Chérif Amrouche; Hamid Bouzit
The scalar Oseen operator $-\Delta + \partial/\partial x_1$ in $\mathbb{R}^2$

Applications of Mathematics, Vol. 53 (2008), No. 1, 41–80

Persistent URL: `http://dml.cz/dmlcz/134698`

THE SCALAR OSEEN OPERATOR $-\Delta + \partial/\partial x_1$ IN $\mathbb{R}^2$

Chérif Amrouche, Pau, Hamid Bouzit, Mostaganem

(Received June 20, 2006)

Abstract. This paper solves the scalar Oseen equation, a linearized form of the Navier-Stokes equation. Because the fundamental solution has anisotropic properties, the problem is set in a Sobolev space with isotropic and anisotropic weights. We establish some existence results and regularities in L^p theory.

Keywords: Oseen equation, weighted Sobolev space, anisotropic weight

MSC 2000: 76D05, 35Q30, 26D15

1. Introduction

Let Ω be an exterior domain of $\mathbb{R}^2$ or the whole space $\mathbb{R}^2$. We consider the following Oseen's problem:

$$
\begin{aligned}
-\nu\Delta \boldsymbol{u} + \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla \pi &= \boldsymbol{f} \quad \text{in } \Omega,\\
\operatorname{div} \boldsymbol{u} &= g \quad \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{u}_* \quad \text{on } \partial\Omega,
\end{aligned} \tag{1.1}
$$

with the condition on $\boldsymbol{u}$ at infinity

$$\lim_{|x|\to+\infty} \boldsymbol{u}(x) = \boldsymbol{u}_\infty. \tag{1.2}$$

The viscosity ν, the external force $\boldsymbol{f}$, the boundary values $\boldsymbol{u}_*$ on $\partial\Omega$ and g are given. The positive coefficient λ corresponds to the Reynolds number. The unknown velocity field $\boldsymbol{u}$ is assumed to converge to a constant vector $\boldsymbol{u}_\infty$, and the scalar function π denotes the unknown pressure. C. W. Oseen [14] obtained (1.1) by linearizing the Navier-Stokes equations, describing the flow of a viscous and incompressible fluid.

Some authors worked on this problem. We can cite Finn [6], [7], more recently Galdi [8], Farwig [3], [4], Farwig and Sohr [5] and Amrouche and Razafison [2]. When $\Omega = \mathbb{R}^2$, the system (1.1) is written as follows

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \lambda \frac{\partial \boldsymbol{u}}{\partial x_1} + \nabla\pi &= \boldsymbol{f} \quad \text{in } \mathbb{R}^2, \\ \operatorname{div} \boldsymbol{u} &= g \quad \text{in } \mathbb{R}^2, \end{aligned} \tag{1.3}$$

with the same condition at infinity. Taking the divergence of the first equation of (1.3), we obtain a decoupled set of equations

$$\Delta\pi = \operatorname{div} \boldsymbol{f} + \nu\Delta g - \lambda\frac{\partial g}{\partial x_1} \quad \text{in } \mathbb{R}^2, \tag{1.4}$$

$$-\nu\Delta \boldsymbol{u} + \lambda\frac{\partial \boldsymbol{u}}{\partial x_1} = \boldsymbol{f} - \nabla\pi \quad \text{in } \mathbb{R}^2. \tag{1.5}$$

We use the results obtained in [1] for the Poisson equation to solve Equation (1.4). Now observe that each component u_j of the velocity satisfies

$$-\nu\Delta u_j + \lambda\frac{\partial u_j}{\partial x_1} = f_j - \frac{\partial \pi}{\partial x_j} \quad \text{in } \mathbb{R}^2. \tag{1.6}$$

Thus, we see that if we solve the scalar equation

$$-\nu\Delta u + \lambda\frac{\partial u}{\partial x_1} = f \quad \text{in } \mathbb{R}^2, \tag{1.7}$$

we can apply to the Oseen problem the results obtained for this last equation. The aim of this paper is then to study the scalar Oseen equation (1.7). Since the fundamental solution of this equation has anisotropic decay properties, see (3.6), (3.9), we will work in Sobolev spaces with an isotropic weight and with the anisotropic weight introduced by Farwig [3] in the particular Hilbertian case ($p = 2$). The case $\lambda = 0$ yields the Laplace equation studied by Amrouche-Girault-Giroire [1] in weighted Sobolev spaces. This paper is divided into five sections. In Section 2, we introduce the functional spaces and we recall some preliminary results. We give also a density result for $\mathcal{D}(\mathbb{R}^2)$ in an anisotropic weighted space and a characterization of homogeneous Sobolev spaces. In Section 3, by adapting a technique used by Stein, we obtained results on Oseen's potential which we use then to solve Equation (1.7), where the left-hand side f is given on the one hand in $L^p(\mathbb{R}^2)$ and on the other hand in $W_0^{-1,p}(\mathbb{R}^2)$. We also look at the case where f belongs at the same moment to two spaces with different powers p and q. We consider, in Section 4, the case where f belongs to spaces L^p with anisotropic weights. Finally, in Section 5, we consider the limit case when λ tends to zero and we compare the limit with the solution of Poisson's equation. The main results of this paper are given by the theorems below.

In Theorem 1, we give (L^p, L^q) continuity properties for the Oseen operators $f \mapsto \mathcal{O} * f$, $f \mapsto \partial\mathcal{O}/\partial x_i * f$, and $f \mapsto \partial^2\mathcal{O}/\partial x_j \partial x_k * f$, where $\mathcal{O}$ is the fundamental scalar Oseen solution, which is defined in Section 3. We observe that the continuity results obtained for the Oseen equation (1.7) are better than the classic properties of the Riesz potential associated to the Laplace operator corresponding to the case $\lambda = 0$.

Theorem 1.1. *Let $f \in L^p(\mathbb{R}^2)$ with $1 < p < \infty$. Then, $\partial^2\mathcal{O}/\partial x_j \partial x_k * f \in L^p(\mathbb{R}^2)$, $\partial\mathcal{O}/\partial x_1 * f \in L^p(\mathbb{R}^2)$ and they satisfy the estimate*

$$\left\| \frac{\partial^2 \mathcal{O}}{\partial x_j \partial x_k} * f \right\|_{L^p(\mathbb{R}^2)} + \left\| \frac{\partial \mathcal{O}}{\partial x_1} * f \right\|_{L^p(\mathbb{R}^2)} \leqslant C \|f\|_{L^p(\mathbb{R}^2)}.$$

Moreover,
1) i) *if $1 < p < 2$, then $\nabla\mathcal{O} * f \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2) \cap \boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)$ and*

$$\|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2)} + \|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)} \leqslant C \|f\|_{L^p(\mathbb{R}^2)}.$$

ii) *If $p = 2$, then $\nabla\mathcal{O} * f \in \boldsymbol{L}^r(\mathbb{R}^2)$ for any $r \geqslant 6$ and the following estimate holds:*

$$\|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^r(\mathbb{R}^2)} \leqslant C \|f\|_{L^p(\mathbb{R}^2)}.$$

iii) *If $2 < p < 3$, then $\nabla\mathcal{O} * f \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2) \cap \boldsymbol{L}^\infty(\mathbb{R}^2)$ and we have the estimate*

$$\|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2)} + \|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^\infty(\mathbb{R}^2)} \leqslant C \|f\|_{L^p(\mathbb{R}^2)}.$$

2) *If $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^{3p/(3-2p)}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and*

$$\|\mathcal{O} * f\|_{L^{3p/(3-2p)}(\mathbb{R}^2)} + \|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^2)} \leqslant C \|f\|_{L^p(\mathbb{R}^2)}.$$

In Theorem 2, we give similar results for the case when f belongs to a negative weighted Sobolev space $W_0^{-1,p}(\mathbb{R}^2)$ and we observe again that we obtain results better than in the case $\lambda = 0$.

Theorem 1.2. *Let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfy the compatibility condition*

$$\langle f, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^2) \times W_0^{1,p'}(\mathbb{R}^2)} = 0, \quad \text{when } 1 < p \leqslant 2.$$

i) *If $1 < p < 3$, then $u = \mathcal{O} * f \in L^{3p/(3-p)}(\mathbb{R}^2)$ is the unique solution of Equation (3.1) such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$ and $\partial u / \partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$. Moreover, we have the estimate*

$$\|u\|_{L^{3p/(3-p)}(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^2)} + \Big\| \frac{\partial u}{\partial x_1} \Big\|_{W_0^{-1,p}(\mathbb{R}^2)} \leqslant C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)},$$

and $u \in L^{2p/(2-p)}(\mathbb{R}^2)$ when $1 < p < 2$, $u \in L^r(\mathbb{R}^2)$ for any $r \geqslant 6$ when $p = 2$, and $u \in L^\infty(\mathbb{R}^2)$ when $2 < p < 3$.

ii) *If $p \geqslant 3$, then Equation (3.1) has a solution $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2)$ that is unique up to a constant, and we have*

$$\inf_{k \in \mathbb{R}} \|u + k\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^2)} \leqslant C \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

Theorem 1.3 is concerned with the case when f belongs to L^p spaces with anisotropic weight.

Theorem 1.3. *Assume that $2 < p < \frac{32}{11}$ and $f \in L^p_{1/2,1/4}(\mathbb{R}^2)$. Then $u = \mathcal{O} * f \in L^p_{-1/2,1/4}(\mathbb{R}^2)$, $\partial u / \partial x_2 \in L^p_{0,1/4}(\mathbb{R}^2)$, $\partial u / \partial x_1 \in L^p_{1/2,1/4}(\mathbb{R}^2)$, and $\nabla^2 u \in (L^p_{1/2,1/4}(\mathbb{R}^2))^{2 \times 2}$. Moreover, we have the estimates*

$$\int_{\mathbb{R}^2} (1+r)^{-p/2} (1+s)^{p/4} |u|^p \, \mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2} (1+r)^{p/2} (1+s)^{p/4} (|\partial u / \partial x_1|^p + |\nabla^2 u|^p) \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_{\mathbb{R}^2} (1+s)^{p/4} \Big| \frac{\partial u}{\partial x_2} \Big|^p \, \mathrm{d}\boldsymbol{x} \leqslant C \int_{\mathbb{R}^2} (1+r)^{p/2} (1+s)^{p/4} |f|^p \, \mathrm{d}\boldsymbol{x},$$

where $r = |\boldsymbol{x}|$, $s = r - x_1 = |\boldsymbol{x}| - x_1$, and the anisotropic weighted L^p spaces are defined in Section 4.

2. Functional spaces and preliminaries

In this paper, p is a real number in the interval $]1, +\infty[$ and its conjugate is denoted by p'. A point in $\mathbb{R}^2$ is denoted $\boldsymbol{x} = (x_1, x_2)$ and we denote as above:

$$r = |\boldsymbol{x}| = (x_1^2 + x_2^2)^{1/2}, \quad \varrho = (1 + r^2)^{1/2}, \quad s = r - x_1,$$
$$s' = r + x_1, \quad \text{for } a, b \in \mathbb{R}, \ \eta_b^a = (1 + r)^a (1 + s)^b.$$

For $R > 0$, B_R denotes the open ball of radius R centered at the origin and $B'_R = \mathbb{R}^2 \setminus \overline{B_R}$. For any $j \in \mathbb{Z}$, $\mathcal{P}_j$ is the space of polynomials of degree lower than or equal to j and if j is negative we set, by convention, $\mathcal{P}_j = 0$. Let B be a Banach space, with dual space B' and a closed subspace X of B. We denote by $B' \perp X$ the subspace of B' orthogonal to X defined by

$$B' \perp X = \{f \in B';\ \forall\, v \in X\colon\ \langle f, v\rangle = 0\}.$$

For $m \in \mathbb{N}^*$, we set

$$k = k(m,p,\alpha) = \begin{cases} -1 & \text{if } \alpha + 2/p \notin \{1, \dots, m\}, \\ m - \alpha - 2/p & \text{if } \alpha + 2/p \in \{1, \dots, m\} \end{cases} \tag{2.1}$$

and we define the weighted Sobolev space

$$W_\alpha^{m,p}(\mathbb{R}^2) = \{u \in \mathcal{D}'(\mathbb{R}^2);\ \forall\, \lambda \in \mathbb{N}^2 :$$
$$\text{if } 0 \leqslant |\lambda| \leqslant k, \text{ then } \varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{-1} \partial^\lambda u \in L^p(\mathbb{R}^2);$$
$$\text{if } k+1 \leqslant |\lambda| \leqslant m, \text{ then } \varrho^{\alpha - m + |\lambda|} \partial^\lambda u \in L^p(\mathbb{R}^2)\},$$

where $\lg \varrho = \ln(1+\varrho)$. It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W_\alpha^{m,p}(\mathbb{R}^2)} = \Big(\sum_{0 \leqslant |\lambda| \leqslant k} \|\varrho^{\alpha - m + |\lambda|} (\lg \varrho)^{-1} \partial^\lambda u\|^p_{L^p(\mathbb{R}^2)} + \sum_{k+1 \leqslant |\lambda| \leqslant m} \|\varrho^{\alpha - m + |\lambda|} \partial^\lambda u\|^p_{L^p(\mathbb{R}^2)} \Big)^{1/p}.$$

Its semi-norm is defined by

$$|u|_{W_\alpha^{m,p}(\mathbb{R}^2)} = \Big(\sum_{|\lambda| = m} \|\varrho^\alpha \partial^\lambda u\|^p_{L^p(\mathbb{R}^2)} \Big)^{1/p}.$$

The logarithmic weight appears only when $\alpha + 2/p \in \{1, \dots, m\}$. We refer to Kufner [11], Hanouzet [9], and Amrouche-Girault-Giroire [1] for a detailed study of the space $W_\alpha^{m,p}(\mathbb{R}^n)$. However, we recall some properties and results that we use in this paper. For any $\lambda \in \mathbb{N}^2$, the mapping

$$u \in W_\alpha^{m,p}(\mathbb{R}^2) \mapsto \partial^\lambda u \in W_\alpha^{m-|\lambda|,p}(\mathbb{R}^2) \tag{2.2}$$

is continuous. When $\alpha + 2/p \notin \{1, \dots, m\}$, we have the following continuous embedding and density

$$W_\alpha^{m,p}(\mathbb{R}^2) \subset W_{\alpha-1}^{m-1,p}(\mathbb{R}^2) \subset \ldots \subset W_{\alpha-m}^{0,p}(\mathbb{R}^2), \tag{2.3}$$

where

$$W_\alpha^{0,p}(\mathbb{R}^2) = \{u \in \mathcal{D}'(\mathbb{R}^2);\ \varrho^\alpha u \in L^p(\mathbb{R}^2)\};$$

also note that the mapping

$$u \in W_\alpha^{m,p}(\mathbb{R}^2) \mapsto \varrho^\gamma u \in W_{\alpha-\gamma}^{m,p}(\mathbb{R}^2) \tag{2.4}$$

is continuous, which is not the case if $\alpha + 2/p \in \{1, \dots, m\}$. The space $W_\alpha^{m,p}(\mathbb{R}^2)$ contains the polynomials of degree lower than or equal to j, denoted $\mathcal{P}_j$, where $j \in \mathbb{N}$ is defined by

$$j = \begin{cases} [m - \alpha - 2/p] & \text{if } \alpha + 2/p \notin \mathbb{Z}^-, \\ m - 1 - \alpha - 2/p & \text{otherwise.} \end{cases} \tag{2.5}$$

The following theorem is fundamental (see [1]).

Theorem 2.1. *Let $m \geqslant 1$ be an integer and α a real number, then there exists a constant C such that*

$$\forall\, u \in W_\alpha^{m,p}(\mathbb{R}^2) \quad \inf_{\mu \in \mathcal{P}_j} \|u + \mu\|_{W_\alpha^{m,p}(\mathbb{R}^2)} \leqslant C|u|_{W_\alpha^{m,p}(\mathbb{R}^2)}, \tag{2.6}$$

where j is the highest degree of a polynomial contained in $W_\alpha^{m,p}(\mathbb{R}^2)$.

