \subseteq N$ so that α induces an o -endomorphisms on the o -group G/N . If $N + g \geq N + ne_\lambda$, then $G_x + g \geq G_x + ne_\lambda$. Hence $N + g < N + ne_\lambda$. Thus $N + g\alpha \leq N + n(e_\lambda \alpha)$, so that $G_x + g\alpha \leq G_x + n(e_\lambda \alpha)$. Then $G^x < G^x + g\alpha \leq G^x + n(e_\lambda \alpha)$, so that $n(e_\lambda \alpha) \notin G^x$ and hence $e_\lambda \alpha \notin G^x$. Thus $e_\lambda \alpha(x)$ is not real, so that $e_\lambda \alpha(x) = +\infty$.

Now suppose that $G_x \alpha \not\subseteq G_x$. Then there is a strictly positive $g \in G_x$ such that $g\alpha \notin G_x$. Thus for all positive integers n we have that $G_x + ng = G_x < G_x + e_\lambda$. Hence $N + ng < N + e_\lambda$ where N is a minimal prime l -ideal contained in G_x . Thus $N + n(g\alpha) \leq N + e_\lambda \alpha$, and hence $G_x < G_x + n(g\alpha) \leq G_x + e_\lambda \alpha$. But G^x/G_x is archimedean so that $e_\lambda \alpha \notin G^x$ and thus $e_\lambda \alpha(x) = +\infty$.

3.2 THEOREM. *Let G be an archimedean l -group, and let $\{e_\lambda : \lambda \in \Gamma\}$ be a maximal disjoint subset of G . Consider G as a l -subgroup of $D(X)$ as in Bernau's Theorem. Then each p -endomorphism of G is a multiplication of G by an element in $D(X)^+$; and conversely if $d \in D(X)^+$ and $Gd \subseteq G$, then the map $g \rightarrow gd$ ($g \in G$) is a p -endomorphism of G .*

Proof. For each $g \in G$, the closure $S(g)$ of the set $\{x \in X : g(x) \neq 0\}$ is the support of g . Now let α be a p -endomorphism of G and let $\lambda \in \Gamma$. Then by 2.1,

$$e_\lambda \alpha \in (e_\lambda \alpha)'' \subset e'' = \{g \in G : S(g) \subseteq S(e_\lambda) = X_\lambda\}.$$

Thus $e_\lambda \alpha$ is zero outside of X_λ and is finite on a dense open subset of X_λ . Thus by 3.1 the set $D_\lambda = \{x \in X_\lambda : G^x \alpha \subseteq G^x \text{ and } G_x \alpha \subseteq G_x\}$ is dense and open. Now for each $x \in D_\lambda$, α induces an o -endomorphism on G^x/G_x . But G^x/G_x is a subgroup of R and hence there is a real number $\alpha_x > 0$ such that if $g \in G$ and $g(x)$ is real, then $g\alpha(x) = g(x)\alpha_x$. Moreover, $e_\lambda \alpha(x) = \alpha_x$ so that map $x \rightarrow \alpha_x$ is a continuous real-valued function defined on D_λ . But this holds for each $\lambda \in \Gamma$, and hence it follows that there is a continuous real-valued function $x \rightarrow \alpha_x$ defined on a dense open subset of X such that $x \in D$, $(g\alpha)(x) = g(x)\alpha_x$ if $g(x)$ is real. But then the map $x \rightarrow \alpha_x$ has a unique extension to an element of $D(X)^+$, say $\bar{\alpha}$; and it follows that $g\alpha = g\bar{\alpha}$ where $g\bar{\alpha}$ denotes the f -ring multiplication in $D(X)$.

Finally, the converse is clear since $D(X)$ is an f -ring.

We remark that a routine computation shows that the map $P(G)$ such that $\alpha \rightarrow \bar{\alpha} \in D(X)$ is an l -isomorphism of l -rings. Hence we have

3.3. COROLLARY. *For archimedean l -groups G and the associated Bernau representation in $D(X)$ we have that $P(G)$ is an f -ring with identity and*

$$P(G) = \{d \in D(X) : Gd \subseteq G\}.$$

Now let F be an archimedean f -ring with identity 1. Then by Bernau [1], F is a subring of $D(X)$ (1 is the identity of $D(X)$) where X is the Stone space associated with l -group $(F, +)$. Moreover, $P(F) = \{d \in D(X) : Fd \subseteq F\}$ and since $1 \in F$ it follows that $P(F) = F$. This together with 3.3 proves

3.4 THEOREM. *The class $\{P(G) : G \text{ is an archimedean } l\text{-group}\}$ is identical with the class of archimedean f -rings with identity.*

We now turn our attention to some consequences of 3.2. We shall, of course, assume that G is an archimedean l -group.

3.5. (i) For $\alpha, \beta \in P^+(G)$ and $g \in G^+$, $g(\alpha \vee \beta) \equiv g\alpha \vee g\beta$ and $g(\alpha \wedge \beta) = g\alpha \wedge g\beta$.

(ii) For $\alpha, \beta \in P^+(G)$ and $g \in G^+ \setminus \{0\}$, $g\alpha\beta = 0$ if and only if $g\alpha \wedge g\beta = 0$; and hence $\alpha\beta = 0$ if and only if $\alpha \wedge \beta = 0$.

(iii) For $\alpha \in P^+(G)$ and $x \in G \setminus \{0\}$, $x\alpha^n = 0$ if and only if $x\alpha = 0$; and hence $\alpha^n = 0$ implies $\alpha = 0$.

