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The Risk-Adjusted Carbon Price*

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Abstract

A popular model of economy and climate change has logarithmic preferences and damages proportional to the carbon stock in which case the certainty-equivalent carbon price is optimal. We allow for different aversions to risk and intertemporal fluctuations, convex damages, uncertainties in economic growth, atmospheric carbon, climate sensitivity and damages, correlated risks, and distributions that are skewed in the longer run to capture climate feedbacks. We derive a non-certainty-equivalent rule for the carbon price, which incorporates precautionary, risk-insurance and risk-exposure, and climate beta effects to deal with future economic and climatic risks. We interpret these effects with a calibrated DSGE model.

Keywords: precaution, insurance, economic, climatic and damage uncertainties, skewness, mean reversion, climate betas, risk aversion, prudence, intergenerational inequality aversion, convex damages, DSGE

JEL codes: H21, Q51, Q54

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Climate policy must take account of the highly uncertain nature of the impact of the atmospheric carbon stock on global mean temperature, of temperature on damages to aggregate output and the highly uncertain nature of future GDP. These uncertainties are in large part due to the long timescales over which today's emissions impact global warming and cause economic damage, but their impact is also affected by skewness of the probability distributions underlying climate uncertainties. To optimally internalize the global warming externality, the price of carbon must be set to the social cost of carbon (SCC),[1] defined as the expected present discounted value of all marginal damages to current and future aggregate production resulting from emitting one ton of CO_2 today.

Our objective is to establish how precautionary and insurance motives affect the optimal risk-adjusted SCC in dynamic stochastic general equilibrium (DSGE). Golosov et al. (2014) provide a pioneering analysis of the optimal SCC in DSGE. Their bold assumptions[2] give a simple rule for the SCC that is proportional to world GDP. However, as they assume logarithmic preferences, economic growth uncertainty does not affect the optimal SCC (Traeger, 2017). We generalize their rule for non-unitary elasticities of intertemporal substitution and coefficient of relative risk aversion, generalized convex damages, uncertainty in the rate of economic growth, carbon stock, climate sensitivity and damages, and skewness and mean reversion in the distributions governing climate sensitivity and damage uncertainties. For this purpose, we adapt a DSGE model of endogenous growth with investment adjustment costs due to Pindyck and Wang (2014) to climate change.

[1] The first best is sustained in a decentralized market economy if the price of carbon is set to the optimal SCC, either via a carbon tax or a competitive permissions market, provided that the global warming externality is the only market failure. From now on, we will use the optimal (risk-adjusted) carbon price and the SCC interchangeably.

[2] These assumptions are logarithmic preferences, a Cobb-Douglas production function, 100% depreciation of capital in each period, and global warming damages either linear in the atmospheric carbon stock if in the utility function or total factor productivity exponential in the atmospheric carbon stock if in the production function.

Motivated by our search for a tractable rule for the SCC, we use power functions to represent the convex dependence of damages on temperature and the concave dependence of temperature on the carbon stock. Furthermore, by using a power-function transformation of a normal variate displaying a variance that grows in time,[3] we capture the significant right-skew evident in the equilibrium climate sensitivity, but not in the transient climate response, whilst capturing the difference in time scales on which these apply. Although we capture skewness[4], we deliberately avoid fat tails and thus Weitzman's (2009) 'dismal theorem'.

We obtain the following insights. First, using scaling and analysis of the key non-dimensional quantities of our model, we identify global warming damages as share of world GDP as the only "small" quantity in which to perform a perturbation analysis (cf. Judd, 1996, 1998; Judd and Guu, 2001).[5] The corresponding optimal price (our Result 1) takes account of an array of uncorrelated and correlated economic, climate and damage risks and can conveniently be evaluated through numerical evaluation of a multi-dimensional integral instead of numerically solving Hamilton-Jacobi-Bellman equations.

Second, by examining only the leading-order effects of various uncertainties, we obtain a leading-order closed-form rule for the optimal SCC, which is proportional to world GDP. This rule is especially simple if the concavity of the temperature-carbon stock relationship is balanced out by the convexity of the damage-temperature relationship so that reduced-form damages are proportional to the atmospheric carbon stock (our Result 2).[6] For a convex dependence of damages on

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[3] Specifically, we use an Ornstein-Uhlenbeck process.

[4] Martin (2013) uses the cumulant generating function to deal with higher moments in the process for the rate of economic growth when analyzing the effects of rare disasters on asset pricing. Pindyck and Wang (2013) also consider skewness and kurtosis of financial markets to model rare disasters.

[5] See Bender and Orszag (1999) for an exposition of these techniques for scientists and engineers.

[6] The factor of proportionality between the price and GDP depends on ethical factors (intergenerational inequality aversion, risk aversion and the rate of impatience), economic factors (the rate of growth of the world economy, its volatility and damages to final production from climate

the carbon stock, correction factors can be applied to Result 2 in the form of simple, one-dimensional deterministic integrals (see Result 2′ in Appendix A).

From these results, we find that precaution about uncertain economic growth outcomes demands a reduction in the risk-adjusted discount rate and a higher optimal risk-adjusted SCC, especially if risk aversion is high, prudence is high and economic growth is volatile. A positive risk-insurance term acts to increase the risk-adjusted discount rate and increasingly so for more risk aversion and larger economic growth volatility.[7] Provided intergenerational inequality aversion exceeds one, the discount rate is adjusted upwards with positive economic growth and downwards with more volatile growth prospects. The risk-adjusted SCC is adjusted downwards with rising economic affluence and upwards with riskier growth prospects. The correction for climate sensitivity uncertainty depends on the combination of the skewness of its equilibrium probability distribution, the (non-climatic) risk-adjusted discount rate and, crucially, on the time scale on which the skew equilibrium probability distribution is reached.

If future damages and GDP are correlated, there is an effect additional to the built-in climate beta of one associated with damages being proportional to GDP. If the correlation is positive, this additional "climate-damages beta" is positive. This occurs if a positive climate damage shock (with negative consequences for GDP) is associated with higher economic growth, not through the proportionality of damages to GDP but through the correlation of GDP with the underlying stochastic processes for climate and damages. We show that provided the coefficient of relative risk aversion is greater than one, the risk insurance effect dominates the risk

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change) and geophysical factors (the share of emissions that stays permanently up in the atmosphere and the rate of decay of atmospheric carbon).

[7] If the elasticity of damages with respect to GDP is not unity, the risk insurance premium is multiplied by this elasticity (cf. Dietz et al., 2018). The case of additive damages corresponds to a zero elasticity, in which case there is no risk insurance premium and thus no upward adjustment of the price of carbon due to the insurance effect.

exposure effect and the optimal carbon price is then reduced (cf. Sandmark and Vennemo, 2007; Daniel et al., 2015). Typically, damages are proportional to GDP and thus the climate-damage beta (close to) one as in Dietz et al. (2018). The latter analyse a "climate beta", which corresponds to an amalgam of the disaggregated climate betas in our model: the built-in climate beta of one associated with the proportionality of damages to GDP and the terms resulting from the correlations of our stochastic processes for climate and damage uncertainties with GDP.

Third, we calibrate our model and decompose the optimal SCC into a deterministic part and parts to deal with uncertain economic growth, carbon stock, climate sensitivity and damages. We thus analyze how ethical discounting, aversion to risk and intertemporal fluctuations, convex damages, skewness of climate sensitivity, and correlated risks affect the components of the optimal risk-adjusted SCC. We find that climate sensitivity and economic growth uncertainties have substantial quantitative impacts on the carbon price, but carbon stock uncertainty has a negligible impact. Climate damage uncertainty is large. Nevertheless, if its distribution is not skewed, it has no effect on the optimal risk-adjusted SCC.

Bretschger and Vinogradova (2018) also analytically examine climate policy in a stochastic environment but focus on the effect of Poisson shocks. Hambel et al. (2017) study numerically the effect of level and growth damages and also separate risk aversion from intertemporal substitution. Jensen and Traeger (2016) show analytically the effect of climate sensitivity on the risk premium in the price of carbon and how this depends on prudence in utility and on the convexity of marginal damages. Although they abstract from a skewed climate sensitivity, they offer an interesting analysis of Bayesian learning of damages. Traeger (2017) puts forward a new IAM with a remarkably detailed climate system and a wide range of objective and epistemological uncertainties, uses cumulant generating functions after transforming to a model that is linear in states with additively separable controls, and shows analytically how the risk-adjusted carbon price depends on the uncertainties. None of these studies analyzes the effects of climate betas.[8]

Lemoine (2017) shows that climate sensitivity, damage, consumption, and temperature uncertainties double the optimal SCC, and that the risk insurance effect dominates the risk exposure effect on the optimal SCC if the coefficient of relative risk aversion is greater than one. In the same vain, we find that the sign of the effects of climate betas on the risk-adjusted SCC depends on whether risk aversion exceeds one. However, we give explicit analytical expressions for the SCC. Our approach differs in three other ways. First, we separate the effects of risk aversion and intertemporal substitution, which is crucial in an analysis of the impact of economic and climatic risks on the SCC. Second, we use a fully specified DSGE model rather than an exogenous stochastic process for consumption growth. Third, we allow for skewness and mean reversion in the distributions of climate sensitivity and damages and for more convex damage functions.

Section I introduces our model. Section II derives the optimal SCC under uncertainty (Result 1). Section III makes additional approximations to derive a closed-form rule for the optimal risk-adjusted SCC (Result 2 and Result 2′ in Appendix A). Section IV discusses the calibration of our model. Section V provides

[8] Earlier work on the effect of uncertainty on the SCC in the IAM developed by Nordhaus (2008) uses Monte Carlo simulations (e.g., Ackerman and Stanton, 2012; Dietz and Stern, 2015), but this assumes that uncertainty is resolved before the first time period and can give misleading results (Crost and Traeger, 2013). Stochastic dynamic programming algorithms do not have this problem, but face limits due to the curse of dimensionality. Recent progress has, however, been impressive. Traeger (2014a) applies stochastic dynamic programming to a 4-state abridged version of DICE. Jensen and Traeger (2014) have Epstein-Zin preferences and study numerically the effect of long-term growth uncertainty. There has also been progress in deriving numerically the optimal SCC when there are tipping risks related to a wide range of catastrophes (e.g., Lemoine and Traeger, 2014; 2016; Lontzek et al., 2015; Cai et al., 2016). Lemoine and Rudik (2017) review the numerical literatures on recursive assessment and Monte Carlo evaluation of climate policy under uncertainty, and discuss the importance of learning. Separately, several theoretical contributions have examined the effects of economic growth uncertainty on the optimal discount rate to use for long-term investment projects (e.g., Gollier, 2012; Traeger, 2014b).

our estimates for the optimal risk-adjusted SCC, including a quantification of the four stochastic determinants and possible climate betas, sensitivity to parameter choice and accuracy of our rule for the risk-adjusted SCC. Section VI concludes.

I. A DSGE Model of Global Warming and the Economy

We use a continuous-time macroeconomic DSGE model with endogenous AK growth model and capital and fossil fuel use as production factors. Fossil fuel use leads to carbon emissions, global warming and damages to aggregate output. We allow for four types of uncertainty: to the growth rate of the economy, the stock of atmospheric carbon, the climate sensitivity, and to global warming damages. We distinguish aversion to risk from aversion to intertemporal or intergenerational differences, so that the coefficient of relative risk aversion, η = CRRA ≥ 0, need not equal the inverse of the elasticity of intertemporal substitution IES. Alternatively, CRRA need not equal the coefficient of relative intergenerational inequality aversion, IIA = 1/EIS = $\gamma \geq 0$. We thus use the continuous-time version of Epstein-Zin (1989) and Kreps-Porteus (1978) recursive preferences following Duffie and Epstein (1992) with the recursive aggregator $f(C,J)$ a function of both consumption C and the value function J, and the rate of pure time preference denoted by $\rho > 0$, so that the representative consumer maximizes

$$(2.1) \qquad J = E_t\left[\int_t^\infty f\left(C(s), J(s)\right) ds\right] \text{ with } f\left(C,J\right) = \frac{1}{1-\gamma}\frac{C^{1-\gamma} - \rho\left(\left(1-\eta\right)J\right)^{\frac{1-\gamma}{1-\eta}}}{\left(\left(1-\eta\right)J\right)^{\frac{1-\gamma}{1-\eta}-1}}.$$

The aggregate capital stock is accumulated according to the stochastic equation

$$(2.2) \qquad dK = \Phi(I,K)dt + \sigma_K K dW_1 \quad \text{with} \quad \Phi(I,K) = I - \frac{1}{2}\omega\frac{I^2}{K} - \delta K,$$

where K denotes the aggregate capital stock, I aggregate investment, $\delta \geq 0$ the depreciation rate of physical capital, and $\omega > 0$ the cost parameter for adjusting investment. [9,10] Adjustment costs are quadratic and homogenous of degree one in capital and investment. Capital is subject to continuous geometric shocks with relative volatility σ_K, and W_1 denotes a Wiener process. Investment is given by $I = Y - C - bF$, where Y is aggregate production, F fossil fuel use, and b the production cost of fossil fuel. Fossil reserves are abundant, and fossil fuel is supplied inelastically at fixed cost. The final goods production function is Cobb-Douglas with constant returns to scale, so $Y = AK^{\alpha} F^{1-\alpha}$ with $0 < \alpha < 1$, where $A \equiv A^*(1-D)$ is total factor productivity (A^*) net of warming damages DA^*. Damages as share of pre-damages aggregate output D increase in the global mean temperature relative to pre-industrial T. We use the power-function specification

$$(2.3) \quad D(T,\lambda) = (T / \Delta T)^{1+\theta_T} \lambda^{1+\theta_\lambda} \quad \text{with} \quad \theta_T \geq -1 \quad \text{and} \quad \theta_\lambda \geq -1,$$

where the stochastic variable λ captures the uncertain nature of damages for a given temperature. The term ΔT is added to ensure that (2.3) is independent of units. Henceforth, we define temperature in degrees Celsius and set $\Delta T = 1^{\circ}\text{C}$. Damages are a convex function of temperature, which requires $\theta_T \equiv TD_{TT} / D_T > 0$. [11] To allow for potential skewness in the impact of damage shocks, we take a power-function transformation of λ with $\theta_\lambda \neq 0$ and specify a symmetric distribution for λ itself.

9 In an AK growth model, shocks to the capital stock and shocks to productivity are equivalent. To avoid an extra state, we introduce volatility directly in the capital accumulation equation (cf. Pindyck and Wang, 2013).

10 Sacrificing formality for ease of presentation, we will first introduce the separate evolution equations for the four stochastic variables before introducing the covariance matrix of the vector of these four variables.

11 We let subscripts denote partial derivatives.

The absolute value of the atmospheric carbon stock is denoted by S. We define the part associated with man-made emissions E as the difference between the current value (S) and the pre-industrial carbon stock, S_{PI}, so that $E \equiv S - S_{PI}$. We let temperature and damages depend on E, not S. Annual carbon emissions from fossil fuel use F are $F\exp(-gt)$, where $\exp(-gt)$ is the emission intensity per unit of fossil fuel used (F, measured in GtC/year), which in accordance with balanced growth we set to decline at the endogenous economic growth rate g. A proportion $0 < \mu < 1$ of fossil fuel emissions ends up in the atmosphere. Atmospheric carbon decays at the rate $\varphi \geq 0$.[12] The dynamics of the carbon stock is given by[13]

(2.4) $$dE = (\mu F e^{-gt} - \varphi E)dt + \sigma_E dW_2,$$

where W_2 denotes a second Wiener process, so that the atmospheric carbon stock is described by an Arithmetic Brownian motion with absolute volatility $\sigma_E \geq 0$. Note that (2.4) ensures that the carbon stock returns to its pre-industrial value in absence of emissions. We have for temperature

(2.5) $$T(E,\chi) = (E / S_{PI})^{1+\theta_E} \chi^{1+\theta_\chi} \Delta T \quad \text{with} \quad \theta_E \geq -1 \quad \text{and} \quad \theta_\chi \geq -1,$$

where the stochastic variable χ captures the uncertain nature of temperature for a given carbon stock. As in (2.3), $\Delta T = 1^{\circ}\text{C}$. The parameter θ_χ allows us to introduce skewness in the impact of stochastic shocks on temperature, and we specify χ itself to have a symmetric distribution. We allow for the effect of lags via the time-

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[12] One could allow for a permanent reservoir and one (Golosov et al., 2014), two (Gerlagh and Liski, 2018) or three (Millar et al., 2017) temporary reservoirs of atmospheric carbon. We show in section IV that our "1-box" model reproduces historical atmospheric carbon stocks relatively well.
[13] Note that (2.4) can theoretically lead to negative carbon stocks, but this is very unlikely.

varying dynamics of the stochastic process for the random variable χ.[14] Temperature is a concave function of the carbon stock, so that $\theta_E \equiv ET_{EE} / T_E < 0$.[15] The climate sensitivity is defined as the temperature increase from doubling the carbon stock from its pre-industrial level; hence, it is $T_2 \equiv T(E = S_{PI}, \chi) = \chi^{1+\theta_\chi}$ and depends on the stochastic climate sensitivity parameter χ. Its normalized skewness $\text{skew}^*\left[T_2\right] \equiv \text{skew}\left[T_2\right] / \left(\text{var}\left[T_2\right]\right)^{3/2}$ is given to leading-order by $\text{skew}^*\left[T_2\right] = 3\theta_\chi (\Sigma_\chi / \mu_\chi)$ (see Appendix F), from which it is evident that skewness is driven by what we thus call the skewness parameter θ_χ and the coefficient of variation of $\chi = \Sigma_\chi / \mu_\chi$. Combining equations (2.3) and (2.5), we get reduced-form damages

$$(2.6) \quad D(E, \chi, \lambda) = (E / S_{PI})^{1+\theta_{ET}} \chi^{1+\theta_{\chi T}} \lambda^{1+\theta_\lambda} \quad \text{with} \quad \theta_{\chi T} \equiv \theta_\chi + \theta_T + \theta_\chi \theta_T .$$

The parameter $\theta_{ET} \equiv \theta_E + \theta_T + \theta_E \theta_T$ captures the combined effect of the concave relationship between temperature and the carbon stock ($\theta_E < 0$) and the convex relationship between damages and temperature ($\theta_T > 0$). It can thus be positive or negative depending on whether the latter effect dominates the former or not. We refer to $\theta_{ET} = 0$ as *proportional* (cf. Golosov et al., 2014) and $\theta_{ET} > 0$ as *convex* (reduced-form) *damages*. The parameter $\theta_{\chi T}$ captures the joint effect of the convex relationship between temperature and climate sensitivity parameter ($\theta_\chi > 0$) and the convex relationship between damages and temperature ($\theta_T > 0$). This parameter is higher if the distribution of climate shocks is more skewed (higher θ_χ). From

[14] We thus include the potential effects of temperature lags due to ocean heating, which are important for estimates of the long-run climate sensitivity (e.g., Roe and Bauman, 2011) (see section IV).
[15] Temperature is often explained by a logarithmic function of the carbon stock (Arrhenius, 1896).

(2.6) total factor productivity and aggregate output decreases in the carbon stock and the shocks to climate sensitivity and damages:

(2.7) $$Y = A(E,\chi,\lambda)K^{\alpha}F^{1-\alpha} \text{ with } A(E,\chi,\lambda) \equiv A^{*}\left(1-(E/S_{PI})^{1+\theta_{ET}}\chi^{1+\theta_{\chi T}}\lambda^{1+\theta_{\lambda}}\right).$$

Finally, the uncertainty in the climate sensitivity and damage parameters are driven by two mean-reverting stochastic Ornstein-Uhlenbeck processes with means $\bar{\chi}$ and $\bar{\lambda}$, coefficients of mean reversion ν_{χ} and ν_{λ}, and volatilities σ_{χ} and σ_{λ}, so[16]

(2.8) $$d\chi = \nu_{\chi}(\bar{\chi}-\chi)dt + \sigma_{\chi}dW_3 \quad \text{and} \quad d\lambda = \nu_{\lambda}(\bar{\lambda}-\lambda)dt + \sigma_{\lambda}dW_4,$$

where W_3 and W_4 are two Wiener processes. Together with $T \propto \chi^{1+\theta_{\chi}}$ in (2.5), the first Ornstein-Uhlenbeck process in (2.8) captures the two essential features of the climate sensitivity distribution. First, the transformation $\chi^{1+\theta_{\chi}}$ of the symmetrically distributed χ allows for positive skewness of the equilibrium climate sensitivity for $\theta_{\chi} > 0$. Second, uncertainty associated with temperature increases with time, reaching a steady state associated with the equilibrium climate sensitivity and its variance and skewness. We calibrate the properties of the equilibrium climate sensitivity to the steady-state variance ($\Sigma_{\chi}^2 = \sigma_{\chi}^2\left(1-\exp(-2\nu_{\chi}t)\right)/2\nu_{\chi} \to \sigma_{\chi}^2/2\nu_{\chi}$ as $t \gg 1/\nu_{\chi}$), so that $1/\nu_{\chi}$ becomes the e-folding time[17] over which this steady-

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[16] For independent stochastic processes, (2.8) has solution $\chi(t) = \chi_0 e^{-\nu_{\chi}t} + \bar{\chi}(1-e^{-\nu_{\chi}t}) + \sigma_{\chi}\int_0^t e^{-\nu_{\chi}(t-s)}dW_3(s)$, and similarly for the stochastic process for λ. The random variables $\chi(t)$ and $\lambda(t)$ are normally distributed with time-varying moments: $\chi(t) \sim \mathrm{N}(\mu_{\chi},\Sigma_{\chi}^2)$ and $\lambda(t) \sim \mathrm{N}(\mu_{\lambda},\Sigma_{\lambda}^2)$. Mean and variance of $\chi(t)$ are $\mu_{\chi} = \chi_0 e^{-\nu_{\chi}t} + \bar{\chi}(1-e^{-\nu_{\chi}t})$ and $\Sigma_{\chi}^2 = \sigma_{\chi}^2\left(1-\exp(-2\nu_{\chi}t)\right)/2\nu_{\chi}$ with stationary limits $\mu_{\chi} \to \bar{\chi}$ and $\Sigma_{\chi}^2 \to \sigma_{\chi}^2/2\nu_{\chi}$.

[17] This is the time it takes for an exponentially growing quantity to increase by a factor e = 2.71828.

state is reached. The vector of all four states can be described by one multi-variate Ornstein-Uhlenbeck process:

$$(2.9)\quad d\mathbf{x} = \boldsymbol{\alpha} - \mathbf{v} \circ (\mathbf{x} - \boldsymbol{\mu})\, dt + \mathbf{S} d\mathbf{W}_t,$$

where $d\mathbf{x} \equiv (dk, dE, d\chi, d\lambda)^T$, $k \equiv \log(K/K_0)$ and $\circ$ denotes the elementwise product of two vectors. The growth rates of this process are

$$(2.10)\quad \boldsymbol{\alpha} \equiv \left(\frac{1}{dt}\frac{E_t[dK]}{K} - \frac{1}{2}\sigma_K^2, \mu F e^{-gt}, 0, 0 \right)^T.$$

The vector of mean reversion rates and the vector of means of this process are

$$(2.11)\quad \mathbf{v} \equiv (0, \varphi, \nu_\chi, \nu_\lambda)^T \quad \text{and} \quad \boldsymbol{\mu} \equiv (0, 0, \bar{\chi}, \bar{\lambda})^T.$$

The covariance matrix $\mathbf{S}\mathbf{S}^T$ of the components of this multivariate process is

$$(2.12)\quad \frac{1}{dt} E_t\left[d\mathbf{x} d\mathbf{x}^T \right] = \mathbf{S}\mathbf{S}^T = \begin{pmatrix} \sigma_K^2 & \rho_{KE}\sigma_K\sigma_E & \rho_{K\chi}\sigma_K\sigma_\chi & \rho_{K\lambda}\sigma_K\sigma_\lambda \\ \rho_{KE}\sigma_K\sigma_E & \sigma_E^2 & \rho_{E\chi}\sigma_E\sigma_\chi & \rho_{E\lambda}\sigma_E\sigma_\lambda \\ \rho_{K\chi}\sigma_K\sigma_\chi & \rho_{E\chi}\sigma_E\sigma_\chi & \sigma_\chi^2 & \rho_{\chi\lambda}\sigma_\chi\sigma_\lambda \\ \rho_{K\lambda}\sigma_K\sigma_\lambda & \rho_{E\lambda}\sigma_E\sigma_\lambda & \rho_{\chi\lambda}\sigma_\chi\sigma_\lambda & \sigma_\lambda^2 \end{pmatrix},$$

where $\rho_{ij}, i \neq j, i, j = K, E, \chi, \lambda$ denote the partial correlation coefficients.