We define the space

$$\boldsymbol{H}_p = \{\boldsymbol{v} \in \boldsymbol{L}^p(\mathbb{R}^2),\ \operatorname{div} \boldsymbol{v} = 0\}.$$

Theorem 2.1 permits to prove that the following divergence operator is an isomorphism (see [1]):

$$\operatorname{div}\colon\ \boldsymbol{L}^{p'}(\mathbb{R}^2)/\boldsymbol{H}_p \longrightarrow W_0^{-1,p'}(\mathbb{R}^2) \perp \mathcal{P}_{[1-2/p]}. \tag{2.7}$$

The next result is a consequence of Theorem 2.1 (see [1]):

Proposition 2.2. *Let $m \geqslant 1$ be an integer and u a distribution such that*

$$\forall\, \lambda \in \mathbb{N}^2 :\ |\lambda| = m,\ \partial^\lambda u \in L^p(\mathbb{R}^2).$$

(i) *If $1 < p < 2$, then there exists a unique polynomial $K(u) \in \mathcal{P}_{m-1}$ such that $u + K(u) \in W_0^{m,p}(\mathbb{R}^2)$, and*

$$\inf_{\mu \in \mathcal{P}_{[m-2/p]}} \|u + K(u) + \mu\|_{W_0^{m,p}(\mathbb{R}^2)} \leqslant C|u|_{W_0^{m,p}(\mathbb{R}^2)}. \tag{2.8}$$

(ii) *If $p \geqslant 2$, then $u \in W_0^{m,p}(\mathbb{R}^2)$ and*

$$\inf_{\mu \in \mathcal{P}_{[m-2/p]}} \|u + \mu\|_{W_0^{m,p}(\mathbb{R}^2)} \leqslant C|u|_{W_0^{m,p}(\mathbb{R}^2)}. \tag{2.9}$$

When $1 < p < 2$, we have the following characterization of the space $W_0^{1,p}(\mathbb{R}^2)$:

$$W_0^{1,p}(\mathbb{R}^2) = \{v \in L^{2p/(2-p)}(\mathbb{R}^2);\ \nabla v \in \boldsymbol{L}^p(\mathbb{R}^2)\}. \tag{2.10}$$

We recall the space introduced in [2]:

$$\widetilde{W}_0^{1,p}(\mathbb{R}^2) = \Big\{u \in W_0^{1,p}(\mathbb{R}^2);\ \frac{\partial u}{\partial x_1} \in W_0^{-1,p}(\mathbb{R}^2)\Big\}, \tag{2.11}$$

which is a Banach space for its natural norm:

$$\|u\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^2)} = \|u\|_{W_0^{1,p}(\mathbb{R}^2)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

Also, we define

$$\widetilde{W}_0^{2,p}(\mathbb{R}^2) = \Big\{u \in W_0^{2,p}(\mathbb{R}^2);\ \frac{\partial u}{\partial x_1} \in L^p(\mathbb{R}^2)\Big\}, \tag{2.12}$$

which is a Banach space for its natural norm:

$$\|u\|_{\widetilde{W}_0^{2,p}(\mathbb{R}^2)} = \|u\|_{W_0^{2,p}(\mathbb{R}^2)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{L^p(\mathbb{R}^2)}.$$

Its dual space denoted $\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)$ can be characterized as follows (see also Remark 2.5).

Proposition 2.3. *Let $f \in \widetilde{W}_0^{-2,p'}(\mathbb{R}^2)$. Then:*

(i) *If $p \neq 2$, there exist $f_0 \in W_2^{0,p'}(\mathbb{R}^2)$, $\boldsymbol{F} \in (W_1^{0,p'}(\mathbb{R}^2))^2$, $\boldsymbol{H} \in (L^{p'}(\mathbb{R}^2))^{2\times 2}$, and $h \in L^{p'}(\mathbb{R}^2)$ such that for all $v \in \widetilde{W}_0^{2,p}(\mathbb{R}^2)$, we have*

$$\langle f, v\rangle_{\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)\times \widetilde{W}_0^{2,p}(\mathbb{R}^2)} = \langle f_0, v\rangle_{W_2^{0,p'}\times W_{-2}^{0,p}} + \langle \boldsymbol{F}, \nabla v\rangle_{W_1^{0,p'}\times W_{-1}^{0,p}} + \langle \boldsymbol{H}, \nabla^2 v\rangle_{\boldsymbol{L}^{p'}\times \boldsymbol{L}^p} + \Big\langle h, \frac{\partial v}{\partial x_1}\Big\rangle_{L^{p'}\times L^p}. \tag{2.13}$$

(ii) *If $p = 2$, then (2.13) holds if we take the weight $\varrho \lg \varrho$ instead of ϱ in the definition of $W_1^{0,p'}(\mathbb{R}^2)$ and $W_{-1}^{0,p}(\mathbb{R}^2)$, and $\varrho^2 \lg \varrho$ instead of ϱ^2 in the definition of $W_2^{0,p'}(\mathbb{R}^2)$ and $W_{-2}^{0,p}(\mathbb{R}^2)$.*

Proof. i) Suppose $p \neq 2$. Let $\boldsymbol{E} = W_{-2}^{0,p}(\mathbb{R}^2) \times (W_{-1}^{0,p}(\mathbb{R}^2))^2 \times (L^p(\mathbb{R}^2))^{2\times 2} \times L^p(\mathbb{R}^2)$, equipped with the norm:

$$\|\boldsymbol{\psi}\|_{\boldsymbol{E}} = \|\psi_0\|_{W_{-2}^{0,p}} + \sum_{i=1}^{n} \|\psi_i\|_{W_{-1}^{0,p}} + \sum_{j,k=1}^{n} \|\psi_{j,k}\|_{L^p} + \|\Omega\|_{L^p},$$

where $\boldsymbol{\psi} = (\psi_0, \psi_i, \psi_{j,k}, \Omega)$. It is clear that the following operator is an isometry

$$T\colon\ v \in \widetilde{W}_0^{2,p}(\mathbb{R}^2) \mapsto \Big(v, \nabla v, \nabla^2 v, \frac{\partial v}{\partial x_1}\Big) \in \boldsymbol{E}.$$

For all $f \in \widetilde{W}_0^{-2,p'}(\mathbb{R}^2)$ the operator defined by $L(h) = \langle f, T^{-1}h\rangle$ is continuous on $T(\widetilde{W}_0^{2,p}(\mathbb{R}^2))$ which is a closed subspace of $\boldsymbol{E}$. Thus, by the Hahn-Banach theorem, we can extend L to an element $\widetilde{L}$ of the dual of $\boldsymbol{E}$. Now, by the Riesz theorem, there exist $f_0 \in W_2^{0,p'}(\mathbb{R}^2)$, $\boldsymbol{F} \in (W_1^{0,p'}(\mathbb{R}^2))^2$, $\boldsymbol{H} \in (L^{p'}(\mathbb{R}^2))^{2\times 2}$ and $h \in L^{p'}(\mathbb{R}^2)$ satisfying (2.13).

ii) If $p = 2$, we take $\varrho \lg \varrho F_i \in L^{p'}(\mathbb{R}^2)$ in the definition of $W_1^{0,p'}(\mathbb{R}^2)$, $\varrho^2 \lg \varrho f_0 \in L^{p'}(\mathbb{R}^2)$ in the definition of $W_2^{0,p'}(\mathbb{R}^2)$ and we proceed as in the case i). Let us note that, when $1 < p < 2$, we can take $\boldsymbol{F} = \boldsymbol{0}$ thanks to Theorem 2.1. □

The last proposition permits to prove the next result.

Proposition 2.4. *$\mathcal{D}(\mathbb{R}^2)$ is dense in $\widetilde{W}_0^{2,p}(\mathbb{R}^2)$.*

Proof. Let $f \in \widetilde{W}_0^{-2,p'}(\mathbb{R}^2)$ be such that

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^2) \quad \langle f, \varphi\rangle_{\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)\times \widetilde{W}_0^{2,p}(\mathbb{R}^2)} = 0. \tag{2.14}$$

i) If $p' \neq 2$, by the previous proposition, there exist $f_0 \in W_2^{0,p'}(\mathbb{R}^2)$, $\boldsymbol{F} \in (W_1^{0,p'}(\mathbb{R})^2)^2$, $\boldsymbol{H} \in (L^{p'}(\mathbb{R}^2))^{2\times 2}$, and $h \in L^p(\mathbb{R}^2)$ satisfying (2.13). In particular, taking $v = \varphi \in \mathcal{D}(\mathbb{R}^2)$ in this equation, we have by (2.14):

$$f_0 - \operatorname{div} \boldsymbol{F} + \operatorname{div}(\operatorname{div} \boldsymbol{H}) - \frac{\partial h}{\partial x_1} = 0,$$

in the sense of distributions. Now, by (2.3), we have the continuous embedding and density $W_0^{1,p}(\mathbb{R}^2) \subset W_{-1}^{0,p}(\mathbb{R}^2)$. Thus, by duality, we have the embedding $W_1^{0,p'}(\mathbb{R}^2) \subset W_0^{1,p'}(\mathbb{R}^2)$, so $\boldsymbol{F} \in (W_0^{-1,p'}(\mathbb{R}^2))^2$, which implies $\operatorname{div} \boldsymbol{F} \in W_0^{-2,p'}(\mathbb{R}^2)$. By the same argument, we deduce that $f_0 \in W_0^{-2,p'}(\mathbb{R}^2)$, thus the last equation yields

$$\frac{\partial h}{\partial x_1} = f_0 - \operatorname{div} \boldsymbol{F} + \operatorname{div}(\operatorname{div} \boldsymbol{H}) \in W_0^{-2,p'}(\mathbb{R}^2) \cap W_0^{-1,p'}(\mathbb{R}^2).$$

So, Equation (2.13) can be written:

$$\langle f, v\rangle_{\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)\times \widetilde{W}_0^{2,p}(\mathbb{R}^2)} = \Big\langle f_0 - \operatorname{div} \boldsymbol{F} + \operatorname{div}(\operatorname{div} \boldsymbol{H}) - \frac{\partial h}{\partial x_1}, v\Big\rangle_{W_0^{-2,p'}(\mathbb{R}^2)\times W_0^{2,p}(\mathbb{R}^2)}.$$

Let $v \in \widetilde{W}_0^{2,p}(\mathbb{R}^2)$. Since $\mathcal{D}(\mathbb{R}^2)$ is dense in $W_0^{2,p}(\mathbb{R}^2)$, there exists a sequence $\varphi_k \in \mathcal{D}(\mathbb{R}^2)$ such that $\varphi_k \longrightarrow v$ in $W_0^{2,p}(\mathbb{R}^2)$. We then obtain

$$\begin{aligned}&\langle f, v\rangle_{\widetilde{W}_0^{-2,p'}(\mathbb{R}^2)\times \widetilde{W}_0^{2,p}(\mathbb{R}^2)} \\ &\quad= \lim_{k\to\infty} \Big\langle f_0 - \operatorname{div} \boldsymbol{F} + \operatorname{div}(\operatorname{div} \boldsymbol{H}) - \frac{\partial h}{\partial x_1}, \varphi_k\Big\rangle_{W_0^{-2,p'}(\mathbb{R}^2)\times W_0^{2,p}(\mathbb{R}^2)} = 0.\end{aligned}$$

ii) If $p = 2$, we take $(\varrho \lg \varrho)\boldsymbol{F} \in \boldsymbol{L}^{p'}(\mathbb{R}^2)$ and $(\varrho^2 \lg \varrho) f_0 \in L^{p'}(\mathbb{R}^2)$ and obtain, by the previous embeddings, $\boldsymbol{F} \in (W_0^{-1,p'}(\mathbb{R}^2))^2$ and $f_0 \in W_0^{-2,p'}(\mathbb{R}^2)$. We can proceed as in the case i); the density result holds and finishes the proof. □

R e m a r k 2.5. Property (2.13) is equivalent to

$$\widetilde{W}_0^{-2,p'}(\mathbb{R}^2) = \Big\{ f \in \mathcal{D}'(\mathbb{R}^2);\ f = f_0 + \operatorname{div} \boldsymbol{F} + \operatorname{div}(\operatorname{div} \boldsymbol{H}) + \frac{\partial h}{\partial x_1} \Big\}, \tag{2.15}$$

where $f_0, \boldsymbol{F}, \boldsymbol{H}$, and h are defined in Proposition 2.3.

Using the same technique as in the proof of the Payne-Weinberger inequality, we get the following:

Lemma 2.6. *Let $u \in \mathcal{D}'(\mathbb{R}^2)$ be such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$.*

i) *If $1 < p < 2$, then there exists a unique constant u_∞ defined by*

$$u_\infty = \lim_{r\to\infty} \frac{1}{2\pi} \int_0^{2\pi} u(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta \tag{2.16}$$

and such that

$$u - u_\infty \in W_0^{1,p}(\mathbb{R}^2). \tag{2.17}$$

Moreover, we have

$$u - u_\infty \in L^{2p/(2-p)}(\mathbb{R}^2), \tag{2.18}$$

with the estimate

$$\|u - u_\infty\|_{L^{2p/(2-p)}(\mathbb{R}^2)} \leqslant C\|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^2)}, \tag{2.19}$$

and

$$\int_0^{2\pi} |u(r\cos\theta, r\sin\theta) - u_\infty|^p\,\mathrm{d}\theta \leqslant C r^{p-2} \int_{\{|\boldsymbol{x}|>r\}} |\nabla u|^p\,\mathrm{d}\boldsymbol{x}. \tag{2.20}$$

ii) *If $p > 2$, then $u \in W_0^{1,p}(\mathbb{R}^2)$ and*

$$|u(\boldsymbol{x})| \leqslant C r^{1-2/p}\|u\|_{W_0^{1,p}(\mathbb{R}^2)} \quad \text{and} \quad r^{(2/p)-1}|u(\boldsymbol{x})| \longrightarrow 0. \tag{2.21}$$

The next result is a corollary of the previous lemma.

Corollary 2.7. *Let $u \in \mathcal{D}'(\mathbb{R}^2)$ be such that $\nabla^2 u \in (L^p(\mathbb{R}^2))^{2\times 2}$. Then:*

i) *If $1 < p < 2$ then there exists a unique vector $\boldsymbol{A} \in \mathbb{R}^2$ such that*

$$\nabla u + \boldsymbol{A} \in \boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2),$$

where $\boldsymbol{A}$ is defined by

$$\boldsymbol{A} = -\lim_{r\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \nabla u(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta. \tag{2.22}$$

Moreover, $u + \boldsymbol{A}\cdot\boldsymbol{x} \in W_0^{2,p}(\mathbb{R}^2)$ and satisfies

$$\inf_{k\in\mathbb{R}} \|u + \boldsymbol{A}\cdot\boldsymbol{x} + k\|_{W_0^{2,p}(\mathbb{R}^2)} \leqslant C\|u\|_{W_0^{2,p}(\mathbb{R}^2)}. \tag{2.23}$$

ii) *If $p \geqslant 2$, then $u \in W_0^{2,p}(\mathbb{R}^2)$ and*

$$\inf_{\mu\in\mathcal{P}_1} \|u + \mu\|_{W_0^{2,p}(\mathbb{R}^2)} \leqslant C\|u\|_{W_0^{2,p}(\mathbb{R}^2)}. \tag{2.24}$$

Now, with these last results, we can give a precise definition of the limit at infinity.

