The Role of Experimental Conditions in Model Validation for Control*

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Abstract. Within a stochastic noise framework, the validation of a model yields an ellipsoidal parameter uncertainty set, from which a corresponding uncertainty set can be constructed in the space of transfer functions. We display the role of the experimental conditions used for validation on the shape of this validated set, and we connect a measure of the size of this set to the stability margin of a controller designed from the nominal model. This allows one to check stability robustness for the validated model set and to propose guidelines for validation design.

1 Introduction

Model validation is the exercise that consists in assessing whether a model of some underlying system is *good enough.* Such quality control step cannot be decoupled from the purpose for which the model is to be used. And just as the research on system identification has, in the last 10 years, focused on issues of *design* in order to obtain a model that suited the objective, so must the validation experiment similarly be designed in such a way that the model is guaranteed to deliver what the model is supposed to deliver. Thus, one must think in terms of “goal-oriented validation”.

In this chapter we focus on the situation where a model is to be validated with the purpose of designing a controller for the underlying system. This is called *model validation for control.*

The assessment of the quality of a model can take a variety of forms, such as a frequency-domain bound on the error between the system and the model transfer functions, or a worst-case bound on such error over all

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frequencies, or the certification of a region in the complex plane in which the system and the model are guaranteed to lie (set membership validation). Depending on the application some of these quality statements can be more useful than others.

Spurred by the strong reliance of robust control theory on specific uncertainty descriptions, the research on model uncertainty estimation and on model validation gathered momentum in the 1990's. Two directions have been pursued.

1. The first consists in estimating uncertainty regions around estimated models. In the stochastic framework, estimates of the total mean square transfer function error were obtained by adopting, for the bias error, a parametrized probability distribution and by estimating the parameters of this distribution from the data, just as is done for the noise error [4]. In the "hard-bound" framework, uncertainty models have been derived under a variety of hard-bound assumptions on the error model and on the noise: see e.g. [3], [5].
2. The second direction consists in reducing a prior set of admissible models by invalidating models on the basis of observed data and prior hard-bound assumptions: see e.g. [11], [9]. The concept of model invalidation, on the basis of an observed incompatibility between a model, prior assumptions and data, was extended to *controller invalidation* in [10].

The validation theory presented in this chapter is inspired by recent validation results of Ljung and collaborators [6], [8], [7] that are based on signal statistics, with essentially no prior assumptions other than some unavoidable *invariance assumption.* To paraphrase Swedish literature [7], one would like to approach the model validation problem 'as naked as possible' and strip off common covers such as *prior assumptions, probabilistic framework, worst case model properties.* What we are then left with are experimental data that we can collect on the true system and compare with simulated data generated by the model, statistics that we can compute from these data, and some invariance assumption that states that the future statistics will not be different from those observed so far.

The key idea of the method proposed by Ljung for the validation of a model $\hat{G}$ is that the residuals ϵ, obtained by substracting simulated outputs from measured outputs, contain information about the model error $G_0 - \hat{G}$. The identification of an unbiased model for the dynamics connecting the input signal u to the residuals ϵ delivers an estimate of the model error $G_0 - \hat{G}$ and a covariance for this estimate.

Our departure from the validation results of Ljung and collaborators, and the new contributions of this chapter, are contained in the following sequence of new ideas and observations whose presentation will form the essence of this chapter.

1. The validation results of Ljung and Guo [8] allow one to define an uncertainty region $\mathcal{D}$ in the frequency domain, that contains G_0, and also

$\hat{G}$ if the model is validated. Our first observation is that different experimental conditions for the collection of validation data, will produce different uncertainty regions $\mathcal{D}_i$, some of which may result in a successful validation and some of which may not. Thus we shall elaborate on **the role of experimental conditions in the validation of a model** and on the concept of **validation design.**

2. We then observe that a model $\hat{G}$ may be validated under closed loop experimental conditions. By collecting data on the closed loop system (G_0, C) with some controller C, one can apply the validation procedure to the closed loop transfer function model $\hat{T} = \frac{\hat{G}C}{1+\hat{G}C}$ of the true closed loop system $T_0 = \frac{G_0 C}{1+G_0 C}$. This defines a closed loop uncertainty set $\mathcal{D}(\hat{T})$, from which the corresponding open loop set $\mathcal{D}(\hat{G})$ can be computed. Thus, we have introduced the concept of **validation in closed loop**.
3. Since each validation experiment leads to a different set of validated models $\mathcal{D}_i(\hat{G})$ that contains G_0, some of these validated regions may be more useful than others, depending on the intended use of the model. This suggests that one should design the validation experiment so that the uncertainty regions are tuned towards the intended use of the model. This leads to **goal-oriented model validation** and to *tuned uncertainty regions.*
4. Vinnicombe [12] has shown that a controller C that stabilizes $\hat{G}$ with a *generalized stability margin* denoted $b_{\hat{G},C}$ stabilizes all plants G for which $\delta_\nu(\hat{G}, G) < b_{\hat{G},C}$, where $\delta_\nu(\hat{G}, G)$ is a metric that measures the distance between $\hat{G}$ and G. Details will be given later in the chapter. We shall introduce the concept of *worst case gap* $\delta_{WC}(\hat{G}, \mathcal{D})$ between a model $\hat{G}$ and all plants in a validated set $\mathcal{D}(\hat{G})$. This leads us to introduce the idea of **model validation for control**: a validation experiment that delivers a validated model set with a smaller worst case gap than another one allows for a larger class of robustly stabilizing controllers.

