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Gábor Szabó

ABSTRACT

We study the topological variant of Rokhlin dimension for topological dynamical systems $(X, \alpha, \mathbb{Z}^m)$ in the case where X is assumed to have finite covering dimension. Finite Rokhlin dimension in this sense is a property that implies finite Rokhlin dimension of the induced action on C^* -algebraic level, as was discussed in a recent paper by Hirshberg, Winter and Zacharias. In particular, it implies under these conditions that the transformation group C^* -algebra has finite nuclear dimension. Generalizing partial results of Lindenstrauss and Gutman, we show that free \mathbb{Z}^m -actions on finite dimensional spaces satisfy a strengthened version of the so-called marker property, which yields finite Rokhlin dimension for such actions.

0. Introduction

The study of group actions and their associated C^* -algebras has always been a central theme in the theory of operator algebras. Topological dynamical systems in particular play a significant role in C^* -algebra theory. The C^* -algebras arising from topological dynamical systems, say from (X, α, G) for a countable discrete group G , a compact metric space X and a continuous action $\alpha : G \curvearrowright X$, give rise to plenty of interesting questions about their structure. From the point of view of C^* -algebra classification theory, this setting begs the question of whether the transformation group C^* -algebras can be classified via their Elliott-invariant, at least under suitable conditions like simplicity.

By now there exist classification results for large classes of crossed products by \mathbb{Z} -actions: early results of Putnam about characterizing crossed products of minimal homeomorphisms on the Cantor set as AT algebras (see [17]) or of Elliott and Evans about irrational rotation algebras (see [2]) have set the stage for this project. In one of the more recent breakthroughs, Toms, Strung and Winter proved that crossed products of uniquely ergodic minimal homeomorphisms on infinite compact metrizable spaces with finite covering dimension are classified by ordered K-theory (see [21, 19]).

The case of \mathbb{Z}^m -actions begins to gather more and more attention and interest as the next step for this long-term study of group actions. Towards a generalization of the celebrated results of [6], Giordano, Matui, Putnam and Skau undertook a deep study on topological orbit equivalence of Cantor minimal \mathbb{Z}^m -actions in [4] and [5]. Going in a more similar direction as this paper, Phillips initiated the study of certain *large subalgebras* inside crossed products by topological \mathbb{Z}^m -actions as a means to study the radius of comparison of these C^* -algebras (see [16]). In particular, an important question concerning the C^* -classification of these crossed products is when they have strict comparison of positive elements.

The current trend in C^* -algebra classification theory is to show certain regularity properties for a class of nuclear C^* -algebras as the first intermediate step towards their K -theoretic classification. Following this philosophy, we focus on finite nuclear dimension (see [22]) as the

regularity property of our choice and show that it is prevalent for transformation group C^* -algebras of free \mathbb{Z}^m -actions on finite dimensional spaces. Looking in the direction of the second step in this classification philosophy, Winter has found a method to deduce classification of simple transformation group C^* -algebras of this type. This works under the prevalence of finite nuclear dimension and a technical condition involving the set of invariant ergodic measures (see [20]). An alternative immediate application is to combine the main results of [12] and [14] to obtain a classification result for free, minimal and uniquely ergodic \mathbb{Z}^m -actions.

The concept of Rokhlin dimension developed in [10] is a natural tool for a systematic approach of showing finite nuclear dimension for crossed products. This notion has been introduced for finite group actions and integer actions on unital C^* -algebras, but can be defined similarly for \mathbb{Z}^m -actions. Integer actions with finite Rokhlin dimension have been shown to behave well with underlying C^* -algebras of finite nuclear dimension. That is, the property of having finite nuclear dimension passes from the underlying C^* -algebra to the crossed product. The aim of the first section is to define Rokhlin dimension for \mathbb{Z}^m -actions on unital C^* -algebras and establish the same permanence property for \mathbb{Z}^m -actions of finite Rokhlin dimension.

In the second section, we introduce a topological variant of Rokhlin dimension for topological \mathbb{Z}^m -actions. In particular, finite Rokhlin dimension in the topological sense is designed in such a way that it implies finite Rokhlin dimension in the C^* -algebraic sense for its induced C^* -action. Moreover, it can be regarded as a topological analogue of the measure theoretic Rokhlin Lemma. The rest of the paper will be devoted to show that finite Rokhlin dimension in the topological sense is satisfied for free \mathbb{Z}^m -action on finite dimensional spaces.

In the third section, we introduce a technical condition that is a stronger version of the small boundary property introduced by Lindenstrauss in [13], namely the bounded topological small boundary property. We show that free actions of countably infinite groups on finite dimensional spaces satisfy said property.

In the fourth section, we define the marker property as in [8] within the more general setting of countable group actions. We also define a stronger variant of this property for \mathbb{Z}^m -actions, namely the controlled marker property. In [8], Gutman has shown that aperiodic homeomorphisms on finite dimensional spaces have the marker property. We generalize this result and show that free actions of countably infinite groups have the marker property whenever the system has the bounded topological small boundary property. In the case of \mathbb{Z}^m -actions, a closer look at the proof will even yield the controlled marker property under these conditions.

In the fifth section, we bring together the results of all the previous sections. Combining the third and fourth section, we can deduce that free \mathbb{Z}^m -actions on finite dimensional spaces satisfy the controlled marker property. The consequence is finite Rokhlin dimension in the topological sense. By the second section, we get finite Rokhlin dimension for the induced C^* -action. Applying the results of the first section, we get the main result:

THEOREM. *Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action. Then the induced C^* -action $\bar{\alpha} : \mathbb{Z}^m \curvearrowright \mathcal{C}(X)$ has finite Rokhlin dimension and the transformation group C^* -algebra $\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m$ has finite nuclear dimension.*

Afterwards, we discuss some possible further generalizations of this result. In particular, we review the overall line of argument towards our main result and explain how one can modify it to get a more general statement, namely where X is merely assumed to be locally compact. We also take a look at possible generalizations towards actions of groups other than \mathbb{Z}^m .

The results of this paper belong to the research of the author's PhD studies. I would like to thank my advisor Wilhelm Winter for many helpful suggestions and the overall guidance.

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1. Preliminaries and C^* -Rokhlin dimension

NOTATION 1.1. Throughout the whole paper, we will stick to the following notations unless specified otherwise:

- X is a compact metric space. In applications, it is often assumed to have finite covering dimension.
- G is a countable and discrete group.
- A is a unital C^* -algebra.
- A completely positive map between C^* -algebras is abbreviated *c.p.* A completely positive and contractive map between C^* -algebras is abbreviated *c.p.c.*
- Either $\alpha : G \curvearrowright X$ is a continuous group action or $\alpha : G \curvearrowright A$ is an action via automorphisms. When $\alpha : G \curvearrowright X$ is a topological action, we denote the induced C^* -action by $\bar{\alpha} : G \curvearrowright \mathcal{C}(X)$. In applications, we often have $G = \mathbb{Z}^m$.
- Suppose M is some set. If $F \subset M$ is a finite subset, we write $F \subset\subset M$.
- For $a, b \in A$ in some C^* -algebra and $\varepsilon > 0$, $a =_\varepsilon b$ means $\|a - b\| \leq \varepsilon$.
- Suppose we have $F, F' \subset\subset A$ in a C^* -algebra and $\varepsilon > 0$. The notation $F =_\varepsilon F'$ means that for all $a \in F$, there exists $b \in F'$ with $a =_\varepsilon b$ and vice versa.
- Given $n \in \mathbb{N}$, we define

$$B_n^m = \{0, \dots, n - 1\}^m \subset \mathbb{Z}^m.$$

Since m will always be known from context to be the rank of \mathbb{Z}^m , we will just write B_n instead.

- Similarly, we define

$$J_n^m = J_n = \{-n + 1, \dots, 0, \dots, n\}^m \subset \mathbb{Z}^m.$$

We begin with a few remarks about almost order zero maps and nuclear dimension.

DEFINITION 1.2. Let A, B be C^* -algebras and $\delta \geq 0$. A *c.p.c.* map $\psi : A \rightarrow B$ has order zero up to δ , if for all $a, b \in A$ with $ab = 0$ we have $\|\psi(a)\psi(b)\| \leq \delta\|a\|\|b\|$. ψ has order zero if $\delta = 0$.

Order zero maps in particular appear in the definition of nuclear dimension and decomposition rank (see [11, 22]), which are important regularity properties for the classification theory of nuclear C^* -algebras. We shall recall a stability property of order zero maps that is satisfied whenever the domain is finite dimensional (see [11, 2.5]).

LEMMA 1.3. Let \mathcal{F} be a finite dimensional C^* -algebra. For all $\varepsilon > 0$, there exists $\delta > 0$ with the following property: Let A be a C^* -algebra and $\varphi : \mathcal{F} \rightarrow A$ a map of order zero up to δ . Then there exists an order zero map $\varphi' : \mathcal{F} \rightarrow A$ such that $\|\varphi - \varphi'\| \leq \varepsilon$.

DEFINITION 1.4 (see [22]). Let A be a C^* -algebra. A is said to have nuclear dimension n , denoted by $\dim_{\text{nuc}}(A) = n$, if n is the smallest natural number with the following property:

For all $F \subset\subset A$ and $\varepsilon > 0$, there exists a finite dimensional C^* -algebra

$$\mathcal{F} = \mathcal{F}^{(0)} \oplus \dots \oplus \mathcal{F}^{(n)}$$

and *c.p.* maps $A \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} A$ such that

- ψ is contractive.

- For every $i = 0, \dots, n$, the map $\varphi^{(i)} := \varphi|_{\mathcal{F}^{(i)}}$ is c.p.c. order zero.
- $\|\varphi \circ \psi(a) - a\| \leq \varepsilon$ for all $a \in F$.

In this context, the triple $(\mathcal{F}, \psi, \varphi)$ is called an n -decomposable c.p. approximation of tolerance ε on F . If no such n exists, we write $\dim_{\text{nuc}}(A) = \infty$.

REMARK 1.5 (see [10, Lemma A.4]). Using 1.3, one can show the following:

If n is a number such that for all $F \subset\subset A$ and $\delta > 0$, there exists a finite dimensional C*-algebra \mathcal{F} such that for all $\eta > 0$, we can choose a c.p. approximation $(\mathcal{F}, \varphi, \psi)$ of tolerance δ on F with φ being decomposable into $n + 1$ maps $\varphi^{(i)}$ of order zero up to η , then $\dim_{\text{nuc}}(A) \leq n$.

