The Role of Transitivity in Devaney's Definition of Chaos

Annalisa Crannell

1. INTRODUCTION. Devaney's definition of chaos for discrete dynamical systems is one of the most popular and most widely known. It says a function f: $M \to M$ is *chaotic* if

(1) f is *transitive*—that is, for any pair of non-empty open sets U and V in M, there is some $k > 0$ with $f^k(U) \cap V \neq \varnothing$;
(2) the periodic points of f are dense in M; and
(3) f displays the famous condition, *sensitive dependence on initial conditions*: there is a number $\delta > 0$ depending only on M and f, so that in every non-empty open subset of M one can find a pair of points whose eventual iterates under f are separated by a distance of at least δ.

Here M is generally a subset of $\mathbf{R}^n$, and f^n means f composed with itself n times —so that, for example, $f^3(x) = f(f(f(x)))$.

One of the ironies of this definition is that, the more popularly understood each hypothesis is, the more redundant it is in relationship to the other two.

For example, sensitive dependence is a condition which is easily understood by mathematicians and non-mathematicians alike. It has been even dubbed "the butterfly effect" in examples of popular literature such as *Jurassic Park* [3], and *The Mathematical Tourist* [7]; the phrase probably dates back to the Ray Bradbury story "A Sound of Thunder", in which a time-traveller changes the course of history by stepping on a prehistoric butterfly [2]. This condition embodies the essence of chaos—the utter unpredictability of what ought to be simple systems—and so there is something popularly pleasing about requiring sensitive dependence on initial conditions.

However, an elegant paper by Banks, Brooks, Cairns, Davis, and Stacey [1] demonstrated that sensitive dependence is assured whenever the function displays transitivity and dense periodic points. That is, despite its popular appeal, sensitive dependence is mathematically redundant—so that in fact, chaos is a property relying only on the topological, and not on the metric, properties of a space.

The requirement that periodic points be dense is slightly less intuitive than requiring sensitive dependence, but it appeals to those who look for patterns within a seemingly random system. Mathematicians in particular instinctively seek symmetry, and the wealth of periodicities within a chaotic system is a wonderful

I am grateful to Bob Gethner, who got me interested in the questions that are asked in this paper, and to Sasha Blokh, who gave me useful and timely advice on how to pursue the answers. In addition I would like to thank Michal Misiurewicz and the reviewer for their helpful suggestions.

mathematical phenomenon. It even allows us to explain, somewhat mystically, that "there is order within chaos". Accordingly, the search for periodic points in an understandable one.

On the other hand, Vellekoop and Berglund [8] recently gave a simple proof of an already-known theorem which says that, on any finite or infinite interval in the Real line, dense periodic points (and hence chaos) follows directly from the condition of transitivity. Moreover, they gave examples which demonstrated that neither dense periodic points nor sensitive dependence is enough to ensure any of the other conditions leading to chaos. Therefore, in one dimension both sensitive dependence and dense periodic points are redundant hypotheses in the definition of chaos.

This leaves us only the study of the transitivity hypothesis, which is required both for historical reasons and for the strength of the condition. Still, it has less intuitive justification—it is harder to explain in nonmathematical terms, and even once it is explained, it seems to follow (morally, although not mathematically) from the sensitivity hypothesis, as both of these hypotheses say that, starting with just about any data, one could eventually get just about any answer. The purpose of this paper is to ask, "why transitivity?—why not something else?" and to provide some conditions which might play the same role as transitivity, but which are slightly more intuitive.

2. A POSSIBLE ALTERNATIVE TO TRANSITIVITY. Perhaps, instead of transitivity, a more philosophically satisfying hypothesis might be one of the following:

Definition. A function $f\colon M \to M$ is *weakly blending* if, for any pair of non-empty open sets U and V in M, there is some $k > 0$ so that $f^k(U) \cap f^k(V) \neq \varnothing$. We say f is *strongly blending* if, for any pair of non-empty open sets U and V in M, there is some $k > 0$ so that $f^k(U) \cap f^k(V)$ contains a non-empty, open subset.

These conditions initially struck the author as an intuitive counterpart to sensitive dependence: *sensitive dependence on initial conditions* thrusts nearby points apart (for the same iterate of f), and *blending* pulls far away points together (again, for the same iterate of f)!