The validated uncertainty regions constructed in this chapter are based on ellipsoidal confidence regions obtained in parameter space from covariance estimates. Thus, all statements about a system belonging to an uncertainty set are understood to be probabilistic; note, however, that the probability level is left to the user to decide.

2 Model and controller validation concepts

We consider that the input-output data that are used to validate a model are generated from a "*true system*":

$$y(t) = G_0(q)u(t) + v(t), \tag{1}$$

where $G_0(q)$ is a linear time-invariant causal operator. We make no special assumptions about the input signal $u(t)$ and the noise $v(t)$. We consider that somebody has delivered to us a model $\hat{G}(q)$ for $G_0(q)$, and our task is to validate that model. We are allowed to perform experiments on the true system by applying N input data $u(t)$ to it and by observing the corresponding N output data $y(t)$. Given this framework, the following particular *validation questions* will be addressed.

Model validation question. On the basis of the data I collect, can I define an uncertainty set $\mathcal{D}$ in which G_0 is guaranteed to lie, at a certain probability level? If $\hat{G} \in \mathcal{D}$, then $\hat{G}$ will be called validated.

Controller validation question. On the basis of the data I collect, can I guarantee that a given controller $C(q)$, typically computed from $\hat{G}(q)$, stabilizes not just $\hat{G}$ but also the true $G_0(q)$? If the answer is positive, the controller is said to be *unfalsified* by the data; in the converse case, it is said to be *falsified* or *invalidated*.

Our results provide a contribution to both of these validation questions, **in a stochastic framework**. Our validation procedure will lead to the validation of sets of transfer functions; it could appropriately be called *set membership validation*. We insist that we do not a posteriori validate an a priori given uncertainty set, but rather the validation of a nominal model $\hat{G}$ under specific experimental conditions determines a validated uncertainty set.

3 The model validation procedure

Consider the true system (1) and a model $\hat{G}$ that requires validation. If we apply some input sequence $U^N = \{u(t), t = 1, \ldots, N\}$ to the system, it generates the noisy output sequence $Y^N = \{y(t), t = 1, \ldots, N\}$ using (1). The corresponding simulated outputs are given by

$$\hat{y}(t) = \hat{G}(q)u(t). \tag{2}$$

Consider now the *model residuals* $\epsilon(t)$ defined as the difference between measured and simulated outputs:

$$\epsilon(t) = y(t) - \hat{y}(t) = y(t) - \hat{G}(q)u(t) \tag{3}$$

Inserting the system equation (1) these residuals can then be written as

$$\epsilon(t) = [G_0(q) - \hat{G}(q)]u(t) + v(t) = \partial G(q)u(t) + v(t). \tag{4}$$

The transfer function ∂G is called the *model error* in [6]. Using the assumption of a linear true system and the independence between $v(t)$ and $u(t)$,[1]

[1] By this we mean that $v(t)$ would not change if we were to change the input signal $u(t)$.

we have thus decomposed the residual error $\epsilon(t)$ into the sum of two independent sources: one, $\partial G(q)u(t)$ that is due to a *model error*, and one, called *disturbance*, that is not due to a model error. The distinction between these two sources of signal error is very fundamental, and has nothing to do with a probabilistic framework. It is at the heart of all validation theories.

Observations

- The essential difference between the two sources of residual error $\epsilon(t)$ is that one can be manipulated by the user by experimenting with $u(t)$ while the other is totally outside the range of experimentation.
- Without any assumption on the disturbance $v(t)$, any observed error $\epsilon(t)$, however large, can always be attributed to the occurrence of a very large disturbance $v(t)$. Thus, one cannot invalidate a model on the basis of an observed data unless some bounded noise assumption is made.
- If an *invariance assumption* is made on the mechanism that generates the disturbance $v(t)$, then one can evaluate whether ∂G is significantly different from zero by estimating an unbiased model for ∂G from $[\epsilon \quad u]$ data.

This last observation is at the heart of the validation procedure proposed by Ljung [6] that we adopt here, with some modifications to account for the added insight gained since the publication of [6].