DEFINITION 1.6. Let A be a unital C*-algebra, and let $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. We say that the action α has (cyclic) Rokhlin dimension d , and write $\dim_{\text{Rok}}^{\text{cyc}}(\alpha) = d$, if d is the smallest natural number with the following property:

For all $F \subset\subset A$, $\varepsilon > 0$, $n \in \mathbb{N}$, there exist positive contractions $(f_v^{(l)})_{v \in B_n}^{l=0, \dots, d}$ in A satisfying the following properties:

- (1) $\|\mathbf{1}_A - \sum_{l=0}^d \sum_{v \in B_n} f_v^{(l)}\| \leq \varepsilon$.
- (2) $\|f_v^{(l)} f_{v'}^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $v \neq v'$ in B_n .
- (3) $\|\alpha^v(f_w^{(l)}) - f_{v+w}^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $v, w \in B_n$.
(Note that $f_v^{(l)}$ denotes $f_{(v \bmod n\mathbb{Z}^m)}^{(l)}$, whenever $v \notin B_n$.)
- (4) $\|[f_v^{(l)}, a]\| \leq \varepsilon$ for all $l = 0, \dots, d$, $v \in B_n$ and $a \in F$.

If there is no such d , we write $\dim_{\text{Rok}}(\alpha) = \infty$.

REMARK. We define (cyclic) Rokhlin dimension with commuting towers of α , written $\dim_{\text{Rok}}^{\text{cyc}, c}(\alpha)$, in the same way with the additional property

- (5) $\|[f_v^{(l)}, f_w^{(l')}] \| \leq \varepsilon$ for all $l, l' = 0, \dots, d$ and $v, w \in B_n$

for the Rokhlin elements $(f_v^{(l)})_{v \in B_n}^{l=0, \dots, d}$.

REMARK. In [10], several possible variants of Rokhlin dimension have been exhibited for integer actions. Inserting $m = 1$ in 1.6 yields the so-called Rokhlin dimension with single towers of [10].

REMARK 1.7. Observe that the finite set $F \subset\subset A$ in the definition above may be replaced by a compact set $K \subset A$. Suppose $K \subset A$ and $\varepsilon > 0$ is given. Then cover K by balls of radius $\varepsilon/3$. By compactness, we have

$$K \subset \bigcup_{a \in F} B_{\varepsilon/3}(a) \quad \text{for a finite set } F \subset\subset K.$$

Choose Rokhlin elements $(f_v^{(l)})_{v \in B_n}^{l=0, \dots, d}$ for the pair $(F, \varepsilon/3)$ and observe that these are in fact Rokhlin elements for the pair (K, ε) .

REMARK 1.8. For $n \in \mathbb{N}$ and $j \in \{-n+1, \dots, n\}$, define

$$d_n(j) = 1 - \frac{|j|}{n}.$$

Recall the notation $J_n = \{-n+1, \dots, n\}^m$. For $m \in \mathbb{N}$, denote $d_n^m : J_n \rightarrow [0, 1]$ the map defined by

$$d_n^m(j_1, \dots, j_m) = \prod_{i=1}^m d_n(j_i).$$

For $a \in \{0, 1\}^m$, let $s_a : J_n \rightarrow J_n$ be the bijection defined by

$$s_a[(j_i)_{i \leq m}] = (j_i + a_i \cdot n \pmod{2n})_{i \leq m}.$$

One can easily see that for all $j \in \{-n+1, \dots, n\}$, one has

$$1 = d_n(j) + d_n(j + n \pmod{2n}).$$

Inductively, one can prove that then for all $v \in J_n$, one has

$$1 = \sum_{a \in \{0, 1\}^m} d_n^m(s_a(v)).$$

REMARK 1.9. Let A be a unital C^* -algebra. Representing it on a Hilbert space $A \subset \mathcal{B}(H)$, one can prove the following by application of the Cotlar-Stein Lemma (see [18, chapter 7, section 2]). Let J be a finite index set. For all $\delta > 0$, there exists $\eta > 0$ such that if $\{b_j \mid j \in J\} \subset A$ is a family of contractions with $\|b_i^* b_j\|, \|b_j b_i^*\| \leq \eta$ for all $i \neq j$ in J , then

$$\left\| \sum_{j \in J} b_j \right\| \leq \delta + \max_{j \in J} \|b_j\|.$$

We now come to the main theorem of this section. Namely, we show that \mathbb{Z}^m -actions of finite Rokhlin dimension preserve finite nuclear dimension of the underlying C^* -algebra.

THEOREM 1.10 (compare to [10, 4.1]). *Let A be a unital C^* -algebra and $\alpha : \mathbb{Z}^m \curvearrowright A$ be a group action via automorphisms. Then we have*

$$\dim_{\text{nuc}}(A \rtimes_{\alpha} \mathbb{Z}^m) \leq 2^m (\dim_{\text{nuc}}(A) + 1) (\dim_{\text{Rok}}^{\text{cyc}}(\alpha) + 1) - 1.$$

Proof. We may assume that both $s = \dim_{\text{nuc}}(A)$ and $d = \dim_{\text{Rok}}^{\text{cyc}}(\alpha)$ are finite, or else the statement is trivial. Let $F \subset A \rtimes_{\alpha} \mathbb{Z}^m$ a finite subset and $\delta > 0$. Then we can find $N \in \mathbb{N}$ and $F' \subset A \rtimes_{\alpha} \mathbb{Z}^m$ such that $F =_{\delta} F'$ and

$$a = \sum_{v \in J_N} a(v) u_v \quad \text{for all } a \in F'.$$

Note that the $\{u_v \mid v \in \mathbb{Z}^m\}$ denote the canonical unitaries in $A \rtimes_{\alpha} \mathbb{Z}^m$ implementing the action α on A . Let $F_1 \subset A$ be the finite subset of all possible coefficients occurring in such a sum. Without loss of generality, we can assume that F_1 consists of contractions. Let $\varepsilon > 0$ and $n \geq N$. As will be specified later, n is a very large number compared to N and ε very small compared to both δ and n . Let

$$\tilde{F} = \bigcup_{v \in J_n} \alpha^{-v}(F_1) \subset A.$$

Choose an s -decomposable c.p. approximation $(\mathcal{F}, \psi, \varphi)$ for \tilde{F} up to $\frac{\delta}{|J_n||J_N|}$, i.e. a finite dimensional C^* -algebra $\mathcal{F} = \mathcal{F}^{(0)} \oplus \dots \oplus \mathcal{F}^{(s)}$ and c.p. maps $A \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} A$ such

that ψ is c.p.c., the maps $\varphi^{(i)} = \varphi|_{\mathcal{F}^{(i)}}$ are c.p.c. order zero and $\|x - (\varphi \circ \psi)(x)\| \leq \frac{\delta}{|J_n||J_N|}$ for all $x \in \tilde{F}$.

Let $\psi_n = \text{id}_{|J_n|} \otimes \psi : M_{|J_n|} \otimes A \rightarrow M_{|J_n|} \otimes \mathcal{F}$ and $\varphi_n = \text{id}_{|J_n|} \otimes \varphi : M_{|J_n|} \otimes \mathcal{F} \rightarrow M_{|J_n|} \otimes A$ denote the amplifications of ψ and φ . Analogously write $\varphi_n^{(i)} = \text{id}_{|J_n|} \otimes \varphi^{(i)}$.

Let $B_{\mathcal{F}}$ be the closed unit ball in \mathcal{F} and define the compact set $K \subset A$ by

$$K = \bigcup_{v \in J_n} \bigcup_{i=0}^s \alpha^{-v}(\varphi^{(i)}(B_{\mathcal{F}})).$$

Using 1.7, we may choose positive contractions $(f_v^{(l)})_{v \in B_{2n}}^{l=0, \dots, d}$ as in 1.6 for the triple $(K, \varepsilon/2, 2n)$. Moreover, we may arrange with a standard functional calculus argument that the relations (2)-(4) are also true for the square roots. By doing an index shift, we get elements $(f_v^{(l)})_{v \in J_n}^{l=0, \dots, d}$ satisfying the analogous properties (1)-(4) (with the square roots in (2)-(4)) up to ε for the index set J_n instead of B_{2n} .

Let A act faithfully on a Hilbert space H and let $A \rtimes_{\alpha} \mathbb{Z}^m$ be canonically embedded into $\mathcal{B}(\ell^2(\mathbb{Z}^m) \otimes H)$. Let $Q \in \mathcal{B}(\ell^2(\mathbb{Z}^m) \otimes H)$ be the projection onto the subspace $\ell^2(J_n) \otimes H$. Then $x \mapsto QxQ$ defines a u.c.p. map $\Psi : A \rtimes_{\alpha} \mathbb{Z}^m \rightarrow M_{|J_n|}(A)$. More specifically, we have for all $a \in A, v \in \mathbb{Z}^m$ that

$$\begin{aligned} \Psi(au_v) &= Q \left[\sum_{w \in \mathbb{Z}^m} e_{w, w-v} \otimes \alpha^{-w}(a) \right] Q \\ &= \sum_{\substack{w \in J_n: \\ w-v \in J_n}} e_{w, w-v} \otimes \alpha^{-w}(a) = \sum_{w \in J_n \cap (v+J_n)} e_{w, w-v} \otimes \alpha^{-w}(a). \end{aligned}$$

Define the diagonal matrix $D \in M_{|J_n|}(\mathbb{C})$ by $D_{v,v} = d_n^m(v)$. Observe that if $v, w \in J_n$, then $|d_n^m(v) - d_n^m(w)| \leq \frac{m\|v-w\|_{\infty}}{n}$. If n is large enough in comparison to N , we can ensure that

$$|\sqrt{d_n^m(v)} - \sqrt{d_n^m(w)}| \leq \delta/|J_N| \quad \text{for all } v, w \in J_n \text{ with } v-w \in J_N.$$