3.6. If $\alpha \in P^+(G)$ and α is onto, then α is one-to-one and $\alpha^{-1} \in P^+(G)$.

Proof. Let $y \in G$ be non-zero. Then since α is onto, there is an $x \in G$ such that $x\alpha = y \neq 0$. But then by 3.5 (iii) $0 \neq x\alpha^2 = y\alpha$ so that α is one-to-one. The rest is routine.

3.7. If $\alpha \in P^+(G)$, then the subgroup of G generated by $\text{Ker } (\alpha)$ and $G\alpha$ is the cardinal sum $\text{Ker } (\alpha) \oplus G\alpha$ and it is an l -subgroup of G . In particular, if $\alpha^2 = \alpha$, then $G = \text{Ker } (\alpha) \oplus G\alpha$.

Proof. First we show that the sum of an l -ideal B and an l -subgroup C of G is an l -subgroup of G . To this end let $x = b + c \in B + C$. Then $B + x = B + c$ and so

$$B + (x \vee 0) = (B + x) \vee B = (B + c) \vee B = (B + c) \vee B = B + c \vee 0$$

since B is an l -ideal of G . Thus $x \vee 0 = b' + c \vee 0$ for some $b' \in B$. Thus $x \vee 0 \in B + C$, and $B + C$ is an l -subgroup of G .

Now $G\alpha$ is an l -subgroup of G and $\text{Ker } (\alpha)$ is an l -ideal of G , so that $\text{Ker } (\alpha) + G\alpha$ is an l -subgroup of G . Now let $y \in \text{Ker } (\alpha) \cap G\alpha$, and let $x \in G$ be such that $y = x\alpha$. Then $0 = y\alpha = x\alpha^2$ so that $x\alpha = 0$ by 3.5 (iii) and hence $y = 0$. Thus $\text{Ker } (\alpha) + G\alpha = \text{Ker } (\alpha) \oplus G\alpha$. To show that $\text{Ker } (\alpha) \oplus G\alpha = \text{Ker } (\alpha) \oplus G\alpha$ it suffices to show that $G\alpha$ is convex in $\text{Ker } (\alpha) \oplus G\alpha$. To this end suppose that $0 < x < y\alpha$ where $x = a + b\alpha$ with $a \in \text{Ker } (\alpha)$ and $b \in G$. Suppose that $a \neq 0$. Then $-b\alpha < a < (y - b)\alpha$, so that

$$((-b) \vee 0)\alpha \leq a \vee 0 \leq ((y - b) \vee 0)\alpha$$

and

$$((-b) \wedge 0)\alpha \leq a \wedge 0 \leq ((y - b) \wedge 0)\alpha.$$

Thus, since $a \vee 0$ and $a \wedge 0$ are in $\text{Ker } (\alpha)$, there is a $c \in \text{Ker } (\alpha)$ and an $h \in G^+$ such that $0 < c < h\alpha$. If M is a minimal prime l -ideal of G such that $c \notin M$, then $M + c \leq M + h\alpha$ so that $h\alpha \notin M$. Now since G is archimedean, there is a positive integer n such that $nc \not\leq h$. Hence there is a minimal prime l -ideal N of G such that $N + nc > N + h$. Now α induces an o -endomorphism $\bar{\alpha}$ of G/N and we have that $N + n\alpha c \geq N + h\alpha$ or $N \geq N + h\alpha$ since $c \in \text{Ker } (\alpha)$. Thus $h\alpha \in N$ since $N \geq N + h\alpha \geq N$. But $c \notin N$ since $N + nc > N + h$ and $h \geq 0$. This contradicts the fact that any minimal prime l -ideal not containing c also does not contain $h\alpha$, and we are done.

3.8. If $\alpha \in P^+(G)$, then α preserves all existing infima and suprema.

Proof. This follows from the fact that in any archimedean f -ring F , $y = \bigwedge \{y_\gamma : \gamma \in \Gamma\}$ and $\alpha \in P^+$ imply $y\alpha = \bigwedge \{y_\gamma \alpha : \gamma \in \Gamma\}$. (See [1] or [7].)

3.9. If $\alpha \in P^+(G)$, then $\text{Ker } (\alpha)$ is a polar and $G/\text{Ker } (\alpha)$ is archimedean.

Proof. By 3.8 it is clear that $\text{Ker } (\alpha)$ is closed. Hence $\text{Ker } (\alpha)$ is a polar [6] and so $G/\text{Ker } (\alpha)$ is archimedean [5].

3.10. If G is complete, then $P(G)$ is complete.

Proof. Let $\{\alpha_\gamma : \gamma \in \Gamma\}$ be an upward directed subset of $P^+(G)$ which is bounded above by $\alpha \in P^+(G)$. Then for $g \in G^+$, $g\alpha_\gamma \leq g\alpha$ for all $\gamma \in \Gamma$. Thus we set $g\beta' = \bigvee \{g\alpha_\gamma : \gamma \in \Gamma\}$. But then for $g, h \in G^+$, it follows since $\{\alpha_\gamma : \gamma \in \Gamma\}$ is directed, that $(g + h)\beta' = g\beta' + h\beta'$, and hence there is an o -endomorphism β of G such that $g\beta = g\beta'$ for $g \in G^+$. Now if $g \wedge h = 0$, then

$$g \wedge h\beta = g \wedge (\bigvee \{h\alpha_\gamma : \gamma \in \Gamma\}) = \bigvee \{g \wedge h\alpha_\gamma : \gamma \in \Gamma\} = 0$$

and it follows that $\beta \in P^+(G)$ and hence $\beta = \bigvee \{\alpha_\gamma : \gamma \in \Gamma\}$. Similarly, the infimum of a downward directed subset of $P^+(G)$ has an infimum, and it follows that $P(G)$ is complete.