3.1 Open loop validation

We compute an *unbiased* estimate $\tilde{G}(\hat{\theta}, q)$ of $\partial G(q)$. Thus, consider a model set $\mathcal{M}_{OL} = \{\tilde{G}(\theta, q) \mid \theta \in D_\theta \subset \mathbf{R}^k\}$, for some subset D_θ, and an independently parametrized noise model. The assumption on unbiasedness implies that $\tilde{G}(\theta_0, q) = \partial G(q)$ for some $\theta_0 \in D_\theta$. Using experimental data $[\epsilon \quad u]$ collected in open loop (see (3)-(4)), one can then compute an unbiased estimate $\tilde{G}(\hat{\theta}, q)$ of $\partial G(q)$, as well as an estimate of the covariance matrix P_θ of $\hat{\theta}$. The true parameter θ_0 then lies with probability $\alpha(k, \chi^2_{ol})$ in the ellipsoidal uncertainty region

$$U_{OL} = \{\theta \mid (\theta - \hat{\theta})^T P_\theta^{-1} (\theta - \hat{\theta}) < \chi^2_{ol}\} \qquad (5)$$

where $\alpha(k, \chi^2_{ol}) = Pr(\chi^2(k) \leq \chi^2_{ol})$ with $\chi^2(k)$ the chi-square probability distribution with k parameters. This parametric uncertainty region U_{OL} defines a corresponding uncertainty region in the space of transfer functions which we denote $\mathcal{D}_{OL}$:

$$\mathcal{D}_{OL} = \{\hat{G}(q) + \tilde{G}(\theta, q) \mid \tilde{G}(\theta, q) \in \mathcal{M}_{OL} \text{ and } \theta \in U_{OL}\} \qquad (6)$$

We then have the following property.

Lemma 1: *$G_0 \in \mathcal{D}_{OL}$ with probability $\alpha(k, \chi^2_{ol})$.*
The proof follows directly from the properties of estimated models when variance errors only are concerned. ■

The importance of Lemma 1 is that our validation procedure has delivered a validated model set $\mathcal{D}_{OL}$, in which the true system is guaranteed to lie, at some probability level. We now introduce the following definition for the validation of the model $\hat{G}$.

Definition : The model $\hat{G}$ is called *validated* if $\hat{G} \in \mathcal{D}_{OL}$ or, equivalently, if there exists $\theta^* \in U_{OL}$ such that $\tilde{G}(\theta^*, q) = 0$.

Comments

1. The estimated model $\tilde{G}$ is a correction to the prior model $\hat{G}$ that is under test. Thus, one could, in the application for which the model is to be used, replace $\hat{G}$ by the better model $\hat{G} + \tilde{G}$, or by a new low order model $\hat{G}$ in the validated set $\mathcal{D}_{OL}$. In the sequel, where we focus on the use of the model for control design, we assume that the control design is based on $\hat{G}$ (possibly a new one), but not on $\hat{G} + \tilde{G}$.
2. In fact, we shall see later that for control design it is not so much the validation of the model $\hat{G}$ that matters but the fact that the validation procedure described above yields a validated region $\mathcal{D}_{OL}$, in which the true system G_0 is known to lie. Thus, even if the model $\hat{G}$ is not validated, the controller design and controller validation procedure described in the sequel of this chapter still apply.
3. The validation procedure just described can be applied to any model $\hat{G}$, whether it is a full order or reduced order model of the true G_0.

3.2 Role of the experimental conditions

The validated model set $\mathcal{D}_{OL}$ depends very much on the experimental conditions under which the validation has been performed. This is perhaps not so apparent in the exact definition (6) of $\mathcal{D}_{OL}$ via the parameter covariance matrix P_θ. However, let us recall that a reasonable approximation for the covariance of the transfer function estimate $\tilde{G}(\hat{\theta}, q)$ is given by:

$$cov(\tilde{G}(\hat{\theta}, e^{j\omega})) \approx \frac{n}{N} \frac{\phi_v(\omega)}{\phi_u(\omega)} \tag{7}$$

This shows clearly the role of the signal spectra $\phi_u(\omega)$ and $\phi_v(\omega)$ in shaping the validated set $\mathcal{D}_{OL}$. Thus, two different validation data sets $[\epsilon^{(1)} \;\; u^{(1)}]$ and $[\epsilon^{(2)} \;\; u^{(2)}]$ will yield two different validated regions $\mathcal{D}_{OL}^{(1)}$ and $\mathcal{D}_{OL}^{(2)}$. The model $\hat{G}$ may well be validated by one of these two experiments and not by the other.

The role of the experimental conditions on the shape of the validated set, and the importance of tuning the validation experiment to the objective to which the model (or the model set) is to be used, are the central themes of this chapter. We shall see, in particular, how the validation experiment can be tuned when the objective is a robust model-based control design.