It follows for all $v \in J_N$ that

$$\begin{aligned} &\|[\sqrt{D}, Qau_vQ]\| \\ &= \left\| \sum_{w \in J_n \cap (v+J_n)} \left(\sqrt{d_n^m(w)} - \sqrt{d_n^m(w-v)} \right) e_{w, w-v} \otimes \alpha^{-w}(a) \right\| \\ &\leq \max \left\{ \left| \sqrt{d_n^m(w)} - \sqrt{d_n^m(w-v)} \right| \mid w \in J_n \cap (v+J_n) \right\} \cdot \|a\| \\ &\leq \frac{\delta}{|J_N|} \|a\|. \end{aligned}$$

Define the c.p.c. map $\mu : A \rtimes_{\alpha} \mathbb{Z}^m \rightarrow M_{|J_n|}(A)$ by $\mu(x) = \sqrt{D}\Psi(x)\sqrt{D}$. By the previous calculation, we have

$$\|\mu(au_v) - DQau_vQ\| \leq \delta/|J_N| \cdot \|a\| \quad \text{for all } v \in J_N \text{ and } a \in A.$$

Now let $p \in \{0, 1\}^m$ and $l \in \{0, \dots, d\}$. Define maps

$$\sigma_p^{(l)} : M_{|J_n|}(A) \rightarrow A \rtimes_{\alpha} \mathbb{Z}^m \quad \text{by} \quad \sigma_p^{(l)}(e_{v,w} \otimes a) = f_{s_p(v)}^{(l)1/2} u_v a u_w^* f_{s_p(w)}^{(l)1/2},$$

see 1.8 for notation. Note that these are indeed c.p. since $\sigma_p^{(l)}(x) = v_{l,p} x v_{l,p}^*$ for the matrix $v_{l,p} \in M_{1,|J_n|}(A \rtimes_{\alpha} \mathbb{Z}^m)$ defined by $v_{l,p} = (f_{s_p(w)}^{(l)1/2} u_w)_{w \in J_n}$.

Now let $w_1, w_2, w_3, w_4 \in J_n$ and $a, b \in K$. In the case $w_2 = w_3$ we have

$$\begin{aligned}
\sigma_0^{(l)}(e_{w_1, w_2} \otimes a) \sigma_0^{(l)}(e_{w_3, w_4} \otimes b) &= f_{w_1}^{(l)1/2} u_{w_1} a u_{w_2}^* f_{w_2}^{(l)1/2} \cdot f_{w_3}^{(l)1/2} u_{w_3} b u_{w_4}^* f_{w_4}^{(l)1/2} \\
&= f_{w_1}^{(l)1/2} u_{w_1} a u_{w_2}^* f_{w_2}^{(l)} u_{w_2} b u_{w_4}^* f_{w_4}^{(l)1/2} \\
&\stackrel{(3)}{=}_{\varepsilon} f_{w_1}^{(l)1/2} u_{w_1} a f_0^{(l)} b u_{w_4}^* f_{w_4}^{(l)1/2} \\
&\stackrel{(4)}{=}_{\varepsilon} f_{w_1}^{(l)1/2} u_{w_1} f_0^{(l)} a b u_{w_4}^* f_{w_4}^{(l)1/2} \\
&\stackrel{(3)}{=}_{\varepsilon} f_{w_1}^{(l)} f_{w_1}^{(l)1/2} u_{w_1} a b u_{w_4}^* f_{w_4}^{(l)1/2} \\
&\stackrel{(2)}{=}_{(|J_n|-1)\varepsilon} f^{(l)} f_{w_1}^{(l)1/2} u_{w_1} a b u_{w_4}^* f_{w_4}^{(l)1/2}.
\end{aligned}$$

For the last step of the above calculation, we denote $f^{(l)} = \sum_{w \in J_n} f_w^{(l)}$ for all $l = 0, \dots, d$. In the case $w_2 \neq w_3$ we have

$$\|\sigma_0^{(l)}(e_{w_1, w_2} \otimes a) \sigma_0^{(l)}(e_{w_3, w_4} \otimes b)\| = \|f_{w_1}^{(l)1/2} u_{w_1} a u_{w_2}^* f_{w_2}^{(l)1/2} \cdot f_{w_3}^{(l)1/2} u_{w_3} b u_{w_4}^* f_{w_4}^{(l)1/2}\| \leq \varepsilon.$$

So in any case, we get

$$\sigma_0^{(l)}(e_{w_1, w_2} \otimes a) \sigma_0^{(l)}(e_{w_3, w_4} \otimes b) =_{(|J_n|+2)\varepsilon} f^{(l)} \sigma_0^{(l)}((e_{w_1, w_2} \otimes a)(e_{w_3, w_4} \otimes b)).$$

It follows that

$$\sigma_0^{(l)}(a) \sigma_0^{(l)}(b) =_{(|J_n|+2)|J_n|^3\varepsilon} f^{(l)} \sigma_0^{(l)}(ab)$$

for all $a, b \in M_{|J_n|}(K)$ and all $l = 0, \dots, d$. The respective statements also hold for $\sigma_p^{(l)}$ in place of $\sigma_0^{(l)}$. So keeping in mind the definition of K , we have that $\sigma_p^{(l)} \circ \varphi_n^{(i)}$ is order zero up to $(|J_n|+2)|J_n|^3\varepsilon$ for all $i = 0, \dots, s$, $p \in \{0, 1\}^m$ and $l = 0, \dots, d$.

Moreover, we have for all $l = 0, \dots, d$, $v, w \in J_n$ and $a \in K$

$$\begin{aligned}
\sigma_0^{(l)}(e_{v, w} \otimes a) &= f_v^{(l)1/2} u_v a u_w^* f_w^{(l)1/2} \\
&\stackrel{(3)}{=}_{\varepsilon} f_v^{(l)1/2} u_v a f_0^{(l)1/2} u_w^* \\
&\stackrel{(4)}{=}_{\varepsilon} f_v^{(l)1/2} u_v f_0^{(l)1/2} a u_w^* \\
&\stackrel{(3)}{=}_{\varepsilon} f_v^{(l)} u_v a u_w^*.
\end{aligned}$$

The respective statements also hold for $\sigma_p^{(l)}$ in place of $\sigma_0^{(l)}$. Let us denote

$$(\star) \quad \sigma_p^{(l)}(e_{v, w} \otimes a) =_{3\varepsilon} f_{s_p(v)}^{(l)} u_v a u_w^* \quad \text{for all } l, v, w, p \text{ and } a \in K.$$

Applying 1.9 and (2), if ε is chosen small enough in comparison to δ , we get for all $v \in J_N$ and $l = 0, \dots, d$ that

$$\begin{aligned}
& \left\| \sum_{w \in J_n \setminus (v+J_n)} d_n^m(w) \sum_{p \in \{0,1\}^m} f_{s_p(w)}^{(l)} \right\| \\
& \leq 2^m \cdot \max_p \left\| \sum_{w \in J_n \setminus (v+J_n)} d_n^m(w) f_{s_p(w)}^{(l)} \right\| \\
& \stackrel{1.9}{\leq} 2^m \cdot \left[\delta/|J_N| + \max \left\{ \|d_n^m(w) f_{s_p(w)}^{(l)}\| \mid w \in J_n \setminus (v+J_n), p \in \{0,1\}^m \right\} \right] \\
& \leq 2^m \cdot \left[\delta/|J_N| + \max \{d_n^m(w) \mid w \in J_n \setminus (v+J_n)\} \right] \\
& \leq 2^m \cdot \left[\delta/|J_N| + (1 - \frac{n-N}{n})^m \right].
\end{aligned}$$

We may assume that n is large enough in comparison to N such that the right side is less or equal $2^{m+1}\delta/|J_N|$. Observe that we have for all l

$$\sum_{w \in J_n} f_w^{(l)} = \sum_{w \in J_n} d_n^m(w) \sum_{p \in \{0,1\}^m} f_{s_p(w)}^{(l)}.$$

It follows for all $v \in J_N$ that

$$\begin{aligned}
& \left\| \mathbf{1}_A - \sum_{l=0}^d \sum_{w \in J_n \cap (v+J_n)} d_n^m(w) \sum_{p \in \{0,1\}^m} f_{s_p(w)}^{(l)} \right\| \\
& \stackrel{(1)}{\leq} \varepsilon + \left\| \sum_{l=0}^d \sum_{w \in J_n \setminus (v+J_n)} d_n^m(w) \sum_{p \in \{0,1\}^m} f_{s_p(w)}^{(l)} \right\| \\
& \leq \varepsilon + (d+1)2^{m+1}\delta/|J_N| \leq \frac{2^{m+2}(d+1)\delta}{|J_N|}.
\end{aligned}$$

Now let $l \in \{0, \dots, d\}$, $a \in \tilde{F}$ and $v \in J_N$. Denote $\sigma := \sum_{l=0}^d \sum_{p \in \{0,1\}^m} \sigma_p^{(l)}$. We have

$$\begin{aligned}
& \sigma \circ \varphi_n \circ \psi_n \circ \mu(a u_v) \\
& = \sum_{l=0}^d \sum_{p \in \{0,1\}^m} \sum_{i=0}^s (\sigma_p^{(l)} \circ \varphi_n^{(i)} \circ \psi_n \circ \mu)(a u_v) \\
& = 2^{m(d+1)(s+1)\delta/|J_N|} \sum_{l=0}^d \sum_p \sum_{i=0}^s (\sigma_p^{(l)} \circ \varphi_n^{(i)} \circ \psi_n)(DQ a u_v Q) \\
& = \sum_{l=0}^d \sum_p \sum_{i=0}^s (\sigma_p^{(l)} \circ \varphi_n^{(i)} \circ \psi_n) \left(\sum_{w \in J_n \cap (v+J_n)} d_n^m(w) \cdot e_{w, w-v} \otimes \alpha^{-w}(a) \right) \\
& = \sum_{l=0}^d \sum_p \sum_{i=0}^s \sigma_p^{(l)} \left(\sum_{w \in J_n \cap (v+J_n)} d_n^m(w) \cdot e_{w, w-v} \otimes (\varphi^{(i)} \circ \psi)(\alpha^{-w}(a)) \right)
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(\star)}{=} 2^{m|J_n|(d+1)(s+1)} \cdot 3\varepsilon \sum_{l=0}^d \sum_p \sum_{i=0}^s \sum_w d_n^m(w) f_{s_p(w)}^{(l)} \cdot u_w[(\varphi^{(i)} \circ \psi)(\alpha^{-w}(a))] u_{v-w} \\
&= \sum_{l=0}^d \sum_p \sum_w d_n^m(w) f_{s_p(w)}^{(l)} \cdot u_w[(\varphi \circ \psi)(\alpha^{-w}(a))] u_{v-w} \\
&= 2^{m(d+1)|J_n| \cdot \frac{\delta}{|J_n||J_N|}} \sum_{l=0}^d \sum_p \sum_w d_n^m(w) f_{s_p(w)}^{(l)} \cdot u_w \alpha^{-w}(a) u_{v-w} \\
&= \left(\sum_{l=0}^d \sum_p \sum_w d_n^m(w) f_{s_p(w)}^{(l)} \right) \cdot a u_v \\
&= 2^{m+2(d+1)\delta/|J_N|} a u_v.
\end{aligned}$$