Remark. If $P(G)$ is complete, G need not be complete. See Example 5 of §8.

We close this section with some results about relationship between $P(G)$ and $P(\hat{G})$ where \hat{G} is the completion of the archimedean l -group G . First we prove

3.11. Let \hat{G} denote the completion of the archimedean l -group G , and let α be an endomorphism of G which preserves all existing infima and suprema. Then there is a unique extension of α to an endomorphism $\hat{\alpha} : \hat{G} \rightarrow \hat{G}$ which preserves all existing infima and suprema.

Proof. Let $h \in \hat{G}$. Then $h = \bigvee \{k \in G : k \leq h\} = \bigwedge \{k \in G : h \leq k\}$. Define $h\hat{\alpha} = \bigvee \{k\alpha : k \in G \text{ and } k \leq h\}$. This supremum exists since α preserves order and the set $\{k \in G : k \leq h\}$ is bounded above by any element of G larger than h . Now let $h' \in \hat{G}$. Then

$$\begin{aligned} h\hat{\alpha} + h'\hat{\alpha} &= \bigvee \{k\alpha : k \in G \text{ and } k \leq h\} + \bigvee \{k'\alpha : k' \in G \text{ and } k' \leq h'\} \\ &= \bigvee \{(k + k')\alpha : k, k' \in G, k \leq h \text{ and } k' \leq h'\} \leq (h + h')\alpha. \end{aligned}$$

To see the reverse inequality, let $x \in G$ be such that

$$x \leq h + h' = \bigvee \{k : k \in G \text{ and } k \leq h\} + \bigvee \{k' : k' \in G \text{ and } k' \leq h'\}.$$

Then

$$x = \bigvee \{(k + k') \wedge x : h, k' \in G, k \leq h, \text{ and } k' \leq h'\},$$

and

$$\begin{aligned} x\alpha &= \bigvee \{(k\alpha + k'\alpha) \wedge x\alpha : k, k' \in G, k \leq h, \text{ and } k' \leq h'\} \\ &\leq \bigvee \{k\alpha + k'\alpha : k, k' \in G, k \leq h, \text{ and } k' \leq h'\} = h\hat{\alpha} + h'\hat{\alpha}. \end{aligned}$$

Since $h + h'$ is the supremum of all such x 's, we have $(h + h')\hat{\alpha} \leq h\hat{\alpha} + h'\hat{\alpha}$. Thus $\hat{\alpha}$ is an endomorphism of \hat{G} . The rest is straightforward.

3.12 PROPOSITION. Let G be an archimedean l -group, and let $\alpha \in P^+(G)$. Then there is a unique extension of α to $\hat{\alpha} \in P^+(\hat{G})$. Moreover, α is one-to-one if and only if $\hat{\alpha}$ is one-to-one; and $\hat{\alpha}$ is onto if α is onto.

Proof. Let $\alpha \in P^+(G)$. Then by 3.8 and 3.11, there is an extension of α to $\hat{\alpha} : \hat{G} \rightarrow \hat{G}$. Now let $h, h' \in \hat{G}$ with $h \wedge h' = 0$. Then

$$h\hat{\alpha} = \bigvee \{k\alpha : k \in G^+ \text{ and } k \leq h\},$$

so that

$$\begin{aligned} h\hat{\alpha} \wedge h' &= \bigvee \{k\alpha : k \in G^+ \text{ and } k \leq h\} \wedge h' = \bigvee \{k\alpha \wedge h' : k \in G^+ \text{ and } k \leq h\} \\ &= \bigvee \{k\alpha \wedge h' : k, k' \in G^+, k \leq h, \text{ and } k' \leq h'\} = 0. \end{aligned}$$

Thus $\hat{\alpha} \in P^+(\hat{G})$.

Now suppose that α is one-to-one on G , and let $g \in \hat{G}^+ \setminus \{0\}$ be such that $g\hat{\alpha} = 0$. Then there is a $g' \in G^+ \setminus \{0\}$ such that $0 < g' < g$. But then $0 \leq g'\hat{\alpha} \leq g\hat{\alpha} = 0$ so that $g'\alpha = 0$. Thus $\hat{\alpha}$ is one-to-one on \hat{G} .

Finally suppose that α is onto, and let $h \in \hat{G}$ with

$$h = \bigvee \{k \in G : k \leq h\} \leq g$$

where $g \in G$. Since α is onto, α is one-to-one by 3.6 and hence α is an l -automorphism of G . Now $k \leq g$ implies $k\alpha^{-1} \leq g\alpha^{-1}$, so that

$$\bigvee \{k\alpha^{-1} : k \in G \text{ and } k \leq h\} \in G.$$

Thus

$$[\bigvee \{k\alpha^{-1} : k \in h \text{ and } k \leq h\}]\hat{\alpha} = \bigvee \{k\alpha^{-1}\hat{\alpha} : k \in G \text{ and } k \leq h\} = h$$

so that $\hat{\alpha}$ is onto.

Remark. Example 11 of §8 shows that $\hat{\alpha} \in P^+(\hat{G})$ can be onto without $\alpha \in P^+(G)$ being onto.

