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The Rooted Tree Embedding Problem into Points in the Plane

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Abstract. In this paper we show that any rooted tree of n vertices can be straight-line embedded into any set S of n points in the plane in general position so that the image of the root is arbitrarily specified.

1. Introduction

Let T be a rooted tree with n vertices and let N be a set $\{p_1, p_2, \ldots, p_n\}$ of n points in general position in R^2, i.e., no three points lie on any line. We denote the root of T by r_1, and the sets of vertices and edges of T by $V(T)$ and $E(T)$, respectively. We consider a bijection φ from $V(T)$ to N and define the image of each edge $uv \in E(T)$ with respect to φ by the line segment $\overline{\varphi(u)\varphi(v)}$. Perles [3] posed the following problem.

Problem. Is there a bijection φ from $V(T)$ to N satisfying the following two conditions?

(C1) $\varphi(r_1) = p_1$.
(C2) For any two nonadjacent edges $u_1v_1, u_2v_2 \in E(T)$, line segments $\overline{\varphi(u_1)\varphi(v_1)}$ and $\overline{\varphi(u_2)\varphi(v_2)}$ are disjoint.

Here we call such a bijection a *rooted tree embedding* (or an *rt-embedding*) of T on N.

It has been proved by Pach and Töröcsik [2] that there is an rt-embedding when the number of the minimum points of N in a closed half-plane containing p_1 satisfies some specific conditions. In this paper we prove the existence of rt-embeddings by giving an algorithm that constructs one, even when p_1 does not satisfy these conditions. Furthermore, our algorithm constructs an rt-embedding in polynomial time with respect to n.

2. Preliminaries

Let N be an n-set $\{p_1, \ldots, p_n\}$ in general position in R^2. We say that a line l containing p_1 is an *(a, b)-separator* of $N - \{p_1\}$ if l does not contain any other point of N, and splits $N - \{p_1\}$ into a points and b points. We also say that l is an *$[a, b)$-separator* of $N - \{p_1\}$ if l contains p_1 and one other point p, and splits $N - \{p_1, p\}$ into $(a - 1)$ points and b points. Now suppose that l is a line containing p_1, such that one open half-plane determined by l contains as many points of N as possible. Let t be the number of points of N in the open half-plane and let $s = (n - 1) - t$. We remark that s corresponds exactly to what is called "the depth of p_1" in [2]. Then the two open half-planes determined by any line containing p_1 contain at least s points of $N - \{p_1\}$. We use the notations s and t to express these minimum and maximum numbers through this paper. The next lemmas follow from the fact that no three points of N lie on any line. Proofs are omitted.

Lemma 2.1. *The following statements hold.*

(1) *For any j with $s \leq j \leq t$, there is a $(j, n - 1 - j)$-separator of $N - \{p_1\}$.*
(2) *For any j with $s < j \leq t$, there is a $[j, n - 1 - j)$-separator of $N - \{p_1\}$.*

Note that $s < j$ in (2) is the strict inequality.

Lemma 2.2. $s + 1 \leq t$.

Let T be a tree rooted at r_1. For each edge $uv \in E(T)$, u is called the *parent* of v and v a *child* of u if u is closer than v to r_1. For any vertex $u \in V(T)$, let $D(u)$ be the set consisting of u and u's descendants. We denote the number of u's children as $ch(u)$. Let $\{v_1, v_2, \ldots, v_{ch(u)}\}$ be the set of u's children, and let $T^i(u)$ denote the subtree of T induced by $D(v_i)$ for $i = 1, \ldots, ch(u)$. We say that v_i is the root of $T^i(u)$, and assume that the order of u's children is specified to satisfy the following condition:

$$|T^1(u)| \geq |T^2(u)|, \ldots, |T^{ch(u)}(u)|, \tag{2.1}$$

where $|T^i(u)|$ denotes the number of vertices of $T^i(u)$. For convenience, we define $T^0(u)$ as the subtree of T induced by $V(T) - D(u)$.

We call r_1 the *first master*, and recursively define the *jth master* as the first child of the $(j-1)$th master when $j \geq 2$. The sequence $\{|T^1(r_j)|: j = 1, 2, \ldots\}$ is a strictly monotone decreasing sequence. Thus let r_k be the master such that $|T^1(r_k)| < t$ and $|T^1(r_{k-1})| \geq t$ if $k \geq 2$. The following lemma holds for master r_k.

Lemma 2.3. $|T^0(r_k)| \leq s + 1$.

Proof. If $k = 1$, then $|T^0(r_1)| = 0 < 1 \leq s + 1$. Assume that $k \geq 2$. Since $V(T)$ is partitioned into $V(T^0(r_k))$ and $V(T^1(r_{k-1}))$ and $|T^1(r_{k-1})| \geq t$, $|T^0(r_k)| = n - |T^1(r_{k-1})| \leq (s + t + 1) - t = s + 1$. □

3. Proof of the Existence of rt-Embeddings

In this section we prove the existence of rt-embeddings. We first deal with the case where p_1 is an extreme point of the convex hull conv(N) of N. Lemmas 3.1 and 3.2 have been discovered independently by Pach and Töröcsik [2]. Here we describe their algorithms and omit proofs.