3.3 Closed loop validation

On the basis of these observations, we now show that a validated set of models $\mathcal{D} = \{\hat{G}(q) + \tilde{G}(\theta, q) |\ \theta \in U\}$ for some parameter set U can alternatively be computed from a closed loop validation experiment. Consider that the feedback control law $u(t) = C(q)[r(t) - y(t)]$ is applied to the true system $G_0(q)$, with some stabilizing controller $C(q)$. The closed loop system is :

$$y(t) = \frac{G_0 C}{1 + G_0 C} r(t) + \frac{1}{1 + G_0 C} v(t) \stackrel{\triangle}{=} T_0 r(t) + n(t) \tag{8}$$

The closed loop model is $\hat{T} = \frac{\hat{G}C}{1+\hat{G}C}$. We can then simulate $\hat{y}(t) = \hat{T} r(t)$ and define the closed loop model error

$$\epsilon(t) = (T_0 - \hat{T}) r(t) + n(t) \stackrel{\triangle}{=} \partial T(q) r(t) + n(t) \tag{9}$$

Consider now a model set $\mathcal{M}_{CL} = \{\tilde{T}(\xi, q) \mid \xi \in D_\xi \subset \mathbf{R}^f\}$, for some subset D_ξ defining stable models, such that $\tilde{T}(\xi_0, q) = \partial T(q)$ for some $\xi_0 \in D_\xi$. Using experimental data $[\epsilon \;\; r]$ collected on the closed loop system, we can then compute an unbiased estimate $\tilde{T}(\hat{\xi}, q)$ of $\partial T(q)$, together with an estimate of the covariance matrix P_ξ of the parameter vector $\hat{\xi}$. The true parameter ξ_0 then lies with probability $\alpha(f, \chi^2_{cl})$ in the ellipsoidal uncertainty region

$$U_{CL} = \{\xi \mid (\xi - \hat{\xi})^T P_\xi^{-1} (\xi - \hat{\xi}) < \chi^2_{cl}\} \tag{10}$$

where $\alpha(f, \chi^2_{cl}) = Pr(\chi^2(f) \leq \chi^2_{cl})$ with $\chi^2(f)$ the chi-square probability distribution with f parameters. This parametric uncertainty region U_{CL} defines a corresponding uncertainty region in the space of closed loop transfer functions $T(\xi, q)$ which we denote $\mathcal{S}_{CL}$:

$$\mathcal{S}_{CL} = \{\hat{T}(q) + \tilde{T}(\xi, q) \mid \tilde{T}(\xi, q) \in \mathcal{M}_{CL} \text{ and } \xi \in U_{CL}\} \tag{11}$$

$\mathcal{S}_{CL}$ is the set of closed loop transfer functions that are validated by our closed loop experiment. From this set (in fact from U_{CL}) we can now define the set $\mathcal{D}_{CL}$ of transfer functions $G(\theta, q)$ that are validated by this closed loop experiment:

$$\mathcal{D}_{CL} = \{\hat{G} + \tilde{G}(\xi, q) \mid \tilde{G}(\xi, q) = \frac{1}{C(q)} \times \frac{\tilde{T}(\xi, q)(1 + \hat{G}C)}{1 - \hat{T} - \tilde{T}(\xi, q)} \text{ and } \xi \in U_{CL}\} \tag{12}$$

The notation $\tilde{G}(\xi, q)$ used in (12) denotes the rational transfer function model whose coefficients are uniquely determined from ξ by the inverse mapping

$$\tilde{G}(\xi, q) = \frac{1}{C(q)} \times \frac{\tilde{T}(\xi, q)(1 + \hat{G}C)}{1 - \hat{T} - \tilde{T}(\xi, q)}. \tag{13}$$

We then have the following property.

Lemma 2: $T_0 \in \mathcal{S}_{CL}$ and $G_0 \in \mathcal{D}_{CL}$ with probability $\alpha(f, \chi^2_{cl})$. ■

Comments

1. Following our earlier definition of a validated model, we observe that the closed loop model $\hat{T}$ is validated if $\hat{T} \in S_{CL}$ or, equivalently, if there exists a $\xi^* \in U_{CL}$ such that $\tilde{T}(\xi^*, q) = 0$. Similarly, the open loop model $\hat{G}$ is validated by this closed loop experiment if $\hat{G} \in D_{CL}$, which is equivalent with the existence of $\xi^* \in U_{CL}$ such that $\tilde{G}(\xi^*, q) = 0$, with $\tilde{G}(\xi^*, q)$ defined by the mapping (13).
2. However, the most useful aspect of this closed loop validation procedure, from a control objective point of view, is not so much the validation of the initial model $\hat{G}$ as it is the validation of the uncertainty set $\mathcal{D}_{CL}$. Sets validated by closed loop experiments typically have properties that allow for a larger set of stabilizing controllers than sets validated in open loop.

4 Controller validation and model validation for control

We now consider the situation where a controller is designed on the basis of the nominal model $\hat{G}$. For the theory that we develop, this model need not necessarily be inside the validated set $\mathcal{D}$, but the typical situation is where $\hat{G} \in \mathcal{D}$. Indeed, if the model $\hat{G}$ has failed a range of validation attempts, any sensible designer will want to replace $\hat{G}$ by a model that is contained in the validated set. We then introduce the concept of controller validation.