As usual let G be an archimedean l -group, and \hat{G} its completion. Then it follows from 3.10 that $P_2 \equiv P(\hat{G})$ is a complete f -ring with identity, and it follows from 3.11 that there is a natural embedding of $P_1 = P(G)$ into P_2 . Example 5, §8 shows that P_2 need not be the completion of the archimedean f -ring P_1 . However we are able to show

3.13 THEOREM. *If G is a divisible archimedean l -group with a strong order unit and a basis, then $P_2 = P(\hat{G})$ is the completion of $P_1 = P_1(G)$.*

Proof. We may assume that $P_1 \subseteq P_2$. Thus since P_2 is complete, it is sufficient to show that for $\beta \in P_2^+ \setminus \{0\}$, there are $\alpha, \gamma \in P_1$ such that $0 < \alpha < \beta < \gamma$ (See [6]).

Now since G is divisible and has a basis, we may assume that

$$\sum \{R_\lambda : \lambda \in P\} \subseteq G \subseteq \prod \{R_\lambda : \lambda \in P\}$$

where $\{R_\lambda : \lambda \in P\}$ is a family of divisible subgroups of the reals. Then [4] \hat{G} is the l -ideal of $\prod \{T_\lambda : \lambda \in \Gamma\}$ ($T_\lambda = R$ for all $\lambda \in \Gamma$) generated by G and

$$\sum \{T_\lambda : \lambda \in \Gamma\} \subseteq \hat{G} \subseteq \prod \{T_\lambda : \lambda \in \Gamma\}.$$

By Theorem 2.8, $\beta \in P_2^+ \setminus \{0\}$ has the form $(\dots, \beta_\lambda, \dots)$ where $\beta_\lambda \in R^+$. Suppose that $\beta_\lambda > 0$ and choose n so that $1/n < \beta_\lambda$ and let

$$\alpha = (0, \dots, 0, 1/n, 0, \dots, 0)$$

where the $1/n$ is in the λ -th place. Then $\alpha \in P_1$ since $G \supseteq \sum \{R_\lambda : \lambda \in \Gamma\}$ and each R_λ is divisible. Moreover, $0 < \alpha < \beta$. Finally, we can assume without loss of generality that $(1, 1, \dots)$ is a strong order unit for G and hence for \hat{G} . But then $(1, 1, \dots)\beta = (\dots, \beta_\lambda, \dots)$ and the set $\{\beta_\lambda : \lambda \in \Gamma\}$ is bounded, by n say. Thus if we set $x\gamma = nx$ for $x \in G$, we have a $\gamma \in P_1$ such that $\beta < \gamma$.

Remarks. 3.13 fails without the assumption of divisibility (See Example 5 of §8). However, we do not know if any of the other assumptions can be omitted.

Note that the intersection of all the laterally complete l -subgroups of $D(X)$ that contain G is the lateral completion of G (see [5]). In particular, each p -endomorphism α of G has a unique extension to a p -endomorphism of the lateral completion of G .

4. A characterization of $P^+(G)$

In this section we prove a single theorem. It shows, for an archimedean l -group G , that the largest f -ring of $B(G)$ containing the identity is $P(G)$. Precisely

4.1 THEOREM. *Let G be an archimedean l -group, and let F be a subring of the ring $B(G)$ of ordered-bounded endomorphisms of G containing the identity 1. Moreover, suppose that F is an f -ring where the order is given by the cone $F^+ = B(G)^+ \cap F$. Then $F^+ \subseteq P^+(G)$ and hence (See 3.3) $P(G)$ is the largest sub- p o ring of $B(G)$ which is an f -ring and contains 1.*

Proof. Consider G as an l -group of almost finite extended real-valued continuous functions on the Stone space X of the Boolean algebra of polars as in §3. Then it is sufficient to show that the support of $g\rho$ is contained in the support of g for each $g \in G^+$ and each $\rho \in F^+$.

To this end suppose that there is an $z \in \text{support}(g\rho)$ which is not in $\text{support}(g)$. Then there is a neighborhood V_1 of z such that $g(V_1) = \{0\}$. However, since $z \in \text{support}(g\rho)$, there is a $y \in V_1^0$ such that $(g\rho)(y) > 0$. Since $g\rho$ is continuous, there is a neighborhood V_2 of y contained in V_1 such that $(g\rho)(\omega) > 0$ for each $\omega \in V_1$. It follows that there is $z_1 \in X$ and a neighborhood V_3 of z_1 such that $g(V_3) = \{0\}$ and $(g\rho)(\omega) > 0$ for all $\omega \in V_3$. Now $g\rho$ is finite on an open dense subset of X and hence there is a $y \in V_3$ such that $0 < (g\rho)(y) < +\infty$. Again, it follows that there is a $z_2 \in X$ and a neighborhood V_4 of z_2 such that $g(V_4) = \{0\}$, and $0 < (g\rho)(\omega) < +\infty$ for all $\omega \in V_4$. Also we have that $(g\rho^2)(y) < +\infty$ for some $y \in V_4$. Consequently there is an $x \in X$ such that $g(x) = 0$, $0 < (g\rho)(x) < +\infty$, and $0 \leq (g\rho^2)(x) < +\infty$.

Pick an integer $n > 0$ such that $ng\rho(x) > g\rho^2(x)$ and let $\delta = \rho \wedge n1$. Then $0 = (\rho - \delta) \wedge (n1 - \delta)$ so that $(\rho - \delta)(n1 - \delta) = 0$ since $\rho - \delta$ and $n1 - \delta$ are disjoint elements of the f -ring F . We obtain a contradiction to our assumption that support $(g\rho)$ is not contained in support (g) by showing that $g(\rho - \delta)(n1 - \delta) \neq 0$. To this end note that

$$g(\rho - \delta)(x) = (g\rho(x) - (g\delta)(x)) = (g\rho)(x)$$

since $0 \leq \delta \leq n1$ and $g(x) = 0$. Thus $g(\rho - \delta) = h$ where $0 < h < g\rho$ and $h(x) = g\rho(x)$. Moreover, $hn1(x) = ng\rho(x)$ and $h\delta(x) \leq h\rho(x) \leq g\rho^2(x) < ng\rho(x)$. Thus

$$g(\rho - \delta)(n1 - \delta)(x)$$

$$= h(n1 - \delta)(x) = hn1(x) - h\delta(x) = ng\rho(x) - h\delta(x) > 0,$$

and we're done.