II. Asymptotic Solutions for the Optimal Risk-Adjusted Price of Carbon

The optimal solution under uncertainty satisfies the Hamilton-Jacobi-Bellman equation corresponding to the recursive utility specification (2.1) which is

$$(3.1)\quad \max_{C,F}\left[f(C, J) + \frac{1}{dt} E_t\left[dJ(t, K, E, \chi, \lambda) \right] \right] = 0,$$

where $\left(1/dt\right)E_t\left[dJ\right]$ is Ito's differential operator applied to J, which depends on time and the four states. Using $I(C,F,K,E,\chi,\lambda)=A(E,\chi,\lambda)K^{\alpha}F^{1-\alpha}-C-bF$ and applying Ito's calculus to (3.2) gives

$$\begin{aligned}
&\max_{C,F}\left[f\left(C,J\right)+J_K\Phi\left(I(C,F,K,E,\chi,\lambda),K\right)+J_E(\mu Fe^{-gt}-\varphi E)\right]+J_t\\
(3.2)\quad &+J_\chi\nu_\chi(\overline{\chi}-\chi)+J_\lambda\nu_\lambda(\overline{\lambda}-\lambda)+\frac{1}{2}J_{KK}K^2\sigma_K^2+\frac{1}{2}J_{EE}\sigma_E^2+\frac{1}{2}J_{\chi\chi}\sigma_\chi^2+\frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2\\
&+J_{KE}K\rho_{KE}\sigma_K\sigma_E+J_{K\chi}K\rho_{K\chi}\sigma_K\sigma_\chi+J_{K\lambda}K\rho_{K\lambda}\sigma_K\sigma_\lambda+J_{E\chi}\rho_{E\chi}\sigma_E\sigma_\chi\\
&+J_{E\lambda}\rho_{E\lambda}\sigma_E\sigma_\lambda+J_{\chi\lambda}\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda=0.
\end{aligned}$$

By differentiating (3.2) with respect to the forward-looking variables C and F, we obtain the optimality conditions $f_C=C^{-\gamma}\left(\left(1-\eta\right)J\right)^{\frac{\gamma-\eta}{\eta-1}}=J_K\Phi_I(I,K)$ (using (2.1)) and $(1-\alpha)Y/F=b+Pe^{-gt}$, where the optimal risk-adjusted SCC is defined by $P\equiv-\mu J_E/J_K\Phi_I(I,K)>0.$ Although solving our problem as a command optimum, the solution corresponds to the outcome in a decentralized market economy provided carbon emissions are priced at an amount equal to the SCC and that there are no other externalities or market failures. Henceforth, we will therefore use the price of carbon and the SCC interchangeably, and we denote these by P.

A. Transforming to non-dimensional form and scaling

Closed-form analytical solutions to the stochastic dynamic optimal control problem (3.1)-(3.2) are not available. Solving this numerically by approximating the value function and its derivatives in 5-dimensional (time and the four states) space is challenging due to the curse of dimensionality and will not give the insight into the mechanisms that determine the stochastic markup of the optimal price of carbon we seek. Instead, we first examine the system for small parameter(s) by transforming to non-dimensional form, then pursue an asymptotic expansion in the thus identified small parameter(s) and only consider leading-order. Out of all the non-dimensional groups (see Appendix B), we identify one group to be small:

$$\text{(3.3)} \qquad \epsilon \equiv D(E_0, \overline{\chi}, \overline{\lambda}) = \overline{\lambda}^{1+\theta_\lambda} \overline{\chi}^{1+\theta_{\chi T}} \left(E_0 / S_{PI}\right)^{1+\theta_{ET}} ,$$

where E_0 denotes the difference between the absolute value of the atmospheric carbon stock at $t = 0$ (S_0) and the pre-industrial carbon stock S_{PI}. The non-dimensional group ϵ becomes the small parameter of our asymptotic expansion, whereas the effects of the other non-dimensional groups are initially analyzed without approximation. The quantity ϵ represents the share of climate damages in total GDP (when climate damage and sensitivity parameters are at their equilibrium values and the atmospheric carbon stock at its initial value). It is typically only a few percentage points and stays well below 10% even at high temperatures (see section IV.C). In the perturbation solutions below, we consider terms up to first order in ϵ. The resulting error scales with $\epsilon^2 \cong 0.01,$ which is small even for the large value of $\epsilon \cong 0.1$, and we deem this to be more than sufficient for accurately estimating the optimal risk-adjusted SCC.

B. Perturbation expansion

To solve our problem, we perform a perturbation expansion[18] in the small parameter ϵ. At each order *n*, the problem is linear in the value function $J^{(n)}$, but remains fully nonlinear in the states, thus retaining risk-aversion and prudence properties without approximation. Mathematically, at each order *n*, the problem is of the form $L\left[\epsilon^n J^{(n)}\right] = \Gamma,$ where L is a linear differential operator in the states and the nonlinear forcing Γ is formed from products or derivatives of lower-order solutions

[18] We emphasize we do not perform a Taylor-series expansion in the state variables around their steady states, as this requires an excessive number of terms due to the large number of states and derivatives needed to capture risk aversion and prudence.

(in *n*), so that the order of the forcing thus obtained (from products or derivatives) is also $O(\epsilon^n)$. We choose the following truncated series solution and restrict our attention to zeroth- and first-order terms in ϵ only, as denoted by the superscripts

$$\begin{aligned} J(K,E,\chi,\lambda,t) &= J^{(0)}\left(K,\epsilon D(E,\chi,\lambda)\right) + \epsilon J^{(1)}(K,E,\chi,\lambda,t,\epsilon D(E,\chi,\lambda)) + O(\epsilon^2), \\ F(K,E,\chi,\lambda,t) &= F^{(0)}\left(K,\epsilon D(E,\chi,\lambda)\right) + \epsilon F^{(1)}(K,E,\chi,\lambda,t,\epsilon D(E,\chi,\lambda)) + O(\epsilon^2), \\ C(K,E,\chi,\lambda,t) &= C^{(0)}\left(K,\epsilon D(E,\chi,\lambda)\right) + \epsilon C^{(1)}(K,E,\chi,\lambda,t,\epsilon D(E,\chi,\lambda)) + O(\epsilon^2). \end{aligned} \tag{3.4}$$

The small parameter ϵ appears both as small parameter of the series solution and as the multiple-scales parameter in front of the dependence on damages. We let total factor productivity be a slowly-varying power-law function of the climate-related variables E, χ and λ: higher derivatives required to model rapid variation are thus automatically ignored at leading order. The zeroth-order value function inherits the properties of the production function (2.7). We then find a consistent leading-order estimate of the SCC from the zeroth and first-order value function:

$$P = -\mu\left(J_E^{(0)} + \epsilon J_E^{(1)}\right)\Big/\phi'(i^{(0)})J_K^{(0)}. \tag{3.5}$$

In the limit $\epsilon \to 0$, climate has no effect, we retain only the zeroth-order solution, and our model reduces to an AK model for which a closed-form solution is available (cf. Pindyck and Wang, 2013). Our derivation of the zeroth-order solution is given in Appendix C with the solution for $J^{(0)}$ given by (C2) (written in terms of the non-dimensional variables introduced in Appendix B). The only difference with Pindyck and Wang (2013) is that ours depends slowly on the climate variables, as determined by the implicit equation for optimal investment (C7) and the dependence of the marginal productivity of capital therein on climate damages. We proceed to derive the first-order solution in Appendix D with the solution for $J^{(1)}$ given by (D3.14)[19]. The first-order value function captures changes to the economy

[19] We only show the solution for $J_E^{(1)}$ as this is needed to evaluate the SCC.

resulting not from climate-induced changes to the marginal productivity of capital (as captured by $J^{(0)}$'s slow dependence the climate-related states), but from direct damages to the economy arising from the three climate-related states.

C. Perturbation solutions

Combining the zeroth- and first-order solutions, we get the following result.

Result 1: The optimal risk-adjusted SCC is (cf. (C3.19)):

$$(3.6)\qquad P = \frac{\mu\, \Theta\left(E,\chi,\lambda\right)Y\big|_{P=0}}{r^*}\left(1-\frac{\Omega(K,E,\chi,\lambda,t)}{E^{\theta_{ET}}\chi^{1+\theta_{ET}}\lambda^{1+\theta_\lambda}K^{1-\eta}}\right)+O(\epsilon^2),$$

where $\Theta \equiv D_E\left(E,\chi,\lambda\right)/\left(1-D\left(E,\chi,\lambda\right)\right)$ denotes what we call the flow damage coefficient, the zeroth-order return on capital corrected for growth is

$$(3.7)\qquad r^* \equiv r^{(0)} - g^{(0)} = \rho + (\gamma-1)(g^{(0)} - \eta\sigma_K^2/2),$$

and $r^{(0)}$ and $g^{(0)}$ are the leading-order expected rates of return on investment and economic growth. Furthermore, we have

$$(3.8)\quad \Omega = E_t\left[\int_t^\infty \Gamma(K,E,\chi,\lambda,s)e^{-r_\Omega(s-t)}ds\right] \text{with } r_\Omega \equiv r^* - (\eta-1)\left(\phi(i^{(0)}) - \eta\sigma_K^2/2\right)+\varphi,$$

$\phi \equiv \Phi/K = i - \omega i^2/2 - \delta$ and $i = I/K$. The scaled forcing in (3.8) is

$$(3.9)\qquad \begin{aligned}\Gamma(K,E,\chi,\lambda,t) \equiv \Big(&(1+\theta_{ET})\varphi \mathrm{X}\Lambda - \nu_\chi(\bar{\chi}-\chi)\mathrm{X}_\chi\Lambda - \nu_\lambda(\bar{\lambda}-\lambda)\mathrm{X}\Lambda_\lambda \\ &-\frac{1}{2}\sigma_\chi^2\mathrm{X}_{\chi\chi}\Lambda - \frac{1}{2}\mathrm{X}\Lambda_{\lambda\lambda}\sigma_\lambda^2 - (1-\eta)\mathrm{X}_\chi\Lambda\rho_{K\chi}\sigma_K\sigma_\chi \\ &-(1-\eta)\mathrm{X}\Lambda_\lambda\rho_{K\lambda}\sigma_K\sigma_\lambda - \mathrm{X}_\chi\Lambda_\lambda\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda\Big)K^{1-\eta}E^{\theta_{ET}} \\ -\theta_{ET}\mu A^{\frac{1}{\alpha}}&\left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}}\mathrm{X}\Lambda K^{2-\eta}E^{\theta_{ET}-1}e^{-g^{(0)}t} - \frac{1}{2}\theta_{ET}(\theta_{ET}-1)\sigma_E^2\mathrm{X}\Lambda K^{1-\eta}E^{\theta_{ET}-2} \\ -\Big(&(1-\eta)\theta_{ET}\mathrm{X}\Lambda\rho_{KE}\sigma_K\sigma_E + \mathrm{X}_\chi\Lambda\rho_{E\chi}\sigma_E\sigma_\chi + \mathrm{X}\Lambda_\lambda\rho_{E\lambda}\sigma_E\sigma_\lambda\Big)K^{1-\eta}E^{\theta_{ET}-1},\end{aligned}$$

where $\mathrm{X} \equiv \chi^{1+\theta_{ET}}$ and $\Lambda \equiv \lambda^{1+\theta_\lambda}$. □

The optimal risk-adjusted SCC in Result 1 is proportional to world GDP and depends directly on the stock of atmospheric carbon through the function $\Theta(E,\chi,\lambda)$. It depends on preferences (ρ, γ and η), geophysical parameters (μ, φ and ν_χ), and the properties of the stochastic processes driving the shocks to GDP, the carbon stock, climate sensitivity and damages. It is evident from Result 1 that the optimal risk-adjusted SCC does not depend directly on the share of fossil fuel in value added or the cost of fossil fuel because of the Cobb-Douglas nature of the production function (2.7). Neither does it depend on adjustment costs for physical capital or the depreciation rate of physical capital. It does depend on the growth-corrected return on capital r^* (3.7), which to leading-order can be approximated by its value in the absence of climate policy ($P=0$). Similarly, the investment rate and growth rate of GDP can, to leading-order, be approximated by their values in the absence of climate policy (cf. (C7)):

$$(3.10)\quad i^{(0)} = Y_K\big|_{P=0} - q^{(0)}\left(\rho + (\gamma-1)\left(\phi(i^{(0)}) - \frac{1}{2}\eta\sigma_K^2\right)\right)$$

with $Y_K\big|_{P=0} = \alpha A(E,\chi,\lambda)^{\frac{1}{\alpha}}\left((1-\alpha)/b\right)^{\frac{1-\alpha}{\alpha}}$ and $g^{(0)} = i^{(0)} - \omega\left(i^{(0)}\right)^2 \Big/ 2 - \delta \equiv \phi(i^{(0)})$.

These follow from the Keynes-Ramsey rule and the capital accumulation equation and subsequently give the price of capital, Tobin's q, as $q(i) = 1/\phi'(i)$. The expected return on investment $r^{(0)}$ equals the sum of the risk-free rate, denoted by $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - (1+\gamma)\eta\sigma_K^2/2,$ and the risk premium $\eta\sigma_K^2$.

Result 1 indicates that the absolute error in our expression for the optimal risk-adjusted SCC is $O(\epsilon^2)$ and that the error as fraction of the SCC (itself an $O(\epsilon)$ quantity) is $O(\epsilon)$. Consistently, we ignore the slow dependence of the discount rate on the atmospheric carbon stock, via the marginal productivity of capital, when evaluating the discounting integral in (3.8). As $\epsilon \to 0,$ the optimal SCC in Result 1 becomes exact. Generally, a closed-form solution to the time integral and the expectations operator over the four stochastic states in (3.8) is unavailable, so that Result 1 must be evaluated numerically. This requires five-dimensional numerical integration over the probability space corresponding to the four states and with respect to time.[20] Although such high-dimensional numerical integration is less challenging and computationally demanding than numerical solution of the partial differential equations describing the value function, we can still make considerable analytical headway while introducing only minimal quantitative errors. To do so, we consider only the leading-order effects of uncertainty in section III. In section V.E we demonstrate the numerical accuracy of the resulting tractable rules by comparing them with the numerically exact evaluation of Result 1.

III. The Optimal Risk-Adjusted SCC: Leading-Order Effects of Uncertainty

To obtain closed-form solutions for the risk-adjusted SCC described by the multi-dimensional integral (3.8) in Result 1, we make two additional assumptions. First, we only consider up to leading-order terms in the climatic and damage uncertainties σ_χ^2 and σ_λ^2 and their covariance terms, including with the capital stock (assumption I) . Second, the stochastic climate sensitivity and climate damage parameters are initially at their equilibrium values ($\hat{\chi}_0 \equiv \chi_0/\bar{\chi} = 1$ and $\hat{\lambda}_0 \equiv \lambda_0/\bar{\lambda}$ $= 1$) (assumption II). Appendix E implements these assumptions to Result 1 to give Result 2′ in Appendix A. Here we present the case with proportional reduced-form damages, normally distributed damage uncertainty and no carbon stock volatility.

[20] If the processes are independent, the integrals over the probability space of states can be evaluated independently.

Result 2: If $\theta_{ET} = 0,\ \theta_{\lambda} = 0$ and $\sigma_E = 0,$ the leading-order optimal SCC is

$$(4.1)\qquad P = \frac{\mu\, \Theta Y\big|_{P=0}}{r^{\star}} \left(1 + \frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\frac{(\sigma_{\chi}/\bar{\chi})^2}{r^{\star}+2\nu_{\chi}} + \Delta_{\text{CK}} + \Delta_{\text{CC}} \right)$$

with $\Delta_{\text{CK}} = -(\eta-1)\sigma_K \left((1+\theta_{\chi T})\frac{\rho_{K\chi}\sigma_{\chi}/\bar{\chi}}{r^{\star}+\nu_{\chi}} + \frac{\rho_{K\lambda}\sigma_{\lambda}/\bar{\lambda}}{r^{\star}+\nu_{\lambda}} \right), \Delta_{\text{CC}} = (1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\sigma_{\chi}\sigma_{\lambda}/\bar{\chi}\bar{\lambda}}{r^{\star}+\nu_{\chi}+\nu_{\lambda}}$

and the discount rate corrected for economic growth, economic growth uncertainty and atmospheric decay is $r^{\star} = \rho + (\gamma-1)(g^{(0)} - \frac{1}{2}\eta\sigma_K^2) + \varphi.$ □

Result 2′ in Appendix A allows for convex damages, skewed damage uncertainty and carbon stock volatility. This leaves the structure of (4.1) intact, but includes correction factors that modify the magnitude of terms, but not the interpretation.

A. The optimal SCC in the absence of economic and climate uncertainty

Without uncertainty, Result 2 gives $P = \mu\, \Theta(E) Y\big|_{P=0} / r^{\star}$ with $r^{\star} = \rho + (\gamma-1)g^{(0)} + \varphi$ for the deterministic optimal SCC. This expression has the same geophysical (μ and φ), economic (Y and g), damage (Θ) and ethical (ρ and γ) determinants as those found in macro models of growth and climate change with Ramsey instead of AK growth. More patience (lower ρ), wealthier future generations (higher $g^{(0)}$ for $\gamma < 1$), and lower intergenerational inequality aversion (lower γ) curb the discount rate and push up the SCC. Rising affluence (higher $g^{(0)}$) pushes up the discount rate, especially if intergenerational inequality aversion is large, and thus reduces the appetite of current generations for ambitious climate policy (the $+\gamma g^{(0)}$ term in $r^{\star}$). Also, with damages proportional to GDP, rising affluence implies a higher growth of damages and a lower growth-corrected discount rate (the $-g^{(0)}$ term in $r^{\star}$), which increases the optimal SCC. Higher economic activity (Y) and a higher flow damage coefficient (Θ) also push up the SCC. A smaller fraction of emissions that goes into the atmosphere (smaller μ) and faster rate of decay of atmospheric carbon (higher φ) depress the SCC.

B. Economic growth uncertainty and the climate beta

Including economic, but not climatic uncertainty, Result 2 gives $P = \mu\,\Theta(E)Y\big|_{P=0} / r^{\star}$ but now with the risk-adjusted discount rate $r^{\star} = \rho + (\gamma - 1)(g^{(0)} - \eta\sigma_K^2/2) + \varphi.$ The estimate of future economic growth is thus reduced to take account of its uncertain nature with risk aversion η. If the rising-affluence dominates the growing-damages effect, growth uncertainty depresses the discount rate and pushes up the risk-adjusted SCC. We decompose the effects on the risk-adjusted discount rate as follows:

$$(4.2)\quad r^{\star} = \underbrace{\rho}_{\text{time impatience}} + \underbrace{\gamma g^{(0)}}_{\text{rising affluence}} - \underbrace{g^{(0)}}_{\text{growing damages}} \underbrace{-\frac{1}{2}(1+\gamma)\eta\sigma_K^2}_{\text{prudence}} + \underbrace{\eta\sigma_K^2}_{\text{insurance}} + \underbrace{\varphi}_{\text{decay atmospheric carbon}}.$$

The first three terms were discussed in section III.A. The prudence term is proportional to the coefficients of relative prudence $\mathrm{CRP} = 1+\gamma$, and risk aversion η, and to economic growth uncertainty (cf. Leland, 1968; Kimball, 1990). The insurance term stems from the perfect correlation between damages and GDP, because damages in our model are proportional to GDP. The insurance term acts to increase the optimal discount and reduce the optimal risk-adjusted SCC, reflecting that positive shocks to damages are associated with positive shocks to GDP and thus less harmful to welfare. This corresponds to a "built-in" climate beta of one.[21] For $\gamma > 1$, the prudence term is dominant, the optimal discount rate reduces, and the

[21] Dietz et al. (2018) use Monte Carlo simulations of DICE (Nordhaus, 2008) and find that the climate beta is close to one if damages are proportional to GDP, but closer to zero if damages are additive. Our section III.D analyzes correlated risks and climate betas more generally.

optimal risk-adjusted carbon price increases with growth uncertainty, and vice-versa for $\gamma < 1$. If utility is logarithmic as in Golosov et al. (2014), $\gamma = \eta = 1$ and $r^{\star} = \rho + \varphi,$ so that uncertainty about the rate of economic growth does not have any impact on the risk-adjusted SCC (cf. Golosov et al., 2014).

If damages are additive and do not rise in proportion to GDP, the correction to the discount rate only consists of the prudence term. More generally, if $0 \le \theta_D \le 1$ denotes the elasticity of damages with respect to GDP, the growth-corrected, risk-adjusted discount rate becomes (Svensen and Traeger, 2014):

$$(4.3) \qquad r^{\star} = \rho + (\gamma - \theta_D) g^{(0)} - \frac{1}{2}(1 + \gamma - 2\theta_D)\eta\sigma_K^2 + \varphi.$$

If the elasticity of damages with respect to GDP is $\theta_D = 1/2,$ the effect of growing damages is halved for $\gamma = 1$ (cf. $-\theta_D g^{(0)}$ in (4.3)). A smaller elasticity of damages with respect to GDP acts to decrease the insurance term and thus induces a lower risk-adjusted discount rate and higher carbon price. Whether GDP uncertainty increases the carbon price depends on whether the coefficient of relative prudence exceeds twice the elasticity of damages with respect to GDP: $\mathrm{CRP} = 1 + \gamma > 2\theta_D$ (Jensen and Traeger, 2014, their equation (9)). With multiplicative damages as in Results 1 and 2 ($\theta_D = 1$), this condition reduces to $\gamma > 1$.

We have abstracted from long-run risk in economic growth (Bansal and Yaron, 2004).[22] It has been shown numerically that including this long-run risk pushes up the optimal risk-adjusted SCC by a factor 2 or 3 if aversion to risk exceeds aversion to intertemporal fluctuations (Bansal et al., 2016).

[22] Epstein et al. (2014) argue that long-run risk and a preference of early resolution of uncertainty implies that the timing premium needed to calibrate asset returns is implausibly high (20-30%).

C. Climate and damage uncertainties

The *climate sensitivity risk correction* $(1/2)\theta_{\chi T}(1+\theta_{\chi T})(\sigma_\chi / \bar{\chi})^2 / (r^\star + 2\nu_\chi)$ in (4.1) depends on $\theta_{\chi T} \equiv \theta_\chi + \theta_T + \theta_\chi \theta_T$, which combines the generally convex dependence of temperature on the normally distributed climate sensitivity parameter ($\theta_\chi > 0$, cf. (2.5)) and the generally convex dependence of damages on temperature ($\theta_T > 0$). The former captures the (positive) skewness of the (equilibrium) climate sensitivity distribution. The climate sensitivity uncertainty correction is thus positive and larger for a more convex damage function, a more (positively) skewed climate sensitivity distribution, a lower rate of mean reversion, greater uncertainty of climate sensitivity, and if the growth-corrected discount rate is smaller (higher θ_T, θ_χ, lower ν_χ, higher σ_χ and lower $r^\star$). There is no corresponding damage uncertainty correction in Result 2, which is consequence of damage uncertainty being normally distributed (i.e., $\theta_\lambda = 0$).

D. Climate betas: correlated climate, damage and economic growth risks

The term in Result 2 correcting for correlations between climate and damage risks, on the one hand, and economic risks, on the other hand, can be rewritten as

$$(4.4) \qquad \Delta_{CK} = -(\eta - 1)\sigma_K^2 \left(\left(1+\theta_{\chi T}\right) \frac{\beta_{K\chi}}{r^\star + \nu_\chi} + \left(1+\theta_\lambda\right) \frac{\beta_{K\lambda}}{r^\star + \nu_\lambda} \right),$$

where $\beta_{K\chi} \equiv \rho_{K\chi}\sigma_\chi / \bar{\chi}\sigma_K$ and $\beta_{K\lambda} \equiv \rho_{K\lambda}\sigma_\lambda / \bar{\lambda}\sigma_K$ denote the climate-sensitivity and climate-damage beta, respectively. These climate betas measure the normalized correlation with shocks to the rate of economic growth in direct analogy with the definition of beta in asset pricing theory (e.g., Lucas, 1978; Breeden, 1979).[23] The

[23] Consistent with our perturbation scheme, the volatility of total GDP is given to leading order by the volatility of the capital stock neglecting the effect of climate damages and thus the carbon stock, climate sensitivity and damage uncertainties.

sign of (4.4) depends on whether relative risk aversion η exceeds one or not, i.e., on whether the *risk-insurance* effect dominates the *risk-exposure* effect or not (cf. Lemoine, 2017). If climate sensitivity and economic growth are positively correlated ($\beta_{K\chi} > 0$), the risk-insurance effect pushes down the risk-adjusted SCC, and more so if relative risk aversion is high, climate sensitivity displays less mean reversion, and the climate-sensitivity beta is large (high η, low ν_χ high $\beta_{K\chi}$). Furthermore, the factor $(1+\theta_{\chi T})$ reflects the increase in the (co-) variance resulting from the power-law dependence of damages on the climate sensitivity parameter χ.. The risk-insurance effect is greater for a more convex damage functions (high θ_T) and a more skew climate sensitivity (high θ_χ), but remains non-zero even for a symmetric climate sensitivity distribution and linear damages ($\theta_\chi = \theta_T = 0$). The risk-exposure effect acts in the opposite direction: if the climate sensitivity parameter χ (and temperature) is high and $\beta_{K\chi} > 0$, the adverse effects (on GDP) are amplified due to the multiplicative nature of damages, requiring a rise in the carbon price. The risk-insurance effect dominates the risk-exposure effect if $\eta > 1$.

Finally, the term $\Delta_{\mathrm{CC}} = (1+\theta_{\chi T})\,\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda \big/ \bar{\chi}\bar{\lambda}\,(r^\star + \nu_\chi + \nu_\lambda)$ in Result 2 captures the correlation between climate sensitivity and damage uncertainty: risk aversion η plays no role as there is no insurance possibility via the economic growth channel. If climate sensitivity uncertainty and damage uncertainty are positively correlated, this term is positive and thus the risk-adjusted SCC is pushed upwards.