Definition : Let the validation procedure of a model $\hat{G}(q)$ result in a validated set $\mathcal{D}$ of transfer function models containing G_0, and let $C(q)$ be a controller designed from $\hat{G}(q)$. Then $C(q)$ is called a *validated controller* for the set $\mathcal{D}$ if it stabilizes all models in $\mathcal{D}$.

Having defined a validated controller, we turn to the question of *model validation for control.* Consider first that two different validation experiments, performed on the same model $\hat{G}$, have led to two different validated sets $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$. The same controller $C(q)$ may be validated for both sets, or for one of them, or for neither. More generally, denote by $\mathcal{C}^{(1)}$ the set of controllers that are validated by the first experiment, and by $\mathcal{C}^{(2)}$ the set of controllers that are validated by the second experiment. By this we mean that, for each $C \in \mathcal{C}^{(1)}$, say, and for each $G \in \mathcal{D}^{(1)}$, the closed loop made up of (G, C) is stable. Then we shall consider that the validated set $\mathcal{D}^{(1)}$ is a better uncertainty set than $\mathcal{D}^{(2)}$ for control design if the set of stabilizing controllers $\mathcal{C}^{(1)}$ is "larger than" the set $\mathcal{C}^{(2)}$ in some sense. Given that the validation results strongly depend on the experimental conditions, this will then lead us to the concept of *validation design for control.* To make these ideas precise, we introduce a metric on the size of the validated set, and we appeal to some basic tools and results of robust control theory.

5 The Vinnicombe gap metric and its stability result

Various measures exist to characterize the distance between two plants. We adopt here the Vinnicombe gap metric ([12,13]) denoted δ_ν. The Vinnicombe gap (or distance) between a scalar plant G and a model $\hat{G}$ is defined as

$$\delta_\nu(\hat{G}, G) = \begin{cases} \max_\omega \kappa\left(\hat{G}(e^{j\omega}), G(e^{j\omega})\right) & if\ (16)\ \ is\ \ satisfied \\ 1 & otherwise \end{cases} \tag{14}$$

where

$$\kappa\left(\hat{G}(e^{j\omega}), G(e^{j\omega})\right) \triangleq \frac{|\hat{G}(e^{j\omega}) - G(e^{j\omega})|}{\sqrt{1 + |\hat{G}(e^{j\omega})|^2}\sqrt{1 + |G(e^{j\omega})|^2}} \tag{15}$$

The condition to be fulfilled in order to have $\delta_\nu(\hat{G}, G) < 1$ is :

$$(1 + \hat{G}^* G)(e^{j\omega}) \neq 0 \quad \forall\omega \text{ and } wno(1 + \hat{G}^* G) + \eta(G) - \tilde{\eta}(\hat{G}) = 0, \tag{16}$$

where $G^*(e^{j\omega}) = G(e^{-j\omega})$, $\eta(G)$ (resp. $\tilde{\eta}(G)$) denotes the number of poles of G in the complement of the closed (resp. open) unit disc, while $wno(G)$ denotes the winding number about the origin of $G(z)$ as z follows the unit circle indented into the exterior of the unit disc around any unit circle pole and zero of $G(z)$.

If the conditions (16) are satisfied, then the distance between two plants has a simple frequency domain interpretation (in the SISO case). Indeed, the quantity $\kappa(\hat{G}(e^{j\omega}), G(e^{j\omega}))$ is the chordal distance between the projections of $\hat{G}(e^{j\omega})$ and $G(e^{j\omega})$ onto the Riemann sphere of unit diameter [12]. The distance $\delta_\nu(\hat{G}, G)$ between $\hat{G}$ and G is therefore, according to (14), the supremum of these chordal distances over all frequencies.

The main interest of the Vinnicombe metric is its use as a tool for the robust stability analysis of feedback systems. Thus, consider a closed loop system made up of the negative feedback connection of a plant G and a controller C. For such feedback system one can define a generalized stability margin [13].

Definition: generalized stability margin.

$$b_{GC} = \begin{cases} \min_\omega \kappa\left(G(e^{j\omega}), -\frac{1}{C(e^{j\omega})}\right) & if\ [C\ G]\ is\ stable \\ 0 & otherwise \end{cases} \tag{17}$$

where $\kappa(G_1, G_2)$ was defined in (15). Note that $0 \leq b_{GC} \leq 1$.

The following is an important robust stability result based on the Vinnicombe metric between plants.

Proposition 1 [12]. Consider a model $\hat{G}$ and a controller C that stabilizes $\hat{G}$ with a stability margin $b_{\hat{G}C}$. Then C stabilizes all G such that

$$\delta_\nu(\hat{G}, G) < b_{\hat{G}C}. \tag{18}$$

■

The condition (18) of Proposition 1 is rather conservative, since $\delta_\nu(\hat{G}, G) = \max_\omega \kappa(\hat{G}(e^{j\omega}), G(e^{j\omega}))$ while $b_{\hat{G}C} = \min_\omega \kappa(\hat{G}(e^{j\omega}), -\frac{1}{C(e^{j\omega})})$. Thus, it is a min-max type condition. A pointwise (i.e. frequency by frequency), and therefore less conservative condition is as follows.