Remark. By a similiar proof, one can show the following result. **Theorem.** Let G be an archimedean l -group and let F be a subring of $B(G)$ which contains 1 , is an l -ring in the partial order $F^+ = B(G)^+ \cap F$, and satisfies: $\alpha, \beta \in F$ and $g \in G^+$ imply $g(\alpha \wedge \beta) = g\alpha \wedge g\beta$. Then $F^+ \subseteq P^+(G)$.

5. The additive subgroup of an archimedean f -ring with identity

The object of this section is to prove

(5.1 THEOREM.) *Let $(G, +)$ be an archimedean l -group with weak order unit e . Then (a) there is at most one multiplication on G so that $(G, +, \cdot)$ is an f -ring with identity e , and (b) such a multiplication exists if and only if*

$$\{e\alpha : \alpha \in P^+(G)\} = G^+.$$

Before proving 5.1 we lay some groundwork. Let G be an archimedean l -group with a weak order unit e , and let $eP^+(G) = \{e\alpha : \alpha \in P^+(G)\}$. For $\alpha, \beta \in P^+(G)$, let $e\alpha + e\beta = e(\alpha + \beta)$, $e\alpha \wedge e\beta = e(\alpha \wedge \beta)$ and $e\alpha \vee e\beta = e(\alpha \vee \beta)$. Then $eP^+(G)$ is a subsemigroup and a sublattice of G^+ . Now for $e\alpha, e\beta \in P^+(G)$, define $(e\alpha)(e\beta) = e\alpha\beta$. Then since e is a weak order unit, it follows from 3.2 that if $e\mu = e\nu$ where $\mu, \nu \in P^+(G)$, then $\mu = \nu$. Hence the multiplication $(e\alpha)(e\beta) = e\alpha\beta$ is well defined. It follows that $eP^+(G)$ is a semiring. Let G^e denote the subgroup of G generated by $eP^+(G)$. Then

$$G^e = \{e\alpha - e\beta : \alpha, \beta \in P(G)\} = \{e\mu - e\nu : \mu, \nu \in P(G) \text{ and } e\mu \wedge e\nu = 0\}.$$

For if $\alpha, \beta \in P(G)$, then $e\alpha = e\alpha \wedge e\beta + s$ and $e\beta = e\alpha \wedge e\beta + t$ where $s \wedge t = 0$. Thus

$$s = e\alpha - e(\alpha \wedge \beta) = e(\alpha - (\alpha \wedge \beta)) = e\mu \quad \text{where } \mu = \alpha - (\alpha \wedge \beta) \in P(G)$$

and

$$t = e\beta - e(\alpha \wedge \beta) = e(\beta - (\alpha \wedge \beta)) = e\nu \quad \text{where } \nu = \beta - (\alpha \wedge \beta) \in P(G).$$

Thus $e\alpha - e\beta = e\mu - e\nu$. In particular, $e\mu = (e\alpha - e\beta)^+$ and $e\nu = (e\alpha - e\beta)^-$. Thus $(e\alpha - e\beta)^+$, $(e\alpha - e\beta)^- \in eP^+(G)$, so that G^e is an l -subgroup of G with positive cone $eP^+(G)$. Now by linearity, one can extend the multiplication on $eP^+(G)$ to G^e , and it then follows easily that the map $\alpha \rightarrow e\alpha$ from $P(G)$ onto G^e is an l -isomorphism. Thus $P(G)$ is l -isomorphic to G^e . Thus if $G^+ = eP^+(G)$, we have that $(G, +)$ can be made into an f -ring with e as identity. This proves one way of part (b) of 5.1. To see the converse of part b), let $g \in G^+$ and assume that $(G, +)$ is an f -ring with e as identity. Then since the map $x \rightarrow xg$ is a p -endomorphism and $eg = g$, we have that $eP^+(G) = G^+$. This completes the proof of part (b) of 5.1.

To see part (a) let \circ and \cdot be two multiplications on G making G^+ into an archimedean f -ring with identity e . Then for $g \in G^+ \setminus \{0\}$, the map $x \rightarrow x \circ g$ and $x \rightarrow x \cdot e$ are p -endomorphisms of G . But a p -endomorphism is determined by its action on a weak order unit and since $e \circ g = g = e \cdot g$, we have that $x \circ g = x \cdot g$ for all $x \in G$ and $g \in G^+$. It follows that $x \circ y = x \cdot y$ for all $x, y \in G$ and we're done.

This theorem is more or less "well known" (see [8]). The novelty is the description in terms of $P(G)$.

6. Contractors on archimedean l -groups

Previously we have considered those endomorphisms of an l -group G which leave invariant all minimal prime subgroups of G (see 2.1). We now turn our attention to those endomorphisms of G which leave invariant all prime subgroups of G .

6.1 DEFINITION. A contractor α on an l -group G is a group endomorphism α of G such that $G^+\alpha \subseteq G^+$ and for each $g \in G^+$ there is an integer $n = n(g)$ such that $g\alpha \leq ng$.³

6.2 PROPOSITION. Let G be an l -group and let α be an endomorphism of G such that $G^+\alpha \subseteq G^+$. Then the following are equivalent:

- (i) α is a contractor on G ;
- (ii) α leaves invariant each convex l -subgroup of G ; and
- (iii) α leaves invariant each prime subgroup of G .

