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THE RM PARACONSISTENT REFUTATION SYSTEM

Abstract. The aim of this paper is to study the refutation system consisting of the refutation axiom $p \wedge \neg p \to q$ and the refutation rules: *reverse substitution* and *reverse modus ponens* (B/A, if $A \to B \in \mathbf{RM}$). It is shown that the refutation system is characteristic for the logic of the 3-element **RM** algebra.

Keywords: refutation systems, paraconsistent logic, relevance logic.

1. Introduction

A refutation system is an inference system consisting of some refutation axioms (which are non-valid formulas) and some refutation rules (which are inference rules preserving non-validity) (see [2]). Refutation systems can be regarded as alternative axiom systems capturing some intuitions about non-valid formulas as well as valid ones. It seems worth investigating such systems in paraconsistent logics, which are defined as non-classical logics rejecting the explosive law (E) := $p \wedge \neg p \to q$ (cf. [3]). In this paper we study the refutation system consisting of the refutation axiom (E) and the refutation rules: *reverse substitution* and *reverse modus ponens* (B/A, where $A \to B \in \mathbf{RM}$). It is shown that this refutation system generates the set of formulas non-valid in the 3-element **RM** algebra. The resulting paraconsistent logic (that is, the set of formulas non-refutable in this system) is simple (3-valued), natural (i.e. (E) is rejected and refutability is justified by derivability in **RM**; a useful standard relevance logic), and maximal.

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2. Preliminaries

Let FOR be the set of formulas generated from a set VAR $= \{p, q, \ldots\}$ of propositional variables by the connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$. We define

$$A \equiv B := (A \rightarrow B) \wedge (B \rightarrow A).$$

RM is the set of formulas provable in the following axiom system.

Axioms:

$$A \rightarrow A$$
$$(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$$
$$A \rightarrow ((A \rightarrow B) \rightarrow B)$$
$$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$$
$$A \rightarrow (A \rightarrow A)$$
$$A \wedge B \rightarrow A$$
$$A \wedge B \rightarrow B$$
$$((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow B \wedge C)$$
$$A \rightarrow A \vee B$$
$$B \rightarrow A \vee B$$
$$((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow (A \vee B \rightarrow C)$$
$$(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)$$
$$(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$$
$$\neg\neg A \rightarrow A$$

Rules:

(*modus ponens*) $$\frac{A \qquad A \rightarrow B}{B}$$

(*adjunction*) $$\frac{A \qquad B}{A \wedge B}$$

RM can be characterized by the matrix $\mathbf{M} = \langle \mathbb{Q}, \mathcal{D}, -, \wedge, \vee, \rightarrow \rangle$ (see [1]), where $\mathbb{Q}$ is the set of rational numbers, $\mathcal{D} := \{x \in \mathbb{Q} : x \geqslant 0\}$, and

$$x \wedge y := \min(x, y),$$
$$x \vee y := \max(x, y),$$
$$x \rightarrow y := \begin{cases} \max(-x, y) & \text{if } x \leqslant y, \\ \min(-x, y) & \text{otherwise.} \end{cases}$$

Thus **RM** is the set of formulas valid in **M**, that is, $A \in \mathbf{RM}$ iff $v(A) \in \mathcal{D}$ for every valuation v in **M**.

We take for granted the following **RM** laws:

$$
\begin{array}{ll}
(1) & (A \to (B \to C)) \to (B \to (A \to C)) \\
 & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) \\
 & A \wedge B \equiv B \wedge A \\
 & A \vee B \equiv B \vee A \\
(2) & (A \to (B \equiv C)) \to ((A \to (C \equiv D)) \to (A \to (B \equiv D))) \\
 & (B \equiv C) \to (D \equiv D(B/C)) \\
(3) & (A \to (B \equiv C)) \to (A \to (D \equiv D(B/C)))
\end{array}
$$

where $D(B/C)$ results from D by replacing some occurrences of B by C.

3. Validity

Let $P := p \wedge \neg p$ and $Q := q \wedge \neg q$.

LEMMA 1. *The following formulas are in* **RM**:

$$
\begin{array}{l}
P \to (\neg Q \equiv \neg Q) \\
P \to (\neg\neg Q \equiv Q) \\
P \to (\neg P \equiv P) \\
P \to (Q \wedge \neg Q \equiv Q) \\
P \to (P \wedge Q \equiv Q) \\
P \to (P \wedge \neg Q \equiv P) \\
P \to (Q \vee \neg Q \equiv \neg Q) \\
P \to (P \vee Q \equiv P) \\
P \to (P \vee \neg Q \equiv \neg Q) \\
P \to ((Q \to Q) \equiv \neg Q) \\
P \to ((P \to P) \equiv P) \\
P \to ((\neg Q \to \neg Q) \equiv \neg Q) \\
P \to ((Q \to \neg Q) \equiv \neg Q) \\
P \to ((\neg Q \to Q) \equiv Q) \\
P \to ((P \to Q) \equiv Q)
\end{array}
$$

$$P \to ((Q \to P) \equiv \neg Q)$$
$$P \to ((P \to \neg Q) \equiv \neg Q)$$
$$P \to ((\neg Q \to P) \equiv Q)$$

PROOF. First we note the following simple facts. Let $x, y \in \mathbb{Q}$. We put

$$X := x \wedge -x, \quad Y := y \wedge -y, \quad \text{and } \mathcal{Z} := \{X, -X, Y, -Y\}.$$

Then we have:

(I) $X \leqslant 0$ and $Y \leqslant 0$.

(II) If $a, b \in \mathcal{Z}$ then $-a, a \wedge b, a \vee b, a \to b \in \mathcal{Z}$.