Proposition 2 [12]. Consider a model $\hat{G}$ and a controller C that stabilizes $\hat{G}$. Then C stabilizes all G such that

$$\kappa\left(\hat{G}(e^{j\omega}), G(e^{j\omega})\right) < \kappa\left(\hat{G}(e^{j\omega}), -\frac{1}{C(e^{j\omega})}\right)\ \forall\ \omega\ \textit{and}\ \delta_\nu(\hat{G}, G) < 1 \tag{19}$$

■

6 The worst case Vinnicombe distance for validated model sets

In the validation context that is of interest to us here, the true system G_0 is unknown, but we have shown that it lies, with probability 0.95 say, in some validated set $\mathcal{D}$. In order to apply the robust stability results of Vinnicombe to our validation results, we introduce the concept of *worst case Vinnicombe distance* between a model $\hat{G}$ and a validated model set $\mathcal{D}$: it corresponds to the largest Vinnicombe distance between the model $\hat{G}$ and any plant inside the set $\mathcal{D}$.

Definition of the worst case Vinnicombe distance: The worst case Vinnicombe distance $\delta_{WC}(\hat{G}, \mathcal{D})$ between a model $\hat{G}$ and a model set $\mathcal{D}$ is defined as

$$\delta_{WC}(\hat{G}, \mathcal{D}) = \max_{G_\mathcal{D} \in \mathcal{D}} \delta_\nu(\hat{G}, G_\mathcal{D}) \tag{20}$$

Another important quantity is now defined: the **worst case chordal distance**. Its computation is the result of a convex optimization problem involving Linear Matrix Inequality (LMI) constraints [1].

Definition of the worst case chordal distance at frequency ω.
At a particular frequency ω, we define $\kappa_{WC}(\hat{G}(e^{j\omega}), \mathcal{D})$ as the maximum chordal distance between the projections on the Riemann sphere of $\hat{G}(e^{j\omega})$ and of the frequency responses of all plants in $\mathcal{D}$ at the same frequency:

$$\kappa_{WC}\left(\hat{G}(e^{j\omega}), \mathcal{D}\right) = \max_{G_\mathcal{D} \in \mathcal{D}} \kappa\left(\hat{G}(e^{j\omega}), G_\mathcal{D}(e^{j\omega})\right) \tag{21}$$

Having extended the distances between plants to worst case distances between a model and a model set, we can now also extend the robust stability results of Vinnicombe to validated model sets.

Theorem 1. Let $\hat{G}$ be a model, C a stabilizing controller for $\hat{G}$ yielding a generalized stability margin $b_{\hat{G}C}$, and $\mathcal{D}$ a validated set of transfer functions containing the true plant G_0. Then C stabilizes all plants in the set $\mathcal{D}$, and hence also G_0, if the following condition holds :

$$\delta_{WC}(\hat{G}, \mathcal{D}) < b_{\hat{G}C}. \tag{22}$$

Proof : It follows immediately from the definitions that for any $G \in \mathcal{D}$, and hence for G_0,

$$\delta_\nu(\hat{G}, G) \leq \delta_{WC}(\hat{G}, \mathcal{D}) < b_{\hat{G}C}$$

and the stability then follows from Proposition 1. ■

Using the pointwise version of the robust stability result of Vinnicombe, we can now state our main stability result for validated model sets.

Theorem 2 (main stability theorem). Let $\hat{G}$ be a model, C a stabilizing controller for $\hat{G}$, and $\mathcal{D}$ a validated set of parametrized transfer functions containing the true plant G_0. Then C stabilizes all plants in the set $\mathcal{D}$, and hence also G_0, if the following condition holds :

$$\kappa_{WC}\left(\hat{G}(e^{j\omega}), \mathcal{D}\right) < \kappa\left(\hat{G}(e^{j\omega}), -\frac{1}{C(e^{j\omega})}\right) \quad \forall\, \omega \in [0, \pi] \tag{23}$$

Proof : It follows from the definition of worst case chordal distance that, at any frequency ω and for any model $G \in \mathcal{D}$, we have

$$\kappa\left(\hat{G}(e^{j\omega}), G(e^{j\omega})\right) \leq \kappa_{WC}\left(\hat{G}(e^{j\omega}), \mathcal{D}\right) < \kappa\left(\hat{G}(e^{j\omega}), -\frac{1}{C(e^{j\omega})}\right).$$

It follows from Proposition 2 that any $G \in \mathcal{D}$ is stabilized by C. ■

We shall illustrate the application of these robust stablity results for validated model sets in Section 8.