Blending has certain obvious disadvantages when compared with transitivity. First and foremost, any function which is blending can not be a homeomorphism, which automatically excludes the study of many interesting multi-dimensional chaotic systems—such as the horeshoe map [4, pp. 180–189]. Moreover, even in low dimensions, functions which are blending are not necessarily transitive, and transitive functions are not necessarily blending. Consider the following two examples:

Example 1. $f\colon S^1 \to S^1$, given by $f(\theta) = \theta + k$, where k/π is irrational. This function is rigid, irrational rotation; it is transitive but not strongly or weakly blending.

Example 2. Any continuous piecewise linear function $f\colon [-1,1] \to [-1,1]$ satisfying:

- $|f'(x)| > 2$ on except at the vertices of f; and
- each vertex of the graph of the function likes alternately on the line $y = \pi/2$ and $y = -\pi/2$ (see the figure below).

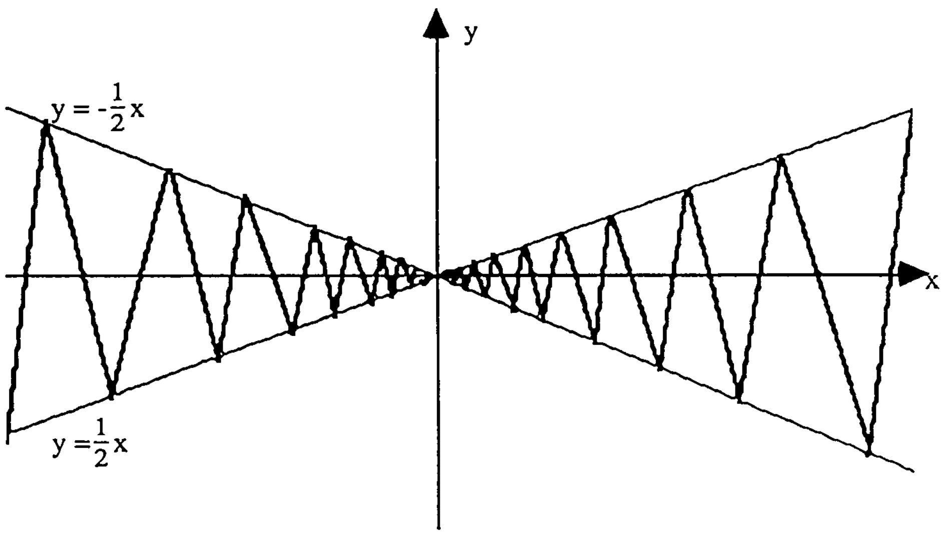

Figure 1. Graph of Example. 2

This function is clearly not transitive; in fact each set is mapped closer to the origin than it had been. At the same time, the large slope of f ensures that if neither I nor $f(I)$ contains $\{0\}$, then $f(I)$ is longer than I. This ensures that every interval is mapped, in a finite number of iterations, to an interval which contains a neighborhood of the origin—the only fixed point. Therefore, the function is strongly blending.

However, a common characteristic of these two examples is that neither has dense periodic points—in fact, the first example has no periodic points at all, and the second example has a lone fixed point. If we include dense periodic points, then the ideas of transitivity and blending in our everyday one-dimensional experience have quite a strong overlap, especially when one is considering chaos. This can be seen in theorems 1 and 2, which are the main theorems of the paper. They show that if periodic points are dense and there's a strongly repelling fixed point, then strong blending $\Rightarrow$ transitivity $\Rightarrow$ weak blending.

3 THE MAIN THEOREMS OF THIS PAPER. One-dimensional dynamical systems are well-understood nowadays, and so there is a wealth of theory on the subject. However, the following theorems will be proved with more simple tools: The link between open sets and continuous functions; the incredible strength of the compactness condition, and induction arguments. These simple proofs are possible because the conditions of transitivity and blending are both topological; the proofs in this section contain many of the ideas that one finds in a Point-Set Topology or an introductory Real Analysis course.

The easier of the two theorems to prove is:

Theorem 1. *Let M be a subjet of $\mathbf{R}^n$, and f: $M \to M$ a continuous function with dense periodic points. Then if f is strongly blending, f is also transitive.*

Proof of Theorem 1. We assume that f is blending and that periodic points of f are dense. Pick two non-empty open sets, U and V. Because of the blending property we know that there is some number $k > 0$ and some non-empty open set $N \subset M$ so that $N \subset f^k(U) \cap f^k(V)$.