Assuming that $3|J_n|\varepsilon \leq \delta/|J_N|$, it follows for all $a \in \tilde{F}$ and $v \in J_N$ that

$$\|a u_v - (\sigma \circ \varphi_n \circ \psi_n \circ \mu)(a u_v)\| \leq 2^{m+3}(d+1)(s+1) \cdot \frac{\delta}{|J_N|}.$$

Hence for $x \in F'$ we get

$$\|x - (\sigma \circ \varphi_n \circ \psi_n \circ \mu)(x)\| \leq 2^{m+3}(d+1)(s+1)\delta.$$

Observe that $\|\sigma\| \leq 2^m(d+2)$ for small enough ε . Hence $\|\sigma \circ \varphi_n\| \leq 2^m(d+2)(s+1)$. Thus, we get for all $x \in F$

$$\begin{aligned}
\|x - (\sigma \circ \varphi_n \circ \psi_n \circ \mu)(x)\| &\stackrel{F \approx_\delta F'}{\leq} \delta + 2^m(d+2)(s+1)\delta + \|y - (\sigma \circ \varphi_n \circ \psi_n \circ \mu)(y)\| \\
&\leq 2^{m+4}(d+1)(s+1)\delta.
\end{aligned}$$

Now let us recall what we got. We have constructed a c.p. approximation

$$(M_{|J_n|}(\mathcal{F})^{2^m(d+1)}, \psi_n \circ \mu, \sigma \circ \varphi_n)$$

of tolerance $2^{m+4}(d+1)(s+1)\delta$ on F , where the map

$$\sigma \circ \varphi_n = \sum_{i=0}^s \sum_{l=0}^d \sum_{p \in \{0,1\}^m} \sigma_p^{(l)} \circ \varphi_n^{(i)}$$

is a direct sum of $2^m(d+1)(s+1)$ c.p.c. maps of order zero up to $(|J_n|+2)|J_n|^3\varepsilon$. Since we can always choose ε as small as we want in relation to n , the maps $\sigma_p^{(l)} \circ \varphi_n^{(i)}$ can be chosen to be order zero up to η for any given $\eta > 0$.

Since m, d, s are constants and $F \subset A \rtimes_\alpha \mathbb{Z}^m$ and $\delta > 0$ were arbitrary, it follows from 1.5 that

$$\dim_{\text{nuc}}(A \rtimes_\alpha \mathbb{Z}^m) \leq 2^m(d+1)(s+1) - 1,$$

which is what we wanted to show. \square

2. The Rokhlin dimension for topological \mathbb{Z}^m -actions

Since we are (at least here) mainly interested in C^* -dynamical systems arising from topological dynamical systems, it is natural to ask whether there exists a variant of Rokhlin dimension

for these actions that is phrased in topological terms. The aim of this section is to give such a variant and to show that finite Rokhlin dimension in the topological sense implies finite Rokhlin dimension in the C^* -algebraic sense.

The definition of the topological variant is due to Wilhelm Winter.

DEFINITION 2.1. Let $(X, \alpha, \mathbb{Z}^m)$ be a topological dynamical system. We say that α has Rokhlin dimension d , and write $\dim_{\text{Rok}}(\alpha) = d$, if d is the smallest natural number with the following property:

For all $n \in \mathbb{N}$, there exists a family of open sets

$$\mathcal{R} = \left\{ U_v^{(l)} \mid l = 0, \dots, d, v \in B_n \right\}$$

in X such that

- $U_v^{(l)} = \alpha^v(U_0^{(l)})$ for all $l = 0, \dots, d$ and $v \in B_n$.
- For all l , the sets $\left\{ \overline{U}_v^{(l)} \mid v \in B_n \right\}$ are pairwise disjoint.
- \mathcal{R} is an open cover of X .

If there is no such d , we write $\dim_{\text{Rok}}(\alpha) = \infty$. In this context, we call \mathcal{R} a Rokhlin cover.

LEMMA 2.2 (cf. [10, Proposition 2.7]). *Let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous action on a compact metric space. Let $\bar{\alpha} : \mathbb{Z}^m \curvearrowright \mathcal{C}(X)$ denote the induced C^* -algebraic action by α . Let $d \in \mathbb{N}$ be a natural number with the properties of 1.6, with the exception that property (1) is replaced by the condition*

$$(1') \quad \sum_{l=0}^d \sum_{v \in B_n} f_v^{(l)} \geq \mathbf{1}_A.$$

Then $\dim_{\text{Rok}}(\bar{\alpha}) \leq d$.

Proof. Let $\varepsilon, \delta > 0$ and $n \in \mathbb{N}$ be given. Choose positive contractions

$$\left\{ h_v^{(l)} \mid l = 0, \dots, d, v \in B_n \right\}$$

satisfying properties (1') and (2)-(4) for δ . If δ is small enough, we can ensure that

$$1 \leq \sum_{l=0}^d \sum_{v \in B_n} h_v^{(l)} \leq d + 2.$$

Denote the function in the middle by S . We have

$$\|S - S \circ \alpha^{-v}\| \leq |B_n|(d+1)\delta \quad \text{for all } v \in B_n.$$

By approximating the function $[1, d+2] \rightarrow [0, 1]$, $t \mapsto t^{-1}$ by polynomials, one gets that

$$\|S^{-1} - S^{-1} \circ \alpha^{-v}\| \leq \varepsilon/2 \quad \text{for all } v \in B_n,$$

if δ is small enough in relation to ε and n . Now define $f_v^{(l)} = S^{-1} \cdot h_v^{(l)}$ for all $l = 0, \dots, d$ and $v \in B_n$. Observe that these elements satisfy the following properties:

- The $f_v^{(l)}$ are positive contractions.
- $\sum_{l=0}^d \sum_{v \in B_n} f_v^{(l)} = \mathbf{1}_A$.
- Property (4) holds trivially.

- Property (2) still holds for δ .
- As for property (3), we get for all $v, w \in B_n$

$$\begin{aligned} \|\bar{\alpha}^v(f_w^{(l)}) - f_{v+w}^{(l)}\| &= \|(S^{-1} \circ \alpha^{-v}) \cdot \bar{\alpha}^v(h_w^{(l)}) - S^{-1} \cdot h_{v+w}^{(l)}\| \\ &\leq \varepsilon/2 + \|S^{-1}\| \cdot \|\bar{\alpha}^v(h_w^{(l)}) - h_{v+w}^{(l)}\| \\ &\leq \varepsilon/2 + \delta. \end{aligned}$$

We see that if δ is chosen small enough in relation to ε and n , then the new functions $f_v^{(l)}$ satisfy the relations (1)-(4) for ε . \square

REMARK 2.3. A slightly more general statement is true by almost the same proof. Namely, if A is a unital C^* -algebra with an action $\alpha : \mathbb{Z}^m \curvearrowright A$, let d be a natural number with the properties (2)-(5) of 1.6 and the property (1'). Then $\dim_{\text{Rok}}^{\text{cyc},c}(\alpha) \leq d$. In other words, in the above theorem, one only needs approximately commuting towers instead of a commutative C^* -algebra.

PROPOSITION 2.4. *Let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a continuous group action on a compact metric space. Let $\bar{\alpha} : \mathbb{Z}^m \curvearrowright \mathcal{C}(X)$ denote the induced C^* -algebraic action by α . Then*

$$\dim_{\text{Rok}}^{\text{cyc}}(\bar{\alpha}) \leq 2^m(\dim_{\text{Rok}}(\alpha) + 1) - 1.$$

Proof. We may certainly assume that the right side is finite. Let $d = \dim_{\text{Rok}}(\alpha)$. For $N \in \mathbb{N}$, let $\{U_v^{(l)} \mid l = 0, \dots, d, v \in B_{2N}\}$ be a Rokhlin cover. Note that by doing an index shift, we may as well assume the form $\{U_v^{(l)} \mid l = 0, \dots, d, v \in J_N\}$ with the corresponding relations, which is notationally more convenient in this proof.

Fix some l and find $\delta > 0$ small enough such that the sets $\{V_v^{(l)} \mid v \in J_N\}$ are still pairwise disjoint for $V_v^{(l)} = \alpha^v(B_\delta(U_0^{(l)}))$. Let $h \in \mathcal{C}_0(V_0^{(l)})$ be a function with $h|_{U_0^{(l)}} = 1$. For all $v \in J_N$, define $g_v^{(l)} \in \mathcal{C}_0(V_v^{(l)})$ via $g_v^{(l)}(x) = h(\alpha^{-v}(x))$.

Observe the following properties:

- $g_v^{(l)} = g_0^{(l)} \circ \alpha^{-v}$ for all $v \in J_N$.
- $g_v^{(l)} \cdot g_w^{(l)} = 0$ for $v \neq w$ in J_N .
- $g_v^{(l)}$ is constantly 1 on $U_v^{(l)}$ for all $v \in J_N$.