Definition 2.8. Let $u \in \mathcal{D}'(\mathbb{R}^2)$ be such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$. We say that u tends to $u_\infty \in \mathbb{R}$ at infinity and we denote

$$\lim_{|\boldsymbol{x}| \to \infty} u(\boldsymbol{x}) = u_\infty,$$

if

$$\lim_{r \to \infty} \int_0^{2\pi} |u(r\cos\theta, r\sin\theta) - u_\infty| \,\mathrm{d}\theta = 0.$$

R e m a r k 2.9. Let $u \in \mathcal{D}'(\mathbb{R}^2)$ be such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$. If $1 < p < 2$, we have the equivalence of the following statements

i) $u - u_\infty \in W_0^{1,p}(\mathbb{R}^2)$,

ii) $\lim_{|\boldsymbol{x}| \to \infty} u(\boldsymbol{x}) = u_\infty$ in the sense of Definition 2.8.

Finally, we recall the following lemma.

Lemma 2.10. *Let r and p be two reals such that $1 < r < \infty$ and $p > 2$. Let $u \in L^r(\mathbb{R}^2)$ and $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$. Then u is a continuous function on $\mathbb{R}^2$ and*

$$\lim_{|\boldsymbol{x}| \to \infty} u(\boldsymbol{x}) = 0.$$

3. The scalar Oseen equation in $\mathbb{R}^2$

In this section, we propose to solve the scalar Oseen equation (1.7). In order to simplify the notation, we assume without loss of generality $\lambda = \nu = 1$:

$$-\Delta u + \frac{\partial u}{\partial x_1} = f \quad \text{in } \mathbb{R}^2, \tag{3.1}$$

$f \in \mathcal{D}'(\mathbb{R}^2)$. To that end, let us define the operator

$$T\colon\; u \mapsto -\Delta u + \frac{\partial u}{\partial x_1}. \tag{3.2}$$

3.1. Study of the kernel

We consider the kernel of the operator T when it is defined on the tempered distributions $\mathcal{S}'(\mathbb{R}^2)$. Let u be an element of the kernel, by Fourier transform we can write

$$4\pi^2 |\boldsymbol{\xi}|^2 \hat{u}(\boldsymbol{\xi}) + 2\pi \mathrm{i} \xi_1 \hat{u}(\boldsymbol{\xi}) = 0.$$

Setting

$$\hat{u}(\boldsymbol{\xi}) = v(\boldsymbol{\xi}) + \mathrm{i}w(\boldsymbol{\xi}),$$

it follows that

$$\begin{cases} 4\pi^2|\boldsymbol{\xi}|^2 v(\boldsymbol{\xi}) - 2\pi\xi_1 w(\boldsymbol{\xi}) = 0, \\ 2\pi\xi_1 v(\boldsymbol{\xi}) + 4\pi^2|\boldsymbol{\xi}|^2 w(\boldsymbol{\xi}) = 0. \end{cases} \tag{3.3}$$

Since the determinant of the above system is $16\pi^4|\boldsymbol{\xi}|^4 + 4\pi^2\xi_1^2$, we deduce that, for $\boldsymbol{\xi} \neq 0$, the support of $\hat{u}$ is included in $\{0\}$. Then we have

$$\hat{u}(\boldsymbol{\xi}) = \sum c_\alpha \delta^{(\alpha)}, \quad c_\alpha \in \mathbb{C}, \quad \text{with a finite sum.}$$

By the inverse Fourier transform, we get

$$u(\boldsymbol{x}) = \sum d_\alpha x^\alpha, \quad d_\alpha \in \mathbb{C}, \quad \text{with a finite sum,}$$

that is, u is a polynomial. Setting for all integers k

$$\mathcal{S}_k = \Big\{ q \in \mathcal{P}_k\,;\ -\Delta q + \frac{\partial q}{\partial x_1} = 0 \Big\}, \tag{3.4}$$

if T is defined on $\mathcal{S}'(\mathbb{R}^2)$, then $\ker T = \mathcal{S}_k$, and we have:

Lemma 3.1. *Let $f \in \mathcal{S}'(\mathbb{R}^2)$ be a tempered distribution and let $u \in \mathcal{S}'(\mathbb{R}^2)$ be a solution of* (3.1). *Then u is uniquely determined up to a polynomial in $\mathcal{S}_k$.*

Let us notice that $\mathcal{S}_0 = \mathbb{R}$ and $\mathcal{S}_1$ is the space of polynomials of degree less than or equal to one and independent of x_1.

3.2. The fundamental solution

Following the idea of [8], we look for the fundamental solution $\mathcal{O}$ of the scalar Oseen equation in the form

$$\mathcal{O}(\boldsymbol{x}) = \mathrm{e}^{x_1/2} f\Big(\frac{r}{2}\Big).$$

We find by direct computations:

$$\Big(-\Delta\mathcal{O} + \frac{\partial \mathcal{O}}{\partial x_1}\Big) = \frac{1}{2\pi r^2}\mathrm{e}^{x_1/2}\Big(\Big(\frac{r}{2}\Big)^2 f''\Big(\frac{r}{2}\Big) + \frac{r}{2} f'\Big(\frac{r}{2}\Big) - \Big(\frac{r}{2}\Big)^2 f\Big(\frac{r}{2}\Big)\Big),$$

where, for $y = \frac{1}{2}r$,

$$y^2 f''(y) + y f'(y) - y^2 f(y) = 0$$

is the modified Bessel equation. Although K_0, the singular solution (at $y=0$) of this equation cannot be given explicitly, we can give an estimate in a neighborhood of zero and also when y is large:

(i) When y is small

$$K_0(y) = \ln\frac{1}{y} + \ln 2 - \gamma + \sigma(y), \tag{3.5}$$

where γ is the Euler constant and σ satisfies

$$\frac{\mathrm{d}^k\sigma}{\mathrm{d}y^k} = o(y^{-k}).$$

Thus, when r is close to zero,

$$\mathcal{O}(\boldsymbol{x}) = -\frac{1}{2\pi}\mathrm{e}^{x_1/2}\Big\{\ln\frac{1}{r} + 2\ln 2 - \gamma + \sigma(r)\Big\}. \tag{3.6}$$

(ii) When $r \longrightarrow +\infty$, using the asymptotic expansion given in [10], we have

$$\begin{aligned} K_0\Big(\frac{r}{2}\Big) &= \Big(\frac{\pi}{r}\Big)^{1/2}\mathrm{e}^{-r/2}\Big[1 - \frac{1}{4r} + O(r^{-2})\Big],\\ K_0'\Big(\frac{r}{2}\Big) &= \Big(\frac{\pi}{r}\Big)^{1/2}\mathrm{e}^{-r/2}\Big[-1 - \frac{3}{4r} + O(r^{-2})\Big]. \end{aligned}$$

As the derivatives of $\mathcal{O}$ are given by

$$\frac{\partial\mathcal{O}}{\partial x_1} = -\frac{1}{4\pi}\mathrm{e}^{x_1/2}\Big[K_0\Big(\frac{r}{2}\Big) + \frac{x_1}{r}K_0'\Big(\frac{r}{2}\Big)\Big], \tag{3.7}$$

$$\frac{\partial\mathcal{O}}{\partial x_2} = -\frac{x_2}{4\pi r}\mathrm{e}^{x_1/2}K_0'\Big(\frac{r}{2}\Big), \tag{3.8}$$

we can deduce the behavior of the fundamental solution $\mathcal{O}$ and these derivatives when r tends to infinity:

$$\mathcal{O}(\boldsymbol{x}) = -\frac{1}{2\sqrt{\pi r}}\mathrm{e}^{-s/2}\Big[1 - \frac{1}{4r} + O(r^{-2})\Big], \tag{3.9}$$

$$\frac{\partial\mathcal{O}}{\partial x_1} = -\frac{1}{4\sqrt{\pi r}}\mathrm{e}^{-s/2}\Big[\frac{s}{r} - \frac{r+3x_1}{8r^2} + O(r^{-2})\Big], \tag{3.10}$$

$$\frac{\partial\mathcal{O}}{\partial x_2} = \frac{x_2}{4r\sqrt{\pi r}}\mathrm{e}^{-s/2}\Big[1 + \frac{3}{4r} + O(r^{-2})\Big]. \tag{3.11}$$

Using the inequality

$$\forall\, b\in\mathbb{R}\quad \mathrm{e}^{-s/2} \leqslant C_b(1+s)^b,$$

we obtain the following anisotropic estimates

$$
|\mathcal{O}(\boldsymbol{x})| \leqslant Cr^{-1/2}(1+s)^{-1}, \quad \Big|\frac{\partial \mathcal{O}}{\partial x_1}(\boldsymbol{x})\Big| \leqslant Cr^{-3/2}(1+s)^{-1}, \quad \Big|\frac{\partial \mathcal{O}}{\partial x_2}(\boldsymbol{x})\Big| \leqslant Cr^{-1}(1+s)^{-1}. \tag{3.12}
$$

Let f and g be two functions defined on an interval $I \subset \mathbb{R}$. We denote $f \sim g$ on $J \subset I$ if there exist two positive constants C_1 and C_2 such that $C_1 g(t) \leqslant f(t) \leqslant C_2 g(t)$ for all t in J.

To study the integrability properties of the fundamental solution and its derivatives, we need the following result.

Lemma 3.2. *Assume that $2-\alpha-\min(\frac{1}{2},\beta)<0$. Then, there exists a constant $C>0$ such that, for all $\mu>1$, we have*

$$
\int_{|\boldsymbol{x}|>\mu} r^{-\alpha}(1+s)^{-\beta}\,\mathrm{d}\boldsymbol{x} \leqslant \begin{cases} C\mu^{2-\alpha-\min(\frac{1}{2},\beta)} & \text{if } \beta \neq \frac{1}{2},\\ C\mu^{3/2-\alpha}\ln r & \text{if } \beta = \frac{1}{2}. \end{cases} \tag{3.13}
$$

Proof. First we prove the following result:

$$
\int_{\partial B_r} r^{-\alpha}(1+s)^{-\beta}\,\mathrm{d}\sigma \sim \begin{cases} r^{1-\alpha-\min(\frac{1}{2},\beta)} & \text{if } \beta \neq \frac{1}{2},\\ r^{\frac{1}{2}-\alpha}\ln r & \text{if } \beta = \frac{1}{2}. \end{cases} \tag{3.14}
$$

Using the polar coordinates, we have for $s = r(1-\cos\theta)$:

$$
I = \int_{\partial B_r} r^{-\alpha}(1+s)^{-\beta}\,\mathrm{d}\sigma = 2r^{1-\alpha}\int_0^{\pi} (1+r(1-\cos\theta))^{-\beta}\,\mathrm{d}\theta.
$$

Since $r^2\sin^2\theta = 2rs - s^2$,

$$
I = 2r^{1-\alpha}\int_0^{2r} (1+s)^{-\beta}(2rs-s^2)^{-1/2}\,\mathrm{d}s.
$$

i) When $0<s\leqslant 1$, $1+s\sim 1$, thus

$$
\int_0^1 (1+s)^{-\beta}(2rs-s^2)^{-1/2}\,\mathrm{d}s \sim r^{-1/2}\int_0^1 s^{-1/2}\,\mathrm{d}s \sim r^{-1/2}.
$$

ii) When $1<s<r$, $1+s\sim s$ and $2rs-s^2 = s(2r-s)\sim rs$, thus

$$
\int_1^r (1+s)^{-\beta}(2rs-s^2)^{-1/2}\,\mathrm{d}s \sim r^{-1/2}\int_1^r s^{-1/2-\beta}\,\mathrm{d}s \sim r^{-\min(\frac{1}{2},\beta)},
$$

and, if $\beta = \frac{1}{2}$, we get

$$\int_1^r (1+s)^{-\beta}(2rs - s^2)^{-1/2}\,\mathrm{d}s \sim r^{-1/2}\ln r.$$

iii) When $r < s < 2r$, $1 + s \sim r$ and $2rs - s^2 \sim r(2r - s)$, thus

$$\int_r^{2r} (1+s)^{-\beta}(2rs - s^2)^{-1/2}\,\mathrm{d}s \sim r^{-1/2-\beta}\int_r^{2r}(2r-s)^{-1/2}\,\mathrm{d}s \sim r^{-\beta}.$$

So,

$$I \sim r^{1-\alpha-\min(\frac{1}{2},\beta)}\big(r^{\min(\frac{1}{2},\beta)-\frac{1}{2}} + 1 + r^{\min(\frac{1}{2},\beta)-\beta}\big) \sim \begin{cases} r^{1-\alpha-\min(\frac{1}{2},\beta)} & \text{if } \beta \neq \frac{1}{2}, \\ r^{\frac{1}{2}-\alpha}\ln r & \text{if } \beta = \frac{1}{2}. \end{cases}$$

By this equivalence we deduce:

$$\int_{|\boldsymbol{x}|>\mu} r^{-\alpha}(1+s)^{-\beta}\,\mathrm{d}\boldsymbol{x} < +\infty \Longleftrightarrow 2 - \alpha - \min\Big(\frac{1}{2},\beta\Big) < 0. \tag{3.15}$$

When this condition is satisfied we obtain our result. □

Using Lemma 3.2 with estimate (3.12), we deduce

$$\forall\, p > 3 \quad \mathcal{O} \in L^p(\mathbb{R}^2) \quad \text{and} \quad \forall\, p \in \Big]\frac{3}{2}, 2\Big[\quad \nabla\mathcal{O} \in \boldsymbol{L}^p(\mathbb{R}^2), \tag{3.16}$$

which means in particular that $\mathcal{O} \in W_0^{1,p}(\mathbb{R}^2)$ for any $\frac{3}{2} < p < 2$. Note also that

$$\mathcal{O} \in L^1_{\mathrm{loc}}(\mathbb{R}^2) \quad \text{and} \quad \nabla\mathcal{O} \in \boldsymbol{L}^1_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.17}$$

and for $\mathcal{B}^R = \mathbb{R}^2 \setminus \overline{\mathcal{B}(\mathbf{0}, R)}$

$$\forall\, p > 3 \quad \mathcal{O} \in L^p(\mathcal{B}^R) \quad \text{and} \quad \forall\, p > \frac{3}{2} \quad \nabla\mathcal{O} \in \boldsymbol{L}^p(\mathcal{B}^R). \tag{3.18}$$

With the weighted L^∞ estimates obtained in [10, Theorems 3.5, 3.7, and 3.8], we get estimates on the convolution of $\breve{\mathcal{O}}$ with a function $\varphi \in \mathcal{D}(\mathbb{R}^2)$ as follows.

Lemma 3.3. *For any $\varphi \in \mathcal{D}(\mathbb{R}^2)$ we have the estimates*

$$|\breve{\mathcal{O}} * \varphi(\boldsymbol{x})| \leqslant C_\varphi \frac{1}{|\boldsymbol{x}|^{1/2}(1+|\boldsymbol{x}|+x_1)^{1/2}}, \tag{3.19}$$

$$\Big|\frac{\partial}{\partial x_1}(\breve{\mathcal{O}} * \varphi)(\boldsymbol{x})\Big| \leqslant C_\varphi \frac{1}{|\boldsymbol{x}|^{3/2}(1+|\boldsymbol{x}|+x_1)^{1/2}}, \tag{3.20}$$

$$\Big|\frac{\partial}{\partial x_2}(\breve{\mathcal{O}} * \varphi)(\boldsymbol{x})\Big| \leqslant C_\varphi \frac{1}{|\boldsymbol{x}|(1+|\boldsymbol{x}|+x_1)}, \tag{3.21}$$

where C_φ depends on the support of φ and $\breve{\mathcal{O}}(\boldsymbol{x}) = \mathcal{O}(-\boldsymbol{x})$.

R e m a r k 3.4. 1) The dependence on $|\boldsymbol{x}|$ of $\breve{\mathcal{O}} * \varphi$ and its first derivatives is the same that of $\breve{\mathcal{O}}$, but the dependence on $1 + s'$ is a little bit different.