Lemma 3.1 [2]. *By using the following algorithm we can find an rt-embedding of T on N when p_1 is an extreme point of* conv(N).

Algorithm 1 (see Fig. 1)

Step 1. Let p_2 be an extreme point of conv(N) adjacent to p_1, and create the total order τ with respect to the angle $\angle p_2 p_1 p$ for $p \in N - \{p_1\}$.
Step 2. According to the total order τ partition $N - \{p_1\}$ into $ch(r_1)$ subsets $N_1, \ldots, N_{ch(r_1)}$ with $|N_i| = |T^i(r_1)|$, and let p_i' be the first point in N_i.
Step 3. Construct recursively an rt-embedding of each subtree $T^i(r_1)$ onto N_i, such that the image of the ith child of r_1 is p_i'.

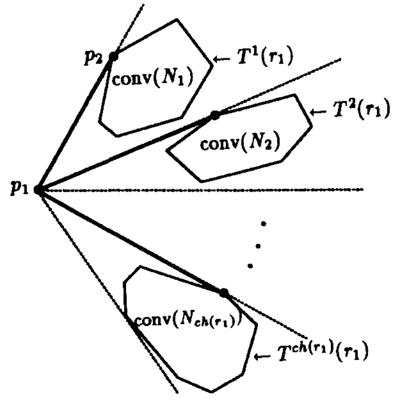

Fig. 1. Algorithm 1.

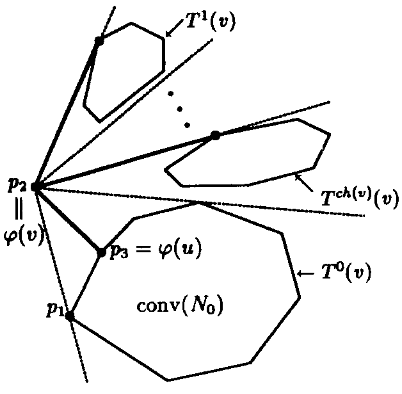

Fig. 2. Algorithm 2.

Lemma 3.2 [2]. *Suppose that* p_1 *is an extreme point of* $\mathrm{conv}(N)$ *and* $n \geq 2$. *Let* p_2 *be an extreme point adjacent to* p_1. *Then the following algorithm constructs an rt-embedding* φ *of* T *on* N *with* $\varphi(v) = p_2$ *for a specified vertex* $v \in V(T) - \{r_1\}$.

Algorithm 2 (see Fig. 2)

Step 1. If v is a child of r_1, then use Algorithm 1 to find an rt-embedding of T on N, otherwise create the total order τ with respect to the angle $\angle p_1 p_2 p$ for $p \in N - \{p_2\}$.

Step 2. Let N_0 be the first $|T_0(v)|$ points of N with respect to τ; $N_1 = N - N_0$; and let p_3 be an extreme point of $\mathrm{conv}(N_0)$ adjacent to p_1 and visible from p_2.

Step 3. Construct recursively an rt-embedding of $T^0(v)$ on N_0 such that v's parent u is mapped to p_3; use Algorithm 1 to construct an rt-embedding of $T(v)$ on N_1 such that v is mapped to p_2

In the rest of the paper we ignore trivial cases when $n \leq 2$. Recall that r_k is the master such that

$$[|T^1(r_k)| < t] \quad \text{and} \quad [|T^1(r_{k-1})| \geq t \text{ if } k \geq 2]. \tag{3.1}$$

Now we consider the following three cases for the master r_k.

Case 1: $k = 1$ (in other cases we assume $k \geq 2$).

Case 2: $h \in \{1, \ldots, ch(r_k)\}$ *exists with* $s \leq |T^1(r_k)| + \cdots + |T^h(r_k)| < t$.

Case 3: otherwise.

Note that Case 1 contains the case when p_1 is an extreme point of $\mathrm{conv}(N)$. Now suppose that $k \geq 2$. From the definitions of s, t, and $T^i(r_k)$, $\sum_{i=0}^{ch(r_k)} |T^i(r_k)| + 1 = n = s + t + 1$ and $|T^0(r_k)| \leq s + 1$ (Lemma 2.3). If $s \leq |T^1(r_k)|$, then we have Case 2. On the other hand if $|T^0(r_k)| = s + 1$, then $s \leq |T^1(r_k)| + \cdots + |T^{ch(r_k)}(r_k)| = t - 1$ (the inequality follows from Lemma 2.2), and again we have Case 2. Thus, from Lemma 2.3 and assumptions (2.1) and (3.1), the following conditions hold in Case 3:

$$|T^0(r_k)| \leq s \quad \text{and} \quad |T^i(r_k)| < s \qquad \text{for} \quad i = 1, 2, \ldots, ch(r_k). \tag{3.2}$$

We first prove that there is an rt-embedding in Cases 1 and 2. We add here that Pach and Töröcsik, by using a different partitioning of T, have given a similar proof for what is essentially Case 2 and part of Case 1 in [2].