Now let $L \in \mathbb{N}$ and $\varepsilon > 0$. Choose n large enough such that $\frac{2}{n} \leq \varepsilon$. Choose a Rokhlin cover $\{U_v^{(l)} \mid l = 0, \dots, d, v \in J_N\}$ for $N = 4Ln$ and choose the functions $\{g_v^{(l)} \mid l = 0, \dots, d, v \in J_N\}$ as above. For all l , define functions $(f_v^{(l)})_{v \in B_L}$ via

$$f_v^{(l)}(x) = \begin{cases} g_w^{(l)}(x) & , \quad \text{if } x \in V_w^{(l)} \text{ for } \|w\|_\infty \leq 2Ln \\ & \text{and } w = v \pmod{L\mathbb{Z}^m} \\ \frac{3Ln - \|w\|_\infty}{Ln} \cdot g_w^{(l)}(x) & , \quad \text{if } x \in V_w^{(l)} \text{ for } 2Ln < \|w\|_\infty \leq 3Ln \\ & \text{and } w = v \pmod{L\mathbb{Z}^m} \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

Now the properties of $g_v^{(l)}$ ensure that

- $f_v^{(l)} \cdot f_w^{(l)} = 0$ for $v \neq w$ in B_L .
- $\sum_{v \in B_L} f_v^{(l)}$ is constantly 1 on $\bigcup_{w \in J_{2Ln}} U_w^{(l)}$.
- For $\|w\|_\infty = L$, we have $\|f_0^{(l)} - f_0^{(l)} \circ \alpha^{-w}\| \leq \frac{1}{n}$.
- For $v \in B_L$, we have $\|f_v^{(l)} - f_0^{(l)} \circ \alpha^{-v}\| \leq \frac{1}{n}$.
- Hence $\|f_w^{(l)} \circ \alpha^{-v} - f_{(v+w)}^{(l)} \bmod L\mathbb{Z}^m\| \leq \frac{2}{n} \leq \varepsilon$ for all $v, w \in B_L$.

Now choose $\{a_j \mid j = 1, \dots, 2^m\} \subset \mathbb{Z}^m$ such that

$$J_N = J_{4Ln} = \bigcup_{j=1}^{2^m} a_j + J_{2Ln}.$$

For $l = 0, \dots, d$, $j = 1, \dots, 2^m$ and $v \in B_L$, define $f_v^{(l,j)} = f_v^{(l)} \circ \alpha^{-a_j}$. We have established that for all l and j , the functions $(f_v^{(l,j)})_{v \in B_L}$ satisfy the relations (2)-(4) of 1.6. Furthermore, $\sum_{v \in B_L} f_v^{(l,j)}$ is constantly 1 on $\bigcup_{w \in (a_j + J_{2Ln})} U_w^{(l)}$, so the choice of the a_j ensures that $\sum_{j=1}^{2^m} \sum_{v \in B_L} f_v^{(l,j)} \geq 1$ on $\bigcup_{w \in J_N} U_w^{(l)}$, hence

$$\sum_{l=0}^d \sum_{j=1}^{2^m} \sum_{v \in B_L} f_v^{(l,j)} \geq 1.$$

We see that the family $\{f_v^{(l,j)} \mid l, j, v\}$ has $2^m \cdot (d+1)$ upper indices, so we are done by Lemma 2.2. \square

We see now that it suffices to study the topological variant of Rokhlin dimension of a topological \mathbb{Z}^m -action in order to study the C^* -algebraic counterpart. A result of [8] enables an easy proof of finite Rokhlin dimension for $m = 1$ in this vein:

THEOREM 2.5 (see [8, theorem 6.1] with proof). *Let X be a compact metric space of finite covering dimension d and $\varphi : X \rightarrow X$ an aperiodic homeomorphism. For all $n \in \mathbb{N}$, we can find an open set $Z \subset X$ such that*

- $\overline{Z} \cap \varphi^j(\overline{Z}) = \emptyset$ for all $j = 1, \dots, n-1$.
- $X = \bigcup_{j=0}^{2^{(d+1)n-1}} \varphi^j(Z)$.

Such an open set is called an n -marker. We will generalize this notion in Section 4 for arbitrary countable group actions.

COROLLARY 2.6. *Let X be a compact metric space and $\varphi : X \rightarrow X$ an aperiodic homeomorphism. Then we have*

$$\dim_{\text{Rok}}(\varphi) \leq 2(\dim(X) + 1) - 1.$$

Proof. We may assume that X has finite covering dimension d , or else there is nothing to show. Let $n \in \mathbb{N}$ be given. Find an open set $Z \subset X$ like in 2.5 for this number. For $j = 0, \dots, n-1$ and $l = 0, \dots, 2d+1$ set

$$U_j^{(l)} = \varphi^{ln+j}(Z).$$

Then it is obvious that $\mathcal{R} = \{U_j^{(l)} \mid l = 0, \dots, 2d + 1, j = 0, \dots, n - 1\}$ is a Rokhlin cover. That is, we have

- $U_j^{(l)} = \varphi^j(U_0^{(l)})$ for all l and j .
- For all l , the sets $\{\overline{U}_j^{(l)} \mid j = 0, \dots, n - 1\}$ are pairwise disjoint.
- \mathcal{R} is an open cover of X .

Hence $\dim_{\text{Rok}}(\varphi) \leq 2d + 1$. □

3. The topological small boundary property

In this section, we define a technical condition that we name the (bounded) topological small boundary property. Weaker versions of this were considered by Lindenstrauss in [13], by Gutman in [7] and [9] and had connections to a dynamical system having mean dimension zero. To the author’s knowledge, Phillips was the first one to link the topological small boundary property to strict comparison for the crossed product, thereby linking it to C^* -classification as well (see [16]). Note also [1], where the topological small boundary property is shown to imply a purely dynamical analogue of strict comparison.

It will turn out that we can in fact assume a stronger bounded variant, whenever we have a finite dimensional underlying space.

DEFINITION 3.1. Let (X, α, G) be a topological dynamical system. Let $M \subset G$ be a subset and $k \in \mathbb{N}$ be some natural number. We say that a set $E \subset X$ is (M, k) -disjoint, if for all distinct elements $\gamma(0), \dots, \gamma(k) \in M$ we have

$$\alpha^{\gamma(0)}(E) \cap \dots \cap \alpha^{\gamma(k)}(E) = \emptyset.$$

We call E topologically α -small if E is (G, k) -disjoint for some k . We call the smallest such k the (topological) smallness constant of E .

DEFINITION 3.2 cf. [13]. Let G be a countably infinite group. A topological dynamical system (X, α, G) has the topological small boundary property, if whenever $A \subset X$ is closed and $U \supset A$ is open, we can find $A \subset V \subset U$ open such that ∂V is topologically α -small.

If we can arrange that each such ∂V has a smallness constant bounded uniformly by a number d , we say that (X, α, G) has the bounded topological small boundary property with respect to d , abbreviated (TSBP $\leq d$).

The main goal of this section is to prove that free actions on finite dimensional spaces have this property. It is important to note that the case $G = \mathbb{Z}$ has been treated by Lindenstrauss in [13]. He has shown that, if $\varphi : X \rightarrow X$ is an aperiodic homeomorphism, then (X, φ) has the bounded topological small boundary property with respect to $\dim(X)$. We merely give a modification of his proof to obtain essentially the same result for free countable group actions.

We would like to quote some well-known facts about properties of covering dimension, since using these will be crucial in some key steps. These statements come up in [13], but for a detailed treatment the reader is referred to [3]. All spaces in question are assumed to be separable metric spaces.

D1 $A \subset B$ implies $\dim(A) \leq \dim(B)$.

D2 If $\{B_i\}_{i \in \mathbb{N}}$ is a family of closed sets in A with $\dim(B_i) \leq k$, then $\dim(\bigcup B_i) \leq k$.

- D3 Let $E \subset A$ be a zero dimensional subspace and $x \in U \subset A$ a point with an open neighbourhood. There exists some open $U' \subset A$ with $x \in U' \subset U$ such that $\partial U' \cap E = \emptyset$.
- D4 If $A \neq \emptyset$, there exists $E \subset A$, which is the countable union of zero dimensional closed subsets in A , such that $\dim(A \setminus E) = \dim(A) - 1$.
- D5 Any countable union of k -dimensional F_σ -sets is a k -dimensional F_σ -set.

LEMMA 3.3. *Let X be a compact metric space and U, V open sets with $\bar{U} \subset V$. Let $E \subset X$ be a zero dimensional subspace. There exists an open set U' with $\bar{U} \subset U' \subset V$ such that $\partial U' \cap E = \emptyset$.*

Proof. Clearly ∂U is compact. For $x \in \partial U$, apply (D3) and find open neighbourhoods $x \in B_x \subset V$ such that $\partial B_x \cap E = \emptyset$. Choose a finite cover $\partial U \subset \bigcup_{i=1}^M B_i$ of such neighbourhoods and set $U' = U \cup \bigcup_{i=1}^M B_i$. It is now easy to see that $\partial U' \subset \bigcup_{i=1}^M \partial B_i$, so we have indeed $\partial U' \cap E = \emptyset$. \square

DEFINITION 3.4. Let X be a compact metric space, G a countable group and $\alpha : G \curvearrowright X$ a continuous group action. For any $g \in G \setminus \{e\}$, set

$$X^g = \{x \in X \mid \alpha^g(x) = x\}.$$

Moreover, let

$$X^G = \bigcup_{g \in G \setminus \{e\}} X^g.$$

We call X^g the g -fixed point set. If $\dim(X^g) \leq 0$ for all $g \neq e$, we say that α has at most zero dimensional fixed point sets. By D5, this is equivalent to $\dim(X^G) \leq 0$. Obviously, free actions fall under this category.

DEFINITION 3.5. Let X be a compact metric space of finite covering dimension n . A family \mathcal{B} of subsets in X is in general position, if for all finite subsets $S \subset \mathcal{B}$ we have

$$\dim\left(\bigcap S\right) \leq \max(-1, n - |S|).$$

LEMMA 3.6. *Let X be a compact metric space of finite covering dimension n together with a continuous group action $\alpha : G \curvearrowright X$ that has at most zero dimensional fixed point sets.*

Let $U, V \subset X$ be open sets with $\bar{U} \subset V$. For any finite subset $F \subset G$, there exists an open set U' with $U \subset U' \subset \bar{U} \subset V$ such that the family $\{\alpha^\gamma(\partial U')\}_{\gamma \in F}$ is in general position and $\partial U' \cap X^G = \emptyset$.