2) By Lemma 3.2 and these last estimates we find that

$$\forall\, q > \frac{3}{2} \quad \breve{\mathcal{O}} * \varphi \in W_0^{1,q}(\mathbb{R}^2). \tag{3.22}$$

3.3. Oseen potential and existence results

Using the *weak-type* (p, q) operators and the Marcinkiewicz Interpolation Theorem, we have the following.

Theorem 3.5. *Let f be given in $L^p(\mathbb{R}^2)$. Then $\partial^2\mathcal{O}/\partial x_j \partial x_k * f \in L^p(\mathbb{R}^2)$, $\partial\mathcal{O}/\partial x_1 * f \in L^p(\mathbb{R}^2)$ and they satisfy the estimate*

$$\Big\|\frac{\partial^2\mathcal{O}}{\partial x_j \partial x_k} * f\Big\|_{L^p(\mathbb{R}^2)} + \Big\|\frac{\partial\mathcal{O}}{\partial x_1} * f\Big\|_{L^p(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.23}$$

Moreover:

i) *If $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^{3p/(3-2p)}(\mathbb{R}^2)$ and*

$$\|\mathcal{O} * f\|_{L^{3p/(3-2p)}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.24}$$

ii) *If $1 < p < 3$, then $(\partial\mathcal{O}/\partial x_i) * f \in L^{3p/(3-p)}(\mathbb{R}^2)$ and*

$$\Big\|\frac{\partial\mathcal{O}}{\partial x_i} * f\Big\|_{L^{3p/(3-p)}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.25}$$

P r o o f. By the Fourier transform, we obtain from Equation (3.1):

$$\mathcal{F}\Big(\frac{\partial^2\mathcal{O}}{\partial x_j \partial x_k} * f\Big) = \frac{-4\pi^2\xi_j\xi_k}{4\pi^2|\boldsymbol{\xi}|^2 + 2\pi \mathrm{i}\xi_1}\mathcal{F}(f).$$

The function $\xi \mapsto m(\xi) = (-4\pi^2\xi_j\xi_k)/(4\pi^2|\boldsymbol{\xi}|^2 + 2\pi i\xi_1)$ is of class $\mathcal{C}^2$ in $\mathbb{R}^2 \setminus \{0\}$ and satisfies for every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$

$$\Big|\frac{\partial^{|\alpha|} m}{\partial \xi^\alpha}(\xi)\Big| \leqslant B|\xi|^{-\alpha},$$

where $|\alpha| = \alpha_1 + \alpha_2$ and B is a constant. Thus, the linear operator

$$T\colon\ f \mapsto \frac{\partial^2 \mathcal{O}}{\partial x_j \partial x_k} * f(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathrm{e}^{2\pi i \boldsymbol{x}\boldsymbol{\xi}} \frac{-4\pi^2\xi_j\xi_k}{4\pi^2|\boldsymbol{\xi}|^2 + 2\pi i\xi_1} \mathcal{F}(f)(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$

is continuos from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$. So, $\partial^2\mathcal{O}/\partial x_j \partial x_k * f \in L^p(\mathbb{R}^2)$ and satisfies

$$\Big\|\frac{\partial^2 \mathcal{O}}{\partial x_j \partial x_k} * f\Big\|_{L^p(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)} \tag{3.26}$$

(see Stein [17, Theorem 3.2, p. 96]). Now, from Equation (3.1), we deduce that $\partial\mathcal{O}/\partial x_1 * f \in L^p(\mathbb{R}^2)$ and the estimate

$$\Big\|\frac{\partial \mathcal{O}}{\partial x_1} * f\Big\|_{L^p(\mathbb{R}^2)} \leqslant C(\|\Delta\mathcal{O} * f\|_{L^p(\mathbb{R}^2)} + \|f\|_{L^p(\mathbb{R}^2)}), \tag{3.27}$$

which proves the first part of the proposition and Estimate (3.23). Next, to prove i) and ii), we adapt the technique used by Stein in [17] which studied the convolution of $f \in L^p(\mathbb{R}^n)$ with the kernel $|\boldsymbol{x}|^{\alpha-n}$. We split the function K into $K_1 + K_\infty$, where

$$\begin{gathered} K_1(\boldsymbol{x}) = K(\boldsymbol{x}) \text{ if } |\boldsymbol{x}| \leqslant \mu \quad \text{and} \quad K_1(\boldsymbol{x}) = 0 \text{ if } |\boldsymbol{x}| > \mu, \\ K_\infty(\boldsymbol{x}) = 0 \text{ if } |\boldsymbol{x}| \leqslant \mu \quad \text{and} \quad K_\infty(\boldsymbol{x}) = K(\boldsymbol{x}) \text{ if } |\boldsymbol{x}| > \mu. \end{gathered}$$

The function K denotes successively $\mathcal{O}$ and $\partial\mathcal{O}/\partial x_i$ and the positive number μ will be fixed in the sequel.

1) Estimate (3.24). According to (3.6), we have $\mathcal{O}_1 \in L^1(\mathbb{R}^2)$ and by (3.16), $\mathcal{O}_\infty \in L^{p'}(\mathbb{R}^2)$, thus $\mathcal{O}_1 * f$ exists almost everywhere and $\mathcal{O}_\infty * f$ exists everywhere, so $\mathcal{O} * f = \mathcal{O}_1 * f + \mathcal{O}_\infty * f$ exists almost everywhere. Next, we shall show that $f \mapsto \mathcal{O} * f$ is of *weak-type* (p, q) with $q = 3p/(3-2p)$ in the sense that:

$$\operatorname{mes}\{\boldsymbol{x};\ |(\mathcal{O} * f)(\boldsymbol{x})| > \lambda\} \leqslant \Big(C_{p,q}\frac{\|f\|_{L^p(\mathbb{R}^2)}}{\lambda}\Big)^q, \quad \text{for all } \lambda > 0. \tag{3.28}$$

We have:

$$\operatorname{mes}\{\boldsymbol{x};\ (\mathcal{O} * f)(\boldsymbol{x}) > 2\lambda\} \leqslant \operatorname{mes}\{\boldsymbol{x};\ (\mathcal{O}_1 * f)(\boldsymbol{x}) > \lambda\} + \operatorname{mes}\{\boldsymbol{x};\ (\mathcal{O}_\infty * f)(\boldsymbol{x}) > \lambda\},$$

and

$$\mathrm{mes}\{\boldsymbol{x};\ |(\mathcal{O}_1 * f)(\boldsymbol{x})| > \lambda\} \leqslant \frac{\|\mathcal{O}_1\|^p_{L^1(\mathbb{R}^2)}\|f\|^p_{L^p(\mathbb{R}^2)}}{\lambda^p},$$
$$\|\mathcal{O}_\infty * f\|_{L^\infty(\mathbb{R}^2)} \leqslant \|\mathcal{O}_\infty\|_{L^{p'}(\mathbb{R}^2)}\|f\|_{L^p(\mathbb{R}^2)}.$$

Note that it is enough to prove the inequality (3.28) for $\|f\|_{L^p(\mathbb{R}^2)} = 1$.

i) Estimate of $I = \int_{|\boldsymbol{x}|<\mu} |\mathcal{O}(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x}$.
If $0 < \mu \leqslant 1$, then by (3.6), $I \leqslant C\mu$.

If $\mu > 1$,

$$I = \int_{|\boldsymbol{x}|<1} |\mathcal{O}(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} + \int_{1<|\boldsymbol{x}|\leqslant\mu} |\mathcal{O}(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x}.$$

Since $\mathcal{O} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$,

$$\int_{|\boldsymbol{x}|<1} |\mathcal{O}(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \leqslant C \leqslant C\mu.$$

Further, from the estimate (3.12) and by using Lemma 3.2, we have

$$\int_{1<|\boldsymbol{x}|<\mu} |\mathcal{O}(\boldsymbol{x})|\,\mathrm{d}\boldsymbol{x} \leqslant C\int_{1<|\boldsymbol{x}|<\mu} r^{-1/2}(1+s)^{-1}\,\mathrm{d}\boldsymbol{x} \leqslant C\mu,$$

thus

$$\forall\,\mu > 0 \quad \|\mathcal{O}_1\|_{L^1(\mathbb{R}^2)} \leqslant C\mu. \tag{3.29}$$

ii) Estimate of $J = \int_{|\boldsymbol{x}|>\mu} |\mathcal{O}(\boldsymbol{x})|^{p'}\,\mathrm{d}\boldsymbol{x}$.
If $\mu > 1$, $|\mathcal{O}(\boldsymbol{x})|^{p'} \sim \mathrm{e}^{-p's/2}r^{-p'/2} \leqslant Cr^{-p'/2}(1+s)^{-p'}$. Thus by Lemma 3.2, for $p' > 3$, we have $J \leqslant C\mu^{3/2-p'/2}$.
If $0 < \mu \leqslant 1$,

$$J = \int_{\mu<|\boldsymbol{x}|<1} |\mathcal{O}(\boldsymbol{x})|^{p'}\,\mathrm{d}\boldsymbol{x} + \int_{|\boldsymbol{x}|>1} |\mathcal{O}(\boldsymbol{x})|^{p'}\,\mathrm{d}\boldsymbol{x} = J_1 + J_2.$$

Proceeding as previously, we get $J_2 \leqslant C \leqslant C\mu^{3/2-p'/2}$. We also have

$$J_1 = \int_{\mu<|\boldsymbol{x}|\leqslant 1} \mathrm{e}^{p'x_1/2}|-\ln r + 2\ln 2 + \gamma + o(r)|^{p'}\,\mathrm{d}\boldsymbol{x} \leqslant C \leqslant C\mu^{3/2-p'/2}.$$

Thus

$$\text{for } p' > 3 \text{ and } \mu > 0 \quad \|\mathcal{O}_\infty\|_{L^{p'}(\mathbb{R}^2)} \leqslant C\mu^{(3-p')/2p'}. \tag{3.30}$$

Setting $\lambda = C\mu^{(3-p')/2p'}$, which implies $\mu = C'\lambda^{2p'/(3-p')} = C'\lambda^{2p/(2p-3)}$, we get

$$\operatorname{mes}\{\boldsymbol{x} \in \mathbb{R}^2\,;\ |(\mathcal{O}_\infty * f)(\boldsymbol{x})| > \lambda\} = 0.$$

So, for $1 < p < \frac{3}{2}$, we have

$$\operatorname{mes}\{\boldsymbol{x} \in \mathbb{R}^2\,;\ (|\mathcal{O} * f)(\boldsymbol{x})| > 2\lambda\} \leqslant C\frac{\|\mathcal{O}_1\|^p_{L^1(\mathbb{R}^2)}}{\lambda^p} \leqslant C\frac{\mu^p}{\lambda^p} \leqslant C\Big(\frac{1}{\lambda}\Big)^{3p/(3-2p)},$$

which proves the inequality (3.28).

2. Estimate (3.25). We also have $K_1 \in L^1(\mathbb{R}^2)$ and $K_\infty \in L^{p'}(\mathbb{R}^2)$, where $K = \partial\mathcal{O}/\partial x_i$, $i = 1, 2$.

i) Estimate of $\int_{|\boldsymbol{x}|>\mu} |\partial\mathcal{O}/\partial x_i(\boldsymbol{x})|^{p'}\,\mathrm{d}\boldsymbol{x}$.
Using estimate (3.12), we get for $\mu \geqslant 1$ and $p < 3$:

$$\int_{|\boldsymbol{x}|>\mu}\Big|\frac{\partial\mathcal{O}}{\partial x_i}(\boldsymbol{x})\Big|^{p'}\,\mathrm{d}\boldsymbol{x} \leqslant C\mu^{3/2-3p'/2} \leqslant C\mu^{3/2-p'}. \tag{3.31}$$

For $\mu < 1$,

$$\int_{|\boldsymbol{x}|>\mu}\Big|\frac{\partial\mathcal{O}}{\partial x_i}(\boldsymbol{x})\Big|^{p'}\,\mathrm{d}\boldsymbol{x} = \int_{\mu<|\boldsymbol{x}|<1}\Big|\frac{\partial\mathcal{O}}{\partial x_i}(\boldsymbol{x})\Big|^{p'}\,\mathrm{d}\boldsymbol{x} + \int_{|\boldsymbol{x}|>1}\Big|\frac{\partial\mathcal{O}}{\partial x_i}(\boldsymbol{x})\Big|^{p'}\,\mathrm{d}\boldsymbol{x}.$$

The case $\mu \geqslant 1$ yields

$$\int_{|\boldsymbol{x}|>1}\Big|\frac{\partial\mathcal{O}}{\partial x_i}(\boldsymbol{x})\Big|^{p'}\,\mathrm{d}\boldsymbol{x} \leqslant C \leqslant C\mu^{3/2-p'}.$$

We also have

$$\begin{aligned}\int_{\mu<|\boldsymbol{x}|<1}\Big|\frac{\partial\mathcal{O}}{\partial x_i}(\boldsymbol{x})\Big|^{p'}\,\mathrm{d}\boldsymbol{x} &\leqslant \int_\mu^1 r^{1-q}\,\mathrm{d}r\int_0^\pi \mathrm{e}^{(p'/2)r\cos\theta}|\sin\theta + C'|^{p'}\,\mathrm{d}\theta\\ &\leqslant C\int_\mu^1 r^{1/2-q}\,\mathrm{d}r \leqslant C\mu^{3/2-p'}.\end{aligned}$$

So, by these two inequalities and (3.31), we get

$$\Big\|\frac{\partial\mathcal{O}}{\partial x_i}\Big\|_{L^{p'}(\mathbb{R}^2)} \leqslant C\mu^{(3-2p')/p'}. \tag{3.32}$$

ii) Estimate of $J = \int_{|\boldsymbol{x}|<\mu} |\partial \mathcal{O}/\partial x_i(\boldsymbol{x})| \,\mathrm{d}\boldsymbol{x}$.