E. Special case: logarithmic preferences

With logarithmic preferences and proportional reduced-form damages, $\gamma = \eta = 1$ and $\theta_{ET} = 0$ (cf. Golosov et al., 2014), we have $\Delta_{\mathrm{CK}} = 0$ and Result 2 becomes

$$(4.5)\qquad P=\frac{\mu\,\Theta Y|_{P=0}}{\rho+\varphi}\left(1+(1+\theta_{\chi T})(\sigma_\chi/\bar{\chi})\left[\frac{1}{2}\theta_{\chi T}\,\frac{(\sigma_\chi/\bar{\chi})}{\rho+\varphi+2\nu_\chi}+\frac{\rho_{\chi\lambda}(\sigma_\lambda/\bar{\lambda})}{\rho+\varphi+\nu_\chi+\nu_\lambda}\right]\right).$$

Hence, economic growth uncertainty and the climate betas do not affect the optimal risk-adjusted SCC, but climatic uncertainty and correlated risks for climate sensitivity and damage uncertainty do.

IV. Calibration

Table 1 summarizes the details of our calibration with further details in Appendix F. To calibrate the non-climatic part of our model to match historical asset returns, we follow Pindyck and Wang (2013) but abstract from catastrophic shocks to economic growth (see Appendices F.1 and F.2). We note that carbon stock volatility is extremely small.[24] By setting $\theta_E=-0.5$ and $\theta_T=1,$ we have proportional reduced-form damages with $\theta_{ET}=0$ (see Appendices F.3 and F.4). By setting $\theta_T=1.5,$ we have convex reduced-form damages with $\theta_{ET}=0.25.$ We let damage uncertainty be normally distributed, so $\theta_\lambda=0,$ with mean μ_λ and standard deviation Σ_λ calibrated to the estimates surveyed in Tol (2009) and associate these with the steady-state distribution, so $\mu_\lambda=\bar{\lambda}$ and $\Sigma_\lambda^2=\sigma_\lambda^2/2\nu_\lambda$ (see Appendix F.5), setting $\nu_\lambda=20\%$/year. The flow damage coefficient $\Theta\equiv D_E/(1-D)$ with μ_χ and θ_χ set to our base case values is approximately constant at 2.6% GDP/TtC for proportional damages but starts at 3.2% of GDP/TtC and then rises with global

[24] Using the same dataset, but considering a Geometric Brownian Motion for the atmospheric carbon concentrations above pre-industrial level instead of the Arithmetic Brownian Motion considered here, Hambel et al. (2017) find a much larger volatility of 0.78 %/year$^{1/2}$. Estimating this volatility, we find 1.4, 0.5 and 0.2 %/year$^{1/2}$ for the periods 1800-2004, 1900-2004 and 1959-2004. This large variation of volatility with time suggest that historical volatility in the atmospheric carbon concentrations is better described by an Arithmetic Brownian Motion, as in (2.4).

warming for convex damages. For comparison, Golosov et al. (2014) have Θ = 3.64% GDP/TtC, which includes a markup for tipping risk.

TABLE 1 – SUMMARY OF THE BASE CASE CALIBRATION

Rate of impatience	ρ = 5.75%/year
Intertemporal substitution (inverse of intergenerational inequality aversion)	EIS = 1/ IIA=1/γ = 0.67
Attitudes to risk	RRA = η = 4.32
World economy	A^* = 0.113 /year, GDP = \$75 trillion/year, $g^{(0)}$ = 2.0 %/year
Investment and adjustment cost	$I_0^{(0)} / Y_0^{(0)}$ = 24.3%, $i^{(0)}$ = 2.75%/year, δ = 0.28%/year, ω = 12.3 year
Asset volatility and returns	σ_K = 12.13%/year$^{1/2}$, $r^{(0)}$ = 7.16%/year, $r_{\text{rf}}^{(0)}$ = 0.80%/year, $r^{(0)} - r_{\text{rf}}^{(0)} = \eta\sigma_K^2$ = 6.36%/year
Share of fossil fuel in value added Production cost of fossil fuel	$1 - \alpha$ = 6.6% b = \$5.4×10^2 /tC
Pre-industrial and 2015 (t = 0) carbon stocks Stochastic carbon stock dynamics Concavity temperature function	E_{PI} = 596 GtC, S_0 = 854 GtC, E_0 = 258 GtC, μ = 1.0, φ = 0.66%/year, σ_E = 0.31 GtC/year$^{1/2}$ θ_E = –0.5
Convexity and mean reversion damages Mean and standard deviation of damage uncertainty	$\theta_\lambda = 0$, ν_λ = 20%/year Proportional damages: $\theta_T = 1$, $\mu_\lambda = 2.2\times10^{-3}$, $\Sigma_\lambda = 1.6\times10^{-3}$ Convex damages: $\theta_T = 1.5$, $\mu_\lambda = 1.6\times10^{-3}$, $\Sigma_\lambda = 1.0\times10^{-3}$
Flow impact global warming damages	Proportional damages: Θ_0 = 2.63% GDP/TtC Convex damages: Θ_0 = 3.16% GDP/TtC
Distribution of the ECS	$\mu_\chi = 1.9$, $\Sigma_\chi = 0.95$, σ_χ =11%/year$^{1/2}$, ν_χ = 0.66 %/year, $\theta_\chi = 0.59$, $\theta_{\chi T} = 2.2$ and 3.0 for proportional and convex damages, respectively
Distribution of the TCR	$\mu_\chi = 1.75$, $\Sigma_\chi = 0.38$, σ_χ =4.5%/year$^{1/2}$, $\nu_\chi \to 0$, $\theta_\chi = 0$, $\theta_{\chi T} = 1.0$ and 1.5 for proportional and convex damages

The equilibrium climate sensitivity (ECS) is defined as the equilibrium change in annual mean global temperature following a doubling of the atmospheric carbon stock relative to pre-industrial levels. From (2.5) the climate sensitivity is $T_2 \equiv T(E = E_{PI}, \chi) = \chi^{1+\theta_\chi}$, where χ is normally distributed with mean μ_χ and standard deviation Σ_χ, and θ_χ is chosen to match the skewness of the climate sensitivity T_2. We fit the distribution of T_2 to a range of estimates for the ECS, thereby getting close to the thin-tailed Gamma distribution of Pindyck (2012). This yields $E\left[T_2\right] = 3.0°\text{C}$, $\text{var}\left[T_2\right] = 4.5°\text{C}^2$ and $\text{skew}\left[T_2\right] = 10°\text{C}^3$, indicating a right-skewed equilibrium distribution. We estimate the rate at which this skew equilibrium distribution is reached at $\nu_\chi = 0.66\%$/year, capturing that the ECS and its associated skewness occurs on time scales of a few centuries. For comparison, we also fit the non-skew transient climate response (TCR), which is defined as the change in annual mean global temperature at the time of doubling following a linear increase in the carbon stock (IPCC, 2013). Matching information from Figure 10.20 and Chapter 10 of IPCC (2013), we obtain $E\left[T_2\right] = 1.75°\text{C}$, $\text{var}\left[T_2\right] = 0.15°\text{C}^2$ and $\text{skew}\left[T_2\right] = 0$, indicating a mean TCR of $1.75\,°\text{C}$ and a normal (non-skew) distribution. Table 2 compares our ECS and TCR calibrations. The skewness, despite being a long-run feature only, is the most important driver of the risk-adjusted SCC, and we adopt the ECS calibration in our base case (Appendix F.6).

TABLE 2 – TWO WAYS OF CALIBRATING CLIMATE SENSITIVITY

	ECS (steady state)	TCR (after 70 years)
$E\left[T_2\right]$	3.0 °C	1.75 °C
$\text{var}\left[T_2\right]$	4.5 °C^2	0.15 °C^2
$\text{skew}\left[T_2\right]$	10 °C^3	0
$\text{skew}^*\left[T_2\right]$	1.0	0

V. Quantification of Effects of Economic and Climatic Risks on Optimal SCC

Table 3 gives the deterministic and the risk-adjusted SCC[25] for the two ways of calibrating climate sensitivity. The effect of atmospheric carbon stock uncertainty is identically zero or numerically negligible with, respectively, proportional and convex reduced-form damages. The markup for climate sensitivity uncertainty on the deterministic optimal carbon price varies from 2% for the TCR calibration and 22% for the ECS calibration for proportional damages or 3% (TCR) and 37% (ECS) for the convex variant. The ECS calibration leads to a larger upward adjustment of the price due to the marked skewness of the distribution in the ECS, which is not present in the distribution of the TCR.[26] The deterministic SCC for the TCR is also much lower due to the lower temperature rise associated with this climate sensitivity and the lower curvature of the flow damage coefficient.[27]

TABLE 3 – EFFECTS OF RISK ON THE OPTIMAL SCC

Damages	Proportional			Convex		
Carbon price ($/t$CO_2$)	base case	TCR	$\nu_\chi = \infty$	base case	TCR	$\nu_\chi = \infty$
Deterministic carbon price	7.27	3.13	7.27	9.38	3.28	9.38
due to economic growth uncertainty	1.99	0.86	1.99	2.11	0.74	2.11
due to carbon stock uncertainty	0.00	0.00	0.00	0.00	0.00	0.00
due to climate sensitivity uncertainty	1.56	0.05	8.45	3.46	0.09	17.93
due to climate damage uncertainty	0	0	0	0	0	0
Total risk-adjusted carbon price (Total risk markup) (Climate risk markup)	10.81 (49%) (22%)	4.04 (29%) (2%)	17.70 (144%) (116%)	14.96 (59%) (37%)	4.11 (25%) (3%)	29.43 (214%) (191%)

[25] We use Result 2′ and refer to the definitions of $r^\star$ and other variables to Appendix A.

[26] If we calibrate the climate sensitivity to the transient response to cumulative emissions (see Appendix F.6.2), we obtain for the case of proportional damages a deterministic SCC of $4.13/t$CO_2$ and markups for economic and climatic risk of 30% and 6%, respectively, thus leading to a risk-adjusted carbon price of 5.71/tCO_2.

[27] For the TCR calibration, Θ_0 is 1.13% (proportional) or 1.11% of GDP/TtC (convex damages).

Although the damage specification of Nordhaus (2008) is associated with a curvature that is approximately constant at $\theta_T = 1$, corresponding to our proportional damages case ($\theta_{ET} = 0$), the curvature of the specification by Ackerman and Stanton (2012) increases rapidly after approximately 1 °C of warming to a value of 4. The effect of the degree of convexity θ_T is considerable, as illustrated by the increase of the markup from 22% to 37% between $\theta_T = 1.0$ and $\theta_T = 1.5$. An even greater convexity of the generally poorly understood and ad-hoc damage function is not inconceivable.

The magnitude of the markup for the uncertain nature of the skew equilibrium climate sensitivity is determined by the time scale over which the equilibrium is reached. In our base case calibration, we have an e-folding time of $1/\nu_\chi = 1.5 \times 10^2$ years, whereas Ricke and Caldeira (2014) argue that temperature rises very quickly, on the time scale of a decade, after a (small) carbon impulse. An upper limit to the climate sensitivity uncertainty correction corresponds to the limit $\nu_\chi \to \infty$, in which the skew equilibrium sensitivity can be thought to arrive instantaneously following emissions. This gives total risk-adjusted carbon prices of \$17.70 and \$29.43 per ton of CO_2 and climate risk markups of 116% and 191%, for proportional and convex damages, respectively. Although climate damages are subject to considerable uncertainty, there seems no evidence for non-zero skewness, resulting in no correction to the optimal carbon price. The main effect of climate damage uncertainty is through correlation with GDP, as discussed in section V.B.

A. Economic determinants of the optimal carbon price

Table 4 shows the effects of preferences and economic growth on the risk-adjusted SCC and its stochastic drivers. We use the more convex damage variant and the ECS calibration as a base. We recall that the optimal discount rate in the absence of stochastic climate corrections is $r^{*} = \rho + (\gamma - 1)(g^{(0)} - \eta\sigma_K^2/2) + (1+\theta_{ET})\varphi$ (from Result 2′ in Appendix A). It is evident that RRA = η has a large downward effect on the discount rate and upward effect on the SCC if economic uncertainty is high (and $\gamma > 1$). If RRA is 10 instead of 4.32 (base case), we obtain a much higher economic uncertainty correction of \$7.46 per ton of CO_2. The correction for climate uncertainty as a share of the deterministic SCC more than doubles. The total risk markup is much higher: 157% instead of 59% (second column). A higher aversion to intergenerational inequality, e.g., an IIA of 3 instead of 1.5, pushes down the deterministic SCC from \$9.38 to \$6.57 per ton of CO_2, but pushes up the correction for economic uncertainty from \$2.11 to \$9.77 per ton of CO_2 (third column), but less so if economic uncertainty is lower (fourth column). For an IIA of 3, the correction for climate sensitivity uncertainty is pushed up from \$3.46 to \$6.53 per ton of CO_2, the risk markup to 248%, and the risk-adjusted SCC becomes \$22.87 per ton of CO_2.

TABLE 4 –IMPACT OF RRA, IIA AND CORRELATED RISKS

Carbon price (\$/tCO_2) (ECS calibration)	base case	RRA = 10	IIA = 3	Annual σ_K = 1.5%	"Climate Beta" θ_D = 0.5	"Climate Beta" $\rho_{K\lambda}$ = 0.5	$\rho_{\chi\lambda}$ = 0.5
Deterministic	9.38	9.38	6.57	9.38	9.38	9.38	9.38
due to economic growth uncertainty	2.11	7.46	9.77	0.02	14.67	2.11	2.11
due to climate sensitivity uncertainty	3.46	7.30	6.53	2.33	13.13	3.46	3.46
due to climate damage uncertainty	0	0	0	0	0	–3.73	2.04
Total risk-adjusted (Total risk markup) (Climate risk markup)	14.96 (59%) (37%)	24.14 (157%) (78%)	22.87 (248%) (100%)	11.74 (25%) (25%)	37.18 (296%) (140%)	11.23 (20%) (-3%)	17.00 (81%) (59%)

Our calibration is based on historical asset returns. If instead, we calibrate based on historical GDP, a much smaller annual volatility of 1.5% is appropriate (cf. Hambel, et al., 2017). Hence, the correction for economic growth uncertainty shrinks from \$2.11 to a mere \$0.02 per ton of CO_2. As a result of the smaller downward correction of the discount rate, the correction to allow for the risk of climate sensitivity uncertainty is cut from \$3.46 to \$2.33 per ton of CO_2 (fourth column) due to an increase in the growth-corrected discount rate $r^\star$. The risk-adjusted SCC drops from \$14.96 to \$11.74 per ton of CO_2, corresponding to a total risk markup of only 25% instead of 59%.

The coefficient of relative risk aversion RRA has a large effect in our base case calibration, which is due to the large volatility of asset returns. For the much smaller volatility of historical GDP of 1.5%, the correction for economic growth uncertainty only increases from \$0.02 to \$0.05 per ton of CO_2 as RRA is increased to 10 (not shown in Table 4). This accords with Crost and Traeger (2013), Ackerman et al. (2013) and Hambel et al. (2017), who all use a small value for economic uncertainty and find that RRA only has a small and that IIA has a large effect on the risk-adjusted SCC (which follows from our growth-corrected discount rate $r^\star = \rho + (\gamma - 1)(g^{(0)} - \eta\sigma_K^2/2) + (1 + \theta_{ET})\varphi$).

B. Climate betas

If damages are not proportional to world GDP, but instead the elasticity of damages with respect to world GDP is only a half, we can compute the effect of setting the "climate beta" θ_D to 0.5 instead of 1 using the ad-hoc modification (4.6). Since damage shocks are no longer automatically insured against by their direct proportionality with GDP, the optimal growth-corrected discount rate $r^\star$ drops, as reflected by a much larger correction for uncertain economic growth (we keep the deterministic price fixed). Because of the lower discount rate, the climate sensitivity uncertainty correction rises significantly. The risk-adjusted SCC thus increases to \$37.18 per tCO_2 corresponding to a very large risk markup of 296% (Table 4, fifth column). A more direct approach to the climate beta is to consider the correlation structure of the uncertain processes themselves. For example, if we set $\rho_{K\lambda} = 0.5$ to capture that adverse damage shocks in the future are more likely if economic growth is high, the SCC must be adjusted downwards by \$3.73 per tCO_2 due to the dominance of the insurance effect for $\text{RRA} > 1$ (Table 4, sixth column). Alternatively, if we set $\rho_{\chi\lambda} = 0.5$ to capture that an adverse future climate sensitivity shock might be associated with an adverse future damage shock, the SCC is pushed up by \$2.04 per tCO_2 (seventh column). Given the very small carbon stock uncertainty, the effects of $\rho_{EK}, \rho_{E\chi}$ and $\rho_{E\lambda}$ are negligible.

C. Comparison with other calibrations

Table 5 compares our base case results to common alternatives, which rely on ethical arguments to use much lower discount rates than derived from asset market returns (e.g., Gollier, 2018).[28] In contrast to the base case, we assume a low economic growth volatility of 1.5 %/year$^{1/2}$, based on GDP instead of asset return volatility, for all alternative calibrations in Table 5 except those in bold. We adopt the calibration based on the ECS and proportional damages for all alternative calibrations except for Stern. As all corrections for damage and carbon stock uncertainty are zero or negligible, we do not show these rows. Golosov et al. (2014) (GHKT) adopt logarithmic utility, IIA = RRA = 1, and ρ = 1.5% per year. From (3.7) then, neither the expected rate of growth nor the uncertainty of the future rate of economic growth influences the optimal SCC. The growth-corrected discount

[28] To analyse this properly, the government should maximize expected welfare using low ethically motivated discount rates, subject to the constraints of the decentralized market economy calibrated to higher asset returns. The optimal carbon price will then typically fall short of the social cost of carbon (Belfiori, 2017; Barrage, 2018).

rate $r^{\star}$ is only 2.16% instead of 5.82% per year, and thus the deterministic SCC is almost fourfold under GHKT. Since the discount rate is so much lower, the adjustments for climate risk are 5-6 times higher, but less so if expressed as share of the deterministic price. The risk-adjusted SCC under GHKT is more than three times as high as in our base ($33.55 instead of $10.81 per tCO_2).

TABLE 5 – COMPARISONS WITH OTHER CALIBRATIONS

Carbon price ($/$tCO_2$)	base case $\theta_T = 1$	base case $\theta_T = 1.5$	GHKT $\theta_T = 1$	Gollier $\theta_T = 1$	Nordhaus $\theta_T = 1$	Stern $\theta_T = 1$	Stern $\theta_T = 1$	Stern $\theta_T = 1.5$	Stern $\theta_T = 1.5$
Deterministic	**7.27**	**9.38**	24.91	20.23	17.58	32.41	**32.41**	46.97	**46.97**
due to economic growth uncertainty	**1.99**	**2.11**	0	0.17	0.04	0.14	**13.19**	0.13	**6.52**
due to climate sensitivity uncertainty	**1.56**	**3.46**	8.64	6.22	4.86	13.21	**22.01**	34.66	**44.33**
Total risk-adjusted	**10.81**	**14.96**	33.55	26.62	22.49	45.77	**67.62**	81.76	**97.82**
(Total risk markup)	**(49%)**	**(59%)**	(35%)	(32%)	(28%)	(41%)	**(109%)**	(74%)	**(108%)**
(Climate risk markup)	**(22%)**	**(37%)**	(35%)	(31%)	(28%)	(41%)	**(68%)**	(74%)	**(94%)**

Key: The base case has proportional damages, ECS calibration and annual economic volatility of 12.1%. GHKT is based on Golosov et al. (2014) and has IIA = RRA = 1 and ρ = 1.5%/year. Gollier (2012) has IIA = RRA = 2, ρ = 0. Nordhaus (2008) has IIA = RRA = 1.45 and ρ = 1.5%/year. Stern (2007) has IIA = RRA = 1.45 and ρ = 0.1 %/year. The last two columns have convex damages with $\theta_T = 1.5$. The columns in bold use an annual asset volatility of 12.1%.

Gollier (2012) focuses on the risk-adjusted discount rate. He suggests using RRA = IIA = 2 and ρ = 0, so that $r^{\star}$ becomes 2.64% per year. As this is more than for GHKT, but less than for our base case, the deterministic SCC is higher than for the base but lower than for GHKT. Intergenerational inequality aversion (IIA) now exceeds one. Thus, there is a positive adjustment for the carbon price to take account of uncertain economic growth, but it is small given that volatility is calculated from GDP instead of asset returns. The adjustment for climate sensitivity uncertainty is approximately 4 times higher than for the base case due to the low discount rate. The risk-adjusted SCC is much higher, but less than under GHKT.

The integrated assessment model DICE developed by Nordhaus (2008) has IIA = RRA = 1.45, ρ = 1.5% per year and thus a growth-corrected discount rate $r^{\star}$ of 3.05% per year ($\theta_T = 1$). As a result, the deterministic SCC, the correction for climate sensitivity risk and the fully risk-adjusted SCC are lower than under Gollier and GHKT, but higher than for the base case. The final four columns change the discount rate to a much lower value, as may be justified on ethical grounds (cf. Stern, 2007). Setting $\rho = 0.1\%$ per year and keeping IIA = RRA = 1.45 gives a risk-adjusted discount rate $r^{\star}$ of 1.65% per year, which gives a risk-adjusted SCC of \$45.77 per tCO_2, as illustrated in the first of these four columns. Focusing on the Stern variant, the next column in bold indicates that, if economic volatility is calculated from asset returns instead of GDP, both the correction for economic growth uncertainty and, due to the lower risk-adjusted discount rate, the correction for climate sensitivity uncertainty rise substantially. As a result, the risk-adjusted SCC is pushed up from \$45.77 to \$67.62 per tCO_2. The last two columns show a more realistic version of the Stern variant, namely with convex damages. These boost the deterministic SCC from \$32.41 to \$46.97 per tCO_2. And the risk-adjusted SCC from \$45.77 to \$81.76 per tCO_2 if economic volatility is calibrated from GDP and from \$67.62 to \$97.82 per tCO_2 if it is calibrated based on asset returns.

D. Accuracy of the tractable rule for the optimal risk-adjusted carbon price

To assess the accuracy of the approximations made in Result 2 and 2′ used in Tables 3-5, we evaluate Result 1 numerically (see Appendix G for details). For all possible calibrations considered, the error is small (less than 2.6% for convex damages but less than 0.3% for proportional damages). Crucially, the effect of ignoring carbon stock uncertainty arising from uncertain future emissions in Result 2′ (see Appendix A) is negligibly small.

VI. Conclusions

Using asymptotic methods, we have derived a tractable rule for the optimal risk-adjusted SCC under a range of economic and climatic uncertainties allowing for the convexity of global warming damages and the skewness of shocks to the climate sensitivity and global warming damages, and the time scales on which they arise. This gives insight into the ethical determinants and the stochastic economic and geophysical drivers of the optimal carbon price and is a very good approximation to our more fundamental result, which only requires that climate damages are a small percentage of world GDP (say, less than 10%). Our rule offers a powerful analytical complement to insights that could hitherto only be derived from numerical solutions of systems of stochastic differential equations.

With damages proportional to the carbon stock, our optimal SCC is also proportional to world GDP. However, if damages are convex, the proportion rises over time as global warming increases. The rate used to discount marginal damages must be corrected for the various economic, climate and damage risks. The risk-adjusted SCC increases in risk aversion but decreases in intergenerational inequality aversion. The effect of risk aversion is quantitatively much smaller. If the elasticity of damages with respect to world GDP is less than one, the climate beta is less than one and the risk-adjusted SCC is higher.

Taking account of uncertainty in the carbon stock dynamics leads to negligible adjustment of the optimal SCC. Uncertain climate sensitivity does act to increase the SCC significantly, especially if allowance is made for the skewness of the equilibrium climate sensitivity distribution. Crucially, only the equilibrium climate sensitivity is associated with significant skewness, and the role this plays in determining the optimal SCC depends strongly on the time horizon over which this equilibrium is reached. Taking account of the uncertainty about the economic impact of damages does not affect the optimal SCC, unless this distribution is skew, for which we have not found a-priori evidence.

Furthermore, our solutions allow insight into the origins of the overall climate beta by allowing for correlated risks in the economic growth rate, the carbon stock, the climate sensitivity and damages. The risk-adjusted SCC is pushed up if stochastic shocks to the climate sensitivity or the climate damage parameter are negatively correlated with shocks to the future rate of economic growth, provided the degree of risk aversion exceeds one. If shocks to damages are negatively correlated with stochastic shocks to the future rate of economic growth, the corresponding climate beta shows by how much more carbon should be priced. The directions of these effects reverse if the risk-exposure effect dominates the risk-insurance effect, i.e., if the coefficient of relative risk aversion is less than one. These effects do not depend on intergenerational inequality aversion and they do not impact the optimal SCC if risk aversion equals one.

Our quantitative results suggest that with convex damages the markups on the deterministic SCC for economic and climate sensitivity uncertainties are 22% and 37%, respectively, giving a risk-adjusted SCC of \$15/tCO_2. However, if the elasticity of damages from global warming with respect to world GDP is ½ instead of 1, these markups are 156% and 140%, respectively. The SCC thus more than doubles to \$37/tCO_2 as world GDP acts less as insurance. If the correlation coefficient between economic and damage uncertainties is ½, capturing that adverse damage shocks in the future are more likely if economic growth is high, the SCC needs to be adjusted downwards by \$4 per tCO_2. But if the correlation coefficient for climate sensitivity and damage uncertainties is ½, the climatic risk markup rises (as the coefficient of relative risk aversion exceeds 1 in our calibration) from 37% to 59%. If a low ethical instead of a market-based discount rate is used as in Stern (2007), the deterministic SCC rises to \$47/tCO_2, the economic and climatic risk markups are 14% and 94%, respectively, and the risk-adjusted SCC becomes \$98/tCO_2. However, if economic volatility is calibrated to the lower volatility of GDP instead of asset returns, the economic and climatic risk markups are negligibly small and 74%, respectively, and the risk-adjusted SCC is still as high as \$82/tCO_2.