Lemma 3.3. *In Case 1 there is an rt-embedding of T on N.*

Proof. We first consider the case when $s \leq |T^1(r_1)|$. From (1) of Lemma 2.1, we can use a clockwise ordering around p_1 to partition $N - N_1$ so that it satisfies

$$\text{conv}(N_i \cup \{p_1\}) \cap \text{conv}(N_j \cup \{p_1\}) = \{p_1\} \qquad \text{if} \quad i \neq j \tag{3.3}$$

(see Fig. 3). Since p_1 is an extreme point of the convex hull of $N_i \cup p_1$ for each i, we can use Algorithm 1 to construct an rt-embedding of T on N.

If $|T^1(r_1)| < s$, then $|T^i(r_1)| < s$ for all i from assumption (2.1). Let p_2 be any point of $N - \{p_1\}$. In the same manner as above, we partition N into $\{N_1, \ldots, N_{ch(r_1)}\}$ with $|N_i| = |T^i(r_1)|$ for $i = 1, \ldots, ch(r_1)$ according to a clockwise ordering on $N - \{p_1\}$ around p_1 beginning from p_2 (see Fig. 4). Since $|T^i(r_1)| < s$ for each i, the partition satisfies (3.3). Thus, we can construct an rt-embedding of T on N. □

Lemma 3.4. *In Case 2 there is an rt-embedding of T on N.*

Proof. Let T^1 and T^0 denote the subtrees of T induced by

$$V(T^1(r_k)) \cup \cdots \cup V(T^h(r_k)) \cup \{r_k\}$$

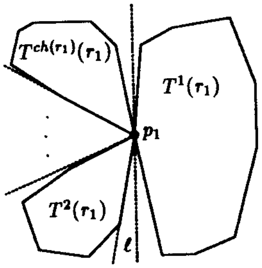

Fig. 3. Case 1.

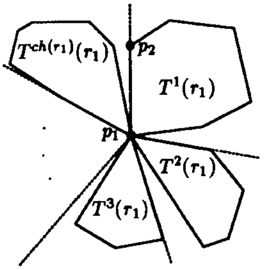

Fig. 4. Case 1.

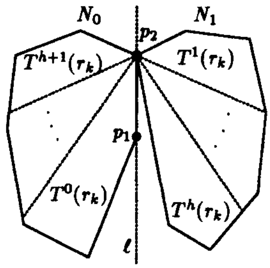

Fig. 5. Case 2.

and $V(T^0(r_k)) \cup V(T^{h+1}(r_k)) \cup \cdots \cup V(T^{ch(r_k)}(r_k)) \cup \{r_k\}$, respectively. Then $E(T)$ is partitioned into $E(T^1)$ and $E(T^0)$. From (2) of Lemma 2.1, there is a $[|T^1|, n-1-|T^1|)$-separator l of $N - \{p_1\}$ because $s < |T^1| \le t$. Suppose that p_2 is the point distinct from p_1 on l. Let N_1 be the subset of N consisting of p_2 and the $(|T^1| - 1)$ points in an open half-plane determined by l, and let N_1 be the points of N in the opposite closed half-plane (see Fig. 5). Then $|N_0| = |T^0|$, $|N_1| = |T^1|$, and $\text{conv}(N_0) \cap \text{conv}(N_1) = \{p_2\}$. Since p_2 is an extreme point of $\text{conv}(N_0)$ adjacent to p_1, we may construct an rt-embedding φ_0 of T^0 on N_0 with $\varphi_0(r_k) = p_2$ by using Algorithm 2. On the other hand, by using Algorithm 1 we may construct an rt-embedding φ_1 of T^1 on N_1 with $\varphi_1(r_k) = p_2$ because p_2 is an extreme point of $\text{conv}(N_1)$. Then the bijection φ from $V(T)$ to N defined by φ_0 and φ_1 is an rt-embedding of T on N. □

Before discussing the proof for Case 3, we give some definitions and a lemma. In Case 3 the inequality $t \le |T^1(r_k)| + \cdots + |T^{ch(r_k)}(r_k)|$ holds; from (3.1) the right-hand side is greater than or equal to s and if it is less than t, then we would have Case 2. Therefore, there is an $h \in \{2, \ldots, ch(r_k)\}$ such that

$$|T^1(r_k)| + \cdots + |T^{h-1}(r_k)| < s < t \le |T^1(r_k)| + \cdots + |T^h(r_k)|. \tag{3.4}$$

Let T^0 and T^1 be the subtrees of T induced by $V(T^0)$ and $V(T^1)$ defined as

$$V(T^0) = \{r_k\} \cup V(T^{h+1}(r_k)) \cup \cdots \cup V(T^{ch(r_k)}(r_k)) \cup V(T^0(r_k)),$$

$$V(T^1) = \{r_k\} \cup V(T^1(r_k)) \cup \cdots \cup V(T^{h-1}(r_k)).$$

From the above definitions, $E(T)$ is partitioned into $E(T^0)$, $E(T^1)$, $E(T^h(r_k))$, and the edge $r_k r$, where r is the root of $T^h(r_k)$ (see Fig. 6). We obtain the next lemma.

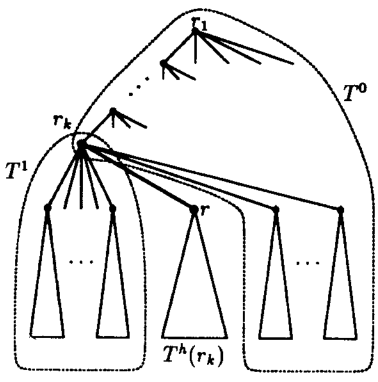

Fig. 6. Decomposition of tree.