Computational aspects

Our two stability theorems are very powerful tools to check the stability of a designed controller C on the system G_0 before it is actually applied to that system. The only requirement is that G_0 be inside the validated region $\mathcal{D}$, which region is itself derived from the covariance matrix of the estimated parameters of the model error model. The results rely heavily on our ability to compute the worst case chordal distance at frequency ω, $\kappa_{WC}(\hat{G}(e^{j\omega}), \mathcal{D})$, between a model $\hat{G}$ and a set $\mathcal{D}$, defined in (21), and/or the worst case Vinnicombe distance $\delta_{WC}(\hat{G}, \mathcal{D})$ between these two objects, defined in (20). This is by no means a trivial matter. The solution to these problems has been obtained using LMI techniques: see [1].

7 Design issues: model validation for control design

We have developed a complete setup from model validation to controller validation, and the computational tools are available to check whether a controller designed from a model stabilizes the true plant, at least at some prespecified probability level. We have also explained the role of the experimental conditions on the shape of the validated sets, and we have developed tools to compute a measure of the size of these validated sets that is directly related to the capability of a controller to stabilize all models in such validated set. These tools can now be used for *validation design.* The following design guidelines can be proposed for the validation of a model that is to be used for control design.

- With some model $\hat{G}$ as the starting point, the validation for control procedure consists of the following steps:
  - Using $\hat{G}$ as the model, perform a validation experiment (see Section 3). This yields a validated set $\mathcal{D}$ containing the true G_0 with probability 0.95%, say. The model $\hat{G}$ may or may not lie in $\mathcal{D}$. Compute the worst case Vinnicombe distance $\delta_{WC}(\hat{G}, \mathcal{D})$ and, possibly also, the worst case chordal distance $\kappa_{WC}(\hat{G}(e^{j\omega}), \mathcal{D})$ at each frequency.
  - Design a controller C and compute its nominal stability margin $b_{\hat{G},C}$ or the chordal distance $\kappa(\hat{G}(e^{j\omega}), -\frac{1}{C(e^{j\omega})})$ at each frequency.
  - Check whether $\delta_{WC}(\hat{G}, \mathcal{D}) < b_{\hat{G},C}$ or, better, whether at each frequency $\kappa_{WC}(\hat{G}(e^{j\omega}), \mathcal{D}) < \kappa(\hat{G}(e^{j\omega}), -\frac{1}{C(e^{j\omega})})$. If so, then C stablizes the true system G_0.
- Given a choice between different experimental conditions for the validation procedure, one should give preference to a validation experiment that yields an uncertainty set $\mathcal{D}$ with the smallest possible worst case Vinnicombe gap.
- The projections of Nyquist plots on the Riemann sphere have maximal resolution around the equator, i.e. where the transfer functions have an amplitude close to one. This has important consequences for "validation for control" design: see [2] for more details on closed loop validation.
- Given a validated set $\mathcal{D}$ and a corresponding worst case gap $\delta_{WC}(\hat{G}, \mathcal{D})$, one can compute a sequence of controllers C_i to drive up the nominal performance of the $(\hat{G}, C_i)$ loop while keeping $b_{\hat{G},C_i} > \delta_{WC}(\hat{G}, \mathcal{D})$. This guarantees stability of the actual closed loop system.

8 A simulation example

Consider the following true system G_0 and model $\hat{G}$, respectively,

$$y = G_0 u + H_0 e = \frac{z^{-1}+0.25z^{-2}}{1-1.4z^{-1}+0.45z^{-2}} u + \frac{1}{1-1.4z^{-1}+0.45z^{-2}} e$$
$$\hat{y} = \hat{G} u = \frac{1.0141z^{-1}+0.2397z^{-2}}{1-1.4237z^{-1}+0.4835z^{-2}} u$$

The actual Vinnicombe distance between $\hat{G}$ and G_0 is $\delta_\nu(\hat{G}, G_0) = 0.0163$. For this model $\hat{G}$, an open-loop and a closed-loop validation were achieved leading to two uncertainty regions $\mathcal{D}_{OL}$ and $\mathcal{D}_{CL}$ correponding to a probability level of 0.95. The controller chosen for closed-loop validation is a proportional controller $C(q) = 1$. The model was validated with 1000 data collected in open-loop and closed-loop, respectively, having the following statistics:

$$\begin{array}{ll} Open-loop: & \sigma_u^2 = 0.2 \;\; and \;\; \sigma_e^2 = 1 \Longrightarrow \sigma_y^2 = 23.4 \\ Closed-loop: & \sigma_r^2 = 10 \;\; and \;\; \sigma_e^2 = 1 \Longrightarrow \sigma_y^2 = 26.9 \end{array}$$

Figure 1 presents the Nyquist plots of G_0, $\hat{G}$ and $\hat{G} + \tilde{G}$, as well as the smallest overbounding ellipsoids of the uncertainty regions $\mathcal{D}_{OL}$ and $\mathcal{D}_{CL}$ at each frequency. Observe that G_0 and $\hat{G}$ lie inside both $\mathcal{D}_{OL}$ and $\mathcal{D}_{CL}$ for all frequencies. Thus, $\hat{G}$ is validated by both experiments here.