Future investigations should be directed at obtaining robust empirical estimates of the climate betas, a largely uncharted territory. Other areas for future research are to extend our analytical approach to the optimal risk-adjusted price of carbon to include compound Poisson shocks to climate sensitivity (e.g., Hambel et al., 2017) or for richer positive feedback processes in the uptake of atmospheric carbon due to the CO_2 absorption capacity of the oceans declining with temperature (Millar et al., 2016). Each of these extensions would push up the optimal price of carbon, as would the risk of tipping points (e.g., Lemoine and Traeger, 2014, 2016; Lontzek et al., 2016; Cai et al., 2016; van der Ploeg and de Zeeuw, 2018). Finally, mean reversion in the stochastic process for the rate of economic growth and a downward-sloping term structure with risk aversion exceeding aversion to intertemporal fluctuations (Gollier and Mahul, 2017) and compound Poisson shocks to capture catastrophic shocks to total factor productivity (cf. Bretschger and Vinogradova, 2018; Bansal et al., 2016) also push up the optimal risk-adjusted SCC. Future work will employ asymptotic methods to identify tractable leading-order solutions for these circumstances.

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Appendix A: Risk-Adjusted Carbon Price with Convex Reduced-Form Damages

We can generalize Result 2 to convex reduced-form damages, skewed damage uncertainty and carbon stock uncertainty (see Appendix E for the derivation). The resulting rule includes additional correction factors, which can be evaluated as simple, one-dimensional integrals along a business-as-usual path. The only additional assumption is that the future atmospheric carbon stock does not inherit any of the uncertainty from new emissions through their dependence on the stochastic capital stock (cf. (E2.3)), which is only associated with a very small error, as evident from section V.D and Appendix G.

Result 2′: In general ($\theta_{ET} \neq 0, \theta_\lambda \neq 0$ and $\sigma_E > 0$), the leading-order optimal SCC is:

(A1) $$P = \frac{\mu\, \Theta(E) Y|_{P=0}}{r^\star} \left(1 + \theta_{ET} \frac{\mu F^{(0)}}{E} \frac{\Upsilon}{r^{\star\star}} + \Delta_{EE} + \Delta_{\chi\chi} + \Delta_{\lambda\lambda} + \Delta_{CK} + \Delta_{CC} \right),$$

where we redefine $r^\star \equiv \rho + (\gamma - 1)(g^{(0)} - \frac{1}{2}\eta\sigma_K^2) + (1+\theta_{ET})\varphi$ to include a more general dependence on φ, $r^{\star\star} \equiv r^\star + (\eta - 1)\sigma_K^2 - \varphi$, and $F^{(0)}$ is shorthand for optimal fossil fuel use without climate policy, $F^{(0)} = \left((1-\alpha)/b\right)^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}} K$, to the zeroth order of approximation. The deterministic correction factor for future emissions and $\theta_{ET} \neq 0$ is

(A2) $$\Upsilon = \frac{r^{\star\star}}{1 - (1+\theta_{ET})\varphi / r^\star} \int_0^\infty \left(\exp(-r^{\star\star} s) - \frac{(1+\theta_{ET})\varphi}{r^\star} \exp(-(r^\star - \varphi)s) \right) e(s)^{\theta_{ET}-1} ds,$$

where $e(s) \equiv 1 + (\mu F^{(0)} / E)(\exp(\varphi s) - 1)/\varphi$ captures new emissions and s is the dummy variable of integration. We also have the corrections for uncertainty in the carbon stock, climate sensitivity and damages, which are now multiplied by new correction factors

(A3) $$\Delta_{EE} = \frac{1}{2}\theta_{ET}(1-\theta_{ET}) \left(\frac{\sigma_E}{E} \right)^2 \frac{1}{r^\star - 2\varphi} \Upsilon_{EE},$$

(A4) $$\Delta_{\chi\chi} = \frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T}) \frac{\left(\sigma_\chi / \bar{\chi}\right)^2}{r^\star + 2\nu_\chi} \Upsilon_{\chi\chi} \quad \text{and} \quad \Delta_{\lambda\lambda} = \frac{1}{2}\theta_\lambda(1+\theta_\lambda) \frac{\left(\sigma_\lambda / \bar{\lambda}\right)^2}{r^\star + 2\nu_\lambda} \Upsilon_{\lambda\lambda} \text{ with}$$

$$\Upsilon_{ij} = 1 + \theta_{ET} \frac{\mu F^{(0)}}{E} \frac{1}{1-(1+\theta_{ET})\varphi/r^\star} \Bigg[\left(1 + \frac{(1+\theta_{ET})\varphi}{\nu_i + \nu_j}\right) \int_0^\infty \exp\left(-(r^\star + \nu_i + \nu_j - \varphi)s\right) e(s)^{\theta_{ET}-1} ds$$

$$- \frac{(1+\theta_{ET})\varphi}{\nu_i + \nu_j} \frac{r^\star + \nu_i + \nu_j}{r^\star} \int_0^\infty \exp(-(r^\star - \varphi)s) e(s)^{\theta_{ET}-1} ds$$

$$+ \frac{r^\star + \nu_i + \nu_j}{\nu_i + \nu_j} \int_0^\infty \left(\exp(-r^{\star\star}s) - \exp(-(r^{\star\star} + \nu_i + \nu_j)s)\right) e(s)^{\theta_{ET}-1} ds \Bigg] \text{ for } i, j = \chi, \lambda.$$

The correction for correlated climate and economic risks is

$$\text{(A5)}\ \Delta_{\text{CK}} = -(\eta - 1)\sigma_K \left(\theta_{ET} \frac{\rho_{KE}\sigma_E}{(r^\star - \varphi)E} \Upsilon_{KE} + (1+\theta_{\chi T}) \frac{\rho_{K\chi} \frac{\sigma_\chi}{\overline{\chi}}}{r^\star + \nu_\chi} \Upsilon_{K\chi} + (1+\theta_\lambda) \frac{\rho_{K\lambda} \frac{\sigma_\lambda}{\overline{\lambda}}}{r^\star + \nu_\lambda} \Upsilon_{K\lambda} \right),$$

with

$$\Upsilon_{Ki} = 1 + \theta_{ET} \frac{\mu F^{(0)}}{E} \frac{1}{1-(1+\theta_{ET})\varphi/r^\star} \Bigg[\left(1 + \frac{(1+\theta_{ET})\varphi}{\nu_i}\right) \int_0^\infty \exp\left(-(\nu_i + r^\star - \varphi)s\right) e(s)^{\theta_{ET}-1} ds$$

$$-(1+\theta_{ET}) \frac{\varphi}{r^\star} \frac{r^\star + \nu_i}{\nu_i} \int_0^\infty \exp\left(-(r^\star - \varphi)s\right) e(s)^{\theta_{ET}-1} ds +$$

$$\frac{2-\eta}{1-\eta} \frac{\nu_i + r^\star}{\nu_i} \int_0^\infty \left(\exp(-r^{\star\star}s) - \exp(-(r^{\star\star} + \nu_i)s)\right) e(s)^{\theta_{ET}-1} ds \Bigg] \text{ for } i = \chi, \lambda.$$

The correction for correlated climate sensitivity and damage risks is

$$\text{(A6)}\quad \begin{aligned} \Delta_{\text{CC}} &= \theta_{ET}(1+\theta_{\chi T}) \frac{\rho_{E\chi}\sigma_E\sigma_\chi/\overline{\chi}}{(r^\star + \nu_\chi)E} \frac{r^\star}{r^\star - \varphi} \Upsilon_{E\chi} \\ &+ (1+\theta_\lambda) \left(\theta_{ET} \frac{\rho_{E\lambda}\sigma_E\sigma_\lambda/\overline{\lambda}}{(r^\star + \nu_\lambda)E} \frac{r^\star}{r^\star - \varphi} \Upsilon_{E\lambda} + (1+\theta_{\chi T}) \frac{\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda/\overline{\chi}\overline{\lambda}}{r^\star + \nu_\chi + \nu_\lambda} \Upsilon_{\chi\lambda} \right), \end{aligned}$$

where $\Upsilon_{\chi\lambda}$ is already defined in (A4) and we do not show $\Upsilon_{E\chi}$ and $\Upsilon_{E\lambda}$. □

Due to the very small magnitude of $\sigma_E^2 / g_0^{(0)}$ (see section IV), we can ignore all terms involving atmospheric carbon stock volatility, as their contributions to the risk-adjusted SCC are negligible in which case $\Delta_{EE} = 0$ and (A5) and (A6) simplify to:

(A5′) $$\Delta_{\mathrm{CK}} = -(\eta - 1)\sigma_K \left((1+\theta_{\chi T}) \frac{\rho_{K\chi}\sigma_\chi / \bar{\chi}}{r^\star + \nu_\chi} \Upsilon_{K\chi} + (1+\theta_\lambda) \frac{\rho_{K\lambda}\sigma_\lambda / \bar{\lambda}}{r^\star + \nu_\lambda} \Upsilon_{K\lambda} \right)$$ and

(A6′) $$\Delta_{\mathrm{CC}} = \left(1+\theta_\lambda\right)(1+\theta_{\chi T}) \frac{\rho_{\chi\lambda}\sigma_\chi \sigma_\lambda / \bar{\chi}\bar{\lambda}}{r^\star + \nu_\chi + \nu_\lambda} \Upsilon_{\chi\lambda}.$$

There are three differences compared to the simpler Result 2. First, convexity of reduced-form damages ($\theta_{ET} > 0$) pushes up the deterministic SCC though the deterministic correction factor $\Upsilon > 0$ in (A2), but also boosts the discount rate $r^\star = \rho + (\gamma - 1) g^{(0)} + (1+\theta_{ET})\varphi$. The net effect is ambiguous, but positive for small decay rates of atmospheric carbon. Second, our power-function specification for damages (2.6) gives $\Theta_E(E) = \left(1/S_{PI}\right)^2 (1+\theta_{ET})\theta_{ET} (E/S_{PI})^{\theta_{ET}-1}$ to leading-order in our small parameter. With convex damages ($\theta_{ET} > 0$), the flow damage coefficient thus rises with the stock of atmospheric carbon. The time path for the carbon price is then steeper than that of world GDP. Third, there is a *climate damage uncertainty correction* $\Delta_{\lambda\lambda} = (1/2)\theta_\lambda (1+\theta_\lambda)(\sigma_\lambda / \bar{\lambda})^2 \Upsilon_{\lambda\lambda} / (r^\star + 2\nu_\lambda)$ in (A4), which adjusts the SCC upwards if the probability density function of damage shocks is right-skewed ($\theta_\lambda > 0$). The upward adjustment is larger if damages are more uncertain and right-skewed, display less mean reversion, and if the growth-corrected discount rate is smaller (higher $\sigma_\lambda, \theta_\lambda$, lower ν_λ, and lower $r^\star$). This correction is separate from the negative effect on the risk-adjusted carbon price of the risk insurance term $\eta\sigma_K^2$, resulting from damages being proportional to GDP (as discussed in section III.B). Finally, the correction factors Υ_{ij} for $i, j = \chi, \lambda$ which appear in Result 2′ are unity for proportional damages ($\theta_{ET} = 0$), but are greater than unity for convex damages ($\theta_{ET} > 0$) and capture the contribution by new emissions

Appendix B: Transformation to Non-Dimensional Form (For Online Publication)

We define the non-dimensional variables

$$\text{(B1)} \qquad \hat{K} \equiv \frac{K}{K_0}, \hat{E} \equiv \frac{E}{E_0}, \hat{\chi} \equiv \frac{\chi}{\overline{\chi}}, \hat{\lambda} \equiv \frac{\lambda}{\overline{\lambda}}, \hat{F} \equiv \frac{F}{F_0}, \hat{C} \equiv \frac{C}{C_0}, \hat{I} \equiv \frac{I}{C_0}, \hat{\Phi} \equiv \frac{\Phi}{C_0}, \hat{t} \equiv g_0 t \text{ and } \hat{J} \equiv g_0 J / (C_0)^{1-\eta},$$

where zero subscripts refer to initial values ($t = 0$), except for $F_0 \equiv A\left(E_0\right)^{\frac{1}{\alpha}}\left(\left(1-\alpha\right)/b\right)^{\frac{1}{\alpha}} K_0$ and $C_0 \equiv g_0 K_0$, so that all hatted quantities are $O\left(1\right)$ initially, assuming $\chi_0 / \overline{\chi} = O\left(1\right)$ and $\lambda_0 / \overline{\lambda} = O\left(1\right)$. We define $g_0 \equiv g\left(E = E_0\right)$ to be the growth rate of the economy without additional climate change, $\phi \equiv \phi / g_0$ and $\hat{i} \equiv i / g_0$, where $i \equiv I / K$. The Hamilton-Jacobi-Bellman equation (3.2) becomes in non-dimensional terms

$$\text{(B2)} \qquad \begin{aligned} 0 \quad = \quad & \max_{\hat{C},\hat{F}} \left[\frac{1}{1-\gamma} \frac{\hat{C}^{1-\gamma} - \hat{\rho}\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}} + \hat{J}_{\hat{t}} + \hat{J}_{\hat{K}}\phi(\hat{i})\hat{K} + \hat{J}_{\hat{E}}(\hat{\mu}\hat{F}e^{-\hat{g}\hat{t}} - \hat{\varphi}\hat{E}) \right. \\ & \left. + \hat{J}_{\hat{\chi}}\hat{\nu}_{\chi}(1-\hat{\chi}) + \hat{J}_{\hat{\lambda}}\hat{\nu}_{\lambda}(1-\lambda) + \frac{1}{2}\hat{J}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2 + \frac{1}{2}\hat{J}_{\hat{E}\hat{E}}\hat{\sigma}_E^2 + \frac{1}{2}\hat{J}_{\hat{\chi}\hat{\chi}}\hat{\sigma}_{\chi}^2 + \frac{1}{2}\hat{J}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 \right] \\ & + \hat{J}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \hat{J}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + \hat{J}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} + \hat{J}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} \\ & + \hat{J}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} + \hat{J}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}, \end{aligned}$$

where $\hat{I} = \hat{Y} - \hat{b}\hat{F} - \hat{C} = \hat{A}(\hat{E},\hat{\chi})\hat{K}^{\alpha}\hat{F}^{1-\alpha} - \hat{b}\hat{F} - \hat{C}$, $\hat{Y} \equiv Y / C_0$ and $\phi = \hat{i} - (1/2)\hat{\omega}\hat{i}^2 - \delta$. The resulting non-dimensional expressions are

$$\text{(B3)} \qquad \begin{aligned} & \hat{\rho} \equiv \frac{\rho}{g_0},\ \hat{b} \equiv \frac{bF_0}{g_0 K_0}, \hat{\omega} \equiv g_0\omega,\ \hat{\delta} \equiv \frac{\delta}{g_0}, \hat{g} \equiv \frac{g}{g_0},\ \hat{\mu} \equiv \frac{\mu F_0}{g_0 E_0},\ \hat{\varphi} \equiv \frac{\varphi}{g_0},\ \hat{\nu}_{\chi} \equiv \frac{\nu_{\chi}}{g_0}, \\ & \hat{\nu}_{\lambda} \equiv \frac{\nu_{\lambda}}{g_0},\ \hat{\sigma}_K \equiv \frac{\sigma_K}{\sqrt{g_0}},\ \hat{\sigma}_E \equiv \frac{\sigma_E}{\sqrt{g_0}E_0},\ \hat{\sigma}_{\chi} \equiv \frac{\sigma_{\chi}}{\sqrt{g_0}\,\overline{\chi}} \text{ and } \hat{\sigma}_{\lambda} \equiv \frac{\sigma_{\lambda}}{\sqrt{g_0}\,\overline{\lambda}}, \end{aligned}$$

with $\hat{A}\left(\hat{E},\hat{\chi}\right) \equiv A(E,\chi)F_0^{1-\alpha} / g_0 K_0^{1-\alpha}$. Damages and total factor productivity become

$$\text{(B4)} \qquad \hat{D}(\hat{E},\hat{\chi},\hat{\lambda}) \equiv \epsilon\hat{\lambda}^{1+\theta_{\lambda}}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}} \text{ and } \hat{A} \equiv \hat{A}^*(1-\hat{D}) = \hat{A}^*(1-\epsilon\hat{\lambda}^{1+\theta_{\lambda}}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}}),$$

where the damage fraction $\hat{D} \equiv D$ is already non-dimensional, $\hat{A}^* \equiv A^* F_0^{1-\alpha} / g_0 K_0^{1-\alpha}$ and the final non-dimensional parameter is

$$\text{(B5)} \qquad \epsilon \equiv \overline{\lambda}^{1+\theta_\lambda} \overline{\chi}^{1+\theta_{\chi T}} \left(\frac{E_0}{S_{PI}} \right)^{1+\theta_{ET}} .$$

The first-order conditions of (B2) with respect to $\hat{C}$ and $\hat{F}$ are, respectively,

$$\text{(B6)} \qquad \frac{\hat{C}^{-\gamma}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}} - \phi'\left(\hat{i}\right)\hat{J}_{\hat{K}} = 0 \;\Rightarrow\; \hat{C} = \left(\phi'\left(\hat{i}\right)\hat{J}_{\hat{K}}\right)^{-\frac{1}{\gamma}} \left((1-\eta)\hat{J}\right)^{-\frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}} \text{ and}$$

$$\text{(B7)} \quad \hat{J}_{\hat{K}}\left((1-\alpha)\hat{A}\left(\hat{E}\right)\hat{K}^{\alpha}\hat{F}^{-\alpha} - \hat{b}\right)\phi'\left(\hat{i}\right) + \hat{J}_{\hat{E}} e^{-\hat{g}\hat{t}}\hat{\mu} = 0 \;\Rightarrow\; \hat{F} = \left(\frac{1-\alpha}{\hat{b} + \hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} K,$$

where we have defined the SCC in non-dimensional terms as

$$\text{(B8)} \qquad \hat{P} \equiv \left(\frac{F_0}{g_0 K_0}\right) P = -\hat{\mu} \frac{\hat{J}_{\hat{E}}}{\phi'(\hat{i})\hat{J}_{\hat{K}}},$$

and use (B7) to write the production function as

$$\text{(A9)} \qquad \hat{Y} = \hat{A}\left(\hat{E}, \hat{\chi}, \hat{\lambda}\right) K^{\alpha} F^{1-\alpha} = \hat{A}\left(\hat{E}, \hat{\chi}, \hat{\lambda}\right)^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{\hat{b} + \hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1-\alpha}{\alpha}} \hat{K}.$$

Appendix C: Derivation of Zeroth-Order Solution (For Online Publication)

In non-dimensional terms, the truncated series solutions for the value function and the forward-looking control variables (3.4) is given by

$$\text{(C1)} \qquad \begin{aligned} \hat{J}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) &= \hat{J}^{(0)}\left(\hat{K}, \epsilon\hat{D}(\hat{E}, \hat{\chi}, \hat{\lambda})\right) + \epsilon\hat{J}^{(1)}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) + O(\epsilon^2), \\ \hat{F}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) &= \hat{F}^{(0)}\left(\hat{K}, \epsilon\hat{D}(\hat{E}, \hat{\chi}, \hat{\lambda})\right) + \epsilon\hat{F}^{(1)}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) + O(\epsilon^2), \\ \hat{C}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) &= \hat{C}^{(0)}\left(\hat{K}, \epsilon\hat{D}(\hat{E}, \hat{\chi}, \hat{\lambda})\right) + \epsilon\hat{C}^{(1)}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) + O(\epsilon^2). \end{aligned}$$

At $O(1)$ the Hamilton-Jacobi-Bellman equation (B2) can be written as

$$
\text{(C2)}\quad
\begin{aligned}
&\frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}^{(0)}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}\frac{\eta-\gamma}{1-\eta}}-\hat{\rho}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}}}{(1-\gamma)\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}-1}}+\hat{J}^{(0)}_{\hat{t}}+\hat{J}^{(0)}_{\hat{K}}\hat{\phi}(\hat{i}^{(0)})\hat{K}\\
&+\frac{1}{2}\hat{J}^{(0)}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}^2_K\hat{J}_{\hat{K}\hat{E}}\hat{K}\hat{E}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E+\hat{J}_{\hat{K}\hat{\chi}}\hat{K}\hat{\chi}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi+\hat{J}_{\hat{K}\hat{\lambda}}\hat{K}\hat{\lambda}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda\\
&\quad+\hat{J}_{\hat{E}\hat{\chi}}\hat{E}\hat{\chi}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi+\hat{J}_{\hat{E}\hat{\lambda}}\hat{E}\hat{\lambda}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda+\hat{J}_{\hat{\chi}\hat{\lambda}}\hat{\chi}\hat{\lambda}\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda=0,
\end{aligned}
$$

where we have substituted for the forward-looking variables $\hat{C}$ and $\hat{F}$ at $O(1)$ from (B6) and (B7) and we have used

$$
\text{(C3)}\quad \underbrace{\frac{1}{d\hat{t}}E_t\left[d\hat{K}\right]}_{O(1)}=\hat{\phi}(\hat{i}^{(0)})\hat{K}\ .
$$

In (C2)-(C3), $\hat{i}^{(0)}$ is the (constant) optimally chosen investment rate. Equation (C2) has a power-law solution of the form $J^{(0)}=\psi_0\hat{K}^{1-\eta}$, and following some manipulation we obtain

$$
\text{(C4)}\quad \hat{J}^{(0)}=\psi_0\hat{K}^{1-\eta}\ \text{ with }\ \psi_0=\frac{1}{1-\eta}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-(1-\eta)}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)})-\frac{1}{2}\eta\hat{\sigma}^2_K\right)\right)^{-\gamma\frac{1-\eta}{1-\gamma}}.
$$

From the first-order condition (B6), we obtain

$$
\text{(C5)}\quad \hat{C}^{(0)}=\hat{c}^{(0)}\hat{K}\ \text{ with }\ \hat{c}^{(0)}=\frac{1}{\hat{\phi}'(\hat{i}^{(0)})}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)})-\frac{1}{2}\eta\hat{\sigma}^2_K\right)\right),
$$

where $q\left(\hat{i}\right)=1/\phi'\left(\hat{i}\right)$ denotes Tobin's q, the price of capital in consumption terms.[29]

We can thus write the value function (C4) as

---

[29] The value of the capital stock is $q\hat{K}$, or dimensionally qK, where $q=1/\hat{\phi}'(\hat{i})=1/\phi'(i)$ is already a fraction and is left unchanged by the scaling (cf. $\omega i=\hat{\omega}\hat{i}$).

(C6) $$J^{(0)} = \frac{1}{1-\eta}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}}\hat{K}^{1-\eta}.$$

Substituting in for $\hat{F}$ from the optimality condition (B7) and for $\hat{Y}$ from (B9), we obtain from $\hat{I} = \hat{Y} - \hat{C} - \hat{b}\hat{F}$:

(C7) $$\hat{i}^{(0)} = \hat{r}^{(0)}_{\text{mpk}} + \hat{\delta} - \hat{c}^{(0)} = \hat{r}^{(0)}_{\text{mpk}} + \hat{\delta} - \hat{q}^{(0)}\left(\hat{\rho} - (1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)}) - \frac{1}{2}\eta\hat{\sigma}_K^2\right)\right),$$

where $\hat{r}^{(0)}_{\text{mpk}} \equiv \hat{Y}_{\hat{K}}\left(\hat{P}=0\right) - \hat{\delta} = \alpha\hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)^{\frac{1}{\alpha}}\left((1-\alpha)/\hat{b}\right)^{\frac{1-\alpha}{\alpha}} - \hat{\delta}$ denotes the marginal productivity of capital net of depreciation.[30] Equation (C7) implicitly defines the optimally chosen investment rate $\hat{i}^{(0)}$. From (C3), the leading-order endogenous growth rate of capital and hence of consumption is given by

(C8) $$\hat{g}^{(0)} = \underbrace{\frac{1}{\hat{K}}\frac{1}{d\hat{t}}E_t\left[d\hat{K}\right]}_{O(1)} = \hat{\phi}(\hat{i}^{(0)}) \quad \text{and hence} \quad \hat{g}^{(0)} = \hat{\phi}\left(\hat{i}^{(0)}\right) = 1.$$

In equilibrium, the marginal propensity to consume $\hat{c}^{(0)}/\hat{q}^{(0)}$ equals the expected return on investment $\hat{r}^{(0)}$ minus the growth rate of the economy $\hat{g}^{(0)}$. In turn, the expected return on investment equals the sum of the risk-free rate $\hat{r}^{(0)}_{\text{rf}}$ and the risk premium $\Delta\hat{r}^{(0)}$. Hence, $\hat{c}^{(0)}/\hat{q}^{(0)} = \hat{r}^{(0)} - \hat{g}^{(0)} = \hat{r}^{(0)}_{\text{rf}} + \Delta\hat{r}^{(0)} - \hat{g}^{(0)}$ and with a risk premium of $\Delta\hat{r}^{(0)} = \eta\hat{\sigma}_K^2$ in the absence of any climate risk at zeroth-order, the risk-free rate is:

(C9) $$\hat{r}^{(0)}_{\text{rf}} = \hat{\rho} + \gamma\hat{g}^{(0)} - (1+\gamma)\eta\hat{\sigma}_K^2/2.$$

Although $\hat{J}^{(0)}_{\hat{E}}$ can be computed from (C6), a consistent leading-order estimate of the optimal SCC also requires $\hat{J}^{(1)}_{\hat{E}}$ and thus the next order in the perturbation expansion, i.e., $\hat{P} = -\hat{\mu}\left(\hat{J}^{(0)}_{\hat{E}} + \epsilon\hat{J}^{(1)}_{\hat{E}}\right)\Big/\phi'(\hat{i}^{(0)})\hat{J}^{(0)}_{\hat{K}}$.