Proof. We prove this by induction in the variable $k = |F|$. The assertion holds for $k = 1$ by application of 3.3. Now assume that the assertion holds for some natural number k . We show that it also holds for $k + 1$.

Let $F = \{\gamma(0), \dots, \gamma(k)\}$ be a set of cardinality $k + 1$ in G . Using the induction hypothesis, there exists an open A_0 with $U \subset A_0 \subset \bar{A}_0 \subset V$, such that the collection $\{\alpha^{\gamma(0)}(\partial A_0), \dots, \alpha^{\gamma(k-1)}(\partial A_0)\}$ is in general position and $\partial A_0 \cap X^G = \emptyset$.

Because of $\bar{A}_0 \subset V$ and the fact that $\partial A_0 \subset X \setminus X^G$, for all $x \in \partial A_0$ we can find $\eta(x) > 0$ such that $\bar{B}_{\eta(x)}(x) \subset V$ and such that $(\alpha^{\gamma(j)}(B_{\eta(x)}(x)))_{j=0, \dots, k}$ are pairwise disjoint. Denote $\bar{B}_x = \bar{B}_{\eta(x)}(x)$ and $B_x = B_{\eta(x)/2}(x)$.

Find some finite subcover $\partial A_0 \subset \bigcup_{i=1}^M B_i$. We will now construct open sets A_i for $i = 0, \dots, M$ (A_0 is already defined) with the following properties:

- (0) $\partial A_i \cap X^G = \emptyset$.
 (1) $\overline{A_i} \subset A_0 \cup \bigcup_{j=1}^M B_j$.
 (2) $A_i \subset A_{i+1} \subset A_i \cup \hat{B}_{i+1}$.
 (3) The collection

$$\mathcal{A}_i = \left\{ \alpha^{\gamma(j)}(\partial A_i) \right\}_{j < k} \cup \left\{ \alpha^{\gamma(k)}(\partial A_i \cap \bigcup_{j=1}^i B_j) \right\}$$

is in general position.

Since by (1) we clearly have $\overline{A_M} \subset V$, combining this with (3) implies that the set $U' = A_M$ has the desired property once we have done this construction. It remains to show how to construct the sets A_i .

So suppose that the set A_i has already been defined for $i < M$. According to D4, for all nonempty subsets $S \subset \mathcal{A}_i$, there exists a zero dimensional F_σ -set $E_S \subset \bigcap S$ such that $\dim(\bigcap S \setminus E_S) = \dim(\bigcap S) - 1$. Define

$$E := \bigcup_{\substack{\emptyset \neq S \subset \mathcal{A}_i \\ 0 \leq j \leq k}} \alpha^{-\gamma(j)}(E_S).$$

By D5, E is a zero dimensional F_σ -set. Use 3.3 to find an open set W such that

$$(\star) \quad \overline{A_i \cap B_{i+1}} \subset W \subset \overline{W} \subset \hat{B}_{i+1} \cap (A_0 \cup \bigcup_{j=1}^M B_j) \quad \text{and} \quad \partial W \cap (E \cup X^G) = \emptyset.$$

Now set $A_{i+1} := A_i \cup W$. This clearly satisfies the properties (0),(1) and (2). To show (3), let $\emptyset \neq S = \{S_1, \dots, S_m\} \subset \mathcal{A}_{i+1}$ correspond to some subset $\{j_1, j_2, \dots, j_m\} \subset \{0, \dots, k\}$. Note that since $\partial A_{i+1} \subset \partial A_i \cup \partial W$, we have either

$$S_l = \alpha^{\gamma(j_l)}(\partial A_{i+1}) \subset \alpha^{\gamma(j_l)}(\partial A_i) \cup \alpha^{\gamma(j_l)}(\partial W) =: S_l^0 \cup S_l^1 \quad (\text{if } j_l \neq k)$$

or

$$\begin{aligned} S_l &= \alpha^{\gamma(j_l)}(\partial A_{i+1} \cap \bigcup_{j=1}^{i+1} B_j) \\ &\subset \alpha^{\gamma(j_l)}((\partial A_i \setminus W) \cap \bigcup_{j=1}^{i+1} B_j) \cup \alpha^{\gamma(j_l)}(\partial W) \\ &\stackrel{(\star)}{\subset} \alpha^{\gamma(j_l)}(\partial A_i \cap \bigcup_{j=1}^i B_j) \cup \alpha^{\gamma(j_l)}(\partial W) \\ &=: S_l^0 \cup S_l^1 \quad (\text{if } j_l = k). \end{aligned}$$

It follows that

$$\bigcap S \subset \bigcup_{a \in \{0,1\}^m} \left(\bigcap_{l=1}^m S_l^{a_l} \right).$$

Since $\overline{W} \subset \hat{B}_{i+1}$, our choice of \hat{B}_{i+1} implies that the sets S_l^1 are pairwise disjoint. So it suffices to consider the case $a = (0, \dots, 0)$ and, since we can change the order without loss of generality, the case $a = (1, 0, \dots, 0)$. For $a = (0, \dots, 0)$, note that $\{S_1^0, \dots, S_m^0\}$ is a subset of \mathcal{A}_i , so we already have

$$\dim \left(\bigcap_{l=1}^m S_l^0 \right) \leq \max(-1, n - m).$$

For $a = (1, 0, \dots, 0)$, define $\hat{S} = \{S_2^0, \dots, S_m^0\}$. This is a subset of \mathcal{A}_i , hence we know that it is in general position. Moreover, considering our choice of the set $E_{\hat{S}}$, recall that

$$\dim(\bigcap \hat{S} \setminus E_{\hat{S}}) \leq \dim(\bigcap \hat{S}) - 1 \leq \max(-1, n - (m - 1)) - 1 \leq \max(-1, n - m).$$

By the choice of W we know that $\partial W \cap E = \emptyset$. Since $\alpha^{-\gamma(j_1)}(E_{\hat{S}}) \subset E$, this implies $E_{\hat{S}} \cap \alpha^{\gamma(j_1)}(\partial W) = \emptyset$. In particular, it follows that

$$S_1^1 \cap \bigcap_{l=2}^m S_l^0 = \alpha^{\gamma(j_1)}(\partial W) \cap \bigcap \hat{S} \subset \bigcap \hat{S} \setminus E_{\hat{S}}.$$

Therefore we have established

$$\dim(S_1^1 \cap \bigcap_{l=2}^m S_l^0) \leq \max(-1, n - m).$$

If we combine these inequalities with D2, it follows that we have $\dim(\bigcap S) \leq \max(-1, n - m)$ as well. So \mathcal{A}_{i+1} is in general position and we are done. \square

LEMMA 3.7. *Let (X, α, G) as before, $F \subset G$ a finite subset and $n \in \mathbb{N}$ a natural number. If a closed subset $E \subset X$ is (F, n) -disjoint, there exists an open neighbourhood V of E such that \overline{V} is (F, n) -disjoint.*

Proof. Note that for all $S \subset F$ with $n = |S|$, we have

$$\emptyset = \bigcap_{\gamma \in S} \alpha^\gamma(E) = \bigcap_{\gamma \in S} \alpha^\gamma \left(\bigcap_{\varepsilon > 0} \overline{B}_\varepsilon(E) \right) = \bigcap_{\varepsilon > 0} \left(\bigcap_{\gamma \in S} \alpha^\gamma(\overline{B}_\varepsilon(E)) \right)$$

By compactness, there must exist some $\varepsilon(S) > 0$ such that

$$\bigcap_{\gamma \in S} \alpha^\gamma(\overline{B}_{\varepsilon(S)}(E)) = \emptyset.$$

If we set $\varepsilon = \min \{\varepsilon(S) \mid S \subset F, n = |S|\}$, the open set $V = B_\varepsilon(E)$ clearly does the trick. \square

THEOREM 3.8. *Let X be a compact metric space with finite covering dimension d . Let G be a countably infinite group with a continuous action $\alpha : G \curvearrowright X$ that has at most zero dimensional fixed point sets. Then (X, α, G) satisfies $(\text{TSBP} \leq d)$. More precisely:*

Let $U, V \subset X$ be open sets with $\overline{U} \subset V$. Then there exists an open set U' with $U \subset U' \subset V$ such that $\partial U'$ is (G, d) -disjoint.

Proof. Since G is countable, choose an increasing sequence $F_1 \subset F_2 \subset F_3 \subset \dots \subset G$ of finite sets such that $G = \bigcup_{k=1}^{\infty} F_k$. We will construct a sequence of open sets $\{U_k\}_{k \in \mathbb{N}}, \{V_k\}_{k \in \mathbb{N}}$ with the following properties for all k :

- (1) $U \subset U_k \subset U_{k+1} \subset V$.
- (2) $\overline{V_{k+1}} \subset V_k$.
- (3) $\overline{U_{k+1}} \subset U_k \cup V_k$.
- (4) \overline{V}_k is (F_k, d) -disjoint.
- (5) $\partial U_k \subset V_k$.

Set $U_0 = U, V_0 = V$. Apply 3.6 to find U_1 such that $U \subset U_1 \subset \overline{U}_1 \subset V$ and ∂U_1 is (F_1, d) -disjoint. Apply 3.7 to find an open neighbourhood V_1 of ∂U_1 such that $\overline{V}_1 \subset V$ and \overline{V}_1 is

(F_1, d) -disjoint. Clearly these sets satisfy (1)-(5) thus far. Suppose that the sets U_k, V_k have been defined for some k . Apply 3.6 to find an open set U_{k+1} such that $U_k \subset U_{k+1} \subset \overline{U_{k+1}} \subset U_k \cup V_k$ and ∂U_{k+1} is (F_{k+1}, d) -disjoint. Since V_k is an open neighbourhood of ∂U_{k+1} , we can find an open neighbourhood V_{k+1} of ∂U_{k+1} such that $\overline{V_{k+1}} \subset V_k$ is (F_{k+1}, d) -disjoint. It is easy to see that these new sets satisfy properties (1)-(5) again.