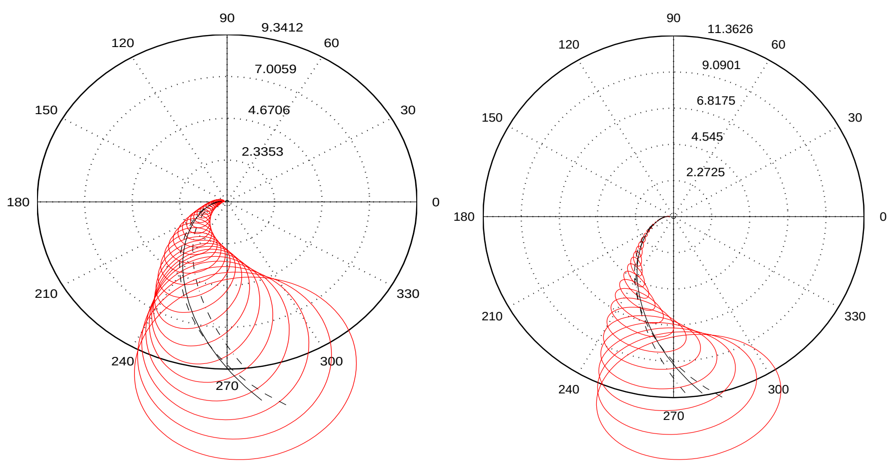

Fig. 1. Nyquist plot of G_0 (solid), $\hat{G}$ (dash), and $\hat{G} + \widetilde{G}$ (dashdot), with ellipsoidal estimates of $\mathcal{D}_{OL}$ and $\mathcal{D}_{CL}$. Left: open loop validation. Right: closed loop validation.

The worst case Vinnicombe distances are:

$$\delta_{WC}(\hat{G}, \mathcal{D}_{OL}) = 0.2604 > \delta_{WC}(\hat{G}, \mathcal{D}_{CL}) = 0.0572 > \delta_\nu(\hat{G}, G_0) = 0.0163.$$

Note that the worst case Vinnicombe distance is much smaller with the closed-loop validated set than with the open-loop set. Thus, the validated set $\mathcal{D}_{CL}$ should allow for less conservative control designs.

We consider a proportional controller $C(q) = 1.5$ which stabilizes the nominal model $\hat{G}$, yielding a nominal stability margin $b_{\hat{G}C} = 0.0461$. This controller also stabilizes G_0, but in practice G_0 is unknown and the stabilization of G_0 by the controller C can only be ascertained by the use of one of the stability theorems of Section 6. We first check whether the Min-Max type condition of Theorem 1 is verified. We have:

$$b_{\hat{G}C} = 0.0461 < \overbrace{\delta_{WC}(\hat{G}, \mathcal{D}_{CL})}^{=0.0572} < \overbrace{\delta_{WC}(\hat{G}, \mathcal{D}_{OL})}^{=0.2604}$$

Thus, the robust stability condition of Theorem 1 is violated with both of the validated regions. We now check the less conservative condition of Theorem 2. Figure 2 compares the worst case chordal distances (for $\mathcal{D}_{OL}$ and $\mathcal{D}_{CL}$) and the pointwise stability margin $\kappa(\hat{G}(e^{j\omega}), -\frac{1}{C(e^{j\omega})})$.

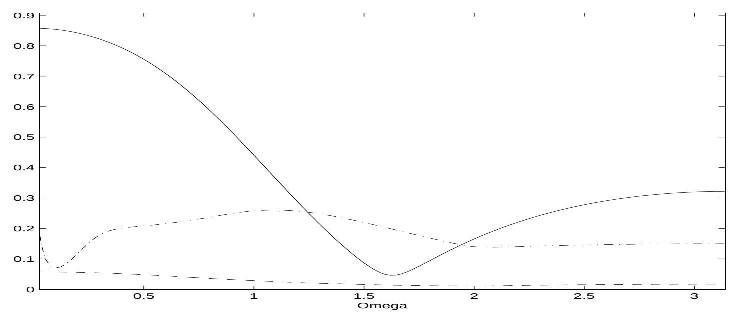

Fig. 2. Frequency by frequency comparison of $\kappa_{WC}(\hat{G}(e^{j\omega}), \mathcal{D}_{OL})$ (dashdot), $\kappa_{WC}(\hat{G}(e^{j\omega}), \mathcal{D}_{CL})$ (dash) and $\kappa(\hat{G}(e^{j\omega}), -\frac{1}{C(e^{j\omega})})$ (solid)

It shows that, even with this less conservative condition, the stability condition of Theorem 2 is violated when the set $\mathcal{D}_{OL}$ is used. However, the stabilization of the true G_0 is guaranteed by the stability condition (23) when the set $\mathcal{D}_{CL}$ is used.

9 Conclusions

We have displayed the role of experimental conditions in the validation of a model. We have then developed tools that allow one to connect some measures of the "size" of a validated model set to the stability margin of a controller designed from the nominal model. This has then led us to propose validation design guidelines, when the validation is performed for the purpose of designing a robust controller.

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