[30] Dimensionally, we have $r^{(0)}_{\text{mpk}} = \hat{r}^{(0)}_{\text{mpk}}g_0$.

Appendix D: Derivation of First-Order Solution (For Online Publication)

D.1. Solution to multi-variate Ornstein-Uhlenbeck process

We define $\hat{k} \equiv k \equiv \log\left(K / K_0\right)$, so the vector of all four states $d\mathbf{x} = \left\{d\hat{k}, d\hat{E}, d\hat{\chi}, d\hat{\lambda}\right\}^T$ can be described by one multi-variate Ornstein-Uhlenbeck process (2.9), which is given in non-dimensional terms by

$$\text{(D1.1)} \quad d\mathbf{x} = \boldsymbol{\alpha} - \mathbf{v} \circ \left(\mathbf{x} - \boldsymbol{\mu}\right) d\hat{t} + \mathbf{S} d\mathbf{W}_{\hat{t}}.$$

The growth rate vector (2.10), relevant to the capital and atmospheric carbon stock processes only, is given in non-dimensional terms by

$$\text{(D1.2)} \quad \boldsymbol{\alpha} = \left(\frac{1}{d\hat{t}} \frac{E_t\left[d\hat{K}\right]}{\hat{K}} - \frac{1}{2}\hat{\sigma}_K^2, \frac{1}{d\hat{t}} E_t\left[d\hat{E}\right], 0, 0 \right)^T = \left(\hat{\phi}(\hat{i}) - \frac{1}{2}\hat{\sigma}_K^2, \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K} e^{-\hat{g}\hat{t}}, 0, 0 \right)^T,$$

the mean reversion rate vector by $\mathbf{v} = \left(0, \hat{\varphi}, \nu_\chi, \nu_\lambda\right)^T$, the vector of means by $\boldsymbol{\mu}^T = (0,0,1,1)^T$, and the covariance matrix $\mathbf{SS}^T$ has the form

$$\text{(D1.3)} \quad \frac{1}{d\hat{t}} E_t\left[d\mathbf{x} d\mathbf{x}^T\right] = \mathbf{SS}^T = \begin{pmatrix} \hat{\sigma}_K^2 & \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda \\ \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \hat{\sigma}_E^2 & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda \\ \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \hat{\sigma}_\chi^2 & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda \\ \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda & \hat{\sigma}_\lambda^2 \end{pmatrix}.$$

We begin by integrating the multi-variate Ornstein-Uhlenbeck process (D1.1), including only terms at zeroth order, so that the coefficients are constant and a closed-form solution is available. Specifically, $\boldsymbol{\alpha}^{(0)} = \left(\hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2, \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K}_0, 0, 0 \right)^T$, where we have relied on the solution for $\hat{K}$ from the zeroth-order problem (cf. (C8)). The slow dependence of productivity $\hat{A}$ on the states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ can be neglected when integrating with respect to time, consistent with the multiple-scales nature of our perturbation expansion. For constant coefficients, (D1.1) can be integrated to give:

(D1.4) $$\mathbf{x}(t) = \boldsymbol{\mu} + \boldsymbol{\alpha} t + e^{\mathbf{v}\hat{t}} \circ \left(\mathbf{x}_0 - \boldsymbol{\mu}\right) + \int_0^{\hat{t}} e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S} d\mathbf{W}_{\hat{u}}.$$

The quantity $\mathbf{x}(t)$ is normally distributed with covariance matrix $\boldsymbol{\Sigma}(t)$:

(D1.5) $$\boldsymbol{\Sigma}(t) = \int_0^{\hat{t}} \left(e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\right)\left(e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\right)^T d\hat{u} =$$

$$\begin{pmatrix}
\hat{\sigma}_K^2 \hat{t} & \frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\nu_\chi}\left(1-e^{-\hat{\nu}_\chi \hat{t}}\right) & \frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\nu}_\lambda}\left(1-e^{-\hat{\nu}_\lambda \hat{t}}\right) \\
\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\hat{\sigma}_E^2}{2\hat{\varphi}}\left(1-e^{-2\hat{\varphi}\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\hat{\nu}_\chi}\left(1-e^{-(\hat{\varphi}+\hat{\nu}_\chi)\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{\nu}_\lambda}\left(1-e^{-(\hat{\varphi}+\hat{\nu}_\lambda)\hat{t}}\right) \\
\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\nu_\chi}\left(1-e^{-\hat{\nu}_\chi \hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\nu_\chi}\left(1-e^{-(\hat{\varphi}+\hat{\nu}_\chi)\hat{t}}\right) & \frac{\hat{\sigma}_\chi^2}{2\nu_\chi}\left(1-e^{-2\hat{\nu}_\chi \hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(1-e^{-(\hat{\nu}_\chi+\hat{\nu}_\lambda)\hat{t}}\right) \\
\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\nu}_\lambda}\left(1-e^{-\hat{\nu}_\chi \hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{\nu}_\lambda}\left(1-e^{-(\hat{\varphi}+\hat{\nu}_\lambda)\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(1-e^{-(\hat{\nu}_\chi+\hat{\nu}_\lambda)\hat{t}}\right) & \frac{\hat{\sigma}_\lambda^2}{2\hat{\nu}_\lambda}\left(1-e^{-2\hat{\nu}_\lambda \hat{t}}\right)
\end{pmatrix}.$$

D.2. Evolution equations for $\hat{K}$ and $\hat{E}$

We consider the expected evolution equations of the states $\hat{K}$ and $\hat{E}$ at $O(\epsilon)$ and $O(1)$, respectively. At this order, we have for the expected evolution of $\hat{K}$:

(D2.1) $$\underbrace{\frac{1}{d\hat{t}} E_t\left[d\hat{K}\right]}_{O(\epsilon)} = \hat{\phi}'(\hat{i}^{(0)})\epsilon \hat{I}^{(1)} = -\hat{\phi}'(\hat{i}^{(0)})\epsilon \hat{C}^{(1)} = \frac{\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}}{\gamma - \frac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{i}^{(0)})}} \hat{K}\left(\frac{\epsilon \hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} + \frac{\eta-\gamma}{1-\eta}\frac{\epsilon \hat{J}^{(1)}}{\hat{J}^{(0)}}\right),$$

where the first identity makes use of the identity $\hat{\Phi} = \epsilon \hat{I}^{(1)} - \hat{\omega}\epsilon \hat{I}^{(1)}\hat{I}^{(0)} / \hat{K} = \epsilon \hat{I}^{(1)}\hat{\phi}'(\hat{i}^{(0)})$ at $O(\epsilon)$. We further note from $\hat{I} = \hat{Y} - \hat{b}\hat{F} - \hat{C}$ that $\hat{I}^{(1)} = -\hat{C}^{(1)}$, since production net of fossil fuel costs is unaffected by the SCC in our formulation:

$$\frac{\partial}{\partial\hat{P}}\left[\hat{Y}-\hat{b}\hat{F}\right]\Big|_{P=0} =$$

(D2.2) $$\frac{\partial}{\partial\hat{P}}\left[\hat{A}^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1-\alpha}{\alpha}}\hat{K}-\hat{b}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}\right]\Bigg|_{P=0} = 0.$$

The identity in (D2.2) relies on the Cobb-Douglas nature of the production function. The third identity in (D2.1) follows from a Taylor-series expansion of $\hat{C}$, given by (B6), with respect to the small parameter ϵ (about $\epsilon = 0$):

(D2.3) $$\hat{c}^{(1)} = \hat{c}^{(0)}\left(-\frac{1}{\gamma}\frac{\phi''}{\hat{\phi}'\left(\hat{i}^{(0)}\right)}\epsilon\hat{i}^{(1)}-\frac{1}{\gamma}\frac{\epsilon\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}}-\frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right).$$

Noting that $\hat{i}^{(1)} = -\hat{c}^{(1)}$, we can rearrange this linear equation to give

(D2.4) $$\hat{c}^{(1)} = \frac{\hat{c}^{(0)}}{1-\frac{1}{\gamma}\frac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{i}^{(0)})}}\left(-\frac{1}{\gamma}\frac{\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}}-\frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}\frac{\hat{J}^{(1)}}{\hat{J}^{(0)}}\right),$$

which is used in the third identity in (D2.1). For $\hat{E}$, we have at $O(1)$:

(D2.5) $$\underbrace{\frac{1}{d\hat{t}}E_t\left[d\hat{E}\right]}_{O(1)} = \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}} - \hat{\varphi}\hat{E}.$$

D.3. The Hamilton-Jacobi-Bellman equation

Substituting for the forward-looking variables $\hat{C}$ from (A6) and $\hat{F}$ from (B7), the Hamilton-Jacobi-Bellman equation (B2) becomes at $O(\epsilon)$:

(D3.1) $$\underset{O(\epsilon)}{\hat{f}^*(\hat{J})}+\epsilon\hat{J}_{\hat{t}}^{(1)}+\epsilon\hat{J}_{\hat{K}}^{(1)}\hat{K}\hat{\phi}(\hat{i}^{(0)})+\frac{\hat{\phi}'\left(\hat{i}^{(0)}\right)\hat{c}^{(0)}}{\gamma-\hat{c}^{(0)}\phi''/\hat{\phi}'(\hat{i}^{(0)})}\left(\hat{K}\hat{J}_{\hat{K}}^{(1)}+(\eta-\gamma)\hat{J}^{(1)}\right)$$

$$+\left(\hat{J}^{(0)}_{\hat{E}}+\epsilon\hat{J}^{(1)}_{\hat{E}}\right)\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}}-\hat{\varphi}\hat{E}\right)+\left(\hat{J}^{(0)}_{\hat{\chi}}+\epsilon\hat{J}^{(1)}_{\hat{\chi}}\right)\hat{\nu}_{\chi}(1-\hat{\chi})$$

$$+\left(\hat{J}^{(0)}_{\hat{\lambda}}+\epsilon\hat{J}^{(1)}_{\hat{\lambda}}\right)\hat{\nu}_{\lambda}(1-\hat{\lambda})+\frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{K}\hat{K}}\hat{K}^{2}\hat{\sigma}^{2}_{K}+\frac{1}{2}\left(\hat{J}^{(0)}_{\hat{E}\hat{E}}+\epsilon\hat{J}^{(1)}_{\hat{E}\hat{E}}\right)\hat{E}^{2}\hat{\sigma}^{2}_{E}$$

$$+\frac{1}{2}\left(\hat{J}^{(0)}_{\hat{\chi}\hat{\chi}}+\epsilon\hat{J}^{(1)}_{\hat{\chi}\hat{\chi}}\right)\hat{\sigma}^{2}_{\chi}+\frac{1}{2}\left(\hat{J}^{(0)}_{\hat{\lambda}\hat{\lambda}}+\epsilon\hat{J}^{(1)}_{\hat{\lambda}\hat{\lambda}}\right)\hat{\sigma}^{2}_{\lambda}+\left(\hat{J}^{(0)}_{\hat{K}\hat{E}}+\epsilon\hat{J}^{(1)}_{\hat{K}\hat{E}}\right)\hat{K}\rho_{KE}\hat{\sigma}_{K}\hat{\sigma}_{E}$$

$$+\left(\hat{J}^{(0)}_{\hat{K}\hat{\chi}}+\epsilon\hat{J}^{(1)}_{\hat{K}\hat{\chi}}\right)\hat{K}\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}+\left(\hat{J}^{(0)}_{\hat{K}\hat{\lambda}}+\epsilon\hat{J}^{(1)}_{\hat{K}\hat{\lambda}}\right)\hat{K}\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda}$$

$$+\left(\hat{J}^{(0)}_{\hat{E}\hat{\chi}}+\epsilon\hat{J}^{(1)}_{\hat{E}\hat{\chi}}\right)\rho_{E\chi}\hat{\sigma}_{E}\hat{\sigma}_{\chi}+\left(\hat{J}^{(0)}_{\hat{E}\hat{\lambda}}+\epsilon\hat{J}^{(1)}_{\hat{E}\hat{\lambda}}\right)\rho_{E\lambda}\hat{\sigma}_{E}\hat{\sigma}_{\lambda}+\left(\hat{J}^{(0)}_{\hat{\chi}\hat{\lambda}}+\epsilon\hat{J}^{(1)}_{\hat{\chi}\hat{\lambda}}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}=0,$$

where we have used the identity $\partial/\partial\hat{k}=\hat{K}\,\partial/\partial\hat{K}$ (chain rule), we substituted the evolution equations for $\hat{K}$ at subsequent orders ((C3) and (D2.1)) and $\hat{E}$ at zeroth-order (D2.5), and defined $\hat{f}^{*}(J)\equiv\hat{f}\left(\hat{C}^{*},\hat{J}\right)$ with $\hat{C}$ optimally chosen. From (2.1) and (B6), $\hat{f}^{*}(J)$ is

(D3.2) $$\hat{f}^{*}=\frac{1}{1-\gamma}\left(\hat{\phi}'(\hat{i})\hat{J}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}-\frac{1-\eta}{1-\gamma}\hat{\rho}\hat{J}.$$

A Taylor-series expansion for $\hat{f}^{*}(J)$ in ϵ (about $\epsilon=0$) gives

(D3.3)

$$\hat{f}^{*}_{O(\epsilon)}=\frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}^{(0)}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\gamma}\left(-\frac{\hat{\phi}''}{\hat{\phi}'(\hat{i}^{(0)})}\epsilon\hat{i}^{(1)}-\frac{\epsilon\hat{J}^{(1)}_{\hat{K}}}{\hat{J}^{(0)}_{\hat{K}}}+\frac{\gamma-\eta}{(1-\gamma)(1-\eta)}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right)-\frac{1-\eta}{1-\gamma}\hat{\rho}\epsilon\hat{J}^{(1)}$$

$$\frac{\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}}{\gamma}\left(-\frac{\hat{\phi}''}{\hat{\phi}'\left(\hat{i}^{(0)}\right)}\frac{\hat{c}^{(0)}}{\gamma-\dfrac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{i}^{(0)})}}\left(\epsilon\hat{K}\hat{J}^{(1)}_{\hat{K}}+(\eta-\gamma)\epsilon\hat{J}^{(1)}\right)-\epsilon\hat{K}\hat{J}^{(1)}_{\hat{K}}+\frac{\gamma-\eta}{(1-\gamma)}\epsilon\hat{J}^{(1)}\right)-\frac{1-\eta}{1-\gamma}\hat{\rho}\epsilon\hat{J}^{(1)},$$

where we have substituted for $\hat{i}^{(1)}=-\hat{c}^{(1)}$ from (C2.4) and used the identity:

(D3.4) $$\frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}^{(0)}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\hat{K}\hat{J}^{(0)}_{\hat{K}}}=\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}.$$

Substituting from (D3.2), two of the terms in (D3.1) simplify to

$$\text{(D3.5)}\quad \underbrace{\hat{f}^{*}}_{O(\epsilon)} + \hat{J}^{(0)}_{\hat{K}} \underbrace{\frac{1}{d\hat{t}} E_t\left[d\hat{K}\right]}_{O(\epsilon)} = -\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}(\eta-\gamma)+(1-\eta)\hat{\rho}\right]\epsilon\hat{J}^{(1)}.$$

Using (D3.5), (D3.1) can be rewritten as a forced equation:

$$\begin{aligned}
&-\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}\left(\eta-\gamma\right)+\left(1-\eta\right)\hat{\rho}\right]\epsilon\hat{J}^{(1)} + \epsilon\hat{J}^{(1)}_{\hat{t}} + \epsilon\hat{J}^{(1)}_{\hat{K}}\hat{K}\hat{\phi}(\hat{i}^{(0)}) \\
&\epsilon\hat{J}^{(1)}_{\hat{E}}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}} - \varphi\hat{E}\right) + \epsilon\hat{J}^{(1)}_{\hat{\chi}}\hat{v}_{\chi}\left(1-\chi\right) + \epsilon\hat{J}^{(1)}_{\hat{\lambda}}\hat{v}_{\lambda}\left(1-\hat{\lambda}\right) \\
\text{(D3.6)}\quad &+\frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2 + +\frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{E}\hat{E}}\hat{E}^2\hat{\sigma}_E^2 + \frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{\chi}\hat{\chi}}\hat{\sigma}_{\chi}^2 + \frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 \\
&\epsilon\hat{J}^{(1)}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \epsilon\hat{J}^{(1)}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + \epsilon\hat{J}^{(1)}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} \\
&+\epsilon\hat{J}^{(1)}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \epsilon\hat{J}^{(1)}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} + \epsilon\hat{J}^{(1)}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda} = -\hat{G}\left(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}\right),
\end{aligned}$$

where the forcing is defined as

$$\begin{aligned}
&\hat{G}(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}) \equiv \hat{J}^{(0)}_{\hat{E}}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}Ke^{-\hat{g}^{(0)}\hat{t}} - \varphi\hat{E}\right) + \hat{J}^{(0)}_{\hat{\chi}}\hat{v}_{\chi}(1-\hat{\chi}) + \\
\text{(D3.7)}\quad &\hat{J}^{(0)}_{\hat{\lambda}}\hat{v}_{\lambda}(1-\hat{\lambda}) + \frac{1}{2}\hat{J}^{(0)}_{\hat{E}\hat{E}}\hat{\sigma}_E^2 + \frac{1}{2}\hat{J}^{(0)}_{\hat{\chi}\hat{\chi}}\hat{\sigma}_{\chi}^2 + \frac{1}{2}\hat{J}^{(0)}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 + \hat{J}^{(0)}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \hat{J}^{(0)}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} \\
&+\hat{J}^{(0)}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} + \hat{J}^{(0)}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \hat{J}^{(0)}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} + \hat{J}^{(0)}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}.
\end{aligned}$$

To obtain derivatives of the zeroth-order value function with respect to $\hat{E},\hat{\chi}$ and $\hat{\lambda}$, we first differentiate with respect to the marginal productivity of capital $\hat{r}^{(0)}_{\text{mpk}}$, which depends on these three variables (via the chain rule of differentiation). From (C6), we obtain:

$$\text{(D3.8)}\quad \frac{\partial\hat{J}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}} = \hat{J}^{(0)}\left(-\left(1-\eta\right)\frac{\hat{\phi}''(\hat{i}^{(0)})}{\hat{\phi}'(\hat{i}^{(0)})} + \gamma\frac{1-\eta}{\hat{c}^{(0)}}\right)\frac{\partial\hat{i}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}.$$

Since the Investment rate is implicitly defined, we get from (C7) by implicit differentiation:

$$\text{(D3.9)}\quad \frac{\partial\hat{i}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}} = \frac{1}{\gamma - \hat{c}^{(0)}\hat{\phi}''(\hat{i}^{(0)})/\hat{\phi}'(\hat{i}^{(0)})}.$$

Combining (D3.8) and (D3.9), we obtain

$$\text{(D3.10)}\quad \frac{\partial \hat{J}^{(0)}}{\partial \hat{r}^{(0)}_{\text{mpk}}} = \hat{J}^{(0)} \frac{1-\eta}{\hat{c}^{(0)}} = \left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}} \left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1} \hat{K}^{1-\eta}.$$

Using the chain rule of differentiation, we find the individual terms that contribute to the forcing (D3.7):

$$\text{(D3.11)}\quad \begin{aligned} \underset{O(\epsilon)}{\hat{J}^{(0)}_{\hat{E}}} &= \left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}} \left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1} \hat{K}^{1-\eta} \frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}} \text{ and} \\ \underset{O(\epsilon)}{\hat{J}^{(0)}_{\hat{E}\hat{E}}} &= \left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}} \left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1} \hat{K}^{1-\eta} \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}^2}, \end{aligned}$$

and similarly for derivatives with respect to $\hat{\chi}$ and $\hat{\lambda}$, as well as cross-derivatives. From the zeroth-order solution $\hat{r}^{(0)}_{\text{mpk}} = \alpha \hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)^{1/\alpha} \left((1-\alpha)/\hat{b}\right)^{(1-\alpha)/\alpha} - \hat{\delta}$ and the non-dimensional total factor productivity (B4) we obtain

$$\text{(D3.12a)}\quad \begin{aligned} \frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}} &= -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*\left(1+\theta_{ET}\right) \hat{E}^{\theta_{ET}} \hat{\mathrm{X}}(\hat{\chi}) \hat{\Lambda}(\hat{\lambda}), \\ \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}^2} &= -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*\theta_{ET}\left(1+\theta_{ET}\right) \hat{E}^{\theta_{ET}-1} \hat{\mathrm{X}}(\hat{\chi}) \hat{\Lambda}(\hat{\lambda}), \end{aligned}$$

$$\text{(D3.12b)}\quad \begin{aligned} \frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\chi}} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} \hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi}) \hat{\Lambda}(\hat{\lambda}), \\ \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\chi}^2} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} \hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}(\hat{\chi}) \hat{\Lambda}(\hat{\lambda}), \end{aligned}$$

$$\text{(D3.12c)}\quad \begin{aligned} \frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\lambda}} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} \hat{\mathrm{X}}(\hat{\chi}) \hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda}), \\ \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\lambda}^2} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}} \hat{\mathrm{X}}(\hat{\chi}) \hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda}), \end{aligned}$$

$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial\hat{E}\partial\hat{\chi}} = -\epsilon\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\hat{A}^*\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),$$

(D3.12d)
$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial\hat{E}\partial\hat{\lambda}} = -\epsilon\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\hat{A}^*\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),$$

$$\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial\hat{\chi}\partial\hat{\lambda}} = -\epsilon\hat{A}^*\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),$$

where have used the following short-hands $\hat{\mathrm{X}}(\hat{\chi}) \equiv \left(\hat{\chi}\right)^{1+\theta_{\chi T}}$ and $\hat{\Lambda}(\hat{\lambda}) \equiv \left(\hat{\lambda}\right)^{1+\theta_{\lambda}}$, so $\hat{D} = \epsilon\hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda})$. Equations (D3.11) and (D3.12) can be substituted into (D3.7):

(D3.13)
$$\begin{aligned}
\hat{G}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) = {} & -\epsilon\hat{A}(\hat{E},\hat{\chi})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\hat{A}^*\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\left(\hat{\phi}'(\hat{\imath}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}} \\
& \Big[\Big(-(1+\theta_{ET})\hat{\varphi}\hat{\mathrm{X}}\hat{\Lambda} + \hat{\nu}_{\chi}(1-\hat{\chi})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda} + \hat{\nu}_{\lambda}(1-\hat{\lambda})\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}} + \frac{1}{2}\hat{\sigma}^2_{\chi}\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}\hat{\Lambda} + \frac{1}{2}\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}^2_{\lambda} \\
& + (1-\eta)\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + (1-\eta)\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} + \hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\Big)\hat{K}^{1-\eta}\hat{E}^{1+\theta_{ET}} \\
& + (1+\theta_{ET})\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}}e^{-\hat{g}^{(0)}\hat{t}} + \frac{1}{2}\theta_{ET}\left(1+\theta_{ET}\right)\hat{\sigma}^2_E\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1} \\
& + \Big((1-\eta)(1+\theta_{ET})\hat{\mathrm{X}}\hat{\Lambda}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + (1+\theta_{ET})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}} \\
& + (1+\theta_{ET})\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}.
\end{aligned}$$

Because we are ultimately interested in $\hat{J}^{(1)}_{\hat{E}}$ for the computation of the social cost of carbon, we first differentiate (D3.6) with respect to $\hat{E}$ and seek a solution for $\hat{J}^{(1)}_{\hat{E}}$ of the form $\hat{J}^{(1)}_{\hat{E}} = \psi_1\left(1+\theta_{ET}\right)\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})$, which gives (from (D3.6)):[31]

(D3.14) $$\hat{J}^{(1)}_{\hat{E}} = \psi_1\left(1+\theta_{ET}\right)\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) \Rightarrow -\hat{r}_{\Omega}\hat{\Omega} + \frac{1}{d\hat{t}}E_t\left[d\hat{\Omega}\right] = -\hat{\Gamma}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}),$$

where we have introduced the effective discount rate

[31] Dimensionally, we have $\Omega = E_0^{\theta_{ET}}\,\overline{\chi}^{1+\theta_{\chi T}}\,\overline{\lambda}^{1+\theta_{\lambda}}\,K_0^{1-\eta}\hat{\Omega}$.

(D3.15) $\hat{r}_{\Omega} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} + (1-\eta)\left(\hat{\phi}(\hat{i}^{(0)}) - \frac{1}{2}\eta\hat{\sigma}_K^2\right) + \hat{\varphi}$,

and the coefficient

(D3.16) $\psi_1 \equiv \hat{A}^{*}\hat{A}(\hat{E},\hat{\chi},\lambda)^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}$.