Proof. That (i) is equivalent to (ii) is clear. That (ii) is equivalent to (iii) follows from the fact [4] that each convex l -subgroup is the intersection of prime convex l -subgroups of G .

Note that the set $C^+(G)$ of contractors on G is closed under multiplication and if G is abelian, it is closed under addition. Thus for abelian l -groups we have that

$$C(G) = \{\alpha - \beta : \alpha, \beta \in C^+(G)\}$$

³ Langford's contractors [10] are our contractors with $n(g) = 1$ for all $g \in G$. Langford asks if contractors on archimedean l -groups commute. Since each contractor is a p -endomorphism the answer is yes.

is a po subring of $P(G)$ with positive cone

$$C(G) \cap P^+(G) = C^+(G).$$

For if $\alpha - \beta \in C(G) \cap P^+(G)$ and g belongs to a convex l -subgroup M of G , then $g\alpha, g\beta \in M$ and so $g(\alpha - \beta) = g\alpha - g\beta \in M$. Therefore $\alpha - \beta \in C^+(G)$.

Now suppose that G is an archimedean l -group and assume as in §3 that $G \subseteq D(X)$ where X is Stone space of the Boolean algebra of polars on G . Moreover, suppose that G has a weak order unit which we can assume is the constant function 1 in $D(X)$. Now let $\alpha \in C(G)$. Then since α is a p -endomorphism of G , α is multiplication by a function $\bar{\alpha}$ of $D(X)$ (Theorem 3.2). Moreover, there is an integer n such that $0 \leq e\bar{\alpha} \leq ne$ and hence for each $x \in X$ we have that $0 \leq \bar{\alpha}(x) \leq n$. Thus $\bar{\alpha} \in C(X)$, the real-valued continuous functions on X . Consequently, $C^+(G)$ is isomorphic to a subring of $C(X)$.

Example 8 of §8 shows that if G does not have a weak order unit, then $C(G)$ need not be contained in $C(X)$. However, we are able to prove

6.3 PROPOSITION. *Let G be an archimedean l -group. Then there is a topological space Y such that $C(G)$ is isomorphic to a subring of $C(Y)$.*

Proof. Let $\{e_\lambda : \lambda \in \Gamma\}$ be a maximal disjoint subset of G and assume that $G \subseteq D(X)$ where X is the Stone space of the Boolean algebra of polars of G as in §3. Let $\{X_\lambda : \lambda \in \Gamma\}$ be a family of compact open subsets of X such that e_λ is the characteristic function of X_λ .

Let $Y = \bigcup \{X_\lambda : \lambda \in \Gamma\}$ and let $\alpha \in C(G)$. Assume that α is represented by the function $\bar{\alpha} \in D(X)$. (Theorem 3.2) Now let $x \in Y$ and let $\lambda \in \Gamma$ be such that $x \in X_\lambda$. Then there is an integer n such that $0 \leq e_\lambda \bar{\alpha} \leq ne_\lambda$ so that $0 \leq \bar{\alpha}(x) \leq n$. Thus $\bar{\alpha}(x)$ is real for each $x \in X_\lambda$, and hence $\bar{\alpha}$ is real on Y . It follows that $C(G)$ is isomorphic to a subring of $C(Y)$.

Now we show that a contractor α on an archimedean l -group G extends to a contractor $\hat{\alpha}$ on \hat{G} , the completion of G . Note that since α is a p -endomorphism, α extends to a p -endomorphism $\hat{\alpha}$ of \hat{G} (see 3.11). Thus we need only show that $\hat{\alpha}$ is a contractor on \hat{G} . First we note that if a, b, c , and d are real numbers such that $0 \leq a \leq b$ and $bc \leq db$, then $ac \leq da$. Hence from Theorem 3.2 it follows that if α is a contractor on an archimedean l -group G and if $a, b \in G^+$ are such that $0 \leq a \leq b$ and $b\alpha \leq nb$ where n is a positive integer, then $a\alpha \leq na$. Now recall that if $h \in \hat{G}$, then $h = \bigvee \{k : k \in G \text{ and } k \leq h\}$ and $h\hat{\alpha}$ is defined by $h\hat{\alpha} = \bigvee \{k\alpha : k \in G \text{ and } k \leq h\}$. Moreover let $g \in G$ be such that $g \geq h$, and let n be an integer such that $g\alpha \leq ng$. Then for $k \leq h$ we have that $k\alpha \leq nk$ and hence

$$\begin{aligned} h\hat{\alpha} &= \bigvee \{k\alpha : k \in G \text{ and } k \leq h\} \leq \bigvee \{nk : k \in G \text{ and } k \leq h\} \\ &= n \bigvee \{k : k \in G \text{ and } k \leq h\} = n\hat{h}. \end{aligned}$$

Hence $\hat{\alpha}$ is a contractor on \hat{G} . Thus we have proven

6.4 PROPOSITION. *Let G be an archimedean l -group, let α be a contractor on*

G , and let \hat{G} denote the completion of \hat{G} . Then if we define $\hat{\alpha} : \hat{G} \rightarrow \hat{G}$ by

$$h\hat{\alpha} = \bigvee \{k\alpha : G \text{ and } k \leq h\}$$

for $h \in \hat{G}$, then $\hat{\alpha}$ is a contractor on \hat{G} .

Remark. It can be shown that if $\{a_\gamma : \gamma \in \Gamma\}$ is a disjoint subset of an l -group G , $a = \bigvee a_\gamma$ and α is a contractor, then $a\alpha = \bigvee (a_\gamma\alpha)$.