For the sake of convenience of notation, we'll let $\tilde{V} = f^{-k}(N) \cap V$; $\tilde{V}$ is the set of points in V which "blend" with those in U.

By the continuity of f, $\tilde{V}$ is open, so our hypotheses allow us to pick a periodic point $x \in \tilde{V}$; let us say that x is of period $p > k$ (it may be that p is a multiple of the prime period of x).

Because of the way we chose $x \in \tilde{V}$, we know that $f^k(x) \in N$, and so there is some $y \in U$ with $f^k(y) = f^k(x)$. From this, the simple computation

$$f^p(y) = f^{(p-k)}(f^k(y)) = f^{(p-k)}(f^k(x)) = f^p(x) = x$$

ensures that $x \in f^p(U) \cap V \neq \emptyset$. ∎

Remark. The assumption we make that N be open is a crucial one, and the theorem does not hold without it. For a counter example, consider the function

$$T(x) = \begin{cases} -(2x-2) & \text{for } -1 \le x \le -\frac{1}{2} \\ 2x & \text{for } |x| < \frac{1}{2} \\ 2-2x & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

defined on the interval $[-1, 1]$. This function is an odd extension of the tent map —its restriction to the interval $[0, 1]$ is well known to be transitive (see for example [5]). Accordingly, T has dense periodic points, and in fact every open interval in the domain eventually maps onto an interval which contains the fixed point at the origin, so that it is weakly blending. However, this function over the entire interval $[-1, 1]$ is not transitive: the interval $(0, 1)$ will never map onto any subinterval of $(-1, 0)$.

Can one hope that the converse is also true: that chaos inevitably blends all sets together (strongly)? The answer is no, unfortunately, as one can see from the following.

Example 3. We can flip the above function and get $F(x) = -T(x)$ on the interval $[-1, 1]$. This is a lovely example of a chaotic function with periodic orbits of all even periods, but no odd periods. (In fact, if x_0 is a periodic point of T with period n, than x_0 is a periodic point of F with period $2n$-so periodic points are dense.) Examining a few iterates of this function will convince the reader that F is, moreover, transitive. On the other hand, if U is an interval to the left of the origin, and V is an interval to the right of the origin, no matter which iterate we examine we will have $F^k(U) \cap F^k(V) = \emptyset$ or $\{0\}$. Therefore, F is only weakly blending.

However, a weaker converse is true:

Theorem 2. *Let I be a compact subset of* $\mathbf{R}$, *and f: $I \to I$ a continuous, transitive function with a repelling fixed point x_0. Then f is weakly blending.*

To prove this theorem, we will use two lemmas:

Lemma 1. *If f and x_0 are as given above, then x_0 has infinitely many eventual pre-images in I.*

Lemma 2. *If f and x_0 are as given above, then the eventual pre-images of x_0 are dense in I.*

In fact, a much stronger version of these lemmas was proved three decades ago in [6]: if f is a piecewise-monotone function, then the set

$$\{y \in I | f^k(y) = x \text{ for some } k\}$$

is dense in I $\forall x \in I$. As this paper needs only the weaker lemmas (with the weaker hypotheses), we will restrict our proofs accordingly.

Proof of Theorem 2. We wish to show that for any two open sets $U, V \subset I$, there is some $n > 0$ with $f^n(U) \cap f^N(V) \neq \emptyset$. Lemma 2 tells us that the eventual preimages of x_0 are dense, and so there exist $u \in U$, $v \in V$, and $j, k > 0$ so that $f^j(u) = x_0 = f^k(v)$. Assume without loss of generality that $k > j$; then because x_0 is fixed, we have $f^k(u) = x_0 = f^k(v)$. Thus, $x_0 \in f^k(U) \cap f^k(V) \neq \emptyset$, and our theorem is proved. ∎

Proof of Lemma 1. We will prove this lemma by induction.

Suppose x_0 is our given repelling fixed point, and we are given a finite set $X_x = \{x_{-n}, \dots, x_{-1}, x_0\}$ with $f(x_k) = x_{k+1}, k = -n, \dots, -1$. If $n = 0$, then we have $X_0 = \{x_0\}$.

Choose an open set $U \subset I$ with $X_n \subset U$ satisfying.

(1) if $y \in U$ then $f(y) \neq x_{-n}$ (unless $n = 0$ and $y = x_0$); and

(2) $f(U \setminus B_\varepsilon) \cap B_\varepsilon = \emptyset$.