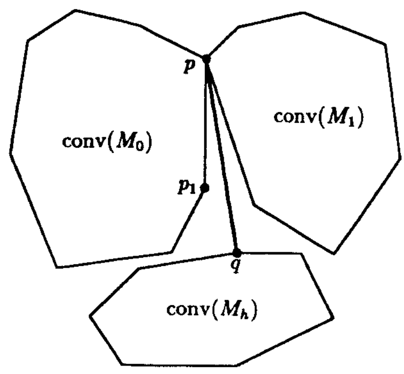

Fig. 7. Decomposition of points.

Lemma 3.5. *If there are subsets M_0, M_1, M_h of N and points p, $q \in N$ such that*

$$|M_0| = |T^0|, \qquad |M_1| = |T^1|, \qquad |M_h| = |T^h(r_k)|, \tag{3.5}$$

$$\mathrm{conv}(M_0) \cap \mathrm{conv}(M_1) = \{p\}, \tag{3.6}$$

$$\mathrm{conv}(M_0) \cap \mathrm{conv}(M_h) = \varnothing, \tag{3.7}$$

$$\mathrm{conv}(M_1) \cap \mathrm{conv}(M_h) = \varnothing, \tag{3.8}$$

$$\mathrm{conv}(M_0) \cap \overline{pq} = \{p\}, \tag{3.9}$$

$$\mathrm{conv}(M_1) \cap \overline{pq} = \{p\}, \tag{3.10}$$

$$\mathrm{conv}(M_h) \cap \overline{pq} = \{q\}, \tag{3.11}$$

$$p \text{ and } p_1 \text{ are extreme points of } \mathrm{conv}(M_0) \text{ adjacent to each other}, \tag{3.12}$$

then there is an rt-embedding of T on N (see Fig. 7).

Proof. We may argue in the following way from the fact that N is in general position. From assumption (3.6), p is an extreme point of conv(M_0), also of conv(M_1). Then there are rt-embeddings φ_0 of T^0 on M_0 and φ_1 of T^1 on M_1 with $\varphi_0(r_k) = \varphi_1(r_k) = p$, by Lemmas 3.1 and 3.2 and by assumptions (3.5) and (3.12). Since q is an extreme point of conv(M_h) from (3.11), an rt-embedding φ_h of $T^h(r_k)$ on M_h exists with $\varphi_h(r) = q$ where r is the root of $T^h(r_k)$, by Lemma 3.1 and assumption (3.5). Then the bijection from $V(T)$ to N defined by φ_0, φ_1, and φ_h is an rt-embedding of T on N by assumptions (3.6)–(3.11). □

In order to prove the existence of rt-embeddings, it is enough to show that N can be distributed so that all the conditions in the above lemma hold.

We select any point p_2 from $N - \{p_1\}$ so that the line $l(p_1p_2)$ passing through p_1 and p_2 is an $[s+1, t-1)$-separator. Without loss of generality, assume that p_2 is directly above p_1 and that the right-hand side of the line belongs to s points of N. Let τ be a clockwise ordering on $N - \{p_1\}$ around p_1 beginning from p_2. According to this total order τ, we can partition $N - \{p_1, p_2\}$ into three subsets $\{N_1, N_h, N_0\}$ such that $|N_1| = |T^1| - 1$, $|N_h| = |T^h(r_k)|$, and $|N_0| = |T^0| - 2$ because $|T^1| + |T^h(r_k)| + |T^0| = n + 1$. Let $M_0 = N_0 \cup \{p_1, p_2\}$, $M_1 = N_1 \cup \{p_2\}$, and $M_h = N_h$. Then (3.5) holds for M_0, M_1, and M_h. We write the first and last points of N_h with respect to τ as p_3 and p_4, respectively. From (3.2) and (3.4), angles $\angle p_2p_1p_3$, $\angle p_2p_1p_4$, and $\angle p_3p_1p_4$ are less than π. More precisely, the following relations hold:

$$\left.\begin{array}{l} p_2 \in (\qquad\qquad\qquad l^+(p_1p_3; p_2) \cap l^+(p_1p_4; p_2)), \\ p_3 \in (l^-(p_1p_2; p_4) \qquad\qquad\qquad \cap\, l^-(p_1p_4; p_2)), \\ p_4 \in (l^+(p_1p_2; p_4) \cap l^-(p_1p_3; p_2) \qquad\qquad\qquad), \\ M_0 \subset (\bar{l}^+(p_1p_2; p_4) \qquad\qquad\qquad \cap\, \bar{l}^+(p_1p_4; p_2)) - \{p_4\}, \\ M_1 \subset (\bar{l}^-(p_1p_2; p_4) \cap \bar{l}^+(p_1p_3; p_2) \qquad\qquad\qquad) - \{p_1, p_3\}, \\ M_h \subset (\qquad\qquad\qquad \bar{l}^-(p_1p_3; p_2) \cap \bar{l}^-(p_1p_4; p_2)) - \{p_1\}. \end{array}\right\} \tag{3.13}$$

Here, for distinct points p, q, $r \in N$, we write the closed (or open) half-plane determined by the line $l(pq)$ including r as $\bar{l}^+(pq; r)$ (or $l^+(pq; r)$) and the opposite closed (or open) half-plane as $\bar{l}^-(pq; r)$ (or $l^-(pq; r)$). From (3.13), it follows that conditions (3.6)–(3.9) and condition (3.12) hold by setting $p = p_2$ and $q = p_3$. The definition of p_3 implies (3.11). Thus the sets M_0, M_1, M_h and the points p_2, p_3 satisfy all the conditions in Lemma 3.5 other than (3.10). When these do not satisfy (3.10), they will be improved to do so.