7. A problem of Birkhoff

In this section E will always denote a complete vector lattice. By a *bounded contractor* on E we shall mean a linear transformation $\alpha : E \rightarrow E$ such that $E^+\alpha \subseteq E^+$ and $x\alpha \leq \lambda x$ for some $\lambda \in R^+$ and all $x \in E^+$.

Recall [2] that if E is a complete vector lattice, then the algebra $B(E)$ of order-bounded linear transformations on E is a complete l -algebra; and if $\{T_\lambda : \lambda \in \Gamma\}$ is a subset of $B(E)$ bounded above, then the supremum of $\{T_\lambda : \lambda \in \Gamma\}$ is the linear transformation $T : E \rightarrow E$ defined by

$$xT = \bigvee \{xT_\lambda : \lambda \in \Gamma\}$$

for $x \in E^+$.

Now let α be a bounded contractor on E . Then the family

$$\{\sum_{n=0}^k \alpha^n / n! : k = 1, 2, \dots\}$$

is bounded above by $e^\lambda 1_E$ where 1_E denotes the identity map of E and $\lambda \in R^+$ is such that $x\alpha \leq \lambda x$ for all $x \in E^+$. Thus the supremum of the family

$$\{\sum_{n=0}^k \alpha^n / n! : k = 1, 2, \dots\}$$

exists in $B(E)$. We denote this supremum by e^α . Birkhoff's problem (Problem 154 of [2]) is now as follows. If α is a bounded contractor on a complete vector lattice E , show that the family $\{e^{t\alpha} : t \in R^+\}$ is a semigroup of operators.

The proof that $\{e^{t\alpha} : t \in R^+\}$ is a semigroup of operators is broken up into a series of steps.

1. As usual let X denote the Stone space of the Boolean algebra of polars of E , and consider E as a sub-vector lattice of $D(X)$ as in §3. Moreover, for $\alpha \in P(E)$, let $\bar{\alpha}$ denote the function in $D(X)$ associated with α as in Theorem 3.2. Thus for $f \in E$, $f\alpha = f\bar{\alpha}$ where the product on the right is the ring multiplication in $D(X)$.

2. Let $f \in D(X)$ and let $R(f) = \{x \in X : f(x) \text{ is real}\}$. Then for each $x \in R(f)$, $[\sum_{n=0}^k f^n / n!](x) \leq e^{f(x)}$ and since the map $x \rightarrow e^{f(x)}$ is continuous on $R(f)$ (and hence extends to an element of $D(X)$ which we denote by $\exp(f)$), we have that the set $\{\sum_{n=0}^k f^n / n! : k = 1, 2, \dots\}$ is bounded above by an element of $D(X)$. Thus the supremum of the set $\{\sum_{n=0}^k f^n / n! : k = 1, 2, \dots\}$ exists; we denote it by e^f . Note that $e^f \leq \exp(f)$. Moreover, suppose that $g \in D(X)$ is $\geq \sum_{n=0}^k f^n / n!$ for each k . Then

$$g(x) \geq \sum_{n=0}^k f^n(x) / n!$$

for each k and each $x \in R(f) \cap R(g)$. It follows that $g(x) \geq e^{f(x)}$ for each $x \in R(g) \cap R(f)$ and hence $g \geq \exp(f)$; and hence $e^f = \exp(f)$. Thus for each $x \in R(f)$, we have that

$$[\vee \{ \sum_{n=0}^k f^n/n! : k = 1, 2, \dots \}](x) = e^{f(x)}.$$

Thus it is clear that for $f, g \in D(X)$, we have that $e^{f+g} = e^f e^g$.

3. If α is a bounded contractor on E , then e^α is also a bounded contractor by the remarks in the first paragraph of this section. Now for $g \in E^+$ we have that

$$\begin{aligned} g e^\alpha &= g \vee \{ \sum_{n=0}^k \alpha^n/n! : k = 1, 2, \dots \} = \vee \{ \sum_{n=0}^k g \alpha^n/n! : k = 1, 2, \dots \} \\ &= \vee \{ \sum_{n=0}^k g \bar{\alpha}^n/n! : k = 1, 2, \dots \} = \vee \{ \sum_{n=0}^k \bar{\alpha}^n/n! \} g : k = 1, 2, \dots \}. \end{aligned}$$

But by 3.8,

$$\begin{aligned} \vee \{ (\sum_{n=0}^k \bar{\alpha}^n/n!)g : k = 1, 2, \dots \} \\ = [\vee \{ \sum_{n=0}^k \bar{\alpha}^n/n! : k = 1, 2, \dots \}]g = e^{\bar{\alpha}}g. \end{aligned}$$

Thus

$$g e^\alpha = e^{\bar{\alpha}}g$$

and hence we have that

$$\bar{e^\alpha} = e^{\bar{\alpha}}.$$

4. For α and β bounded contractors on E , we have that $\alpha + \beta$ is a bounded contractor and

$$e^{(\alpha+\beta)} = e^{\overline{\alpha+\beta}} = e^{\bar{\alpha}+\bar{\beta}} = e^{\bar{\alpha}}e^{\bar{\beta}} = \overline{e^\alpha e^\beta} = \overline{e^\alpha} \overline{e^\beta}$$

so that $e^{\alpha+\beta} = e^\alpha e^\beta$.