If $0 < \mu < 1$,

$$J = \int_{|\boldsymbol{x}|<\mu} \Big| \mathrm{e}^{x_1/2} \frac{x_2}{r^2} + o\Big(\frac{1}{r}\Big) \Big| \,\mathrm{d}\boldsymbol{x} = \int_0^{\mu} \int_{-\pi}^{\pi} \mathrm{e}^{(r/2)\cos\theta} |\sin\theta + C'| \,\mathrm{d}r \,\mathrm{d}\theta$$

$$\leqslant C \int_0^{\mu} \mathrm{d}r \leqslant C\mu \leqslant C\mu^{1/2}.$$

If $\mu \geqslant 1$,

$$J = \int_{|\boldsymbol{x}|<1} \Big| \frac{\partial \mathcal{O}}{\partial x_i} \Big| \,\mathrm{d}\boldsymbol{x} + \int_{1<|\boldsymbol{x}|<\mu} \Big| \frac{\partial \mathcal{O}}{\partial x_i} \Big| \,\mathrm{d}\boldsymbol{x} = J_1 + J_2.$$

The preceding case yields $J_1 \leqslant C \leqslant C\mu^{1/2}$. By Estimate (3.12) and Lemma 3.2 we have

$$J_2 \leqslant C \int_{|\boldsymbol{x}|<\mu} \frac{\mathrm{d}\boldsymbol{x}}{r(1+s)} \leqslant C \int_0^{\mu} r^{-1/2} \,\mathrm{d}r \leqslant C\mu^{1/2}.$$

We obtain then

$$\Big\| \frac{\partial \mathcal{O}}{\partial x_i} \Big\|_{L^1(\mathbb{R}^2)} \leqslant C\mu^{1/2}. \tag{3.33}$$

As previously, we have, for $1 < p < 3$ and all $\lambda > 0$:

$$\operatorname{mes}\Big\{ \boldsymbol{x} \in \mathbb{R}^2 \,;\; \Big| \Big(\frac{\partial \mathcal{O}}{\partial x_i} * f \Big)(\boldsymbol{x}) \Big| > 2\lambda \Big\} \leqslant C \Big(\frac{1}{\lambda} \Big)^{3p/(3-p)}.$$

Now, using the Marcinkiewicz Theorem, the operator $R\colon\ f \mapsto \mathcal{O} * f$ is continuous from $L^p(\mathbb{R}^2)$ into $L^{3p/(3-2p)}(\mathbb{R}^2)$ and $R_i\colon\ f \mapsto \partial \mathcal{O}/\partial x_i * f$ is continuous from $L^p(\mathbb{R}^2)$ into $L^{3p/(3-p)}(\mathbb{R}^2)$. $\square$

R e m a r k 3.6. i) We can prove that $\mathcal{O} \in L^{3,\infty}(\mathbb{R}^2)$, i.e.

$$\sup_{\mu>0} \mu^3 \operatorname{mes}\{ \boldsymbol{x} \in \mathbb{R}^2 \,;\; |\mathcal{O}(\boldsymbol{x})| > \mu \} < +\infty. \tag{3.34}$$

So that, thanks to the weak Young inequality (cf. Reed and Simon [16]):

$$\|\mathcal{O} * f\|_{L^{3p/(3-2p),\infty}(\mathbb{R}^2)} \leqslant C \|\mathcal{O}\|_{L^{3,\infty}(\mathbb{R}^2)} \|f\|_{L^{p,\infty}(\mathbb{R}^2)}. \tag{3.35}$$

This estimate shows that if $1 < p < \frac{3}{2}$, then there exist p_0 and p_1 such that $1 < p_0 < p < p_1 < \frac{3}{2}$ and such that the operator

$$T\colon\ f \mapsto \mathcal{O} * f$$

is continuous from $L^{p_0}(\mathbb{R}^2)$ into $L^{(3p_0)/(3-2p_0),\infty}(\mathbb{R}^2)$ as well as from $L^{p_1}(\mathbb{R}^2)$ into $L^{(3p_1)/(3-2p_1),\infty}(\mathbb{R}^2)$. The Marcinkiewicz interpolation theorem allows again to conclude that the operator $T\colon\ L^p(\mathbb{R}^2) \longrightarrow L^{3p/(3-2p)}(\mathbb{R}^2)$ is continuous.

ii) The same remark is true for $\nabla\mathcal{O}$ which belongs to $\boldsymbol{L}^{3/2,\infty}(\mathbb{R}^2)$.

By Theorem 3.5 and the Sobolev embedding we easily obtain the following result.

Theorem 3.7. *Let $f \in L^p(\mathbb{R}^2)$ with $1 < p < \infty$. Then, $\partial^2\mathcal{O}/\partial x_j\partial x_k * f \in L^p(\mathbb{R}^2)$, $\partial\mathcal{O}/\partial x_1 * f \in L^p(\mathbb{R}^2)$ and they satisfy the estimate*

$$\left\|\frac{\partial^2\mathcal{O}}{\partial x_j\partial x_k} * f\right\|_{L^p(\mathbb{R}^2)} + \left\|\frac{\partial\mathcal{O}}{\partial x_1} * f\right\|_{L^p(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.36}$$

Moreover:
1) i) *If $1 < p < 2$, then $\nabla\mathcal{O} * f \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2) \cap \boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)$ and*

$$\|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2)} + \|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.37}$$

ii) *If $p = 2$, then $\nabla\mathcal{O} * f \in \boldsymbol{L}^r(\mathbb{R}^2)$ for any $r \geqslant 6$ and the following estimate holds*

$$\|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^r(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.38}$$

iii) *If $2 < p < 3$, then $\nabla\mathcal{O} * f \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2) \cap \boldsymbol{L}^\infty(\mathbb{R}^2)$ and we have the estimate*

$$\|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2)} + \|\nabla\mathcal{O} * f\|_{\boldsymbol{L}^\infty(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.39}$$

2) *If $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^{3p/(3-2p)}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and*

$$\|\mathcal{O} * f\|_{L^{3p/(3-2p)}(\mathbb{R}^2)} + \|\mathcal{O} * f\|_{L^\infty(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.40}$$

Remark 3.8. i) Applying the Young Inequality and (3.16), we verify that if $f \in L^p(\mathbb{R}^2)$ with $1 < p < \frac{3}{2}$, then $\mathcal{O} * f \in L^q(\mathbb{R}^2)$ for all $q \in]3p/(3-2p), +\infty[$, a property a little weaker than (3.40).
ii) The same remark is true for $\nabla\mathcal{O} * f$.

By using Theorem 3.7 and Lemma 3.1, it is clear that if $f \in L^p(\mathbb{R}^2)$, then the solutions of Equation (3.1) are of the form

$$u = \mathcal{O} * f + Q, \quad \text{with } Q \in \mathcal{S}_{[2-3/p]}. \tag{3.41}$$

This means that $\mathcal{O} * f$ is the solution of Equation (3.1): unique if $1 < p < \frac{3}{2}$, up to a constant if $\frac{3}{2} \leqslant p < 3$, and up to an element of $\mathcal{S}_1$ if $p \geqslant 3$.
By Theorem 3.7, we have the following result for a given $f \in L^p(\mathbb{R}^2)$.

Theorem 3.9. *Let $f \in L^p(\mathbb{R}^2)$, then Equation* (3.1) *has at least a solution u of the form* (3.41) *such that $\nabla^2 u \in (L^p(\mathbb{R}^2))^{2\times 2}$, $\partial u/\partial x_1 \in L^p(\mathbb{R}^2)$, and*

$$\|\nabla^2 u\|_{\boldsymbol{L}^p(\mathbb{R}^2)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{L^p(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.42}$$

Moreover:
1) *If $1 < p < \frac{3}{2}$, then $u \in L^{3p/(3-2p)}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, $\nabla u \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2) \cap \boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)$ and they satisfy*

$$\begin{aligned}\|u\|_{L^{3p/(3-2p)}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2)}\\ + \|\nabla u\|_{\boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}.\end{aligned} \tag{3.43}$$

2) i) *If $\frac{3}{2} \leqslant p < 2$, then $\nabla u \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2) \cap \boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)$ and*

$$\|\nabla u\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.44}$$

ii) *If $p = 2$, then $\nabla u \in \boldsymbol{L}^r(\mathbb{R}^2)$ for any $r \geqslant 6$ and the following estimate holds:*

$$\|\nabla u\|_{\boldsymbol{L}^r(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.45}$$

iii) *If $2 < p < 3$, then $\nabla u \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2) \cap \boldsymbol{L}^\infty(\mathbb{R}^2)$ and*

$$\|\nabla u\|_{\boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^\infty(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.46}$$

3) *If $p \geqslant 3$, then $u \in W_0^{2,p}(\mathbb{R}^2)$ and we have the estimate*

$$\inf_{\lambda \in \mathcal{S}_1} \|u + \lambda\|_{W_0^{2,p}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{3.47}$$

Remark 3.10. Another demonstration of Theorem 3.9 consists in using the Fourier approach. Let $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^2)$ be a sequence converging to f in $L^p(\mathbb{R}^2)$. Then the sequence (u_j) given by

$$u_j = \mathcal{F}^{-1}(m_0(\boldsymbol{\xi})\hat{f}_j), \quad m_0(\boldsymbol{\xi}) = (4\pi|\boldsymbol{\xi}|^2 + 2\mathrm{i}\pi\xi_1)^{-1}, \tag{3.48}$$

is a solution of Equation (3.1) with the right-hand side f_j. Let us recall now the

Lizorkin Theorem (see [12]). *Let $D = \{\boldsymbol{\xi} \in \mathbb{R}^2;\ |\xi_1| > 0, |\xi_2| > 0\}$ and $m\colon D \longrightarrow \mathbb{C}$ be a continuous function such that its derivatives $\partial^k m/\partial\xi_1^{k_1}\partial\xi_2^{k_2}$ are continuous and satisfy*

$$|\xi_1|^{k_1+\beta}|\xi_2|^{k_2+\beta}\Big|\frac{\partial^k m}{\partial\xi_1^{k_1}\partial\xi_2^{k_2}}\Big| \leqslant M, \tag{3.49}$$

where $k_1, k_2 \in \{0, 1\}$, $k = k_1 + k_2$, and $0 \leqslant \beta < 1$. Then, the operator

$$T\colon\ g \mapsto \mathcal{F}^{-1}(m_0\mathcal{F}(g)), \quad m_0(\boldsymbol{\xi}) = \frac{1}{4\pi^2|\boldsymbol{\xi}|^2 + 2\mathrm{i}\pi\xi_1}$$

is continuous from $L^p(\mathbb{R}^2)$ into $L^r(\mathbb{R}^2)$ with $1/r = 1/p - \beta$.

Applying this continuity property with $f_j \in L^p(\mathbb{R}^2)$ and $\beta = \frac{2}{3}$, we show that (u_j) is bounded in $L^{3p/(3-2p)}(\mathbb{R}^2)$ if $1 < p < \frac{3}{2}$, so this sequence admits a subsequence still denoted (u_j) which converges weakly to a solution u of Equation (3.1) with right-hand side f. For the derivative of u_j with respect to x_1, the multiplier which intervenes is of the form $m(\boldsymbol{\xi}) = 2\mathrm{i}\pi\xi_1(4\pi^2|\boldsymbol{\xi}|^2 + 2\mathrm{i}\pi\xi_1)^{-1}$, so that (3.49) is satisfied for $\beta = 0$, so $r = p$. The same property takes place for the second derivatives with $m(\boldsymbol{\xi}) = -4\pi^2\xi_1\xi_2(4\pi^2|\boldsymbol{\xi}|^2 + 2\mathrm{i}\pi\xi_1)^{-1}$. Finally, we verify, with $\beta = \frac{1}{3}$, that the first derivative of (u_j) with respect to x_2 is bounded in $L^{3p/(3-p)}(\mathbb{R}^2)$, which implies that $\partial u/\partial x_2 \in L^{3p/(3-p)}(\mathbb{R}^2)$.

In order to study Equation (3.1) with a right-hand side $f \in W_0^{-1,p}(\mathbb{R}^2)$, we give the following definition of the convolution of f with the fundamental solution $\mathcal{O}$:

$$\forall\,\varphi \in \mathcal{D}(\mathbb{R}^2) \quad \langle \mathcal{O} * f, \varphi\rangle =: \langle f, \breve{\mathcal{O}} * \varphi\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)},$$

where $\breve{\mathcal{O}}(\boldsymbol{x}) = \mathcal{O}(-\boldsymbol{x})$.

Theorem 3.11. *Let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfy the compatibility condition*

$$\langle f, 1\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)} = 0, \quad \text{when } 1 < p \leqslant 2. \tag{3.50}$$

i) *If $1 < p < 3$, then $u = \mathcal{O} * f \in L^{3p/(3-p)}(\mathbb{R}^2)$ is the unique solution of Equation (3.1) such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$ and $\partial u/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$. Moreover, we have the estimate*

$$\|u\|_{L^{3p/(3-p)}(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^2)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{W_0^{-1,p}(\mathbb{R}^2)} \leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}, \tag{3.51}$$

and $u \in L^{2p/(2-p)}(\mathbb{R}^2)$ if $1 < p < 2$, $u \in L^r(\mathbb{R}^2)$ for any $r \geqslant 6$ if $p = 2$, and $u \in L^\infty(\mathbb{R}^2)$ if $2 < p < 3$.

ii) *If $p \geqslant 3$, then Equation* (3.1) *has a solution $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2)$ that is unique up to a constant, and we have*

$$\inf_{k\in\mathbb{R}} \|u+k\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^2)} \leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{3.52}$$

Proof. Let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfy the condition (3.50). Thanks to Lemma 3.3 and Remark 3.4, if $\varphi \to 0$ in $\mathcal{D}(\mathbb{R}^2)$, we have $\check{\mathcal{O}} * \varphi \to 0$ in $W_0^{1,p'}(\mathbb{R}^2)$ for all $p \in]1,3[$ which implies that $\mathcal{O} * f \in \mathcal{D}'(\mathbb{R}^2)$. We also know, by the isomorphism (2.7), that there exists $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^2)$ such that

$$f = \operatorname{div} \boldsymbol{F} \quad \text{and} \quad \|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{3.53}$$

i) Suppose now that $1 < p < 3$. Then,

$$\begin{aligned}
\Big\langle \frac{\partial}{\partial x_j}(\mathcal{O} * f), \varphi \Big\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)} &= -\Big\langle \mathcal{O} * f, \frac{\partial \varphi}{\partial x_j} \Big\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)} \\
&= \Big\langle \boldsymbol{F}, \nabla\Big(\check{\mathcal{O}} * \frac{\partial \varphi}{\partial x_j}\Big) \Big\rangle_{\boldsymbol{L}^p(\mathbb{R}^2)\times\boldsymbol{L}^{p'}(\mathbb{R}^2)} \\
&= \Big\langle \boldsymbol{F}, \nabla \frac{\partial}{\partial x_j}(\check{\mathcal{O}} * \varphi) \Big\rangle_{\boldsymbol{L}^p(\mathbb{R}^2)\times\boldsymbol{L}^{p'}(\mathbb{R}^2)}.
\end{aligned}$$

Moreover, by (3.23),

$$\begin{aligned}
\Big|\Big\langle \frac{\partial}{\partial x_j}(\mathcal{O} * f), \varphi \Big\rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)}\Big| &\leqslant \|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \Big\|\nabla \frac{\partial}{\partial x_j}(\check{\mathcal{O}} * \varphi)\Big\|_{\boldsymbol{L}^{p'}(\mathbb{R}^2)} \\
&\leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} \|\varphi\|_{L^{p'}(\mathbb{R}^2)}.
\end{aligned}$$

That is,

$$\Big\|\frac{\partial}{\partial x_j}(\mathcal{O} * f)\Big\|_{L^p(\mathbb{R}^2)} \leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

With the same condition on p as in the previous case, for all $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\langle \mathcal{O} * f, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)} = -\langle \boldsymbol{F}, \nabla(\check{\mathcal{O}} * \varphi) \rangle_{\boldsymbol{L}^p(\mathbb{R}^2)\times\boldsymbol{L}^{p'}(\mathbb{R}^2)},$$

and by (3.25)

$$\begin{aligned}
|\langle \mathcal{O} * f, \varphi \rangle_{\mathcal{D}'(\mathbb{R}^2)\times\mathcal{D}(\mathbb{R}^2)}| &\leqslant \|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \Big\|\frac{\partial}{\partial x_j}(\check{\mathcal{O}} * \varphi)\Big\|_{\boldsymbol{L}^{p'}(\mathbb{R}^2)} \\
&\leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} \|\varphi\|_{L^{3p/(4p-3)}(\mathbb{R}^2)}.
\end{aligned}$$

Note that $1 < p < 3 \iff 1 < 3p/(4p-3) < 3$. Consequently, $\mathcal{O} * f \in L^{3p/(3-p)}(\mathbb{R}^2)$ and

$$\|\mathcal{O} * f\|_{L^{3p/(3-p)}(\mathbb{R}^2)} \leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

Moreover, by the Sobolev embedding, $\mathcal{O} * f \in L^{2p/(2-p)}(\mathbb{R}^2)$ if $1 < p < 2$, $\mathcal{O} * f$ belongs to $L^r(\mathbb{R}^2)$ for all $r \geqslant 6$ if $p = 2$ and belongs to $L^\infty(\mathbb{R}^2)$ if $2 < p < 3$. We have thus showed that if $1 < p < 3$, the operator

$$\text{(3.54)} \qquad \begin{aligned} R\colon W_0^{-1,p}(\mathbb{R}^2) \perp \mathcal{P}_{[1-2/p']} &\longrightarrow W_0^{1,p}(\mathbb{R}^2) \cap L^{3p/(3-p)}(\mathbb{R}^2), \\ f &\mapsto \mathcal{O} * f, \end{aligned}$$

is continuous.