Let q_2 be the extreme point of $\text{conv}(M_1)$ which is next to p_2 with respect to the counterclockwise ordering on the extreme points of $\text{conv}(M_1)$. We consider three subcases of Case 3.

Case 3.1: $p_3 \in l^+(p_2q_2; p_1)$ (Fig. 8).

Case 3.2: $p_3 \in l^-(p_2q_2; p_1)$ and $q_2 \in l^-(p_1p_4; p_2)$ (Fig. 9).

Case 3.3: $p_3 \in l^-(p_2q_2; p_1)$ and $q_2 \in l^+(p_1p_4; p_2)$ (Fig. 10).

We show the existence of an rt-embedding of T on N in each case.

Lemma 3.6. *In Case* 3.1 *there is an rt-embedding of* T *on* N.

Proof. From the definition of q_2, $M_1 \in \bar{l}^-(p_2q_2; p_1)$. The assumption of Case 3.1 implies (3.10). From the above discussion, the assertion follows from Lemma 3.5. □

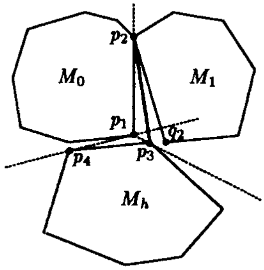

Fig. 8. Case 3.1.

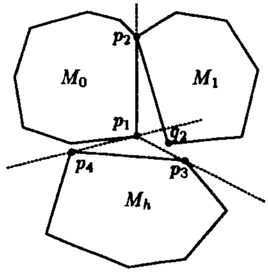

Fig. 9. Case 3.2.

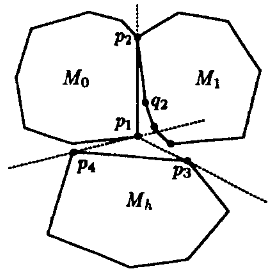

Fig. 10. Case 3.3.

Lemma 3.7. *In Case* 3.2 *there is an rt-embedding of T on N.*

Proof. Figures 11 and 12 may help the reader to understand this proof. In this case M_1, p_2, and p_3 do not satisfy (3.10). Suppose that p is the point of $(M_1 \cup M_h) - \{q_2\}$ which is the $|M_1|$th point among $(M_1 \cup M_h) - \{q_2\}$ with respect to the total order determined by angles $\angle p_2 q_2 q$ for $q \in (M_1 \cup M_h) - \{q_2\}$. Let M'_1 be the set of the first $|M_1|$ points of $(M_1 \cup M_h) - \{q_2\}$ and let $M'_h = (M_1 \cup M_h) - M'_1$.

Roughly speaking, the plane is partitioned into four convex regions: the triangle $\Delta p_1 p_2 q_2$, above the polygonal line $p_4 p_1 p_2$, below the polygonal line $p_4 p_1 q_2 p$, and the rest. The last three unbounded regions include M_0, M'_h, and M'_1, respectively. More precisely, the following relations hold:

$$
\begin{aligned}
\overline{p_2 q_2} &\subset (\bar{l}^-(p_1 p_2; p_4) \cap \bar{l}^+(p_2 q_2; p_1) \cap \bar{l}^+(p_1 q_2; p_2)),\\
M_0 &\subset (\bar{l}^+(p_1 p_2; p_4) \cap \bar{l}^+(p_1 p_4; p_2)) - \{p_4\},\\
M'_1 &\subset (\bar{l}^-(p_1 p_2; p_4) \cap \bar{l}^+(q_2 p; p_2) \cap \bar{l}^-(p_2 q_2; p_1)) - \{p_1, q_2\},\\
M'_h &\subset (\bar{l}^-(p_1 p_4; p_2) \cap \bar{l}^-(q_2 p; p_2) \cap \bar{l}^-(p_1 q_2; p_2)) - \{p_1, p\}.
\end{aligned}
$$

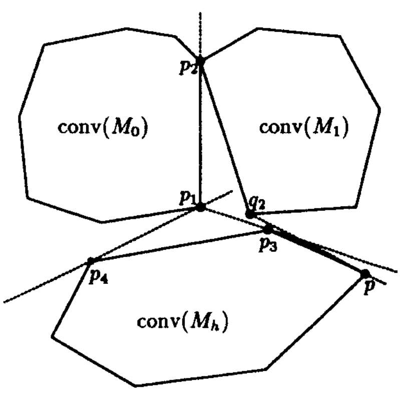

Fig. 11. Case 3.2.