Now set $U' = \bigcup_{k=0}^{\infty} U_k$. It follows immediately from (1) that $U \subset U' \subset V$. From condition (1), (2) and (3) it follows that $U_{k+r} \cup V_k = U_k \cup V_k$ for all k and $r > 0$, so in particular $U_k \subset U' \subset U_k \cup V_k$ for all k . It follows that

$$\partial U' \subset \overline{U_k \cup V_k} \setminus U_k \subset \overline{V_k}.$$

Since V_k is (F_k, d) -disjoint, we have that $\partial U'$ is (F_k, d) -disjoint for all k , hence it is (G, d) -disjoint. □

4. The controlled marker property

The aim of this section is to use the bounded topological small boundary property to obtain a generalization of Gutman's marker property (see 2.5) for free countable group actions and free \mathbb{Z}^m -actions in particular. Very similarly to 2.6, it will follow that free \mathbb{Z}^m -actions have finite Rokhlin dimension. First we have to introduce the notion of markers, the marker property, and, in particular, the controlled marker property.

DEFINITION 4.1. Let (X, α, G) be a topological dynamical system and $F \subset G$ a finite subset. We call an open set $Z \subset X$ an F -marker, if

- The family of sets $\{\alpha^g(\overline{Z}) \mid g \in F\}$ is pairwise disjoint.
(Or in the notation of the previous section, \overline{Z} is $(F, 1)$ -disjoint.)
- $X = \bigcup_{g \in G} \alpha^g(Z)$.

We say that α has the marker property if there exist F -markers for all $F \subset G$.

DEFINITION 4.2. Let $(X, \alpha, \mathbb{Z}^m)$ be a topological dynamical system, $F \subset \mathbb{Z}^m$ a finite subset and $L \in \mathbb{N}$ a natural number. We call Z an L -controlled F marker, if Z is an F -marker such that the second condition can be strengthened to

$$X = \bigcup_{l=1}^L \bigcup_{v \in F} \alpha^{v_l+v}(Z) \quad \text{for certain } v_1, \dots, v_L \in \mathbb{Z}^m.$$

We say that α has the L -controlled marker property, if there exist L -controlled B_n -markers for all n .

The marker property has been shown for all aperiodic homeomorphisms (i.e. free \mathbb{Z} -actions) on finite dimensional spaces by Gutman in [8]. Careful reading of that proof, however, yields the controlled marker property in the form of 2.5. It is important to note that although the marker property is trivial if the action is assumed to be minimal, Gutman's proof gives a uniform bound (in relation to F) of how many copies one needs to cover the space with an F -marker, which is something new even in the minimal case. We would like to build on his ideas in the case $G = \mathbb{Z}$ to construct a proof for the general case of countable group actions.

LEMMA 4.3. Let X be a compact metric space, G a countably infinite group, $d \in \mathbb{N}$ a natural number and $\alpha : G \curvearrowright X$ a free continuous action that satisfies (TSBP $\leq d$).

Let $F \subset G$ a finite subset and let $g_1, \dots, g_d \in G$ be group elements with the property that the sets

$$F^{-1}F, g_1F^{-1}F, \dots, g_dF^{-1}F$$

are pairwise disjoint. Using the notation $g_0 = e$, set $M = \bigcup_{l=0}^d g_lF^{-1}F$.

Let $U, V \subset X$ be open sets such that

- The family of sets $\{\alpha^g(\overline{U}) \mid g \in F\}$ is pairwise disjoint.
- The family of sets $\{\alpha^g(\overline{V}) \mid g \in M^{-1}\}$ is pairwise disjoint.

Then there exists an open set $W \subset X$ such that $U \subset W, V \subset \bigcup_{g \in M} \alpha^g(W)$ and the family of sets $\{\alpha^g(\overline{W}) \mid g \in F\}$ is pairwise disjoint.

Proof. First observe that there is $\varepsilon > 0$ such that $\overline{B}_\varepsilon(U)$ is $(F, 1)$ -disjoint. Applying (TSBP $\leq d$), we can enlarge U to a new open set within $B_\varepsilon(U)$ such that ∂U is (M, d) -disjoint. Showing the lemma for this larger set gives certainly no loss of generality, so we may just assume that ∂U was (M, d) -disjoint to begin with.

Set $R = \overline{V} \setminus \bigcup_{g \in M} \alpha^g(U)$. Observe that R is closed and $(M^{-1}, 1)$ -disjoint, so choose $\rho > 0$ such that $\overline{B}_\rho(R)$ is $(M^{-1}, 1)$ -disjoint as well. We now claim that there exists a $\delta > 0$ such that

$$(\star) \quad |\{g \in M \mid \alpha^g(\overline{U}) \cap \overline{B}_\delta(x) \neq \emptyset\}| \leq d \quad \text{for all } x \in R.$$

Assume that this is not true. Let $x_n \in R$ be elements with $\delta_n > 0$ such that $\delta_n \rightarrow 0$ and

$$|\{g \in M \mid \alpha^g(\overline{U}) \cap \overline{B}_{\delta_n}(x_n) \neq \emptyset\}| \geq d + 1 \quad \text{for all } n.$$

By compactness, we can assume that x_n converges to some $x \in R$ by passing to a subsequence. Moreover, since M has only finitely many subsets, we can also assume (again by passing to a subsequence if necessary) that there are distinct $\gamma(0), \dots, \gamma(d) \in M$ such that $\alpha^{\gamma(l)}(\overline{U}) \cap \overline{B}_{\delta_n}(x_n) \neq \emptyset$ for all n and all $l = 0, \dots, d$. But then $\delta_n \rightarrow 0$ implies

$$x \in R \cap \bigcap_{l=0}^d \alpha^{\gamma(l)}(\overline{U}) \subset \bigcap_{l=0}^d \alpha^{\gamma(l)}(\partial U) = \emptyset.$$

So this gives a contradiction to ∂U being (M, d) -disjoint. So we may choose a number $\delta \leq \rho$ satisfying (\star) . Moreover, choose some finite covering

$$R \subset \bigcup_{i=1}^s B_\delta(z_i) \quad \text{for some } z_1, \dots, z_s \in R.$$

Note that the right side is $(M^{-1}, 1)$ -disjoint by our choice of ρ . Now observe that (\star) and the fact that the sets $\{g_lF^{-1}F \mid l = 0, \dots, d\}$ are pairwise disjoint, enables us to define a map $c : \{1, \dots, s\} \rightarrow \{0, \dots, d\}$ such that

$$\alpha^g(\overline{U}) \cap \overline{B}_\delta(z_i) = \emptyset \quad \text{for all } g \in g_{c(i)}F^{-1}F.$$

Finally, set

$$W = U \cup \bigcup_{i=1}^s \alpha^{g_{c(i)}^{-1}}(B_\delta(z_i)).$$

Obviously we have $U \subset W$. Moreover, we have

$$\begin{aligned} V &\subset \bigcup_{g \in M} \alpha^g(U) \cup R \\ &\subset \bigcup_{g \in M} \alpha^g(U) \cup \bigcup_{i=1}^s \underbrace{B_\delta(z_i)}_{=\alpha^{g_{c(i)}}(\alpha^{g_{c(i)}^{-1}}(B_\delta(z_i)))} \\ &\subset \bigcup_{g \in M} \alpha^g(U) \cup \bigcup_{i=1}^s \alpha^{g_{c(i)}}(W) \subset \bigcup_{g \in M} \alpha^g(W) \end{aligned}$$

At last we have to show that \overline{W} is $(F, 1)$ -disjoint. Suppose that $\alpha^a(\overline{W}) \cap \alpha^b(\overline{W}) \neq \emptyset$ for some $a \neq b$ in F . That is, there exist $x, y \in \overline{W}$ such that $\alpha^a(x) = \alpha^b(y)$. Let us go through all the possible cases:

- $x, y \in \overline{U}$ is obviously impossible.
- $x \in \alpha^{g_{c(i_1)}^{-1}}(\overline{B}_\delta(z_{i_1}))$ and $y \in \alpha^{g_{c(i_2)}^{-1}}(\overline{B}_\delta(z_{i_2}))$ for some $1 \leq i_1, i_2 \leq s$. It follows that

$$\alpha^a(x) = \alpha^b(y) \in \alpha^{ag_{c(i_1)}^{-1}}(\overline{B}_\delta(z_{i_1})) \cap \alpha^{bg_{c(i_2)}^{-1}}(\overline{B}_\delta(z_{i_2})),$$

so

$$\emptyset \neq \alpha^{b^{-1}ag_{c(i_1)}^{-1}}(\overline{B}_\delta(z_{i_1})) \cap \alpha^{g_{c(i_2)}^{-1}}(\overline{B}_\delta(z_{i_2})) \subset \alpha^{b^{-1}ag_{c(i_1)}^{-1}}(\overline{B}_\rho(R)) \cap \alpha^{g_{c(i_2)}^{-1}}(\overline{B}_\rho(R)).$$

Observe that by $a \neq b$, we have $b^{-1}ag_{c(i_1)}^{-1} \neq g_{c(i_2)}^{-1}$ in M^{-1} . Since $\overline{B}_\rho(R)$ is $(M^{-1}, 1)$ -disjoint, the right side of the above is empty. So this is impossible.

- $x \in \overline{U}$ and $y \in \alpha^{g_{c(i)}^{-1}}(\overline{B}_\delta(z_i))$ for some $1 \leq i \leq s$. Then it follows that

$$\alpha^a(x) = \alpha^b(y) \in \alpha^a(\overline{U}) \cap \alpha^{bg_{c(i)}^{-1}}(\overline{B}_\delta(z_i)) \neq \emptyset.$$

Or equivalently, $\alpha^{g_{c(i)}b^{-1}a}(\overline{U}) \cap \overline{B}_\delta(z_i) \neq \emptyset$, a contradiction to the definition of $c(i)$.

So the sets in the family $\{\alpha^g(\overline{W}) \mid g \in F\}$ are indeed pairwise disjoint. □

PROPOSITION 4.4. *Let G be a countable group, X a compact metric space, $d \in \mathbb{N}$ a natural number and $\alpha : G \curvearrowright X$ be a free continuous group action satisfying $(\text{TSBP} \leq d)$. Let F, g_1, \dots, g_d, M be as in 4.3. Then X admits an F -marker Z with the property $X = \bigcup_{g \in M} \alpha^g(Z)$.*

Proof. For all $x \in X$, choose a neighbourhood U_x such that the family of sets $\{\alpha^g(\overline{U}_x) \mid g \in M^{-1}\}$ is pairwise disjoint. Note that this is possible because the action is free.