(Here B_ε is assumed to be the ball of radius ε centered at x_{-n}.) In the case $n = 0$, we use the fact that x_0 is repelling to satisfy the second of these two assumptions.

From here, we will use transitivity to show that f must send the exterior of the set U arbitrarily close to x_{-n}: that is, for every $\varepsilon > 0$, $f(u^c) \cap B_\varepsilon \neq \emptyset$.

We can choose U sufficiently small that U^c contains an open set. By the transitivity of f, we know that $f^{k+1}(U^c) \cap B_\varepsilon \neq \emptyset$ for some $k \geq 0$; we're trying to show that $k = 0$.

Let Y be the set of points which start in the compliment of U and which are first mapped into B_ε on the $k + 1^{st}$ iteration. That is,

$$Y = \{y \in U^c | f^{k+1}(y) \in B_\varepsilon: \quad f^j(y) \notin B_\varepsilon \text{ if } 1 \leq j \leq k\}.$$

Then clearly $f^k(Y) \cap B_\varepsilon = \emptyset$. Moreover, we must have $f^k(y) \in U^c$ if $y \in Y$, for if it were otherwise, assumption (2) would give us

$$f^{k+1}(y) = f(f^k(y)) \in f(U \setminus B_\varepsilon) \subset (B_\varepsilon)^c$$

for some $y \in T$. This contradicts the definition of Y. Therefore, we see that $f^k(Y) \subset U^c$ and that $f(f^k(Y)) \cap B_\varepsilon \neq \emptyset$—so $f^k(Y)$ is the subset of U^c which proves our claim.

The rest of the proof of Lemma 1 follows easily, for the claim holds regardless of the size of ε, and therefore the compactness of U^c tells us that there is some point y in U^c with $f(y) = x_{-n}$.

This argument gives us an infinite sequence $\{x_{-k}\}_{k=0}^{\infty}$ with $f^k(x_{-k}) = x_0$, and so completes the proof of Lemma 1. ∎

Proof of Lemma 2. Let $X = \{y \in I | f^k(y) = x_0 \text{ for some } k\}$. We want to show that X is dense in I. Because f is transitive, it follows that if X is anywhere dense, then X must be everywhere dense. Let's assume that opposite: that X is totally disconnected.

If such is the case, the X^c must be open, so we can write $X^c = \bigcup_{k=1}^{\infty} I_k$, where the I_k's are distinct, open intervals in I. Lemma 1 notes that X is infinite, so, in fact, there must be an infinite number of such intervals.

Note, moreover, that at least one interval has x_0 as an endpoint; call this interval I_1. Because x_0 is fixed, we have $f^2(I_1) \cap I_1 \neq \varnothing$—in fact, because of the construction of the I_k's, we have $f^2(I_1) \subseteq I_1$.

On the other hand, transitivity prohibits exactly such a cycle, for I_1 must visit each of the infinite number of intervals—a contradiction. This contradiction arose from assuming that X is not dense in I, and so our final lemma is proved. ■

REFERENCES

1. J. Banks, J. Brooks, G. Gairns, G. David, and R. Stacey, *On Devaney's definition of chaos*, American Mathematical Monthly **99** (1992), 332–334.
2. Ray Bradbury, *A Sound of Thunder*, Golden Apples of the Sun, Doubleday & Company, Inc., 1953.
3. Michael Crichton, *Jurassic Park: a novel*, Knopf, New York, 1990.
4. Bob Devaney, *An Introduction to Chaotic Dynamical Systems*, second edition, Addisson-Wesley, 1989.
5. Steven N. MacEachern and L. Mark Berliner, *Aperiodic chaotic orbits*, American Mathematical Monthly **100** (1993), 237–241.
6. William Parry, *Symbolic dynamics and transformations of the unit interval*, Transactions of the American Mathematical Society **122** (1966), 368–378.
7. Ivars Peterson, *The mathematical tourist: snapshots of modern mathematics*, Freeman, New York, 1988.
8. M. Vellekoop and R. Berglund, *On Intervals, Transitivity = Chaos*, American Mathematical Monthly **101** (1994), 353–355.

Department of Mathematics
Franklin & Marshall College
Lancaster, PA 17604
A_Crannell@ACAD.FandM.edu

This paper gives wrong solutions to trivial problems. The basic error, however, is not new.

—Clifford Truesdell

Mathematical Reviews 12, p. 561.