$\Longrightarrow$

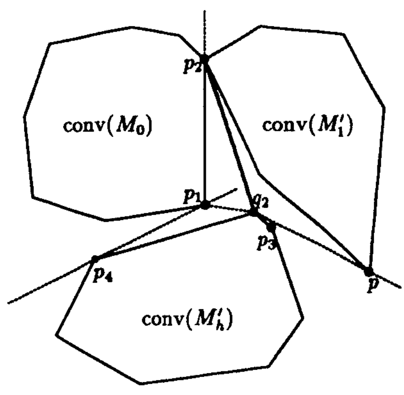

Fig. 12. New partition: M'_1 and M'_h.

Hence M_0, M'_1, M'_h, p_2, and q_2 satisfy conditions (3.5)–(3.12). By Lemma 3.5, there is an rt-embedding of T on N.

In the rest of the proof we prove the above relations. Since $M_1 \cup \{p_3\} \subset \bar{l}^-(p_1p_2; p_4) \cap \bar{l}^-(p_2q_2; p_1)$, $M'_1 \subset \bar{l}^-(p_1p_2; p_4) \cap \bar{l}^-(p_2q_2; p_1) \cap \bar{l}^+(q_2p; p_2)$ and $p \in l^-(p_2q_2; p_1)$. The assumption that $q_2 \in l^-(p_1p_4; p_2)$ and the fact that $q_2 \in l^+(p_1p_3; p_2)$ imply that $M_h \subset l^-(p_1q_2; p_2) \cap \bar{l}^-(p_1p_4; p_2)$. Then $p \in l^-(p_1q_2; p_2)$ because $|(M_1 \cup M_h) \cap l^+(p_1q_2; p_2)| < |M_1|$. Since $p \in l^-(p_1q_2; p_2) \cap l^-(p_2q_2; p_1)$, $p_1 \in l^-(q_2p; p_2)$ holds. From the definition of M'_h,

$$M'_h \subset \bar{l}^-(p_1q_2; p_2) \cap \bar{l}^-(p_1p_4; p_2) \cap \bar{l}^-(q_2p; p_2)$$

because $M_h \subset l^-(p_1q_2; p_2) \cap \bar{l}^-(p_1p_4; p_2)$ and $p_1 \in l^-(q_2p; p_2)$. To summarize the discussion, the above relations can be shown. □

Lemma 3.8. *In Case* 3.3 *there is an rt-embedding of T on N.*

Proof. We first modify the sets M_0 and M_1 (see Figs. 13 and 14). Now we consider the counterclockwise sequence σ of extreme points of conv(M_1) beginning from p_2. Since $|M_h| < s$, there is at least one extreme point of conv(M_1) in $l^-(p_1p_4; p_2)$. Let q'_3 be the first point in σ with $q'_3 \in l^-(p_1p_4; p_2)$, and let q'_2 be the previous point of q'_3 in σ. Suppose that τ is the clockwise ordering on $(M_0 \cup M_1) - \{q'_2\}$ around q'_2 beginning from p_1. Let M'_0 be the set which consists of q'_2 and the first $(|M_0| - 1)$ points of $(M_0 \cup M_1) - \{q'_2\}$ with respect to τ, and let M'_1 be the set consisting of q'_2 and the last $(|M_1| - 1)$ points with respect to τ. Then $M'_0 \cup M'_1 = M_0 \cup M_1$ and the first point p' among M'_1 is the $|M_0|$th point among $(M_0 \cup M_1) - \{q'_2\}$. As in Fig. 14, M'_0 and M'_1 are included in two convex regions: the left-hand side of the polygonal line $p'q'_2p_1p_4$ and the right-hand side of the polygonal line $p'q'_2q'_3$, respectively. We first prove this.

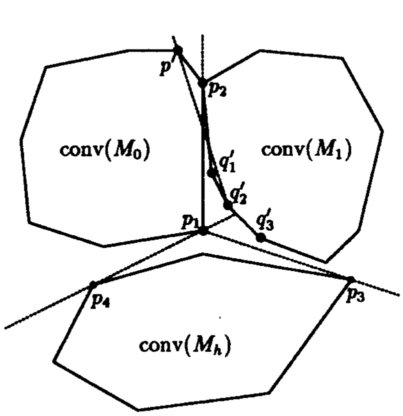

Fig. 13. Case 3.3.

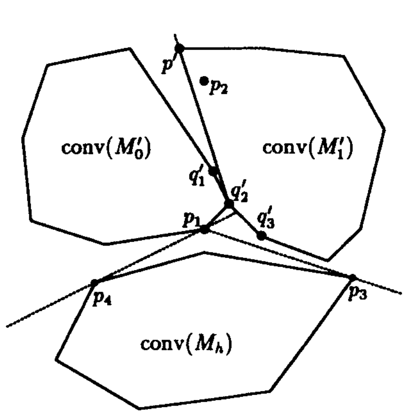

Fig. 14. New partition: M'_0 and M'_1.