Choose a finite subcovering $X = \bigcup_{i=0}^s U_i$. Apply 4.3 (with respect to $U = U_0, V = U_1$) to find an open set W_1 such that $U_0 \subset W_1, U_1 \subset \bigcup_{g \in M} \alpha^g(W)$ and such that \overline{W}_1 is $(F, 1)$ -disjoint. Clearly we have $U_0 \cup U_1 \subset \bigcup_{g \in M} \alpha^g(W_1)$.

Now carry on inductively. If W_k is already defined, apply 4.3 (with respect to $U = W_k, V = U_{k+1}$) to find W_{k+1} such that $W_k \subset W_{k+1}$ and $U_{k+1} \subset \bigcup_{g \in M} \alpha^g(W_{k+1})$ and such that \overline{W}_{k+1} is $(F, 1)$ -disjoint. Note also that if W_k had the property that

$$U_0 \cup \dots \cup U_k \subset \bigcup_{g \in M} \alpha^g(W_k),$$

then it follows that

$$\begin{aligned} U_0 \cup \dots \cup U_k \cup U_{k+1} &\subset \bigcup_{g \in M} \alpha^g(W_k) \cup U_{k+1} \\ &\subset \bigcup_{g \in M} \alpha^g(W_k) \cup \bigcup_{g \in M} \alpha^g(W_{k+1}) \\ &= \bigcup_{g \in M} \alpha^g(W_{k+1}). \end{aligned}$$

So set $Z = W_s$. \bar{Z} is $(F, 1)$ -disjoint by construction, and indeed an F -marker with

$$X = U_0 \cup \dots \cup U_s \subset \bigcup_{g \in M} \alpha^g(Z).$$

□

REMARK 4.5. Let $m \in \mathbb{N}$. For all n and $a \in \{0, 1\}^m$, defining

$$w_a = \left(\delta_{1, a_j} \cdot n \right)_{j=1, \dots, m} \in \mathbb{Z}^m$$

yields 2^m distinct elements with the property that $B_{2n} = \bigcup_a (w_a + B_n)$.

THEOREM 4.6. Let X be a compact metric space, $d \in \mathbb{N}$ a natural number and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous action satisfying $(\text{TSBP} \leq d)$. Then $(X, \alpha, \mathbb{Z}^m)$ has the $2^m(d+1)$ -controlled marker property.

Proof. For $n \in \mathbb{N}$, choose $v_1, \dots, v_d \in \mathbb{Z}^m$ such that (note $v_0 := 0$)

$$B_n - B_n, v_1 + (B_n - B_n), \dots, v_d + (B_n - B_n)$$

are pairwise disjoint. Define $M = \bigcup_{l=0}^d (v_l + (B_n - B_n))$. Apply 4.4 to find a B_n -marker Z such that $X = \bigcup_{v \in M} \alpha^v(Z)$. Observing that B_{2n} is a translate of $B_n - B_n$, use the previous remark to choose $w_1, \dots, w_{2^m} \in \mathbb{Z}^m$ so that $B_n - B_n = \bigcup_j (w_j + B_n)$. It follows that

$$M = \bigcup_{l=0}^d (v_l + (B_n - B_n)) = \bigcup_{l=0}^d \bigcup_{j=1}^{2^m} ((v_l + w_j) + B_n),$$

so

$$X = \bigcup_{v \in M} \alpha^v(Z) = \bigcup_{l=0}^d \bigcup_{j=1}^{2^m} \bigcup_{v \in B_n} \alpha^{(v_l + w_j) + v}(Z).$$

So we see that Z is a $2^m(d+1)$ -controlled B_n -marker. □

5. Conclusion and further remarks

Finally we can combine the results from the previous sections to get our main result.

COROLLARY 5.1. Let X be a compact metric space, $d \in \mathbb{N}$ a natural number and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous action satisfying $(\text{TSBP} \leq d)$. Then

$$\dim_{\text{Rok}}(\alpha) \leq 2^m(d+1) - 1.$$

Proof. Let $n \in \mathbb{N}$ be given. By 4.6, we may apply the $2^m(d+1)$ -controlled marker property and choose a B_n -marker Z such that there exist

$$\{v_l \mid l = 0, \dots, 2^m(d+1) - 1\} \subset \mathbb{Z}^m$$

with

$$X = \bigcup_{l=0}^{2^m(d+1)-1} \bigcup_{v \in B_n} \alpha^{v_l+v}(Z).$$

For $l = 0, \dots, 2^m(d+1) - 1$ and $v \in B_n$, define $U_v^{(l)} = \alpha^{v_l+v}(Z)$. Then

$$\mathcal{R} = \left\{ U_v^{(l)} \mid l = 0, \dots, 2^m(d+1) - 1, v \in B_n \right\}$$

obviously forms a Rokhlin cover with the desired properties. □

COROLLARY 5.2. *Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous action. Then*

$$\dim_{\text{Rok}}(\alpha) \leq 2^m(\dim(X) + 1) - 1.$$

Proof. This follows directly from 3.8 and 5.1. □

THEOREM 5.3. *Let X be a compact metric space of finite covering dimension and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free continuous group action. Then the induced C^* -algebraic action $\bar{\alpha}$ on $\mathcal{C}(X)$ has finite Rokhlin dimension, and the transformation group C^* -algebra $\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m$ has finite nuclear dimension. More specifically, we have*

$$\dim_{\text{Rok}}^{\text{cyc}}(\bar{\alpha}) \leq 2^{2m}(\dim(X) + 1) - 1$$

and

$$\dim_{\text{nuc}}(\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m) \leq 2^{3m}(\dim(X) + 1)^2 - 1.$$

Proof. Combining 5.2 and 2.4 yields

$$\dim_{\text{Rok}}^{\text{cyc}}(\bar{\alpha}) \leq 2^{2m}(\dim(X) + 1) - 1.$$

Remembering that $\dim_{\text{nuc}}(\mathcal{C}(X)) = \dim(X)$, we apply 1.10 to obtain

$$\dim_{\text{nuc}}(\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m) \leq 2^{3m}(\dim(X) + 1)^2 - 1. □$$

Now let us address possible further generalizations and open questions.

REMARK 5.4. The results of this paper apply more generally to free \mathbb{Z}^m -actions on locally compact metric spaces. Since the author is not aware of immediate applications for open problems in the non-compact case, that case was dropped to make proofs shorter. However, this may change in the future, so we sketch how one would have to modify the approach to handle the non-compact case.

- One can define an analogous notion of C^* -Rokhlin dimension for actions on non-unital C^* -algebras. It is defined just as in 1.6, with the exception that condition (1) is replaced

by

$$(1') \quad \|a - a\left(\sum_{l=0}^d \sum_{v \in B_n} f_v^{(l)}\right)\| \leq \varepsilon \quad \text{for all } l = 0, \dots, d, v \in B_n, a \in F.$$

In other words, instead of the Rokhlin elements almost summing up to $\mathbf{1}$, the sum behaves like an approximate unit on elements of F .

- 1.10 generalizes to actions on non-unital C^* -algebras. In the proof, condition (1) is used only once, and it is obvious that (1') is actually sufficient in that step.
- Similarly, one can define an analogous notion of Rokhlin dimension for topological actions on locally compact metric spaces. It is defined just as in 2.1, with the exception that we only require \mathcal{R} to cover an arbitrarily given compact subset $K \subset X$.
- 2.2 and 2.4 can then be generalized in a straightforward manner.
- Definitions 3.1, 3.2 and 3.5 also make sense for actions on non-compact spaces. Note that in 3.2, the set denoted by A should be required to be compact rather than just closed.
- Every single statement of section 3 is true in the locally compact case with the same proofs, if we restrict them to statements about bounded open sets. Therefore 3.8 generalizes to countable group actions on locally compact metric spaces having at most zero dimensional fixed point sets.
- For group actions on locally compact spaces, we can modify the definition of markers. For $F \subset G$ and a compact subset $K \subset X$, an (F, K) -marker is a bounded open set $Z \subset X$ such that

- * $\alpha^g(\overline{Z}) \cap \alpha^h(\overline{Z}) = \emptyset$ for all $g \neq h$ in F .
- * $K \subset \bigcup_{g \in G} \alpha^g(Z)$.

Similarly we can define controlled (F, K) -markers in the case of \mathbb{Z}^m -actions. We say that an action has the L -controlled marker property if for all n and all compact $K \subset X$, there exists an L -controlled (B_n, K) -marker.

- 4.3, 4.4 and 4.6 are generalized to the non-compact case in a straightforward manner, using $(\text{TSBP} \leq d)$ for free actions on locally compact metric spaces. Hence all free \mathbb{Z}^m -actions with $(\text{TSBP} \leq d)$ have the $2^m(d+1)$ -controlled marker property.
- Combining all these generalized statements, we get the corresponding statements of 5.1, 5.2 and 5.3 for the non-compact case.

REMARK 5.5. Let G be a countable and locally finite group, i.e. $G = \bigcup_n H_n$ for an increasing sequence of finite subgroups H_n . Then the results of this paper also apply to free $(\mathbb{Z}^m \times G)$ -actions on (locally) compact metric spaces. Since the generalization is very straightforward, we do not give a sketch on how to modify all the important steps like above. Instead, we remark that the set B_n has to be replaced by $B_n \times H_n$ in order to get the right definitions for $(\mathbb{Z}^m \times H)$ -actions. The proofs are then almost identical.

QUESTION 5.6. What is the right notion of Rokhlin dimension for actions of a larger class of countable, non-abelian, discrete, amenable groups?

Keep in mind that finite Rokhlin dimension should be regarded as a topological version of the measure theoretic Rokhlin lemma. In view of existing results on groups that can satisfy such a Rokhlin lemma (see [15] in particular), can we define Rokhlin dimension for actions of monitilable groups?

QUESTION 5.7. Suppose that one can successfully generalize Rokhlin dimension to actions of a larger class of countable, discrete, amenable groups. For which groups is it automatic that free actions on finite dimensional spaces have finite Rokhlin dimension? Lemma 4.3 seems to suggest that this might be the case for finitely generated groups whose unit balls in some length metric have some form of bounded growth condition similar to \mathbb{Z}^m . Is polynomial growth sufficient?

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