We thus have proven

7.1 THEOREM. *Let E be a complete vector lattice and let α be a linear transformation such that $E^+ \alpha \subseteq E^+$ and $x\alpha \leq \lambda x$ for some $\lambda \in R^+$ and all $x \in E^+$. Then the family $\{e^{t\alpha} : t \in R^+\}$ is a semigroup of operators where*

$$e^{t\alpha} = \vee \{ \sum_{n=0}^k t^n \alpha^n/n! : k = 1, 2, \dots \}.$$

8. Examples

In examples (1), (2), (3) and (4), let $G = R \oplus R$ be the cardinal sum of two copies of the reals R . Then each o -endomorphism of G is linear,

$$B(G)^+ = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \geq 0 \}$$

and $B(G)$ is the full ring of linear transformations on G . Moreover,

$$P^+(G) = C(G) = \{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in R^+ \}.$$

1. Let $\tilde{S} = \{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in R \}$. The \tilde{S} is a sub- f -ring of l -endomorphisms of $B(G)$ not contained in $P(G)$.

2. $S = \{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in R^+ \setminus \{0\} \}$ is a subsemiring of l -endomorphisms of $B(G)$ which contains I . But $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in (\tilde{S})^+ \setminus S$.

3. Let $\tilde{S} = \{ \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} : a, x \in R \}$. Then \tilde{S} is a commutative subring of $B(G)$ which contains 1_G and is generated by the l -endomorphisms $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Now $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \wedge 1_G = 0$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} 1_G = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ so that \tilde{S} is not an f -ring. Also $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not an l -endomorphism since $(1, 0) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (1, 1)$ and $(0, 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (0, 1)$ are not disjoint. Finally \tilde{S} is not real representable since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = 0$.

4. Let $S = \{ \begin{pmatrix} x & 0 \\ 0 & 2x+y \end{pmatrix} : x, y \in R^+ \}$. Then S is a subsemiring of $P^+(G)$ and $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1_G \in \tilde{S} \setminus S$. In particular, $(\tilde{S})^+ \not\subseteq S$.

5. An example where $P(G)$ is complete and proper subring of $P(\hat{G})$. Let G be the additive group of all rational numbers with square free denominators. Then clearly $\hat{G} = R$ and so $P(\hat{G}) = R^+$, but $P^+(G) = Z^+$. For if the map $x \rightarrow kx$ belongs to $P^+(G)$, then $1 \rightarrow k$ so that $k \in G$ and also $k^2 \in G$ since $k \rightarrow k^2$. Now write $k^2 = m^2/p_1^{2e_1} \cdots p_n^{2e_n}$ where p_{e_i} are primes and m is an integer. Then it follows since $k^2 \in G$ that k must be an integer and hence $P^+(G) = Z^+$.

6. A non-archimedean o -group G for which $P^+(G)$ is commutative. Let $G = Z \oplus Q$ where $(n, q) \leq (0, 0)$ if $q > 0$ or $q = 0$ and $n \geq 0$. Then all o -endomorphisms commute since o -endomorphisms are of the form $(n, q) \rightarrow (kn, pq)$ where $k \in Z^+$ and $p \in Q^+$.

7. A contractor which is an l -automorphism but whose inverse is not a contractor. Let G be the direct product of countably many copies of the reals. For $(x_1, x_2, \dots) \in G$, let $(x_1, x_2, \dots)\alpha = (x_1, x_2/2, x_3/3, \dots)$. Then

$$(1, 1, \dots)\alpha^{-1} = (1, 2, 3, \dots) \not\subseteq n(1, 1, 1, \dots)$$

for any integer n .

8. A contractor in $D(X) \setminus C(X)$. Let G be the direct sum of countably many copies of the reals and let $\rho = (1, 2, 3, \dots)$. Then since the Stone space X of the Boolean algebra of polars of G is the Stone-Cech compactification of the rational numbers, it is clear that $\rho \in D(X) \setminus C(X)$.

9. An archimedean l -group, a p -endomorphism α , and a prime l -ideal P such that $P\alpha \not\subseteq P$. Let G be the direct product of countably many copies of the integers and let α be the p -endomorphism given by $(1, 2, 3, \dots)$. Then if $G(1, 1, 1, \dots)$ denotes the l -ideal generated by $(1, 1, 1, \dots)$ we have that $(1, 1, 1, \dots)\alpha \notin G(1, 1, 1, \dots)$. Thus there is a prime l -ideal P of G containing $G(1, 1, 1, \dots)$ but not containing $(1, 1, 1, \dots)\alpha$. Thus $P\alpha \not\subseteq P$.

10. An o -endomorphism of the l -group of a vector lattice which is not linear. Let $G = R \oplus R$ where $(x, y) \geq (0, 0)$ if $x > 0$ or $x = 0$ and $y \geq 0$. Then $\begin{pmatrix} a & f \\ 0 & b \end{pmatrix}$, where $a, b \in R^+$ and f is a non-linear group endomorphism of R , is a non-linear o -endomorphism of G .

11. An archimedean l -group G , and a p -endomorphism α of G such that $\hat{\alpha} : \hat{G} \rightarrow \hat{G}$ is onto but α is not onto. Let $G = \sum_{i=1}^{\infty} Q t^i$ where Q denotes the rationals and t is a transcendental number. Then $G \subseteq R$ and we put the

natural induced order on G . Now $\hat{G} = R$ and the map $\alpha : G \rightarrow G$ given by $y \rightarrow ty$ is not onto but $\hat{\alpha} : \hat{G} \rightarrow \hat{G}$ is onto.

Added in proof. There is some overlap between the theory developed here, and that in A. Bigard and K. Keimel, *Sur les endomorphismes conservant les polaires d'un groupe réticulé archimédien*, Bull. Soc. Math. France, vol. 97 (1969) pp. 381-398.

Their paper was submitted after but published before ours.

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