ii) Suppose now that $p \geqslant 3$ and let $f \in W_0^{-1,p}(\mathbb{R}^2)$. Then we have the relation (3.53). Now, since $\boldsymbol{\mathcal{D}}(\mathbb{R}^2)$ is dense in $\boldsymbol{L}^p(\mathbb{R}^2)$, there exists a sequence $\boldsymbol{F}_m \in \boldsymbol{\mathcal{D}}(\mathbb{R}^2)$ such that $\boldsymbol{F}_m \to \boldsymbol{F}$ in $\boldsymbol{L}^p(\mathbb{R}^2)$. Set $f_m = \operatorname{div} \boldsymbol{F}_m$ and $\psi_m = \mathcal{O} * f_m$. For all $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\Big\langle \frac{\partial \psi_m}{\partial x_j}, \varphi \Big\rangle = \Big\langle \boldsymbol{F}_m, \nabla \frac{\partial}{\partial x_j}(\breve{\mathcal{O}} * \varphi) \Big\rangle.$$

Thus, according to the inequality (3.36), we have

$$\text{(3.55)} \qquad \begin{aligned} \Big|\Big\langle \frac{\partial \psi_m}{\partial x_j}, \varphi \Big\rangle\Big| &\leqslant C\|\boldsymbol{F}_m\|_{\boldsymbol{L}^p(\mathbb{R}^2)}\|\varphi\|_{L^{p'}(\mathbb{R}^2)} \\ &\leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}\|\varphi\|_{L^{p'}(\mathbb{R}^2)}. \end{aligned}$$

Hence, $\nabla\psi_m$ is bounded in $\boldsymbol{L}^p(\mathbb{R}^2)$. We can apply Theorem 2.1: for each m, there exists a constant C_m such that $\psi_m + C_m \in W_0^{1,p}(\mathbb{R}^2)$ and

$$\|\psi_m + C_m\|_{W_0^{1,p}(\mathbb{R}^2)} \leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}.$$

From this it follows that $\psi_m + C_m$ converges weakly to some function $u \in W_0^{1,p}(\mathbb{R}^2)$ and

$$-\Delta u + \frac{\partial u}{\partial x_1} = f,$$

so that Equation (3.1) admits a solution u and, moreover, $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2)$. $\square$

R e m a r k 3.12. i) If $1 < p < 2$, then, since the solution u of Equation (3.1) given by Theorem 3.11 belongs in particular to $W_0^{1,p}(\mathbb{R}^2)$, we deduce that

$$\lim_{|\boldsymbol{x}|\to\infty} u(\boldsymbol{x}) = 0$$

in the sense of Definition 2.8. Consequently, for any given constant u_∞, the distribution $v = u + u_\infty$ is the unique solution of Equation (3.1) that is such that $\nabla v \in \boldsymbol{L}^p(\mathbb{R}^2)$, $\partial v/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$, and

$$\lim_{|\boldsymbol{x}|\to\infty} v(\boldsymbol{x}) = u_\infty.$$

ii) If $2 < p < 3$, then, by Lemma 2.10, the same result holds with pointwise convergence.

Corollary 3.13. *Assume $1 < p < 3$. If u is a distribution such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$ and $\partial u/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$, then there exists a unique constant k such that $u + k \in L^{3p/(3-p)}(\mathbb{R}^2)$ and*

$$\|u+k\|_{L^{3p/(3-p)}(\mathbb{R}^2)} \leqslant C\Big(\|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^2)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{W_0^{-1,p}(\mathbb{R}^2)}\Big). \tag{3.56}$$

Moreover, if $1 < p < 2$, then $u+k \in L^{2p/(2-p)}(\mathbb{R}^2)$ and $u(\boldsymbol{x})$ tends to the constant $-k$ when $|\boldsymbol{x}|$ tends to infinity in the sense of Definition 2.8. If $p = 2$, then $u + k$ belongs to $L^r(\mathbb{R}^2)$ for any $r \geqslant 6$. If $2 < p < 3$, then u belongs to $L^\infty(\mathbb{R}^2)$, is continuous in $\mathbb{R}^2$, and tends to $-k$ pointwise.

Proof. Set $g = -\Delta u + \partial u/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$. Since $\mathcal{P}_{[1-2/p']}$ contains at most the constants and according to the density of $\mathcal{D}(\mathbb{R}^2)$ in $\widetilde{W}_0^{1,p}(\mathbb{R}^2)$, g satisfies the compatibility condition (3.50). By the previous theorem, there exists a unique $v \in L^{3p/(3-p)}(\mathbb{R}^2)$ such that $\nabla v \in \boldsymbol{L}^p(\mathbb{R}^2)$ and $\partial v/\partial x_1 \in L^p(\mathbb{R}^2)$, and satisfying both $T(u - v) = 0$ (T is the Oseen operator, see (3.2)) and the estimate

$$\begin{aligned}\|v\|_{L^{3p/(3-p)}(\mathbb{R}^2)} &\leqslant C\Big(\|\Delta u\|_{W_0^{-1,p}(\mathbb{R}^2)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{W_0^{-1,p}(\mathbb{R}^2)}\Big)\\ &\leqslant C\Big(\|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^2)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{W_0^{-1,p}(\mathbb{R}^2)}\Big).\end{aligned} \tag{3.57}$$

Setting $w = u - v$, we have for $i = 1, 2$ that $\partial w/\partial x_i \in L^p(\mathbb{R}^2)$ and $T(\partial w/\partial x_i) = 0$. We deduce then by Lemma 3.1 that $\nabla u = \nabla v$, thus there exists a unique constant k such that $v = u + k$. The last properties are consequences of Lemma 2.6 and Lemma 2.10.

Remark 3.14. Let $u \in \mathcal{D}'(\mathbb{R}^2)$ be such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$.

i) When $1 < p < 2$, thanks to Proposition 2.2, we know that there exists a unique constant k such that $u + k \in L^{2p/(2-p)}(\mathbb{R}^2)$. Here, by the fact that in addition $\partial u/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$, we have even $u + k \in L^{3p/(3-p)}(\mathbb{R}^2)$.

ii) When $2 \leqslant p < 3$, u is only in $W_0^{1,p}(\mathbb{R}^2)$ but it is in no space $L^r(\mathbb{R}^2)$. But, if moreover $\partial u/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$, then $u + k \in L^{3p/(3-p)}(\mathbb{R}^2)$ for some unique constant k, and $u \in L^r(\mathbb{R}^2)$ for any $r \geqslant 6$ if $p = 2$, while $u \in L^\infty(\mathbb{R}^2)$ otherwise.

As a consequence of Theorems 3.9 and 3.11, we solve Equation (3.1) when the data f belongs to the intersection of two weighted spaces. We have then the two following results.

Proposition 3.15. *Suppose that $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_0^{-1,q}(\mathbb{R}^2)$ with $1 < p < q < \infty$ and that f satisfies the compatibility condition* (3.50). *Then, Equation* (3.1) *has a solution $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2) \cap \widetilde{W}_0^{1,q}(\mathbb{R}^2)$ satisfying*

$$\|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^q(\mathbb{R}^2)} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{W_0^{-1,p}} + \Big\|\frac{\partial u}{\partial x_1}\Big\|_{W_0^{-1,q}(\mathbb{R}^2)} \leqslant C(\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} + \|f\|_{W_0^{-1,q}(\mathbb{R}^2)}). \tag{3.58}$$

Moreover:

i) *The solution u is unique if $p < 3$ and unique up to a constant if $p \geqslant 3$. It is equal to $\mathcal{O} * f$ if $p < 3$.*

ii) *If $p < q < 2$, then $u \in L^{3p/(3-p)}(\mathbb{R}^2) \cap L^{2q/(2-q)}(\mathbb{R}^2)$ and*

$$\|u\|_{L^{3p/(3-p)}(\mathbb{R}^2)} + \|u\|_{L^{2q/(2-q)}(\mathbb{R}^2)} \leqslant C(\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} + \|f\|_{W_0^{-1,q}(\mathbb{R}^2)}). \tag{3.59}$$

iii) *If $p < q = 2$, then $u \in L^r(\mathbb{R}^2)$ for any $r \geqslant 3p/(3-p)$ and*

$$\|u\|_{L^r(\mathbb{R}^2)} \leqslant C(\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} + \|f\|_{W_0^{-1,q}(\mathbb{R}^2)}). \tag{3.60}$$

iv) *If $p < 3$ and $q > 2$ then $u \in L^{3p/(3-p)}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with the estimate*

$$\|u\|_{L^{3p/(3-p)}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)} \leqslant C(\|f\|_{W_0^{-1,p}(\mathbb{R}^2)} + \|f\|_{W_0^{-1,q}(\mathbb{R}^2)}). \tag{3.61}$$

Proof. Let $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_0^{-1,q}(\mathbb{R}^2)$ satisfy the compatibility condition (3.50) with $1 < p < q < \infty$. We know that there exist $u \in \widetilde{W}_0^{1,p}(\mathbb{R}^2)$ and $v \in \widetilde{W}_0^{1,q}(\mathbb{R}^2)$ which are solutions of (3.1). Moreover, by a uniqueness argument we have necessarily $\nabla u = \nabla v$ and Estimate (3.58) comes from (3.51).

i) If $p \geqslant 3$, then $u - v = k$, where k is an arbitrary constant, so $u = v + k \in \widetilde{W}_0^{1,p}(\mathbb{R}^2) \cap \widetilde{W}_0^{1,q}(\mathbb{R}^2)$. If $p < 3$, then $u = \mathcal{O} * f$.

ii) Suppose that $q < 2$, then we know that $u = \mathcal{O} * f \in L^{3p/(3-p)}(\mathbb{R}^2)$ and $u = v \in L^{2p/(2-p)}(\mathbb{R}^2)$ and u satisfies Estimate (3.59).

iii) If $q = 2$, then, by Theorem 3.11, $u = \mathcal{O} * f \in L^{3p/(3-p)}(\mathbb{R}^2)$ and $u = v \in L^r(\mathbb{R}^2)$ for any $r \geqslant 3p/(3-p)$.

iv) If $p < 3$ and $q > 2$, we know that $u = \mathcal{O} * f \in L^{3p/(3-p)}(\mathbb{R}^2)$. Since $\nabla u \in \boldsymbol{L}^q(\mathbb{R}^2)$ with $q > 2$, it follows that $u \in L^\infty(\mathbb{R}^2)$ and we have Estimate (3.61). □

R e m a r k 3.16. When $f \in W_0^{-1,q}(\mathbb{R}^2)$ with $q \geqslant 3$, we have seen that $\mathcal{O} * f$ is not necessarily defined. But if moreover $f \in W_0^{-1,p}(\mathbb{R}^2)$ with $p < 3$, and satisfies the compatibility condition (3.50), then $\mathcal{O} * f$ makes sense in $\widetilde{W}_0^{1,p}(\mathbb{R}^2)$ and belongs to $\widetilde{W}_0^{1,q}(\mathbb{R}^2)$.

Proposition 3.17. *Let $f \in L^p(\mathbb{R}^2) \cap W_0^{-1,p}(\mathbb{R}^2)$ satisfy the compatibility condition* (3.50)*. Then Equation* (3.1) *has a solution $u = \mathcal{O} * f$ such that $\nabla u \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^2)$, $\partial u / \partial x_1 \in W_0^{1,p}(\mathbb{R}^2) \cap W_0^{-1,p}(\mathbb{R}^2)$ and*

$$\|\nabla u\|_{\boldsymbol{W}^{1,p}} + \left\| \frac{\partial u}{\partial x_1} \right\|_{W^{1,p}} + \left\| \frac{\partial u}{\partial x_1} \right\|_{W^{-1,p}} \leqslant C(\|f\|_{L^p} + \|f\|_{W_0^{-1,p}}). \tag{3.62}$$

Moreover:

i) *If $p < \frac{3}{2}$, then u is unique, belongs to $L^{3p/(3-2p)}(\mathbb{R}^2) \cap W^{1,3p/(3-p)}(\mathbb{R}^2)$ and satisfies the estimate*

$$\|u\|_{L^{3p/(3-2p)}(\mathbb{R}^2)} + \|u\|_{W^{1,3p/(3-p)}(\mathbb{R}^2)} \leqslant C(\|f\|_{L^p} + \|f\|_{W_0^{-1,p}}). \tag{3.63}$$

ii) *If $\frac{3}{2} \leqslant p < 3$, then u is unique in $W^{1,3p/(3-p)}(\mathbb{R}^2)$ and satisfies the estimate*

$$\|u\|_{W^{1,3p/(3-p)}(\mathbb{R}^2)} \leqslant C(\|f\|_{L^p(\mathbb{R}^2)} + \|f\|_{W_0^{-1,p}(\mathbb{R}^2)}). \tag{3.64}$$

iii) *If $p \geqslant 3$, then u belongs to $W_0^{2,p}(\mathbb{R}^2) \cap \widetilde{W}_0^{1,p}(\mathbb{R}^2)$, is unique up to a constant, and*

$$\inf_{k \in \mathbb{R}} (\|u + k\|_{W_0^{2,p}(\mathbb{R}^2)} + \|u + k\|_{\widetilde{W}_0^{1,p}(\mathbb{R}^2)}) \leqslant C(\|f\|_{L^p} + \|f\|_{W_0^{-1,p}}). \tag{3.65}$$

P r o o f. The proof is the same as the one given for the previous proposition. □

Now we take f more regular, for example $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_1^{0,q}(\mathbb{R}^2)$, and we find what regularity we obtain for the solution u.

Proposition 3.18. *Let p and q be two real numbers such that $1 < p < \infty$, $q > 2$, and $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$. Suppose that $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_1^{0,q}(\mathbb{R}^2)$ and satisfies the compatibility condition* (3.50)*. Then the unique solution of Equation* (3.1) *given by Proposition* 3.15 *possesses the additional properties*

$$\nabla^2 u \in (W_1^{0,q}(\mathbb{R}^2))^{2\times 2} \quad \text{and} \quad \frac{\partial u}{\partial x_1} \in W_1^{0,q}(\mathbb{R}^2).$$

P r o o f. From the relation $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$ we have $1 < p < 2$, and since $q > 2$,

$$\mathcal{P}_{[1-2/q']} = \mathcal{P}_{[1-2/p]} = \{0\}.$$

Since $W_1^{0,q}(\mathbb{R}^2) \subset W_0^{-1,q}(\mathbb{R}^2)$, it follows that $f \in W_0^{-1,p}(\mathbb{R}^2) \cap W_0^{-1,q}(\mathbb{R}^2)$ and satisfies the compatibility condition (3.50) for p and q.

i) If $2 < q < 3$, then Equation (3.1) has a unique solution $u \in L^{3p/(3-p)}(\mathbb{R}^2) \cap L^{3q/(3-q)}(\mathbb{R}^2)$ such that $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2) \cap \boldsymbol{L}^q(\mathbb{R}^2)$ and $\partial u/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2) \cap W_0^{-1,q}(\mathbb{R}^2)$. Further,

$$-\Delta\Big(\varrho\frac{\partial u}{\partial x_j}\Big) + \frac{\partial}{\partial x_1}\Big(\varrho\frac{\partial u}{\partial x_j}\Big) = \varrho\frac{\partial f}{\partial x_j} - 2\nabla\varrho\nabla\Big(\frac{\partial u}{\partial x_j}\Big) - \frac{\partial u}{\partial x_j}\Delta\varrho + \frac{\partial u}{\partial x_j}\frac{\partial \varrho}{\partial x_1} =: F.$$