Next we consider the above formulas. They are of the form

$$P \to A(P, Q)$$

Now let v be any valuation in **M**. Then, by (II), we have

$$(*) \quad v(A(P, Q)) \in \{v(P), -v(P), v(Q), -v(Q)\}.$$

For v we consider two cases.

Case 1. $v(P) \leqslant v(Q)$. Then, by (I) and $(*)$, we get $v(P) \leqslant v(A(P, Q))$. Hence $v(P \to A(P, Q)) = \max(-v(P), v(A(P, Q))) \geqslant 0$.

Case 2. $v(P) > v(Q)$. Then it is easy to check that

$$(**) \quad v(A(P, Q)) \in \{v(P), -v(P), -v(Q)\}.$$

We give details only for the cases eighth, fourteenth, and eighteenth; the other ones being similar.

$v(P \vee Q \equiv P) = v(P \vee Q \to P) \wedge v(P \to P \vee Q) = v(P \to P) \wedge v(P \to P) = \max(-v(P), v(P)) = -v(P)$, because $v(P \vee Q) = v(P)$.

$v((\neg Q \to Q) \equiv Q) = v((\neg Q \to Q) \to Q) \wedge v(Q \to (\neg Q \to Q)) = v(Q \to Q) \wedge v(Q \to Q) = -v(Q)$, because $-v(Q) > v(Q)$.

$v((\neg Q \to P) \equiv Q) = v((\neg Q \to P) \to Q) \wedge v(Q \to (\neg Q \to P)) = v(Q \to Q) \wedge v(Q \to Q) = -v(Q)$, because $-v(Q) > v(P)$.

Therefore, by (I) and $(**)$, $v(P) \leqslant v(A(P, Q))$, and so $v(P \to A(P, Q)) = \max(v(-P), v(A(P, Q))) \geqslant 0$.

Thus, for any valuation v in **M** we have $v(P) \leqslant v(A(P, Q))$, and so $v(P \to A(P, Q)) \geqslant 0$ which gives the result. ⊣

4. Refutability

Let **3** be the submatrix $\langle\{-1,0,1\},\{0,1\},-,\wedge,\vee,\rightarrow\rangle$ of **M**. We put: $G_{-1} := Q$, $G_0 := P$, and $G_1 := \neg Q$. For any valuation v in **3**, let s_v be the following substitution:

$$\mathrm{s}_v(A) = G_{v(A)} \qquad \text{(for any } A \in \mathrm{VAR}).$$

Lemma 2. *For any* $B \in \mathrm{FOR}$ *we have* $P \rightarrow (\mathrm{s}_v(B) \equiv G_{v(B)}) \in \mathbf{RM}$.

Proof. By induction on the complexity of B.

Let $B \in \mathrm{VAR}$. Then this is true, because $\mathrm{s}_v(B) = G_{v(B)}$ and $v(\mathrm{s}_v(B) \equiv G_{v(B)}) \geqslant 0$.

Let $B \notin \mathrm{VAR}$. We assume that the lemma holds for formulas simpler than B. Then

$$B \in \{\neg C, C \wedge D, C \vee D, C \rightarrow D\}$$

and by the induction hypothesis we have

$$P \rightarrow (\mathrm{s}_v(C) \equiv G_{v(C)}) \in \mathbf{RM},$$
$$P \rightarrow (\mathrm{s}_v(D) \equiv G_{v(D)}) \in \mathbf{RM}.$$

Hence, by (3) and *modus ponens*, we get

$$P \rightarrow (\neg \mathrm{s}_v(C) \equiv \neg G_{v(C)}) \in \mathbf{RM},$$
$$P \rightarrow ((\mathrm{s}_v(C) \otimes \mathrm{s}_v(D)) \equiv (G_{v(C)} \otimes G_{v(D)})) \in \mathbf{RM},$$

where $\otimes \in \{\wedge, \vee, \rightarrow\}$. Since by Lemma 1 we have

$$P \rightarrow (\neg G_{v(C)} \equiv G_{v(\neg C)}) \in \mathbf{RM},$$
$$P \rightarrow ((G_{v(C)} \otimes G_{v(D)}) \equiv (G_{v(C \otimes D)})) \in \mathbf{RM},$$

by (2) and *modus ponens* we obtain

$$P \rightarrow (\mathrm{s}_v(B) \equiv G_{v(B)}) \in \mathbf{RM}$$

as required. ⊣

We say that a formula is *refutable* iff it is derivable in the following refutation system.

Refutation axiom:

(E) $p \wedge \neg p \to q$

Refutation rules:

(*reverse substitution*) B/A, if B is a substitution instance of A.

(*reverse modus ponens*) B/A, if $A \to B \in \mathbf{RM}$.

THEOREM. *A formula is refutable if and only if it is not valid in* **3**.

PROOF. ($\Rightarrow$) This follows from the fact that (E) is not valid in **3** and the refutation rules preserve non-validity in **3**.

($\Leftarrow$) Assume that A is not valid in **3**. Then $v(A) = -1$ for some valuation v in **3**, so $G_{v(A)} = G_{-1} := q \wedge \neg q$. By Lemma 2 we have

$$P \to (\mathrm{s}_v(A) \equiv q \wedge \neg q) \in \mathbf{RM}.$$

Hence

$$P \to (\mathrm{s}_v(A) \to q) \in \mathbf{RM},$$

so, by (1) and *modus ponens*, we obtain

$$\mathrm{s}_v(A) \to (p \wedge \neg p \to q) \in \mathbf{RM}.$$

Therefore $\mathrm{s}_v(A)$ is refutable, by *reverse modus ponens* and (E), and so A is refutable, by *reverse substitution*, which was to be shown. ⊣

References

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