The scaled forcing is defined by[32]

(D3.17) $$\Gamma(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) \equiv \Big((1+\theta_{ET})\hat{\varphi}\hat{\mathrm{X}}\hat{\Lambda} - \hat{\nu}_{\chi}(1-\hat{\chi})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda} - \hat{\nu}_{\lambda}(1-\hat{\lambda})\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}} - \frac{1}{2}\hat{\sigma}_{\chi}^2\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}\hat{\Lambda}$$
$$-\frac{1}{2}\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 - (1-\eta)\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} - (1-\eta)\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} - \hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}$$
$$-\theta_{ET}\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}-1}e^{-\hat{g}^{(0)}\hat{t}} - \frac{1}{2}(\theta_{ET}-1)\theta_{ET}\hat{\sigma}_E^2\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-2}$$
$$-\theta_{ET}\Big((1-\eta)\hat{\mathrm{X}}\hat{\Lambda}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}\Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1}.$$

Equation (D3.14) has the closed-form solution:

(D3.18) $\hat{\Omega} = E_t\left[\int_{\hat{t}}^{\infty}\hat{\Gamma}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{s})e^{-\hat{r}_{\Omega}(\hat{s}-\hat{t})}d\hat{s}\right]$.

We can now compute the SCC according to $\hat{P} = -\hat{\mu}\left(\hat{J}_{\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{E}}^{(1)}\right)\Big/\phi'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}$:

(D3.19) $\hat{P} = \frac{\hat{\mu}\,\hat{\Theta}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{Y}\big|_{\hat{P}=0}}{\hat{r}^{*}}\left(1 - \frac{\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})}{\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda})\hat{K}^{1-\eta}}\right)$ with $\hat{\Theta} \equiv \frac{\hat{D}_{\hat{E}}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)}{1-\hat{D}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)}$,

where we introduced $\hat{r}^{*} \equiv \hat{r}^{(0)} - \hat{g}^{(0)}$. Dimensionally, (D3.19) corresponds to Result 1.

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[32] Dimensionally, we have $\Gamma = E_0^{1+\theta_{ET}}\,\overline{\chi}^{1+\theta_{\chi T}}\,\overline{\lambda}^{1+\theta_{\lambda}}\,K_0^{1-\eta}\,g_0\hat{\Gamma}$.

Appendix E: Leading-Order Effects of Uncertainty (For Online Publication)

Following the additional assumptions outlined at the start of section III, this appendix derives closed-form solutions for the optimal risk-adjusted SCC based on Result 1.

E.1. Carbon stock dynamics

The expected value of the carbon stock is governed by the differential equation (D2.5) with solution

$$\text{(E1.1)} \quad E_t\left[\hat{E}(\hat{s})\right] = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s}) + \hat{\mu}^*\hat{K}(\hat{t})\left[1-\exp(-\hat{\varphi}\Delta\hat{s})\right]/\hat{\varphi} = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s})\hat{e}(\hat{t}),$$

where $\hat{\mu}^* \equiv \hat{\mu}\left((1-\alpha)/\hat{b}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}$, $\Delta\hat{s} \equiv \hat{s}-\hat{t}$ and $\hat{e}(\Delta\hat{s}) = 1+(\hat{\mu}^*\hat{K}(\hat{t})/\hat{E}(\hat{t}))(\exp(\hat{\varphi}\Delta\hat{s})-1)/\hat{\varphi}$.

Dimensionally, we define μ^* so that $\mu F^{(0)} = \mu^* K$, where μ does not have units and μ^* has units of TtC/\$year. We can then obtain $\mu^* = \mu\left(A\left(1-\alpha\right)/b\right)^{1/\alpha}$ or $\hat{\mu}^* = \left(K_0/g_0E_0\right)\mu^*$.

E.2. Leading-order forcing

To identify leading-order terms only, we expand in $\Delta\hat{\chi} \equiv \hat{\chi}-\hat{\mu}_\chi$, $\Delta\hat{\lambda} \equiv \hat{\lambda}-\hat{\mu}_\lambda$ and $\Delta E \equiv E - E_t\left[E\right]$ with the corresponding covariance matrix given by (D1.5) (assumption II). We begin by considering terms that only involve capital stock uncertainty, which can be evaluated without further approximation. The probability density function for time $\hat{s}$, but with the expectation operator evaluated at time $\hat{t}$, is

$$\text{(E2.1)} \quad f_k = \frac{1}{\sqrt{2\pi\hat{\sigma}_K^2(\hat{s}-\hat{t})}}\exp\left(-\frac{1}{2}\left(\frac{(\hat{k}-\hat{\alpha}_k\hat{s})^2}{\hat{\sigma}_K^2(\hat{s}-\hat{t})}\right)\right),$$

where $\hat{\alpha}_k = \hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2$. Combining with the discount factor in (D3.18) and an additional factor accounting for the decay of the atmospheric carbon stock, we have without further approximation

$$\text{(E2.2)} \quad \begin{aligned} &E_t\left[\left(\hat{K}(\hat{s})\right)^{1-\eta}\right]\exp\left(-\left(\hat{r}_\Omega+\theta_{ET}\hat{\varphi}\right)(\hat{s}-\hat{t})\right) = \left(\hat{K}(\hat{t})\right)^{1-\eta}\exp(-\hat{r}^\star\Delta\hat{s}) \quad \text{and} \\ &E_t\left[\left(\hat{K}(\hat{s})\right)^{2-\eta}\right]\exp\left(-(\hat{r}_\Omega+\hat{g}^{(0)}+(\theta_{ET}-1)\hat{\varphi})\Delta\hat{s}\right) = \left(\hat{K}(\hat{t})\right)^{2-\eta}\exp(-\hat{r}^{\star\star}\Delta\hat{s}),\end{aligned}$$

where we have introduced the new short-hands $\hat{r}^{\star} \equiv \hat{r}^{*} + (1+\theta_{ET})\hat{\varphi} = \hat{r}^{(0)} - \hat{g}^{(0)} + (1+\theta_{ET})\hat{\varphi}$ and $\hat{r}^{\star\star} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} - (1-\eta)\hat{\sigma}_K^2 + \theta_{ET}\hat{\varphi} = \hat{r}^{\star} - (1-\eta)\hat{\sigma}_K^2 - \hat{\varphi}$ and note the use of alternative star symbols to denote rates corrected for atmospheric carbon stock decay. To leading order, we have for the terms involving the carbon stock:

(E2.3)
$$E_t\left[\hat{E}^{\theta_{ET}}\right] = \left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}}\left[1 + \frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right] + O(\hat{\Sigma}_E^4),$$
$$E_t\left[\hat{E}^{\theta_{ET}-1}\right] = \left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}-1}\left[1 + \frac{1}{2}\left(\theta_{ET}-1\right)\left(\theta_{ET}-2\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right] + O(\hat{\Sigma}_E^4),$$
$$E_t\left[\hat{E}^{\theta_{ET}-2}\right] = \left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}-2}\left[1 + \frac{1}{2}\left(\theta_{ET}-2\right)\left(\theta_{ET}-3\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right] + O(\hat{\Sigma}_E^4),$$

where we let the subscript on Σ^2 denote the relevant elements of the covariance matrix $\mathbf{\Sigma}$ (D.1.5) and we have ignored any contributions to uncertainty from new emissions through their dependence on uncertain future GDP. The following terms make a contribution to the forcing (D3.17): $\mathrm{X}\Lambda$, $\mathrm{X}_{\hat{\chi}}\Lambda$, $(\hat{\chi}-\hat{\mu}_{\chi})\mathrm{X}_{\hat{\chi}}\Lambda$, $\mathrm{X}\Lambda_{\lambda}$, $(\hat{\lambda}-\hat{\mu}_{\lambda})\mathrm{X}\Lambda_{\lambda}$, $\mathrm{X}_{\hat{\chi}\hat{\chi}}\Lambda$ and $\mathrm{X}\Lambda_{\hat{\lambda}\hat{\lambda}}$. Keeping only those terms contributing at leading order, we have

(E2.4a)
$$E_t\left[\mathrm{X}(\hat{\chi})\right] = \hat{\mu}_{\chi}^{1+\theta_{\chi T}}\left[1 + \frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right] + O(\hat{\Sigma}_{\chi}^4),$$
$$E_t\left[\mathrm{X}_{\hat{\chi}}(\hat{\chi})\right] = \hat{\mu}_{\chi}^{\theta_{\chi T}}\left[(\theta_{\chi T}+1) + \frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}(\theta_{\chi T}-1)\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right] + O(\hat{\Sigma}_{\chi}^4),$$

(E2.4b)
$$E_t\left[(\hat{\chi}-\hat{\mu}_{\chi})\mathrm{X}_{\hat{\chi}}\left(\hat{\chi}\right)\right] = \hat{\mu}_{\chi}^{1+\theta_{\chi T}}\left[(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_{\chi}}{\hat{\mu}_{\chi}}\right)^2\right] + O(\hat{\Sigma}_{\chi}^4),$$
$$E_t\left[\mathrm{X}_{\hat{\chi}\hat{\chi}}(\hat{\chi})\right] = \hat{\mu}_{\chi}^{\theta_{\chi T}-1}\left[(\theta_{\chi T}+1)\theta_{\chi T}\right] + O(\hat{\Sigma}_{\chi}^2),$$

$$E_t\left[\Lambda(\hat\lambda)\right]=\hat\mu_\lambda^{1+\theta_\lambda}\left[1+\frac{1}{2}(\theta_\lambda+1)\theta_\lambda\left(\frac{\hat\Sigma_\lambda}{\hat\mu_\lambda}\right)^2\right]+O(\hat\Sigma_\lambda^4),$$

(E2.5a)

$$E_t\left[\Lambda_{\hat\lambda}(\hat\lambda)\right]=\hat\mu_\lambda^{\theta_\lambda}\left[(\theta_\lambda+1)+\frac{1}{2}(\theta_\lambda+1)\theta_\lambda(\theta_\lambda-1)\left(\frac{\hat\Sigma_\lambda}{\hat\mu_\lambda}\right)^2\right]+O(\hat\Sigma_\lambda^4),$$

(E2.5b)

$$E_t\left[(\hat\lambda-\hat\mu_\lambda)\Lambda_{\hat\lambda}(\hat\lambda)\right]=\hat\mu_\lambda^{1+\theta_\lambda}\left[(\theta_\lambda+1)\theta_\lambda\left(\frac{\hat\Sigma_\lambda}{\hat\mu_\lambda}\right)^2\right]+O(\hat\Sigma_\lambda^4),$$

$$E_t\left[\Lambda_{\hat\lambda\hat\lambda}(\hat\lambda)\right]=\hat\mu_\lambda^{\theta_\lambda-1}\left[(\theta_\lambda+1)\theta_\lambda\right]+O(\hat\Sigma_\lambda^2),$$

where all the terms have been evaluated be at their equilibrium values, namely $\hat\chi=1$ and $\lambda=1,$ so that $\mathrm{X}=1$ and, an assumption that will be made henceforth (assumption I). Using (E2.2)-(E2.5), we now consider the terms in the forcing (D3.17) consecutively and let the subscript indices correspond to the sequence of terms in (D3.17) (left to right). To consider the covariance terms in the forcing (D3.17), we also expand in $\Delta\hat k\equiv\hat k-\left(\hat\phi(\hat i^{(0)})-\hat\sigma_K^2/2\right)\hat t$ and only consider deviations from the zeroth-order mean consistent with our search for leading-order terms only. Considering the forcing terms in (D3.17) consecutively, the following terms arise:

(E2.6)

$$E_t\left[\Gamma_1\right]=(1+\theta_{ET})\hat\varphi\left[1+\frac{1}{2}\theta_{ET}\left(1+\theta_{ET}\right)\frac{\hat\sigma_E^2}{\left(E_t\left[\hat E\right]\right)^2}\frac{1-\exp\left(-2\hat\varphi\Delta\hat s\right)}{2\hat\varphi}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\hat\sigma_\chi^2\frac{1-\exp\left(-2\hat\nu_\chi\Delta\hat s\right)}{2\hat\nu_\chi}+\frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\hat\sigma_\lambda^2\frac{1-\exp\left(-2\hat\nu_\lambda\Delta\hat s\right)}{2\hat\nu_\lambda}$$

$$+\left(1-\eta\right)\left(1+\theta_{ET}\right)\frac{\rho_{KE}\hat\sigma_K\hat\sigma_E}{E_t\left[\hat E\right]}\frac{1-\exp\left(-\hat\varphi\Delta\hat s\right)}{\hat\varphi}+\left(1-\eta\right)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat\sigma_K\hat\sigma_\chi\frac{1-\exp\left(-\hat\nu_\chi\Delta\hat s\right)}{\hat\nu_\chi}$$

$$+\left(1-\eta\right)\left(1+\theta_\lambda\right)\rho_{K\lambda}\hat\sigma_K\hat\sigma_\lambda\frac{1-\exp\left(-\hat\nu_\lambda\Delta\hat s\right)}{\hat\nu_\lambda}+\left(1+\theta_{ET}\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat\sigma_E\hat\sigma_\chi}{E_t\left[\hat E\right]}\frac{1-\exp\left(-\left(\hat\varphi+\hat\nu_\chi\right)\Delta\hat s\right)}{\hat\varphi+\hat\nu_\chi}$$

$$+\left(1+\theta_{ET}\right)\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat\sigma_E\hat\sigma_\lambda}{E_t\left[\hat E\right]}\frac{1-\exp\left(-\left(\hat\varphi+\hat\nu_\lambda\right)\Delta\hat s\right)}{\hat\varphi+\hat\nu_\lambda}$$

$$+\left(1+\theta_{\chi T}\right)\left(1+\theta_{\lambda}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}}\Bigg]E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{1+\theta_{ET}} \text{ and thus}$$

$$\frac{\hat{\Omega}_1}{\hat{K}^{1-\eta}\hat{E}^{1+\theta_{ET}}}=(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(1+(1+\theta_{ET})\hat{\mu}^*\frac{\hat{K}}{\hat{E}}\frac{1}{\hat{r}^*}-(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\right)$$

$$(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\frac{1}{2}\theta_{ET}\left(\theta_{ET}+1\right)\frac{\hat{\sigma}_E^2}{\hat{E}^2}\frac{1}{\hat{r}^*}+(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\frac{\hat{\sigma}_{\chi}^2}{\hat{r}^*+2\nu_{\chi}}$$

$$+(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\frac{1}{2}\theta_{\lambda}\left(\theta_{\lambda}+1\right)\frac{\hat{\sigma}_{\lambda}^2}{\hat{r}^*+2\hat{\nu}_{\lambda}}+(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(1-\eta\right)\left(\theta_{ET}+1\right)\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{E}}\frac{1}{\hat{r}^*}$$

$$+(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(1-\eta\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi}}{\hat{r}^*+\hat{\nu}_{\chi}}+(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(1-\eta\right)\left(1+\theta_{\lambda}\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}}{\hat{r}^*+\hat{\nu}_{\lambda}}$$

$$+(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(\theta_{ET}+1\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}}{\hat{E}}\frac{1}{\hat{r}^*+\hat{\nu}_{\chi}}$$

$$+(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(\theta_{ET}+1\right)\left(1+\theta_{\lambda}\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}}{\hat{E}}\frac{1}{\hat{r}^*+\hat{\nu}_{\lambda}}+(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(1+\theta_{\chi T}\right)\left(1+\theta_{\lambda}\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}}{\hat{r}^*+\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}},$$

(E2.7)

$$E_t\left[\Gamma_2\right]=\left(1+\theta_{\chi T}\right)\hat{\nu}_{\chi}\Bigg(\theta_{\chi T}\hat{\sigma}_{\chi}^2\frac{1-\exp\left(-2\hat{\nu}_{\chi}\Delta\hat{s}\right)}{2\hat{\nu}_{\chi}}$$

$$\left(1-\eta\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi}\frac{1-\exp\left(-\hat{\nu}_{\chi}\Delta\hat{s}\right)}{\hat{\nu}_{\chi}}+\left(1+\theta_{ET}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_{\chi}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_{\chi}}$$

$$+\left(1+\theta_{\lambda}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}}\Bigg)E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{1+\theta_{ET}}, \text{ and thus}$$

$$\frac{\hat{\Omega}_2}{K^{1-\eta}\hat{E}^{1+\theta_{ET}}}=\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\hat{\sigma}_{\chi}^2\left(\frac{1}{\hat{r}^*}-\frac{1}{\hat{r}^*+2\hat{\nu}_{\chi}}+(1+\theta_{ET})\left(\hat{\mu}^*\frac{\hat{K}}{\hat{E}}-\hat{\varphi}\right)\left(\frac{1}{\hat{r}^{*2}}-\frac{1}{\left(\hat{r}^*+2\hat{\nu}_{\chi}\right)^2}\right)\right)$$

$$+\left(1+\theta_{\chi T}\right)\left(1-\eta\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi}\left(\frac{1}{\hat{r}^*}-\frac{1}{\hat{r}^*+\hat{\nu}_{\chi}}+\left(1+\theta_{ET}\right)\left(\hat{\mu}^*\frac{\hat{K}}{\hat{E}}-\hat{\varphi}\right)\left(\frac{1}{\hat{r}^{*2}}-\frac{1}{\left(\hat{r}^*+\hat{\nu}_{\chi}\right)^2}\right)\right)$$

$$+\left(1+\theta_{ET}\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{E}}\left(\frac{1}{\hat{r}^*}-\frac{1}{\hat{r}^*+\hat{\nu}_\chi}+\theta_{ET}\left(\hat{\mu}^*\frac{\hat{K}}{\hat{E}}-\hat{\varphi}\right)\left(\frac{1}{\hat{r}^{*2}}-\frac{1}{\left(\hat{r}^*+\hat{\nu}_\chi\right)^2}\right)-\frac{\hat{\nu}_\chi}{\left(\hat{r}^*+\hat{\nu}_\chi\right)^2}\frac{\hat{\varphi}}{\hat{r}^*}\right)$$

$$+\left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\frac{\hat{\nu}_\chi\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(\frac{1}{\hat{r}^*}-\frac{1}{\hat{r}^*+\hat{\nu}_\chi+\hat{\nu}_\lambda}+\left(1+\theta_{ET}\right)\left(\hat{\mu}^*\frac{\hat{K}}{\hat{E}}-\hat{\varphi}\right)\left(\frac{1}{\hat{r}^{*2}}-\frac{1}{\left(\hat{r}^*+\hat{\nu}_\chi+\hat{\nu}_\lambda\right)^2}\right)\right)\Bigg),$$

where we have expanded, for example, $1\big/\left(\hat{r}^*+\theta_{ET}\hat{\varphi}\right)=\left(1/\hat{r}^*\right)\left(1-\left(\theta_{ET}\hat{\varphi}/\hat{r}^*\right)\right)+O\left(\hat{\varphi}^2\right)$ for consistency and continue to do so. The subsequent terms in (D3.17) give to leading order

$$E_t\left[\Gamma_3\right]=\left(1+\theta_\lambda\right)\hat{\nu}_\lambda\Bigg(\theta_\lambda\hat{\sigma}_\lambda^2\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$\left(1-\eta\right)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda}+\left(1+\theta_{ET}\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda}$$

(E2.8) $$\left(1+\theta_{\chi T}\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Bigg)E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{1+\theta_{ET}},$$

and thus

(E2.9) $$E_t\left[\Gamma_1\right]=(1+\theta_{ET})\hat{\varphi}\Bigg[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(E_t\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}}$$

$$+\frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\hat{\sigma}_\chi^2\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}+\frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\hat{\sigma}_\lambda^2\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+(1-\eta)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}}+(1-\eta)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi}$$

$$+(1-\eta)\left(1+\theta_\lambda\right)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda}+\theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi}$$

$$+\theta_{ET}\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda}$$

$$+\left(1+\theta_{\chi T}\right)\left(1+\theta_{\lambda}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\frac{1-\exp\left(-\left(\hat{v}_{\chi}+\hat{v}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{v}_{\chi}+\hat{v}_{\lambda}}\Bigg]E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},$$

and

(E2.10)
$$E_t\left[\Gamma_2\right]=\hat{v}_{\chi}(1+\theta_{\chi T})\Bigg(\theta_{\chi T}\hat{\sigma}_{\chi}^2\frac{1-\exp\left(-2\hat{v}_{\chi}\Delta\hat{s}\right)}{2\hat{v}_{\chi}}$$
$$\left(1-\eta\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi}\frac{1-\exp\left(-\hat{v}_{\chi}\Delta\hat{s}\right)}{\hat{v}_{\chi}}+\theta_{ET}\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_{\chi}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_{\chi}}$$
$$+\left(1+\theta_{\lambda}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\frac{1-\exp\left(-\left(\hat{v}_{\chi}+\hat{v}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{v}_{\chi}+\hat{v}_{\lambda}}\Bigg)E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},$$

(E2.11)
$$E_t\left[\Gamma_3\right]=\hat{v}_{\lambda}\left(1+\theta_{\lambda}\right)\Bigg(\theta_{\lambda}\hat{\sigma}_{\lambda}^2\frac{1-\exp\left(-2\hat{v}_{\lambda}\Delta\hat{s}\right)}{2\hat{v}_{\lambda}}$$
$$\left(1-\eta\right)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}\frac{1-\exp\left(-\hat{v}_{\lambda}\Delta\hat{s}\right)}{\hat{v}_{\lambda}}+\theta_{ET}\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_{\lambda}}$$
$$\left(1+\theta_{\chi T}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\frac{1-\exp\left(-\left(\hat{v}_{\chi}+\hat{v}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{v}_{\chi}+\hat{v}_{\lambda}}\Bigg)E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},$$

(E2.12) $$E_t\left[\Gamma_4\right]=-\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\hat{\sigma}_{\chi}^2E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},$$

(E2.13) $$E_t\left[\Gamma_5\right]=-\frac{1}{2}\left(1+\theta_{\lambda}\right)\theta_{\lambda}\hat{\sigma}_{\lambda}^2E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},$$

(E2.14) $$E_t\left[\Gamma_6\right]=-\left(1-\eta\right)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi}E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},$$

(E2.15) $$E_t\left[\Gamma_7\right]=-\left(1-\eta\right)\left(1+\theta_{\lambda}\right)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},$$

(E2.16) $$E_t\left[\Gamma_8\right]=-\left(1+\theta_{\chi T}\right)\left(1+\theta_{\lambda}\right)\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},$$

(E2.17) $$E_t\left[\Gamma_9\right]=-\theta_{ET}\hat{\mu}^*\left[1+\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\hat{\sigma}_\chi^2\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\right.$$

$$+\frac{1}{2}\theta_\lambda\left(\theta_\lambda+1\right)\hat{\sigma}_\lambda^2\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}+\left(2-\eta\right)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi}$$

$$+\left(2-\eta\right)\left(1+\theta_\lambda\right)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda}$$

$$\left.+\left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\right]E_t\left[\hat{K}^{2-\eta}\right]e^{-\hat{g}^{(0)}\Delta\hat{s}}\left(E_t\left[\hat{E}\left(\hat{s}\right)\right]\right)^{\theta_{ET}-1},$$

(E2.18) $$E_t\left[\Gamma_{10}\right]=-\frac{1}{2}\theta_{ET}(\theta_{ET}-1)\hat{\sigma}_E^2E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-2},$$

(E2.19) $$E_t\left[\Gamma_{11}\right]=-(1-\eta)\theta_{ET}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_EE_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1},$$

(E2.20) $$E_t\left[\Gamma_{12}\right]=-(1+\theta_{\chi T})\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1},$$

(E2.21) $$E_t\left[\Gamma_{13}\right]=-\theta_{ET}(1+\theta_\lambda)\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1},$$

where elements of the covariance matrix have been substituted from (D1.5).