The point q'_2 is distinct from p_2 because of the assumption of Case 3.3, that is, $q'_2 \in l^-(p_1p_2; p_4)$. Furthermore, q'_2 belongs to $l^+(p_1p_4; p_2)$ and to $l^+(p_1p_3; p_2)$. This implies $\overline{q'_2p_3} \in \bar{l}^-(p_1q'_2; p_4) \cap \bar{l}^+(p_1p_3; p_2)$. The fact that

$$q'_2 \in l^+(p_1p_4; p_2) \cap \bar{l}^-(p_1p_2; p_4)$$

implies $M_0 \subset \bar{l}^+(p_1p_4; p_2) \cap \bar{l}^+(p_1q'_2; p_4)$. Since $\bar{l}^+(p_1p_4; p_2) \cap \bar{l}^+(p_1q'_2; p_4)$ contains at least $|M_0|$ points of $(M_0 \cup M_1) - \{q'_2\}$, $p' \in l^+(p_1q'_2; p_4)$ and $M'_0 \subset \bar{l}^+(p_1p_4; p_2) \cap \bar{l}^+(p_1q'_2; p_4) \cap \bar{l}^+(q'_2p'; p_1)$. For the previous point q'_1 of q'_2 in σ, the open half-plane $l^+(q'_2q'_1; p_1)$ does not contain any point of M_1, particularly, does not contain p_2. Then $\bar{l}^+(p_1q'_2; p_4) \cap l^+(q'_2q'_1; p_1)$ contains at most $(|M_0| - 1)$ points of $(M_0 \cup M_1) - \{q'_2\}$. Hence p' belongs to $\bar{l}^-(q'_2q'_1; p_1)$. The angle $\angle p'q'_2q'_3$ is less than π. Thus $M'_1 \subset \bar{l}^-(q'_2p'; p_1) \cap \bar{l}^-(q'_2q'_3; p_1)$.

We consider two cases: the former (Case 3.3(a)) assumes $M_h \subset l^+(q'_2q'_3; p_1)$ and the latter (Case 3.3(b)) assumes $M_h \not\subset l^+(q'_2q'_3; p_1)$.

Case 3.3(a). The above discussion says that

$$\begin{aligned}
\overline{q'_2p_3} &\subset (\bar{l}^+(q'_2q'_3; p_1) \cap \bar{l}^-(p_1q'_2; p_4) \cap \bar{l}^+(p_1p_3; p_2)),\\
M'_0 &\subset (\bar{l}^+(p_1p_4; p_2) \cap \bar{l}^+(q'_2p'; p_1) \cap \bar{l}^+(p_1q'_2; p_4)) - \{p_4, p'\},\\
M'_1 &\subset (\bar{l}^-(q'_2p'; p_1) \cap \bar{l}^-(q'_2q'_3; p_1)),\\
M_h &\subset (\bar{l}^-(p_1p_4; p_2) \cap l^+(q'_2q'_3; p_1) \cap \bar{l}^-(p_1p_3; p_2)) - \{p_1\}.
\end{aligned}$$

These relations guarantee that M'_0, M'_1, M_h, q'_2, and p_3 satisfy all conditions in Lemma 3.5.

Case 3.3(b). In this case there is an instance such that M'_0, M'_1, M_h, q'_2, and p_3 satisfy all the conditions in Lemma 3.5. However, we modify M'_1 and M_h in order that all the conditions in Lemma 3.5 hold (see Figs. 15 and 16). Let p'' be the $|M'_1|$th

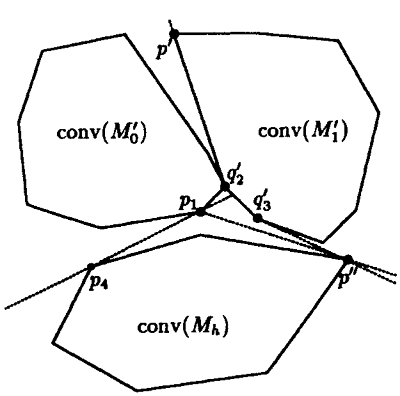

Fig. 15. Case 3.3(b).

⟹

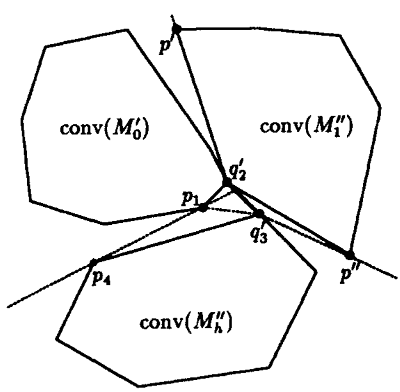

Fig. 16. New partition: M''_1 and M''_h.

point among $(M'_1 \cup M_h) - \{q'_3\}$ with respect to the clockwise ordering τ' on $(M'_1 \cup M_h) - \{q'_3\}$ around q'_3 beginning from q'_2. Let M''_1 be the first $|M'_1|$ points among $(M'_1 \cup M_h) - \{q'_3\}$ with respect to τ' and let $M''_h = (M'_1 \cup M_h) - M''_1$. The set M''_h contains q'_3. Then the plane is partitioned into four convex regions: the triangle $\triangle p_1 q'_2 q'_3$, the left-hand side of the polygonal line $p' q'_2 p_1 p_4$, the right-hand side of the polygonal line $p' q'_2 q'_3 p''$, and below the polygonal line $p_4 p_1 q'_3 p''$. The last three unbounded regions include M'_0, M''_1, and M''_h, respectively (see Fig. 16). We prove this fact below.