Since $\nabla u \in \boldsymbol{L}^q(\mathbb{R}^2)$, in view of (2.3), (2.2), and (2.4), the terms $\varrho\,\partial f/\partial x_j$, $\nabla\varrho\,\nabla(\partial u/\partial x_j)$, and $\partial u/\partial x_j\,\Delta\varrho$ belong to $W_0^{-1,q}(\mathbb{R}^2)$. On the other hand, since $\nabla u \in \boldsymbol{L}^p(\mathbb{R}^2)$, the term $\partial u/\partial x_j \cdot \partial\varrho/\partial x_1$ belongs to $L^p(\mathbb{R}^2)$. By the Sobolev embedding and the relation between p and q, $L^p(\mathbb{R}^2) \subset W_0^{-1,q}(\mathbb{R}^2)$ because $W_0^{1,q'}(\mathbb{R}^2) \subset L^{p'}(\mathbb{R}^n)$, and we deduce that $F \in W_0^{-1,q}(\mathbb{R}^2)$. Thus, by Theorem 3.11, there exists a unique $v_j \in L^{3q/(3-q)}(\mathbb{R}^2)$, such that $\nabla v_j \in \boldsymbol{L}^q(\mathbb{R}^2)$ and $\partial v_j/\partial x_1 \in W_0^{-1,q}(\mathbb{R}^2)$, satisfying

$$-\Delta\Big(v_j - \varrho\frac{\partial u}{\partial x_j}\Big) + \frac{\partial}{\partial x_1}\Big(v_j - \varrho\frac{\partial u}{\partial x_j}\Big) = 0.$$

We deduce that $w_j = v_j - \varrho\,\partial u/\partial x_j$ is a polynomial. Since $\nabla v_j \in \boldsymbol{L}^q(\mathbb{R}^2)$ and $q > 2$, we have, by Proposition 2.2, $v_j \in W_0^{1,q}(\mathbb{R}^2) \subset W_{-1}^{0,q}(\mathbb{R}^2)$. We have also $\varrho\,\partial u/\partial x_j \in W_{-1}^{0,q}(\mathbb{R}^2)$, so $w_j \in \mathcal{P}_{[1-2/q]} = \mathcal{P}_0$. Thus, there exists a constant k such that $\varrho\,\partial u/\partial x_j = v_j + k \in W_0^{1,q}(\mathbb{R}^2)$, which implies $\partial u/\partial x_j \in W_1^{1,q}(\mathbb{R}^2)$ and so $\nabla^2 u \in (W_1^{0,q}(\mathbb{R}^2))^{2\times 2}$. The same argument proves that $\partial u/\partial x_1 \in W_1^{0,q}(\mathbb{R}^2)$.

ii) If $q \geqslant 3$, then Equation (3.1) has, in view of Proposition 3.15 ii), a unique solution $u \in \widetilde{W}_0^{1,q}(\mathbb{R}^2) \cap \widetilde{W}_0^{1,p}(\mathbb{R}^2)$. The right-hand side F also belongs to $W_0^{-1,q}(\mathbb{R}^2)$ and we proceed as previously. □

4. Study in anisotropic weighted spaces

In this section we consider the case when the weight is anisotropic, of the form $r^\alpha(1+s)^\beta$ or $\eta_\beta^\alpha = (1+r)^\alpha(1+s)^\beta$. Note that the behavior at infinity of these weights is not uniform. In fact, in the parabola $s = 1$ we have $r^\alpha(1+s)^\beta \sim \eta_\beta^\alpha \sim r^\alpha$ and out of a sector $\mathcal{S}_{\lambda,R} = \{x \in \mathbb{R}^2;\ x_1 > \lambda r, 0 < \lambda < 1\}$ we have $r^\alpha(1+s)^\beta \sim \eta_\beta^\alpha \sim r^{\alpha+\beta}$. It is for this reason that these functions are called anisotropic weights. For $R > 0$, we denote by B_R the ball centered at the origin with the radius R, $B'_R = \mathbb{R}^2 \setminus \overline{B_R}$, and we define the space

$$L^p_{\alpha,\beta}(\Omega) = \{v \in \mathcal{D}'(\Omega);\ \eta_\beta^\alpha v \in L^p(\Omega)\},$$

where $\Omega = \mathbb{R}^2$ or any open domain of $\mathbb{R}^2$. We begin by studying the problem

$$\begin{aligned} -\Delta z + \frac{\partial z}{\partial x_1} + a_0 z = g \quad \text{in } B'_R, \\ z = 0 \quad \text{on } \partial B'_R, \end{aligned} \tag{4.1}$$

where $g \in L^p_{1/2,0}(B'_R)$ and

$$a_0 = \frac{1}{8r}\frac{2s^2+s+2}{(1+s)^2}. \tag{4.2}$$

First we have the following.

Lemma 4.1. *Let p be such that $2 < p < \frac{32}{11}$ and let $g \in L^p_{1/2,0}(B'_R)$. There exists $R^* > 0$ such that if $R > R^*$, then Problem* (4.1) *has a unique solution $z \in L^p_{-1/2,0}(B'_R)$ such that $\nabla^2 z \in (L^p(B'_R))^{2\times 2}$ and $\partial z/\partial x_1 \in L^p(B'_R)$. Moreover, there exists $C > 0$ such that*

$$\|z\|_{L^p_{-1/2,0}(B'_R)} + \left\|\frac{\partial z}{\partial x_1}\right\|_{L^p(B'_R)} + \|\nabla^2 z\|_{\boldsymbol{L}^p(B'_R)} \leqslant C\|g\|_{L^p_{1/2,0}(B'_R)}. \tag{4.3}$$

Proof. For all $\varepsilon > 0$, since $g \in L^p_{1/2,0}(B'_R)$ and $a_0 > 0$, the problem

$$\begin{aligned} -\Delta z_\varepsilon + \frac{\partial z_\varepsilon}{\partial x_1} + a_0 z_\varepsilon + \varepsilon z_\varepsilon = g \quad \text{in } B'_R, \\ z_\varepsilon = 0 \quad \text{on } \partial B'_R \end{aligned} \tag{4.4}$$

has a unique solution $z_\varepsilon \in W^{2,p}(B'_R)$. By multiplying the first equation of (4.4) by $r^{1-p/2}|z_\varepsilon|^{p-2}z_\varepsilon$ and since in two dimensions $\Delta(r^{1-p/2}) = (1-\frac{1}{2}p)^2 r^{-1-p/2}$, we get after integration by parts in B'_R

$$\begin{aligned} (p-1)\int_{B'_R} r^{1-p/2}|z_\varepsilon|^{p-2}|\nabla z_\varepsilon|^2\,\mathrm{d}\boldsymbol{x} + \int_{B'_R} a_0 r^{1-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} + \varepsilon\int_{B'_R} r^{1-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} \\ = \frac{1}{p}\Big(1-\frac{1}{2}p\Big)^2\int_{B'_R} r^{-1-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} + \Big(\frac{1}{p}-\frac{1}{2}\Big)\int_{B'_R}|z_\varepsilon|^p\frac{x_1}{r}r^{-p/2}\,\mathrm{d}\boldsymbol{x} \\ + \int_{B'_R} r^{1-p/2}|z_\varepsilon|^{p-2}z_\varepsilon g\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Note that $a_0 \geqslant \frac{5}{32r}$, thus

$$\begin{aligned} \Big(\frac{5}{32} - \Big|\frac{1}{p}-\frac{1}{2}\Big|\Big)\int_{B'_R} r^{-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} \\ \leqslant \frac{1}{p}\Big(1-\frac{1}{2}p\Big)^2\int_{B'_R} r^{-1-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} + \int_{B'_R} r^{1-p/2}|z_\varepsilon|^{p-1}|g|\,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.5}$$

Moreover, since $r > R$,

$$\frac{1}{p}\Big(1-\frac{1}{2}p\Big)^2\int_{B'_R} r^{-1-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} \leqslant \frac{1}{pR}\Big(1-\frac{1}{2}p\Big)^2\int_{B'_R} r^{-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x}. \tag{4.6}$$

Inequalities (4.5) and (4.6) give

$$\Big(\frac{5}{32}-\Big|\frac{1}{p}-\frac{1}{2}\Big|-\frac{1}{pR}\Big(1-\frac{1}{2}p\Big)^2\Big)\int_{B'_R} r^{-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} \leqslant \int_{B'_R} r^{1-p/2}|z_\varepsilon|^{p-1}|g|\,\mathrm{d}\boldsymbol{x}.$$

Since $2 < p < \frac{32}{11}$, we have $\frac{5}{32}-|\frac{1}{p}-\frac{1}{2}|-\frac{1}{pR}(1-\frac{1}{2}p)^2 > 0$, if $R > R^*$, with R^* sufficiently large. Thus, from the previous inequality we obtain

$$\begin{aligned}\int_{B'_R} r^{-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} &\leqslant C_1\int_{B'_R} r^{1-p/2}|z_\varepsilon|^{p-1}|g|\,\mathrm{d}\boldsymbol{x}\\ &\leqslant C_1\Big(\int_{B'_R} r^{p/2}|g|^p\,\mathrm{d}\boldsymbol{x}\Big)^{1/p}\Big(\int_{B'_R} r^{-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x}\Big)^{(p-1)/p}.\end{aligned}$$

Thus

$$\int_{B'_R} r^{-p/2}|z_\varepsilon|^p\,\mathrm{d}\boldsymbol{x} \leqslant C\int_{B'_R} r^{p/2}|g|^p\,\mathrm{d}\boldsymbol{x},$$

where the constant C is independent of R and ε. The sequence (z_ε) is thus bounded in $L^p_{-1/2,0}(B'_R)$, which is a reflexive space, so $z_\varepsilon \rightharpoonup z$ in $L^p_{-1/2,0}(B'_R)$, and

$$\|z\|_{L^p_{-1/2,0}(B'_R)} \leqslant \liminf_{\varepsilon\to 0}\|z_\varepsilon\|_{L^p_{-1/2,0}(B'_R)} \leqslant C\|g\|_{L^p_{1/2,0}(B'_R)},$$

where z satisfies the equation

$$-\Delta z + \frac{\partial z}{\partial x_1} = g - a_0 z \quad \text{in } B'_R.$$

Let us show that $\nabla^2 z \in (L^p(B'_R))^{2\times 2}$ and $\partial z/\partial x_1 \in L^p(B'_R)$. Now, the fact that the function $g - a_0 z_\varepsilon$ is bounded in $L^p_{1/2,0}(B'_R)$ implies that it is bounded in $L^p(B'_R)$. Since $\nabla^2 z_\varepsilon$ remains bounded in $L^p(B'_R)$, it follows that $\nabla^2 z \in (L^p(B'_R))^{2\times 2}$ and

$$\|\nabla^2 z\|_{\boldsymbol{L}^p(B'_R)} \leqslant \liminf_{\varepsilon\to 0}\|\nabla^2 z_\varepsilon\|_{\boldsymbol{L}^p(B'_R)} \leqslant C\|g\|_{L^p_{1/2,0}(B'_R)}. \tag{4.7}$$

Thus, $\partial z/\partial x_1 \in L^p(B'_R)$ and we have Estimate (4.3). It remains to prove that $z = 0$ on $\partial B'_R$. Since $\nabla^2 z_\varepsilon$ is bounded in $L^p(B'_R)$, if Ω is a bounded domain such that $\overline{B_R} \subset \Omega$, setting $\Omega = \Omega \cap B'_R$, we have

$$z_\varepsilon \rightharpoonup v \quad \text{in } W^{2,p}(\Omega).$$

Since $z_\varepsilon = 0$ on $\partial B'_R$, it follows that $v = 0$ on $\partial B'_R$. Moreover, since $z_\varepsilon \rightharpoonup z$ in $L^p_{-1/2,0}(B'_R)$, it follows that $v = z|_{\overline{\Omega}}$ and so $z = 0$ on $\partial B'_R$. □

We know, according to Proposition 3.18, that for f given in $W_0^{-1,p}(\mathbb{R}^2)\cap W_1^{0,q}(\mathbb{R}^2)$, where p and q satisfy the relation $\frac{1}{p}=\frac{1}{q}+\frac{1}{2}$, we obtain that $\nabla^2 u$ and $\partial u/\partial x_1$ belong to $W_1^{0,q}(\mathbb{R}^2)$. But if f is only given in $W_1^{0,p}(\mathbb{R}^2)$, we cannot find the same regularity on $\nabla^2 u$ and $\partial u/\partial x_1$. Then we look at f in $L^p_{\alpha,\beta}$, with $\alpha+\beta$ close to 1. Moreover, taking account of the conditions put by Kračmar, Novotný, and Pokorný in [10] on α and β, one takes $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{4}$.

Theorem 4.2. *Assume* $2<p<\frac{32}{11}$ *and* $f\in L^p_{1/2,1/4}(\mathbb{R}^2)$. *Then,* $\mathcal{O}*f\in L^p_{-1/2,1/4}(\mathbb{R}^2), \partial/\partial x_2(\mathcal{O}*f)\in L^p_{0,1/4}(\mathbb{R}^2)$, $\partial/\partial x_1(\mathcal{O}*f)\in L^p_{1/2,1/4}(\mathbb{R}^2)$, *and* $\nabla^2(\mathcal{O}*f)\in(L^p_{1/2,1/4}(\mathbb{R}^2))^{2\times 2}$. *Moreover, we have the estimate*

$$\|\mathcal{O}*f\|_{L^p_{-1/2,1/4}(\mathbb{R}^2)}+\Big\|\frac{\partial}{\partial x_2}(\mathcal{O}*f)\Big\|_{L^p_{0,1/4}(\mathbb{R}^2)}+\Big\|\frac{\partial}{\partial x_1}(\mathcal{O}*f)\Big\|_{L^p_{1/2,1/4}(\mathbb{R}^2)} + \|\nabla^2(\mathcal{O}*f)\|_{\boldsymbol{L}^p_{1/2,1/4}(\mathbb{R}^2)}\leqslant C\|f\|_{L^p_{1/2,1/4}(\mathbb{R}^2)}. \tag{4.8}$$

P r o o f. From [10], we have $\mathcal{O}*f\in L^p_{-1/2-\varepsilon,1/4}(\mathbb{R}^2)$, $\partial/\partial x_2(\mathcal{O}*f)\in L^p_{0,1/4}(\mathbb{R}^2)$, $\partial/\partial x_1(\mathcal{O}*f)\in L^p_{1/2-\varepsilon,1/4}(\mathbb{R}^2)$, for all $\varepsilon>0$. It remains to prove that $\mathcal{O}*f\in L^p_{-1/2,1/4}(\mathbb{R}^2)$ and $\partial/\partial x_1(\mathcal{O}*f)\in L^p_{1/2,1/4}(\mathbb{R}^2)$. For $R>R^*$, we use the following partition of unity

$$\begin{gathered}\varphi_1,\varphi_2\in\mathcal{C}^\infty(\mathbb{R}^2),\quad 0\leqslant\varphi_1,\varphi_2\leqslant 1,\quad \varphi_1+\varphi_2=1\ \text{ in }\ \mathbb{R}^2,\\ \varphi_1=1\ \text{ in }\ B_R\quad\text{and}\quad \operatorname{Supp}\varphi_1\subset B_{R+1}.\end{gathered}$$

We set $u=\mathcal{O}*f$ and we split u into $u=u_1+u_2$, where $u_1=\varphi_1 u$ and $u_2=\varphi_2 u$. Since $\operatorname{Supp}u_1\subset B_{R+1}$, $u_1\in L^p_{-1/2,1/4}(\mathbb{R}^2)$ and

$$\|u_1\|_{L^p_{-1/2,1/4}(\mathbb{R}^2)}\leqslant C\|f\|_{L^p_{1/2,1/4}(\mathbb{R}^2)}.$$