E.3. Leading-order solution

Combining all the leading-order terms in the forcing equation (E2.9)-(E2.21) and substituting into (D3.19), we obtain after considerable manipulation

(E3.1) $$\hat{P}=\frac{\hat{\mu}\;\hat{\Theta}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{Y}\Big|_{\hat{P}=0}}{\hat{r}^\star}\left(1+\theta_{ET}\hat{\mu}^*\frac{\hat{K}}{\hat{E}}\frac{1}{\hat{r}^{\star\star}}\Upsilon+\Delta_{EE}+\Delta_{\chi\chi}+\Delta_{\lambda\lambda}+\Delta_{CK}\right),$$

where

(E3.2) $$\Delta_{EE}\equiv\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\hat{E}^2}\frac{1}{\hat{r}^\star-2\hat{\varphi}}\Upsilon_{EE},\quad \Delta_{\chi\chi}\equiv\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\frac{\hat{\sigma}_\chi^2}{\hat{r}^\star+2\hat{\nu}_\chi}\Upsilon_{\chi\chi},$$
$$\Delta_{\lambda\lambda}\equiv\frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2}{\hat{r}^\star+2\hat{\nu}_\lambda}\Upsilon_{\lambda\lambda},$$

$$\Delta_{CK} \equiv -(\eta-1)\hat{\sigma}_K \left(\theta_{ET} \frac{\rho_{KE}\hat{\sigma}_E}{\hat{E}\left(\hat{r}^\star - \hat{\varphi}\right)} \Upsilon_{KE} + \left(1+\theta_{\chi T}\right) \frac{\rho_{K\chi}\hat{\sigma}_\chi}{\hat{r}^\star + \hat{v}_\chi} \Upsilon_{K\chi} + \left(1+\theta_\lambda\right) \frac{\rho_{K\lambda}\hat{\sigma}_\lambda}{\hat{r}^\star + \hat{v}_\lambda} \Upsilon_{K\lambda} \right),$$

$$\Delta_{CC} \equiv \theta_{ET}\left(1+\theta_{\chi T}\right) \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{E}} \frac{\hat{r}^\star}{\hat{r}^\star - \hat{\varphi}} \frac{1}{\hat{r}^\star + \hat{v}_\chi} \Upsilon_{E\chi}$$

$$+\left(1+\theta_\lambda\right)\left(\left(1+\theta_{\chi T}\right) \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{r}^\star + \hat{v}_\lambda + \hat{v}_\lambda} \Upsilon_{\chi\lambda} + \theta_{ET} \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{E}} \frac{\hat{r}^\star}{\hat{r}^\star - \hat{\varphi}} \frac{1}{\hat{r}^\star + \hat{v}_\lambda} \Upsilon_{E\lambda} \right),$$

and the correction factors are given by

(E3.3)

$$\Upsilon = \frac{\hat{r}^{\star\star}}{1-(1+\theta_{ET})\hat{\varphi}/\hat{r}^\star} \int_0^\infty \left(\exp(-\hat{r}^{\star\star}\Delta\hat{s}) - \frac{(1+\theta_{ET})\hat{\varphi}}{\hat{r}^\star} \exp(-(\hat{r}^\star - \hat{\varphi})\Delta\hat{s}) \right) \left(\hat{e}(\Delta\hat{s})\right)^{\theta_{ET}-1} d\Delta\hat{s},$$

$$\Upsilon_{Ki} = 1 + \theta_{ET}\hat{\mu}^* \frac{\hat{K}}{\hat{E}} \frac{1}{1-(1+\theta_{ET})\hat{\varphi}/\hat{r}^\star} \left[\left(1 + \frac{(1+\theta_{ET})\hat{\varphi}}{\hat{v}_i} \right) \int_0^\infty \exp\left(-(\hat{v}_i + \hat{r}^\star - \hat{\varphi})\Delta\hat{s}\right) \left(\hat{e}(\Delta\hat{s})\right)^{\theta_{ET}-1} d\Delta\hat{s} \right.$$

$$-(1+\theta_{ET}) \frac{\hat{\varphi}}{\hat{r}^\star} \frac{\hat{r}^\star + \hat{v}_i}{\hat{v}_i} \int_0^\infty \exp(-(\hat{r}^\star - \hat{\varphi})\Delta\hat{s}) \left(\hat{e}(\Delta\hat{s})\right)^{\theta_{ET}-1} d\Delta\hat{s} +$$

$$\left. \frac{2-\eta}{1-\eta} \frac{\hat{v}_i + \hat{r}^\star}{\hat{v}_i} \int_0^\infty \left(\exp(-\hat{r}^{\star\star}\Delta\hat{s}) - \exp(-(\hat{r}^{\star\star} + \hat{v}_i)\Delta\hat{s}) \right) \left(\hat{e}(\Delta\hat{s})\right)^{\theta_{ET}-1} d\Delta\hat{s} \right] \text{ for } i = \chi, \lambda,$$

$$\Upsilon_{ij} = 1 + \theta_{ET}\hat{\mu}^* \frac{\hat{K}}{\hat{E}} \frac{1}{1-(1+\theta_{ET})\hat{\varphi}/\hat{r}^\star} \left[\left(1 + \frac{(1+\theta_{ET})\hat{\varphi}}{\hat{v}_i + \hat{v}_j} \right) \int_0^\infty \exp\left(-(\hat{r}^\star + \hat{v}_i + \hat{v}_j - \hat{\varphi})\Delta\hat{s}\right) \left(\hat{e}\left(\Delta\hat{s}\right)\right)^{\theta_{ET}-1} d\Delta\hat{s} \right.$$

$$-\frac{(1+\theta_{ET})\hat{\varphi}}{\hat{v}_i + \hat{v}_j} \frac{\hat{r}^\star + \hat{v}_i + \hat{v}_j}{\hat{r}^\star} \int_0^\infty \exp(-(\hat{r}^\star - \hat{\varphi})\Delta\hat{s}) \left(\hat{e}\left(\Delta\hat{s}\right)\right)^{\theta_{ET}-1} d\Delta\hat{s}$$

$$\left. + \frac{\hat{r}^\star + \hat{v}_i + \hat{v}_j}{\hat{v}_i + \hat{v}_j} \int_0^\infty \left(\exp(-\hat{r}^{\star\star}\Delta\hat{s}) - \exp(-(\hat{r}^{\star\star} + \hat{v}_i + \hat{v}_j)\Delta\hat{s}) \right) \left(\hat{e}\left(\Delta\hat{s}\right)\right)^{\theta_{ET}-1} d\Delta\hat{s} \right] \text{ for } i, j = \chi, \lambda,$$

and we do not explicitly give the correction factors for the terms involving carbon stock uncertainty, as these terms are negligibly small.

Dimensionally, (E3.1) together with (E3.2) and (E3.3) gives, using the definitions summarized in (A.1) and (A.3) and $\hat{\mu}^*\hat{K} = \mu F^{(0)} / (g_0 E_0)$), Result 2′ stated in Appendix A. This generalizes Result 2 to convex reduced-form damages, non-zero carbon stock volatility and potentially skewed damage uncertainty.

Appendix F: Calibration (For Online Publication)

F.1. Asset returns, risk aversion and intertemporal substitution

To calibrate the non-climatic part of our model to match historical asset returns, we follow the calibration of Pindyck and Wang (2013), but ignore the effect of catastrophic shocks considered by these authors.[33,34] Using monthly asset data from the S&P 500 for the period 1947-2008, we obtain an annual return on assets (capital gains plus dividends) of $r^{(0)}$ = 7.2%/year with annual volatility of $\sigma_K = 12\%$. For a return on safe assets of 0.80 %/year based on the annualized monthly return on 3-months T-bills, we obtain a risk premium of $\Delta r^{(0)} \equiv r^{(0)} - r_{\mathrm{rf}}^{(0)} = 6.4\%$/year and calibrate the coefficient of relative risk aversion as η = 4.3 (*cf.* $\Delta r^{(0)} = \eta \sigma_K^2$). Taking the growth rate to be equal to the historical growth rate of $g^{(0)}$ = 2.0%/year, the equation $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - (1+\gamma)\eta \sigma_K^2 / 2$ (cf. (B9)) defines the combinations of ρ and γ that are consistent with historical asset returns. Setting the coefficient of elasticity of intertemporal substitution $\mathrm{EIS} = 2/3$, we obtain $\gamma = \mathrm{EIS}^{-1} = 1.5$ and thus a rate of time preference is ρ = 5.8%/year.

E.2. Productivity, fossil fuel, adjustment costs and the depreciation rate

To calibrate total factor productivity, we consider the production function in the absence of climate damage that can be obtained by setting $P = 0$ (*i.e.* at zeroth order), namely

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[33] Pindyck and Wang (2013) use Poisson shocks to capture small risks of large disasters (cf. Barro, 2016) and thus match skewness and kurtosis of asset returns. These shocks are responsible for approximately 1%-point of the risk premium. We furthermore calibrate the zeroth-order or non-climatic part of our model based on historical GDP data, which may have included the (small) effects of climate in reality.

[34] The alternative is to calibrate our AK model to the observed volatility of consumption or output (cf. Gollier, 2012), which are generally much less volatile than capital (asset returns). Because the volatilities of capital, consumption and output are equal to the volatility of capital in an AK model, this alternative calibration gives a much lower volatility and, consequently, a higher coefficient of relative risk aversion to match the equity premium (see also the discussion in Pindyck and Wang, 2013). Historical data for the growth rate of world GDP for 1961-2015 imply an annual volatility of σ_K = 1.5 % and thus a much higher value of risk aversion of $\eta = 2.8 \times 10^2$ for an equity premium of 6.4 %/year. Kocherlota (1996) obtains σ_K = 3.6%/year from US annual consumption growth during 1889-1978, which gives $\eta = 49$.

$Y^{(0)} = A^* K$ with $A^* = A^{1/\alpha} \left((1-\alpha)/b\right)^{(1-\alpha)/\alpha}$ (cf. (A9)). Pindyck and Wang (2013) use empirical estimates of the physical, human and intangible capital stocks and find $A^* = 0.113$ /year, which we adopt. Based on emissions of $F_0^{(0)} = 9.1$ GtC/year in 2015, energy costs making up a share $1-\alpha = 6.6$ % of world GDP at \$75 trillion/year, we obtain an estimate of the cost of fossil fuel of $b = Y_0^{(0)}(1-\alpha)\big/F_0^{(0)} = \$5.4\times 10^2$ /tC.[35] From this, we estimate the gross marginal productivity of capital $Y_K^{(0)}\big|_{t=0} = \alpha A^* = 0.106$ /year.[36] Using Pindyck and Wang's (2013) consumption-investment ratio $c^{(0)}\big/i^{(0)} = 2.84$ and the identity $\alpha A^* = c^{(0)} + i^{(0)}$, we obtain initial values of $c^{(0)} =$ 7.84%/year and $i^{(0)} = 2.76$%/year. Using $q^{(0)} = c^{(0)}\big/(r^{(0)} - g^{(0)}) =$ 1.51 and $q^{(0)} = (1-\omega i^{(0)})^{-1}$, we obtain for the adjustment-cost parameter $\omega =$ 12.2 year. Finally, we find the depreciation rate that is consistent with the assumed rate of economic growth: $\delta = i^{(0)} - \omega (i^{(0)})^2\big/2 - g^{(0)} = 0.30$ %/year.

E.3. Atmospheric carbon stock and uncertainty

We calibrate our stylized carbon stock model (2.4) to the Law Dome Ice Core 2000-year data set and historical emissions. The first column of Figure F1 shows maximum-likelihood estimates of the two parameters of our simple atmospheric carbon stock model (2.4) for different time periods, from which it is evident that estimates displaying a certain linear relationship between φ and μ are of comparable likelihood.[37] These loci of maximum

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[35] We estimate the share of energy costs from data for energy use and energy costs from BP Statistical Review of World Energy 2017. Data for emissions are obtained from the same source available online at https://www.bp.com/en/global/corporate/energy-economics/statistical-review-of-world-energy.html. Our estimate of energy costs as a percentage of GDP is in good agreement with data from the U.S. Energy Information Administration available online at https://www.eia.gov/totalenergy/data/annual/showtext.php?t=ptb0105.

[36] This is in line with Caselli and Feyrer (2007), who estimate annual marginal products of capital of 8.5% for rich countries and 6.9% for poor countries, and an observed annual risk premium of 5-7%. They use a depreciation rate of 6.0% to calculate the capital stock from investment, include the share of reproducible capital rather than the share of total capital, account for differences in prices between capital and consumption goods and correct for inflation.

[37] Annual data from the Law Dome firn and ice core records and the Cape Grim record are available online at ftp://ftp.ncdc.noaa.gov/pub/data/paleo/icecore/antarctica/law/law2006.txt.This data is based on spline fits to different dataset with different spline windows across time reflecting changes

likelihood are shown separately in Figure F2, with the overall maximum denoted by a red circle and corresponding values given in Table F.1.

TABLE F.1 – ATMOSPHERIC CARBON STOCK CALIBRATION

Time	μ	φ [%/year]	σ_E [GtC/year$^{1/2}$]	σ_E / S_0 [%/year$^{1/2}$]	σ_E / E_0 [%/year$^{1/2}$]
1750-2004	1.0	0.66	0.31	0.036	0.12
1800-2004	0.75	0.00	0.26	0.029	0.10
1900-2004	0.59	0.00	0.21	0.025	0.081
1959-2004	0.79	0.91	0.23	0.027	0.089

The remaining columns in Figure F.1 show the predicted and observed rate of change of the atmospheric carbon stock (second column), the predicted and observed atmospheric carbon stock (third column) and the remaining variability (fourth column).[38] For our base case, we set $\mu = 1.0$ and $\varphi = 0.66$%/year. Figure F.1 indicates that our simple model (2.4) captures the observed variation in the atmospheric carbon stock reasonably well, including for very long time periods. The final column in Table F.1 shows our estimates of the volatility as a percentage of the initial carbon stock, from which it is evident that the stochastic carbon stock correction to the optimal carbon price (4.1a) will be extremely small.[39]

in the temporal resolution of the data. The discrete nature of the fitted data is evident for the early years. Annual carbon emissions from fossil fuel consumption and cement production are available online at http://cdiac.ornl.gov/trends/emis/tre_glob_2013.html.

[38] By setting $\varphi = 0$, we can estimate the fraction μ of emissions that stays in the atmosphere forever, whilst the remainder is instantaneously absorbed by the oceans and other carbon sinks. Calibrating to this data, we find $\mu = 0.68$, 0.64, 0.56 and 0.43 for the periods 1750-2004, 1800-2004, 1900-2004 and 1959-2004, respectively. Performing a similar analysis, Le Quéré et al. (2009) find that, between 1959 and 2008, 43% of each year's CO_2 emissions remained in the atmosphere on average.

[39] We set the initial atmospheric carbon concentration at $t = 0$ to $S_0 =$ 401 ppm of CO_2 (May 2015), corresponding to 0.854 TtC or 3.13 $TtCO_2$, and the pre-industrial atmospheric carbon concentration to 280 ppm CO_2, 0.596 TtC or 2.19 $TtCO_2$, so that $E_0 = 121$ ppm CO_2, 0.258 TtC or 0.94 $TtCO_2$. Updated and historical values can be found online at http://www.esrl.noaa.gov/gmd/ccgg/trends/global.html. We use the conversion factors: 1 ppm of CO_2 corresponds to 2.13 GtC and 1 GtC corresponds to 3.664 $GtCO_2$.

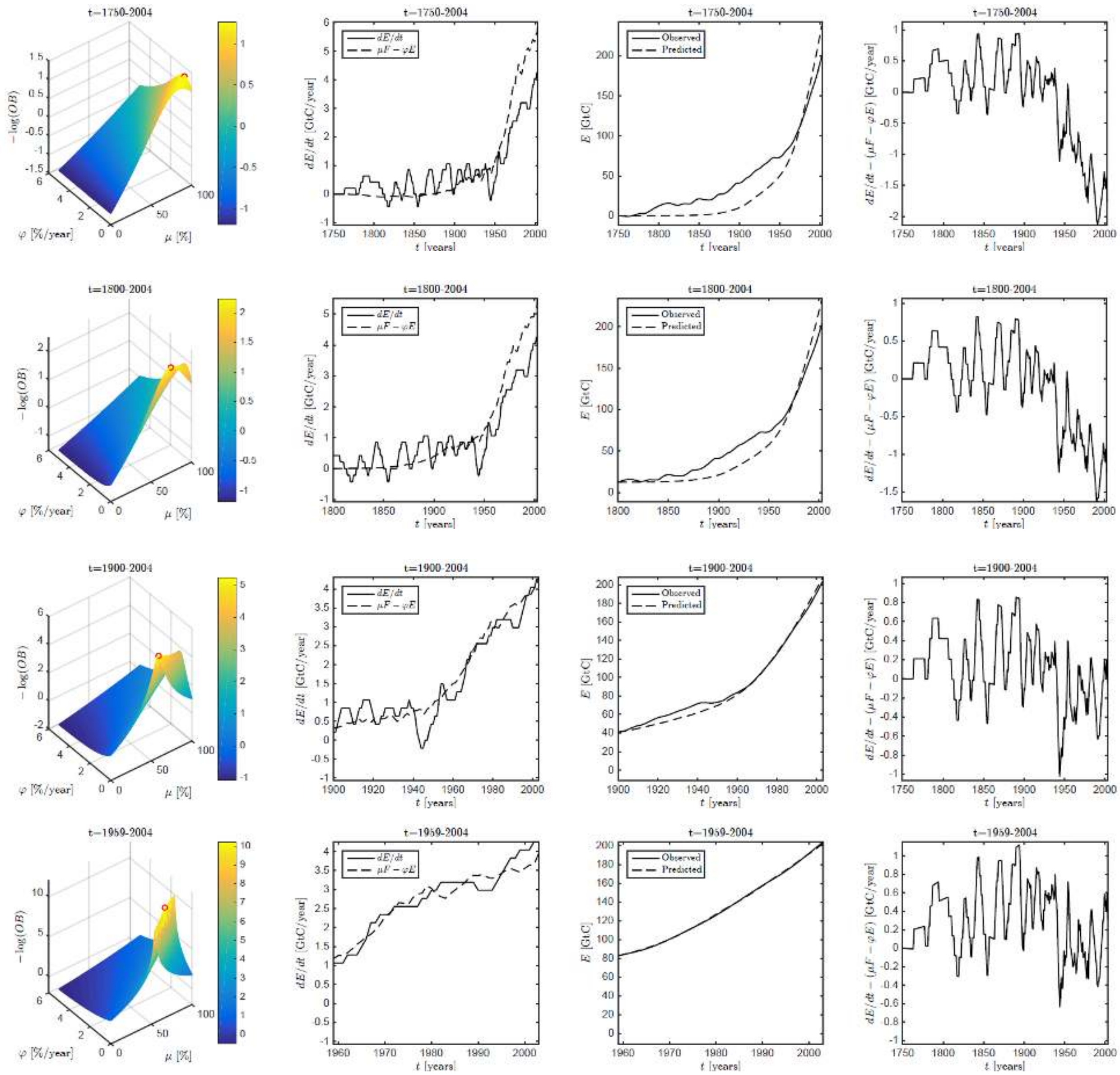


FIGURE F.1. ATMOSPHERIC CARBON STOCK CALIBRATION

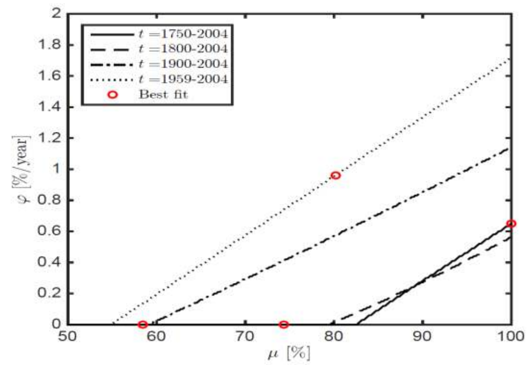


FIGURE F.2. LOCI OF BEST FIT OF TMOSPHERIC STOCK CALIBRATION

F.4. Calibration of the curvature of the temperature-carbon stock relationship

It is common to assume a logarithmic relationship between temperature and atmospheric carbon stock (Nordhaus, 2008; Golosov et al., 2014; Hambel et al., 2016), thus introducing concavity. In our model, the normalized curvature of the temperature relationship (2.5) is constant: $\theta_E \equiv ET_{EE}(E,\chi)/T_E(E,\chi)$. The radiative law for global mean temperature, $T \propto \ln(S/S_{PI})/\ln(2) \propto \ln\left((E+S_{PI})/S_{PI}\right)/\ln(2)$ (Arrhenius, 1854)[40] gives $\theta_E = -E/(E+S_{PI})$. If we evaluate (2.5) at double (quadruple) the pre-industrial stock $E = S_{PI}$ ($E = 3S_{PI}$), we obtain $\theta_E = -0.50$ ($\theta_E = -0.75$).[41] For $S_0 = 0.854$ TtC or $E_0 =$ 0.258 TtC (given $S_{PI} = 0.596$ TtC), we obtain $\theta_E = -0.30$. Alternatively, using the simulations reported in Allen et al. (2009) [42] for the peak CO_2-induced warming as a function of cumulative emissions shown in Figure F.3, which we take to be equivalent to

[40] In their table 6.2, IPCC (2001) propose a logarithmic relationship for radiative forcing as a function CO_2, also given in IPCC (1990, chapter 2, where original sources are cited), among two other non-logarithmic, but generally concave parametrizations. IPCC (1990, chapter 2, page 51) note that for "low/moderate/high concentrations, the form is well approximated by a linear/square-root/logarithmic dependence", where the limit of validity of the logarithmic calibration is said to be 1000 ppm. For other greenhouse gases alternative parametrizations are proposed: a square-root dependence for methane and a linear dependence for halocarbons.

[41] Whereas the normalized curvature of Arrhenius's (1854) logarithmic radiative law with respect to the atmospheric carbon stock *S*, namely $ST_{SS}(S)/T_S(S)$ is constant and equal to -1, this limit is only reached for large carbon stock in our case, in which $\theta_E \equiv ET_{EE}(E,\chi)/T_E(E,\chi)$.

[42] The black crosses are digitized from the white crosses in Figure 2 of Allen et al. (2009) corresponding to their best fit.

the transient climate response to cumulative emissions (TCRE), we estimate the curvature of our temperature-carbon stock relationship to be $\theta_E = -0.45$. We set $\theta_E = -0.50$ for our base case calibration.

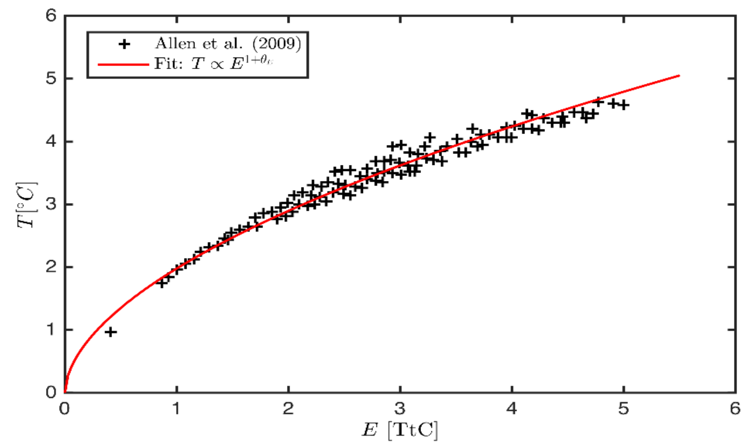
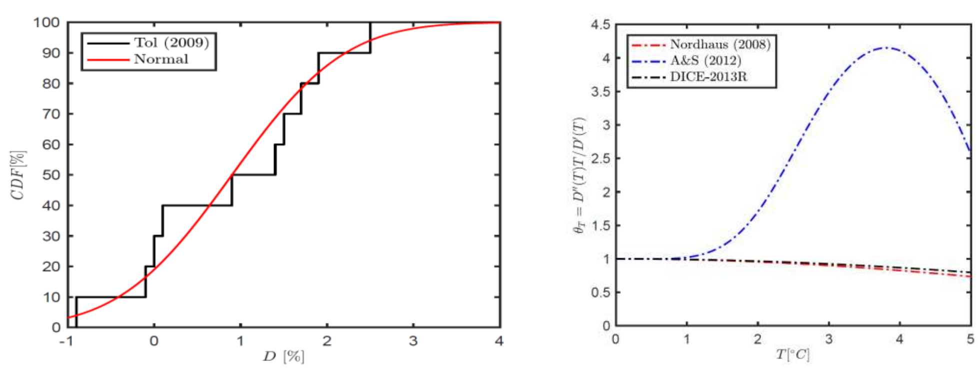


FIGURE F.3. TEMPERATURE-CARBON STOCK RELATIONSHIP FOR THE TCRE

F.5. Climate damages and climate damage uncertainty

The empirical distribution function corresponding to the 10 damages estimates at $T = 2.5$ ºC reported by Tol (2009) is plotted in Figure F.4. We use the 14 damages estimates reported by Tol (2009) to estimate the mean μ_λ and the 10 damages estimates for $T = 2.5$ ºC to estimate the standard deviation Σ_λ for three values of $\theta_T = 1$ namely 0.5, 1 and 1.5 and report the results in Figure F.5 and Table F.2.



(a) CUMULATIVE DENSITY at $T = 2.5$ ºC (b) CURVATURE

FIGURE F.4. ASPECTS OF THE CLIMATE DAMAGE CALIBTRATION

TABLE F.2 – CLIMATE DAMAGE CALIBRATION

	Concave damages	Proportional damages	Convex damages
θ_T	0.50	1.0	1.5
θ_{ET}	-0.25	0	0.25
μ_λ	3.1×10^{-3}	2.2×10^{-3}	1.6×10^{-3}
Σ_λ	2.6×10^{-3}	1.6×10^{-3}	1.0×10^{-3}
$\Sigma_\lambda/\mu_\lambda$	0.83	0.76	0.66

With $\theta_E = -0.5$, we thus have concave damages ($\theta_T = 0.5$, $\theta_{ET} = -0.25$), proportional damages ($\theta_T = 1$, $\theta_{ET} = 0$) and convex damages ($\theta_T = 1.5$, $\theta_{ET} = 0.25$), emphasizing their dependence on the carbon stock. Figure F.4(b) also shows the curvature $\theta_T \equiv TD''(T)/D'(T)$ of three commonly used damage specifications by Nordhaus (2008), Ackerman and Stanton (2012) and Weitzman (2012) (A&S (2012)), and by DICE2013R, the last also based on the survey by Tol (2009):[43]

$$\text{(F1)} \qquad D(T) = \begin{cases} 1 - 1/\left(1 + 0.00284T^2\right) & \text{Nordhaus (2008),} \\ 1 - 1/\left(1 + 0.00245T^2 + 5.021\times10^{-6}T^{6.76}\right) & \text{A\&S (2012),} \\ 1 - 1/\left(1 + 0.002131T^2\right) & \text{DICE2013R.} \end{cases}$$

The grey arrows in Figure 3 correspond to the ranges of damage estimates of Dietz and Stern (2008): 0.5-2% of GDP for $T = 3$°C, 1-5% for $T = 4$°C, and 1-8% for $T = 5$°C (also used by Pindyck (2012)).[44]Finally, we discuss the implications for the flow damage coefficient $\Theta \equiv D_E\left(E,\chi,\lambda\right)/\left(1 - D\left(E,\chi,\lambda\right)\right)$, the constant of proportionality between the optimal carbon price and GDP in Results 2 and 3. Figure F.6 shows its value as function of the atmospheric carbon stock, with the values of χ and λ set to their mean

[43] These damage functions turn from convex to concave at 10.8ºC, 5.8ºC and 12.5ºC, respectively. Our power-law damage function has constant curvature making assessment of the effects of uncertainty more straightforward.
[44] Nordhaus and Sztorc (2013) also report a range of 1-5% of GDP at 4ºC.

values μ_{χ} and μ_{λ} for the three values of θ_T considered in Figure F.5 and Table F.2. We set the values of μ_{χ} and θ_{χ} corresponding to the ECS, our base case (cf. Table 1).