Since $q'_3 \in l^-(p_1p_4; p_2) \cap l^+(p_1p_3; p_2)$, $M_h \subset l^-(p_1q'_3; q'_2)$. Then $p'' \in l^-(p_1q'_3; q'_2)$ because $|(M'_1 \cup M_h) \cap l^+(p_1q'_3; q'_2)| < |M'_1|$. Since

$$M_h \subset \bar{l}^-(p_1p_4; p_2) \cap \bar{l}^-(p_1q'_3; q'_2)$$

and $p'' \in l^-(p_1q'_3; q'_2)$, the definition of M''_h implies

$$M''_h \subset \bar{l}^-(p_1p_4; p_2) \cap \bar{l}^-(p_1q'_3; q'_2) \cap \bar{l}^+(q'_3p''; p_1).$$

From the assumption of Case 3.3(b), $\bar{l}^-(q'_2q'_3; p_1)$ contains at least $(|M'_1| + 1)$ points of $M'_1 \cup M_h$. Then $p'' \in l^-(q'_2q'_3; p_1)$ and

$$M''_1 \subset \bar{l}^-(q'_2p'; p_1) \cap \bar{l}^-(q'_2q'_3; p_1) \cap \bar{l}^-(q'_3p''; p_1).$$

The above discussion says that

$$\begin{aligned}
\overline{q'_2q'_3} &\subset (\bar{l}^-(p_1q'_2; p_4) \cap \bar{l}^+(q'_2q'_3; p_1) \cap \bar{l}^+(p_1q'_3; q'_2)),\\
M'_0 &\subset (\bar{l}^+(p_1p_4; p_2) \cap \bar{l}^+(q'_2p'; p_1) \cap \bar{l}^+(p_1q'_2; p_4)) - \{p_4, p'\},\\
M''_1 &\subset (\bar{l}^-(q'_2p'; p_1) \cap \bar{l}^-(q'_3p''; p_1) \cap \bar{l}^-(q'_2q'_3; p_1)) - \{q'_3\},\\
M''_h &\subset (\bar{l}^-(p_1p_4; p_2) \cap \bar{l}^+(q'_3p''; p_1) \cap \bar{l}^-(p_1q'_3; q'_2) - \{p_1, p''\}.
\end{aligned}$$

Then M'_0, M''_1, M''_h, q'_2, and q'_3 satisfy conditions (3.5)–(3.12). □

Lemmas 3.3, 3.4, 3.6, 3.7, and 3.8 indicate the existence of rt-embeddings of any rooted tree with n vertices on any set of n points in general position in the plane. An rt-embedding can be constructed of a given rooted tree on any given set of n points. Moreover, from the discussion of this paper, we can do this in polynomial time with respect to n.

Theorem 3.9. *Let T be any rooted tree with n vertices $V(T)$ and let N be a set of n points in R^2. If no three points of N lie on any line, there is an rt-embedding of $V(T)$ on N. Furthermore, some rt-embedding can be constructed in polynomial time with respect to n.*

4. Complexity

In this section we discuss the time complexity of our algorithm for finding an rt-embedding. Our arguments are limited to overall results and we refer the reader to [4] and [1] for details. We assume that Algorithms 1 and 2 require $f(n)$ and $f'(n)$ times, respectively. These have been analyzed to require $O(n^2)$ time in [2].

First we must reconstruct T to satisfy condition (2.1) and enumerate the numbers s and t. By using the postorder traversal for trees, we can modify T in $O(n)$ time. To find the numbers s and t, we first sort $N - \{p_1\}$ around p_1 in $O(n \log n)$ time. Then s and t can be found in $O(n)$ time. $O(n)$ time is also sufficient to find the master r_k satisfying (3.1), and to determine whether a given instance belongs to Case 1, 2, or 3. The procedure for Case 1 requires $O(n + \sum_{i=1}^{ch(r_1)} f(|T^i(r_1)| + 1))$ time because $N - \{p_1\}$ are already sorted around p_1. In the same way, an rt-embedding is found in $O(n + f(n) + f'(n))$ time in Case 2. In Case 3 M_0, M_1, M_h, p_2, p_3, p_4, and q_2 are enumerated in $O(n)$ time. In constant time we can determine which among Cases 3.1, 3.2, and 3.3 holds. In Case 3.1 the time complexity is $O(2f(n) + f'(n))$. In Case 3.2 we can enumerate M'_1 and M'_h by selecting the $|M_1|$th point p in linear time, and, hence, an rt-embedding is found in $O(n + 2f(n) + f'(n))$ time. In Case 3.3 the sequence σ of extreme points of $\text{conv}(M_1)$ is found in linear time by using the incremental method and the clockwise ordering on M_1 around p_1. In the same way as Case 3.2, M'_0, M'_1, and p' are found in linear time, and then M''_1, M''_h, and p'' are also found in linear time in Case 3.3(b). Hence an rt-embedding is found in $O(n + 2f(n) + f'(n))$ time in Case 3.3. To summarize the above discussion, the time complexity of our algorithm is $O(\max\{n \log n, \sum_{i=1}^{ch(r_1)} f(|T^i(r_1)| + 1), f(n), f'(n)\})$. Hence the time complexity of our algorithm is $O(n^2)$, and obviously the space complexity is $O(n)$. We remark that any speedup in the computation when p_1 is an extreme point would immediately shorten the overall computation time.

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