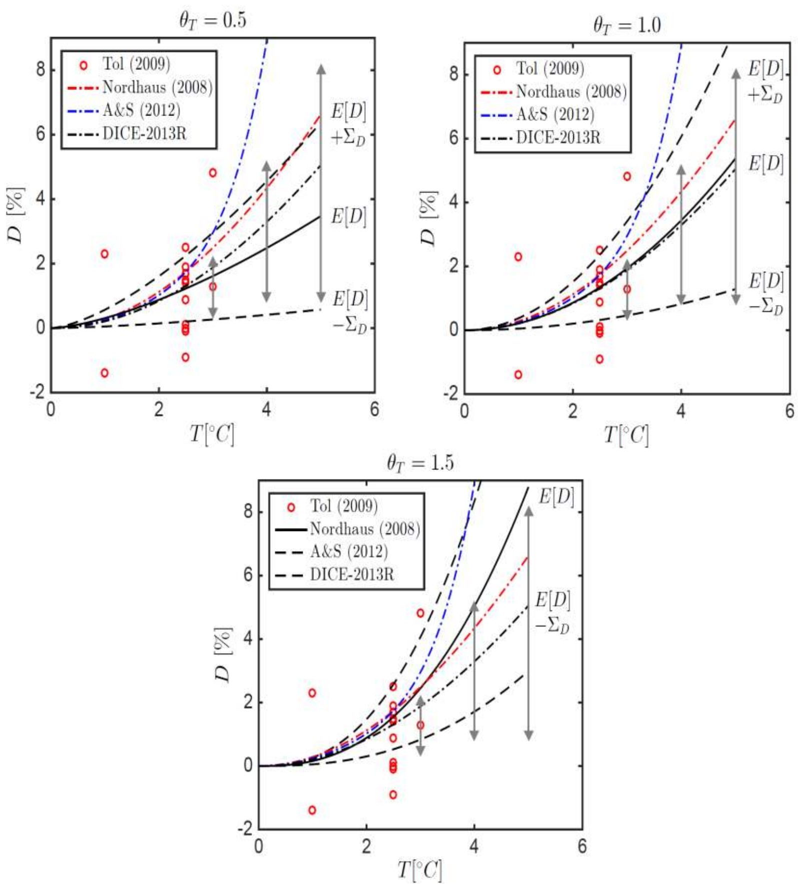


FIGURE F.5. CLIMATE CALIBRATION

The solid line in Figure F.6 shows that $\Theta(E) \equiv D'(E)/(1-D)$ is approximately constant at 2.6% GDP/TtC for proportional damages ($\theta_T = 1, \theta_{ET} = 0$). For convex damages ($\theta_T = 1.5, \theta_{ET} = 0.25$) the flow damage coefficient starts at 3.1% GDP/TtC and then rises gradually with increasing emissions. With concave damages ($\theta_T = 0.5, \theta_{ET} = -0.25$), the flow damage coefficient starts at 2.1% of GDP/TtC and then gradually falls with increasing emissions. With convex (concave) damages, the optimal carbon price rises faster (slower) than GDP in our model. Golosov et al. (2014), on the other hand, have a constant flow damage coefficient of $\Theta = 3.64\%$ of GDP/TtC, which includes a markup for tipping risk.[45]

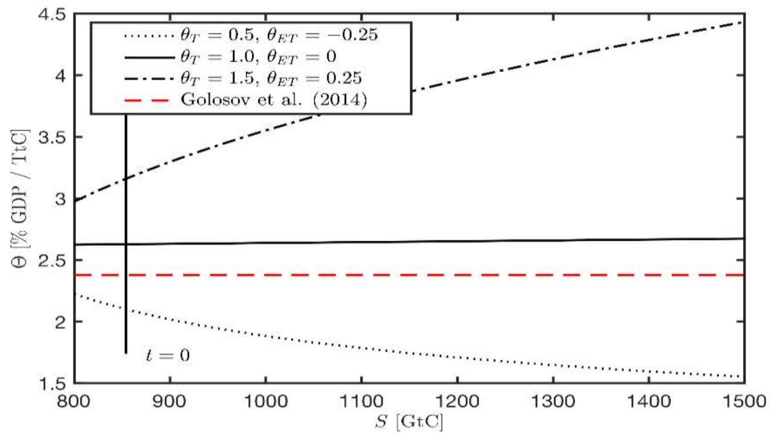


FIGURE F.6. FLOW DAMAGE COEFFICIENT $\Theta(E)$

E.6. Climate sensitivity and uncertainty

We consider three measures of the climate sensitivity: the equilibrium climate sensitivity (ECS), the transient climate response (TCR), and the transient climate response to cumulative carbon emissions (TCRE), where the latter is defined as the change in annual mean global temperature per unit of cumulated carbon emissions in a scenario with continuing emissions.[46] Let χ be normally distributed with mean μ_χ and standard deviation Σ_χ.[47] The parameter θ_χ is chosen to match the potentially positively-skewed probability density functions of the climate sensitivity T_2 described by

$$\text{(F2)} \qquad f_{T_2}\left(T_2;\mu_\chi,\Sigma_\chi,\theta_\chi\right) = \frac{1}{\sqrt{2\pi}\Sigma_\chi\left(1+\theta_\chi\right)} T_2^{-\frac{\theta_\chi}{1+\theta_\chi}} \exp\left(-\left(T_2^{\frac{1}{1+\theta_\chi}} - \mu_\chi\right)^2 / 2\Sigma_\chi^2\right).$$

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[45] Without tipping risk, $\Theta = 2.38\%$ GDP/TtC, as shown in Figure F.6.

[46] For low to medium estimates of climate sensitivity, the TCRE is nearly identical to the peak climate response to cumulative carbon emissions (IPCC, 2013).

[47] Since the power-law transformation $T_2 = \chi^{1+\theta_\chi}$ does not allow negative values of χ, we should use a truncated normal distribution with zero probabilities for negative values of χ. In practice, these probabilities are negligibly small without truncation and we ignore this complexity, further motivated by our consideration of leading-order terms (5.2a-d). As a result, there is a small probability atom at $T_2 = 0$°C in Figure F.4a (2.6 %), which we ignore.

Unlike for fat-tailed distributions, which typically have algebraically-decaying tails, all moments of (F2) are defined due to its exponential tail (for $\theta_\chi \geq -1$), so that Weitzman's (2009) 'dismal theorem' does not apply. Positive values of θ_χ result in a positively-skewed (non-Gaussian) distribution with more probability mass at high temperatures. Leading-order central moments of climate sensitivity can be obtained from performing Taylor-series expansions of $T_2 = \chi^{1+\theta_\chi}$ about its mean μ_χ:

(F2a) $$E[T_2] = \mu_\chi^{1+\theta_\chi}\left(1+\frac{1}{2}\theta_\chi(1+\theta_\chi)(\Sigma_\chi/\mu_\chi)^2\right)+O(\Sigma_\chi^4),$$

(F2b) $$\text{var}[T_2] \equiv E\left[\left(T_2 - E[T_2]\right)^2\right] = (1+\theta_\chi)^2\mu_\chi^{2(1+\theta_\chi)}(\Sigma_\chi/\mu_\chi)^2 + O(\Sigma_\chi^4),$$

(F2c) $$\text{skew}[T_2] \equiv E\left[\left(T_2 - E[T_2]\right)^3\right] = 3\theta_\chi(1+\theta_\chi)^3\mu_\chi^{3(1+\theta_\chi)}(\Sigma_\chi/\mu_\chi)^4 + O(\Sigma_\chi^6),$$

(F2d) $$\text{skew}^*[T_2] \equiv \text{skew}[T_2]/\left(\text{var}[T_2]\right)^{3/2} = 3\theta_\chi(\Sigma_\chi/\mu_\chi) + O(\Sigma_\chi^3).$$

We fit this distribution to the ECS, TCR and TCRE, respectively. Table F.3 reports the results.

Table F.3 – THREE WAYS OF CALIBRATING CLIMATE SENSITIVITY

	ECS	TCR	TCRE
$E[T_2]$	3.0°C	1.75°C	1.9°C
$\text{var}[T_2]$	4.5°C^2	0.15°C^2	0.11°C^2
$\text{skew}[T_2]$	10°C^3	0	0.0071°C^3
$\text{skew}^*[T_2]$	1.0	0	0.19
μ_χ	1.9	1.75	1.5
Σ_χ	0.95	0.38	0.17
σ_χ	11%/year$^{1/2}$	4.5%/year$^{1/2}$	-
ν_χ	0.66%/year	$\to 0$	$\to \infty$
θ_χ	0.59	0	0.57
$\theta_{\chi T}$ $(\theta_T = 0.5)$	1.4	0.5	1.4
$\theta_{\chi T}$ $(\theta_T = 1.0)$	2.2	1.0	2.1
$\theta_{\chi T}$ $(\theta_T = 1.5)$	3.0	1.5	2.9

F.6.1. Climate sensitivity and uncertainty for the ECS

To calibrate (F.2) for the ECS, we compare to the (thin-tailed) Gamma distribution proposed by Pindyck (2012), who considers a three-parameter Gamma distribution:[48]

(F3) $$f_{T_2,\mathrm{P}}\left(T_2; r_P, \lambda_P, \theta_P\right) = \frac{\left(\lambda_P\right)^{r_P}}{\Gamma(r_P)}\left(T_2-\theta_P\right)^{r_P-1} e^{-\lambda_P\left(T_2-\theta_P\right)},$$

where $\Gamma(r_P)=\int_0^\infty s^{r_P-1}e^{-s}ds$ is the Gamma function, and the mean, variance and skewness are $r_P/\lambda_P+\theta_P$, r_P/λ_P^2 and $2r_P/\lambda_P^3$, respectively. For sufficiently large temperatures, the tail in (F.3) always decays exponentially, so that all moments are defined. By fitting a thin-tailed Gamma distribution to a mean of 3ºC, and 5% and 1% exceedance probabilities corresponding to 7ºC and 10ºC, respectively, Pindyck (2012) obtains a variance and skewness of 4.5 and 9.8. We fit the mean, variance and skewness of (F2) to Pindyck's (2012) values and obtain $\mu_\chi = 1.9$, $\Sigma_\chi = 0.95$ and $\theta_\chi = 0.59$. Figure F.7 compares Pindyck's (2012) distribution (F3) and our fitted distribution (F2). The mean features of Pindyck's (2012) skewed, but thin-tailed distribution Gamma distribution, notably its high-temperature tail, are captured well by our probability distribution. It can readily be verified that (F2a)-(F2d) provide very accurate approximations to the first three moments.[49] Also shown in Figure F.7 is the fat-tailed distribution proposed by Roe and Baker (2007).

By setting $T_2 = \Delta T/\left(1-\chi\right)$ to reflect a standard linear feedback process, the fat-tailed distribution of Roe and Baker (2007) can be obtained by transformation of the normally distributed process χ:[50]

[48] Pindyck (2012) proposes a three-parameter gamma distribution, which allows for non-zero probability of negative temperature change. In his calibration, this probability is 2.3%. We do not allow for negative temperatures. We use the same parameter symbols as Pindyck (2012) and let subscript *P* denote symbols relevant to this equation only.
[49] The mean (ºC), variance ($^{\circ}C^2$) and skewness ($^{\circ}C^3$) are, respectively: 3.0, 4.5 and 9.8 (Pindyck, 2012); 3.5, 4.5 and 10 (our fit) and 3.0, 4.8 and 9.7 (leading-order approximations (E2a), (E2b) and (E2c)).
[50] Roe and Baker (2007) use *f* instead of χ to denote the underlying normally distributed process.

$$\text{(F4)} \qquad f_{T_2,\mathrm{RB}}\left(T_2;\mu_{\chi,\mathrm{RB}},\Sigma_{\chi,\mathrm{RB}},\Delta T_{\mathrm{RB}}\right)=\frac{1}{\sqrt{2\pi}\Sigma_{\chi,\mathrm{RB}}}\frac{\Delta T_{\mathrm{RB}}}{T_2^2}\exp\left(-\frac{\left(1-\mu_{\chi,\mathrm{RB}}-\Delta T_{\mathrm{RB}}/T_2\right)^2}{2\Sigma_{\chi,\mathrm{RB}}^2}\right),$$

where we no longer automatically set $\Delta T = 1$°C. The slowly decaying tail causes all its moments, including mean and variance, to diverge, and the distribution can be said to be truly fat-tailed. For the Roe and Baker (2007) distribution (F.4), we rely on the calibration by Newbold and Daigneault (2009), who calibrate to exceedance probabilities of 1.2°C (95%), 2.4°C (50%) and 7.5°C (5%) by setting $\Delta T_{\mathrm{RB}} = 2.04$, $\mu_{\chi,\mathrm{RB}} = 0.14$, $\Sigma_{\chi,\mathrm{RB}} = 0.39$. As our intention here is not to illustrate the dismal effect of fat tails, we proceed in the following completely ad-hoc fashion. To avoid unbounded moments, we truncate the probability distribution at 10°C to obtain a mean of 2.6°C, a variance of 2.1°C^2, a skewness of 6.8°C^3 and thus a standardized skewness of 2.3. We fit our distribution to these moments and obtain $\mu_P = 1.0$, $\Sigma_P = 3.9\times10^{-6}$ and $\theta_P = 1.5\times10^5$. The red dashed line in Figure F.7 illustrates that our distribution (F.2) cannot meaningfully reproduce the tat tails of Roe and Baker (2007). Nevertheless, both distributions are within the consensus range of IPCC (2013), who conclude that the equilibrium climate sensitivity is "extremely unlikely" (0-5 %) <1 °C, "likely" (66-100 %) 1.5-4.5°C and "very unlikely" (0-10 %) >6°C.[51]

Crucially, the equilibrium sensitivity distribution is not reached instantaneously. From the data in Figure F.8 (taken from IPCC, 2001)[52], we estimate a coefficient of mean reversion of $\nu_\chi =$ 0.66%/year in the scenario where CO_2 is doubled and 0.91%/year if CO_2 is quadrupled. We calibrate the ECS distribution to the large-time (or fast-mean-reversion)

[51] Summing up the information presented in its Figure 10.20 and Chapter 10. The percentages in brackets correspond to the probabilities IPCC (2013) assigns to the respective measures of likelihood.
[52] We use data from the AOGCM simulation GFDL_R15_a reported in Figure 9.1 of IPCC (2001). More recent reports no longer include the simple scenario's 2xCO_2 and 4xCO_2, making direct estimation of ν_χ harder. The order of magnitude of our estimate of the coefficient of mean reversion agrees with more recent model runs (see IPCC, 2013, Box 12.2). In reality, the response to small emissions is much faster and on a decadal scale (Ricke and Caldeira, 2014) than the response to larger emissions (Zickfeld and Herrington, 2015), reflecting non-linearity in the system, which is not captured by our Ornstein-Uhlenbeck process.

limit $\Sigma_\chi = \sigma_\chi / \sqrt{2\nu_\chi}$ of our Ornstein-Uhlenbeck process, relevant for the ECS (cf. $\Sigma_\chi = \sigma_\chi \sqrt{\left(1-\exp\left(-2\nu_\chi t\right)\right)/2\nu_\chi}$ generally). Adopting $\nu_\chi = 0.66\%$/year and using the value of Σ_χ reported in Table 2, the fast-mean-reversion limit gives $\sigma_\chi = 11\%/\text{year}^{1/2}$.

(a) PROBABILITY DENSITY FUNCTION (b) EXCEEDANCE PROBABILITY

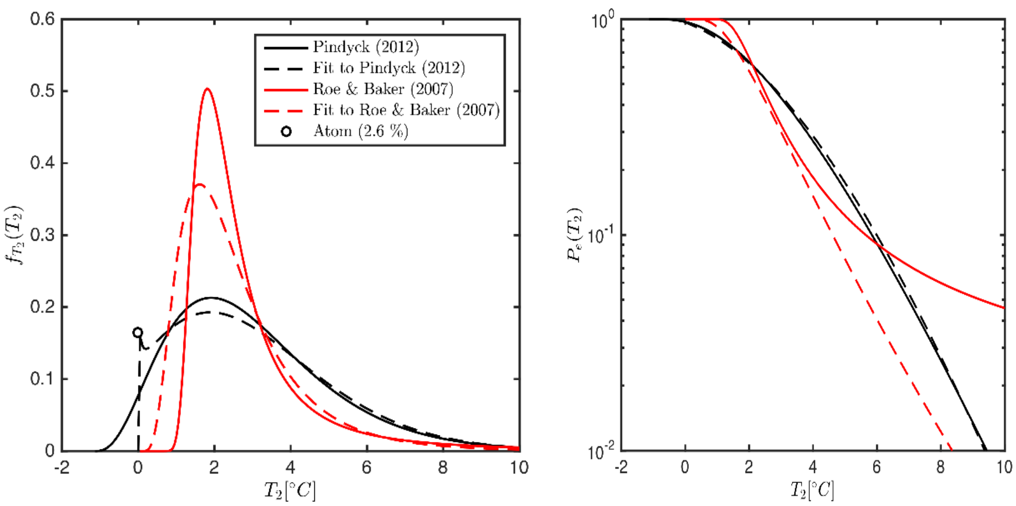

FIGURE F.7. EQUILIBRIUM CLIMATE SENSITIVITY DISTRIBUTION

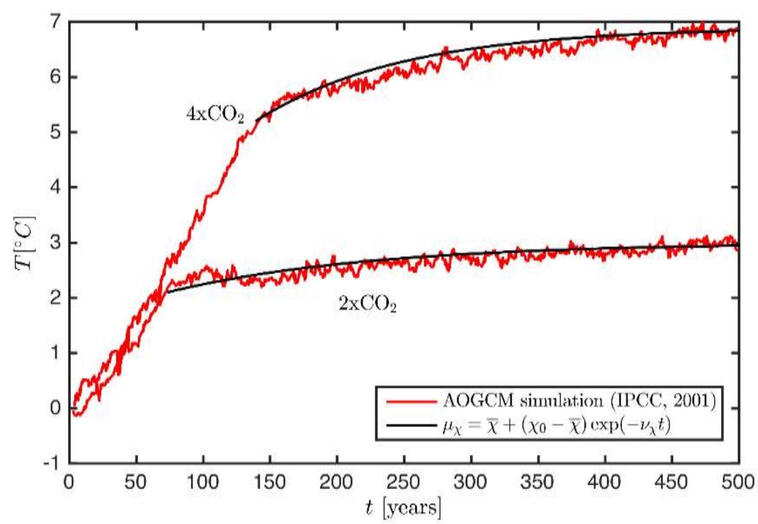

FIGURE F.8. EQUILIBRIUM CLIMATE DYNAMICS AND MEAN REVERSION

F.6.2. Climate sensitivity uncertainty based on the TCR and TCRE

To calibrate (F2) for the TCR, we use information from Figure 10.20 and Chapter 10 of IPCC (2013), indicating that the TCR is "likely" (66-100%) between 1.0ºC and 2.5ºC and "extremely unlikely" (0-5%) to be greater than 3ºC. Since there is no evidence for skewness in the TCR distribution, we set $\theta_\chi = 0$. We let the interval 1-2.5ºC correspond to the 95% confidence interval of a normal distribution with mean 1.75ºC, giving $\Sigma_\chi = 0.38$. Using the small-time (or slow-mean-reversion) limit $\Sigma_\chi = \sigma_\chi \sqrt{t}$, relevant for the TCR, and a period of 70 years required for doubling the atmosphere carbon stock with a rate of increase of 1.0%/year, we estimate a volatility of $\sigma_\chi = 4.5\%/\text{year}^{1/2}$.[53]

To calibrate (5.2) for the TCRE, we use the 5-95% ranges reported in IPCC (2013): 0.7-2.0ºC/TtC (Gillett et al., 2013), 1.0-2.1ºC/TtC (Matthews et al., 2009) and 1.4-2.5ºC/TtC (Allen et al., 2009). Using the values in Allen et al. (2009), who also report a best-guess of 1.9ºC/TtC, we find a slightly positively-skewed distribution with $\mu_\chi = 1.5$, $\Sigma_\chi = 0.17$ and $\theta_\chi = 0.57$.[54] To reflect the transient nature of the response, we set $\nu_\chi \to \infty$, so that the large-time limit, in which volatility reaches a steady state, is instantaneously reached. To apply our model to the TCRE, one would set μ = 1 and $\varphi = 0$, so that emissions stay in the atmosphere forever.[55]

[53] With the large-time limit $\Sigma_\chi = \sigma_\chi \big/ \sqrt{2\nu_\chi}$ and the value $\Sigma_\chi = 0.95$ for the ECS, we have an alternative way to compute mean reversion implied by the difference in volatilities of the TCR and ECS. We obtain $\nu_\chi = (0.045/0.95)^2/2 = 0.11$ %/year, which is much smaller than the value of $\nu_\chi = 0.66$ %/year.

[54] We set the best guess of 1.9 ºC/TtC to equal the temperature that is most likely ($\partial f_{T_2} \big/ \partial T_2 = 0$) and use the 5% and 95% exceedance probabilities to fit our distribution (5.1). We thus set $\Delta T = \left(1/(0.596)\right)^{-(1+\theta_E)} = 0.75$ ºC with θ_E = -0.45 and calibrate at E = 1 GtC, noting the non-linear dependence of T on E that is retained.

[55] The TCRE depends on the TCR and the cumulative airborne fraction (CAF), defined as the ratio of the increased mass of CO_2 in the atmosphere to cumulative CO_2 emissions. The CAF is equivalent to our μ. Zickfeld et al. (2013) estimate values of the CAF to the time of CO_2 doubling of 0.4-0.7, which are in line with some of the values in Table F.1 for shorter time periods.

Appendix G: Accuracy of Results 2 and 2′ (For Online Publication)

Result 1 is evaluated numerically by discretization in time before evaluating the expectation operator numerically exactly and summing up the discounted contributions of every time step. Whereas the stochastic processes for χ and λ are autonomous, the stochastic process for K remains autonomous in Result 1, and all three have (independent) probability distributions available in closed form, the probability distribution of E at any time period in the future must combine all uncertain emissions (proportional to K) before that time. As the time integral of a Geometric Brownian motion does not have a closed-form solution, we update the probability distribution function of E every time step with the stochastic emissions and the decay in that period according to the differential equation for E and project on a fixed grid for E to enable transfer of the probability density function between time periods. Of course, the validity of Result 1 itself still relies on the parameter ϵ being small. Consistent with our perturbation scheme, all our optimal risk-adjusted carbon prices in Results 1 and 2 or 2 ′are evaluated along the business-as-usual path for which $P = 0$. We assess the accuracy of Result 2′ (or its special case Result 2) for our base line calibration and for the Gollier calibration ($\eta = \gamma =$ 2, $\rho = 0$%/year and $\sigma_K = 1.5$%/year$^{1/2}$), as its lower discount rate $r^\star$ compared to our base case calibration makes for a more demanding test on the accuracy of Result 2′. Three factors determine the accuracy of Result 2′, as discussed below.

Table G.1 – ACCURACY OF RESULT 2 AND 2′ COMPARED TO RESULT 1

	Base case calibration		Gollier calibration	
	Proportional damages	Convex damages	Proportional damages	Convex damages
Error due to discretization	0.3%	1.0%	0.3%	2.6%
Error due to deterministic carbon stock	0.0%	0.0%	0%	0.1%
Error due to truncation of climate sensitivity distribution	0.1%	1.0%	0.2%	2.3%

First, the small error due to discretization (in the states and *t*) associated with the evaluation of Result 1 on a grid with step size Δ*E* = 0.5 GtC and Δ*t* = 1 year can be estimated from grid convergence of Result 1. Second, we ignore any uncertainty in the atmospheric carbon stock that arises because of the uncertain nature of future economic growth and thus of future emissions. For our base case calibration with flat damages ($\theta_{ET} = 0$), the stochastic nature of E does not lead to a change in the net present value of expected damages and thus the carbon price is unchanged. For convex damages ($\theta_{ET} > 0$), the effect remains negligibly small. Third, in Result 2′ we only consider leading-order terms in the climate sensitivity uncertainty. To obtain an upper limit to the error associated with this, we numerically evaluate the expectation of $\chi^{1+\theta_{\chi T}}$ in the steady-state limit of the equilibrium sensitivity calibration and compare to the leading-order approximation used in Result 2′, $E[\chi^{1+\theta_{\chi T}}] = \mu_{\chi}^{1+\theta_{\chi T}}(1+\theta_{\chi T}(1+\theta_{\chi T})(\Sigma_{\chi} / \mu_{\chi})^2)$. We obtain a relative error of -0.8% and -7.2% for $\theta_{ET} = 0$ (proportional damages) and $\theta_{ET} = 0.25$ (convex damages), respectively. Due to discounting over the horizon over which the climate sensitivity reaches its steady state, only part of this error manifests itself in the final estimate of the SSC, as is evident from Table G.1.