Furthermore, u_2 is a solution of the following problem

$$-\Delta u_2+\frac{\partial u_2}{\partial x_1}=\widetilde{f}\quad\text{in }\ \mathbb{R}^2,$$

where $\widetilde{f}=\varphi_2 f+u\Delta\varphi_1+2\nabla u\nabla\varphi_1-u(\partial\varphi_1/\partial x_1)$. Since the regularity of $\varphi_2 f$ determines that of $\widetilde{f}$, it follows that $\widetilde{f}\in L^p_{1/2,1/4}(\mathbb{R}^2)$. Setting $v=(1+s)^{1/4}u_2$, we have $v\in L^p_{-1/2-\varepsilon,0}(\mathbb{R}^2)$, and v satisfies the equation

$$-\Delta v+\frac{\partial v}{\partial x_1}=(1+s)^{1/4}\widetilde{f}-2\nabla u_2\cdot\nabla(1+s)^{1/4}-u_2\Big[\Delta(1+s)^{1/4}-\frac{\partial}{\partial x_1}(1+s)^{1/4}\Big].$$

A simple calculation yields

$$\Big(\Delta - \frac{\partial}{\partial x_1}\Big)(1+s)^{1/4} = \frac{1}{8r}(2s^2+s+2)(1+s)^{-7/4},$$

thus $u_2[\Delta(1+s)^{1/4} - \partial/\partial x_1(1+s)^{1/4}] = a_0 v$, where a_0 is defined in (4.2). Hence, v satisfies Problem (4.1), where $g = (1+s)^{1/4}\widetilde{f} - 2\nabla u_2 \cdot \nabla(1+s)^{1/4} \in L_{1/2,0}(B'_R)$. Applying Lemma 4.1, there exists a unique solution $w \in L^p_{-1/2,0}(B'_R)$ of this problem. Setting $z = v - w$, we have $z \in L^p_{-1/2-\varepsilon,0}(\mathbb{R}^2)$, and z satisfies

$$-\Delta z + \frac{\partial}{\partial x_1} + a_0 z = 0 \quad \text{in } \ \mathbb{R}^2.$$

Then $z = 0$, which implies that $v \in L^p_{-1/2,0}(\mathbb{R}^2)$ and

$$\|v\|_{L^p_{-1/2,0}(\mathbb{R}^2)} \leqslant C\|g\|_{L^p_{1/2,0}(B'_R)} \leqslant C\|f\|_{L^p_{1/2,1/4}(\mathbb{R}^2)}.$$

Hence $u_2 \in L^p_{-1/2,1/4}(\mathbb{R}^2)$ and

$$\|u_2\|_{L^p_{-1/2,1/4}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p_{1/2,1/4}(\mathbb{R}^2)},$$

which proves that $u \in L^p_{-1/2,1/4}(\mathbb{R}^2)$ and

$$\|u\|_{L^p_{-1/2,1/4}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p_{1/2,1/4}(\mathbb{R}^2)}. \tag{4.9}$$

Now, using the fact that u_2 satisfies

$$-\Delta(\eta^{1/2}_{1/4}u_2) + \frac{\partial}{\partial x_1}(\eta^{1/2}_{1/4}u_2) =: F,$$

where

$$F = \eta^{1/2}_{1/4}f - u\Delta(\eta^{1/2}_{1/4}\,\varphi_2) - 2\nabla u \cdot \nabla(\eta^{1/2}_{1/4}\varphi_2) + u\frac{\partial}{\partial x_1}(\eta^{1/2}_{1/4}\varphi_2) \in L^p(\mathbb{R}^2),$$

we obtain by Theorem 3.9 that there exists a function v such that $\nabla^2 v \in (L^p(\mathbb{R}^2))^{2\times 2}$ and $\partial v/\partial x_1 \in L^p(\mathbb{R}^2)$, satisfying

$$-\Delta v + \frac{\partial v}{\partial x_1} = -\Delta(\eta^{1/2}_{1/4}u_2) + \frac{\partial}{\partial x_1}(\eta^{1/2}_{1/4}\,u_2).$$

Moreover,

$$\|\nabla^2 v\|_{\boldsymbol{L}^p(\mathbb{R}^2)} + \Big\|\frac{\partial v}{\partial x_1}\Big\|_{L^p(\mathbb{R}^2)} \leqslant C\|F\|_{L^p(\mathbb{R}^2)} \leqslant C\|f\|_{L^p_{1/2,1/4}(\mathbb{R}^2)}. \tag{4.10}$$

We set $w = \nabla^2 v - \nabla^2(\eta_{1/4}^{1/2} u_2)$; since $\nabla^2 u \in \bigcap_{\varepsilon>0} (L^p_{1/2-\varepsilon,1/4}(\mathbb{R}^2))^{2\times 2}$, we have

$$w \in \bigcap_{\varepsilon>0} L^p_{-\varepsilon,0}(\mathbb{R}^2) \quad \text{and} \quad -\Delta w + \frac{\partial w}{\partial x_1} = 0 \quad \text{in } \mathbb{R}^2.$$

Thus $w = 0$, which implies that

$$\nabla^2(\eta_{1/4}^{1/2} u) \in (L^p(\mathbb{R}^2))^{2\times 2}.$$

We thus obtain

$$\nabla^2 u \in (L^p_{1/2,1/4}(\mathbb{R}^2))^{2\times 2}, \quad \frac{\partial u}{\partial x_1} \in L^p_{1/2,1/4}(\mathbb{R}^2),$$

and the estimate

$$\|\nabla^2 u\|_{\boldsymbol{L}^p_{1/2,1/4}(\mathbb{R}^2)} + \left\|\frac{\partial u}{\partial x_1}\right\|_{L^p_{1/2,1/4}(\mathbb{R}^2)} \leqslant C\|f\|_{L^p_{1/2,1/4}(\mathbb{R}^2)}. \tag{4.11}$$

This finishes the proof. □

Let us set

$$K^p_{\alpha,\beta}(\Omega) = \{v \in \mathcal{D}'(\Omega);\ r^\alpha(1+s)^\beta \in L^p(\Omega)\},$$

which is a reflexive Banach space when it is equipped with its natural norm. With the same arguments as above we can prove the following result. The case $\beta = \frac{1}{4}$ corresponds to Theorem 4.2.

Theorem 4.3. *Assume $2 \leqslant p < 8/(3-\beta)$ and $0 < \beta < \frac{1}{4}$. Then, for $f \in K^p_{1/2,\beta}(\mathbb{R}^2)$, we have $\mathcal{O} * f \in K^p_{-1/2,\beta}(\mathbb{R}^2)$, $\partial/\partial x_2(\mathcal{O} * f) \in K^p_{0,\beta}(\mathbb{R}^2)$, $\partial/\partial x_1(\mathcal{O} * f) \in K^p_{1/2,\beta}(\mathbb{R}^2)$, and $\nabla^2(\mathcal{O} * f) \in (K^p_{1/2,\beta}(\mathbb{R}^2))^{2\times 2}$. Moreover, we have the estimates*

$$\begin{aligned} \|\mathcal{O} * f\|_{K^p_{-1/2,\beta}(\mathbb{R}^2)} + \left\|\frac{\partial}{\partial x_2}(\mathcal{O} * f)\right\|_{K^p_{0,\beta}(\mathbb{R}^2)} + \left\|\frac{\partial}{\partial x_1}(\mathcal{O} * f)\right\|_{K^p_{1/2,\beta}(\mathbb{R}^2)} \\ + \|\nabla^2(\mathcal{O} * f)\|_{\boldsymbol{K}^p_{1/2,\beta}(\mathbb{R}^2)} \leqslant C\|f\|_{K^p_{1/2,\beta}(\mathbb{R}^2)}. \end{aligned} \tag{4.12}$$

For $\alpha, \beta \in \mathbb{R}$ we denote

$$L^p_{\alpha,\beta(s')}(\mathbb{R}^2) = \{v \in \mathcal{D}'(\Omega);\ \varrho^\alpha(1+s')^\beta v \in L^p(\mathbb{R}^2)\},$$

which is a reflexive Banach space when it is equipped with its natural norm

$$\|v\|_{L^p_{\alpha,\beta(s')}(\mathbb{R}^2)} = \|\varrho^\alpha(1+s')^\beta v\|_{L^p(\mathbb{R}^2)}.$$

Proposition 4.4. *For any given $f \in L^2_{1/2,((\delta-1)/2)(s')}(\mathbb{R}^2)$, with $\delta > 0$ close to zero, Equation* (3.1) *has a unique solution $u \in K^2_{\delta/2-1,0}(\mathbb{R}^2)$ such that $\nabla u \in \boldsymbol{L}^2_{\delta/4-1/2,0}(\mathbb{R}^2)$. Moreover, there exists a constant $C > 0$ such that*

$$\|u\|_{K^2_{\delta/2-1,0}(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^2_{\delta/4-1/2,0}(\mathbb{R}^2)} \leqslant C\|f\|_{L^2_{1/2,((\delta-1)/2)(s')}(\mathbb{R}^2)}. \tag{4.13}$$

Proof. By the density of $\mathcal{D}(\mathbb{R}^2)$ in $L^2_{1/2,((\delta-1)/2)(s')}(\mathbb{R}^2)$ (see [2]), there exists a sequence (f_k) of $\mathcal{D}(\mathbb{R}^2)$ such that $f_k \to f$ in $L^2_{1/2,((\delta-1)/2)(s')}(\mathbb{R}^2)$. Since $f_k \in \mathcal{D}(\mathbb{R}^2)$, we have $f_k \in K^2_{1/2,\beta}(\mathbb{R}^2)$, $0 < \beta < \frac{1}{4}$. Thus, from Theorem 4.2, the equation

$$-\Delta u_k + \frac{\partial u_k}{\partial x_1} = f_k \quad \text{in } \mathbb{R}^2, \tag{4.14}$$

has a solution $u_k = \mathcal{O} * f_k \in K^2_{-1/2,0}(\mathbb{R}^2)$ such that $\nabla u_k \in \boldsymbol{K}^2_{0,\beta}(\mathbb{R}^2)$, $\nabla^2 u_k \in (K^2_{1/2,\beta}(\mathbb{R}^2))^{2\times 2}$ and $\partial u_k/\partial x_1 \in K^2_{1/2,\beta}(\mathbb{R}^2)$. Multiplying Equation (4.14) by hu_k where $h = \breve{\mathcal{O}} * r^{\delta-2}$ with $\delta > 0$ and $\breve{\mathcal{O}}$ is the fundamental solution of the operator $-\Delta - \partial/\partial x_1$, we obtain after two integrations by parts

$$\int_{\mathbb{R}^2} |\nabla u_k|^2 h \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^2} u_k^2\Big(-\Delta h - \frac{\partial h}{\partial x_1}\Big)\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2} f_k h u_k \,\mathrm{d}\boldsymbol{x}. \tag{4.15}$$

Since $-\Delta h - \partial h/\partial x_1 = r^{\delta-2}$, we have

$$\int_{\mathbb{R}^2} |\nabla u_k|^2 h \,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_{\mathbb{R}^2} u_k^2 r^{\delta-2}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2} f_k h u_k \,\mathrm{d}\boldsymbol{x},$$

and as $h \geqslant 0$ we then get the two inequalities

$$\int_{\mathbb{R}^2} u_k^2 r^{\delta-2}\,\mathrm{d}\boldsymbol{x} \leqslant 2\int_{\mathbb{R}^2} f_k h u_k \,\mathrm{d}\boldsymbol{x}, \tag{4.16}$$

$$\int_{\mathbb{R}^2} |\nabla u_k|^2 h \,\mathrm{d}\boldsymbol{x} \leqslant \int_{\mathbb{R}^2} f_k h u_k \,\mathrm{d}\boldsymbol{x}. \tag{4.17}$$

A simple calculation yields

$$\Big(-\Delta - \frac{\partial}{\partial x_1}\Big)(1+r)^{\delta/2-1} = \frac{2-\delta}{4}(1+r)^{\delta/2-2}\Big(\frac{4-\delta}{1+r} - \frac{1}{r} - \frac{x_1}{r}\Big),$$

thus

$$\Big(-\Delta - \frac{\partial}{\partial x_1}\Big)(h - M(1+r)^{\delta/2-1}) \geqslant \frac{1}{r^{2-\delta}} - M\frac{2-\delta}{2r}(1+r)^{\delta/2-1} \geqslant 0,$$

for $0 < M \leqslant 2^{2+\delta/2}/(2-\delta)\cdot((1-\delta)/(2+\delta))^{1+\delta/2}$. Thus, there exists $M > 0$ such that $h(x) \geqslant M(1+r)^{\delta/2-1}$, so from the inequality (4.17), we obtain

$$M\int_{\mathbb{R}^2}(1+r)^{\delta/2-1}|\nabla u_k|^2\,\mathrm{d}\boldsymbol{x} \leqslant \int_{\mathbb{R}^2} f_k h u_k\,\mathrm{d}\boldsymbol{x}. \tag{4.18}$$

The Cauchy-Schwarz inequality gives

$$\int_{\mathbb{R}^2} f_k h u_k\,\mathrm{d}\boldsymbol{x} \leqslant \left(\int_{\mathbb{R}^2} f_k^2 h^2 r^{2-\delta}\,\mathrm{d}\boldsymbol{x}\right)^{1/2}\left(\int_{\mathbb{R}^2} r^{\delta-2}u_k^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

Hence, from the inequalities (4.16) we get

$$\int_{\mathbb{R}^2} r^{\delta-2}u_k^2\,\mathrm{d}\boldsymbol{x} \leqslant 4\int_{\mathbb{R}^2} f_k^2 h^2 r^{2-\delta}\,\mathrm{d}\boldsymbol{x} = 4\int_{\mathbb{R}^2} f_k^2\frac{1+r}{(1+s')^{1-\delta}}h^2 r^{1-\delta}(1+s')^{1-\delta}\,\mathrm{d}\boldsymbol{x}.$$

We adapt the result of Theorem 3.5 obtained in [10]: we have $h^2 r^{1-\delta}(1+s')^{1-\delta} \in L^\infty(\mathbb{R}^2)$, thus $u_k \in K^2_{\delta/2-1,0}(\mathbb{R}^2)$ and there exists $C > 0$ such that

$$\|u_k\|_{K^2_{\delta/2-1,0}(\mathbb{R}^2)} \leqslant C\|f_k\|_{L^2_{1/2,(\delta/2-1/2)(s')}} \leqslant C\|f\|_{L^2_{1/2,(\delta/2-1/2)(s')}}. \tag{4.19}$$

Now, using the inequalities (4.18) and (4.19), we deduce that $\nabla u_k \in \boldsymbol{L}^2_{\delta/4-1/2,0}(\mathbb{R}^2)$ and

$$\|\nabla u_k\|_{\boldsymbol{L}^2_{\delta/4-1/2,0}(\mathbb{R}^2)} \leqslant C\|f_k\|_{L^2_{1/2,(\delta/2-1/2)(s')}} \leqslant C\|f\|_{L^2_{1/2,(\delta/2-1/2)(s')}}. \tag{4.20}$$

So, the sequences u_k and $\boldsymbol{v}_k = \nabla u_k$ remain bounded in $K^2_{\delta/2-1,0}(\mathbb{R}^2)$ and in $\boldsymbol{L}^2_{\delta/4-1/2,0}(\mathbb{R}^2)$, respectively. These spaces are reflexive, therefore extracting a subsequence if necessary, we have

$$u_k \rightharpoonup u \text{ in } K^2_{\delta/2-1,0}(\mathbb{R}^2) \quad\text{and}\quad \nabla u_k \rightharpoonup \nabla u \text{ in } \boldsymbol{L}^2_{\delta/4-1/2,0}(\mathbb{R}^2)$$

with the estimates

$$\|u\|_{K^2_{\delta/2-1,0}(\mathbb{R}^2)} \leqslant \liminf_{k\to\infty}\|u_k\|_{K^2_{\delta/2-1,0}(\mathbb{R}^2)} \leqslant C\|f\|_{L^2_{1/2,(\delta/2-1/2)(s')}}, \tag{4.21}$$

$$\|\nabla u\|_{\boldsymbol{L}^2_{\delta/4-1/2,0}(\mathbb{R}^2)} \leqslant \liminf_{k\to\infty}\|\nabla u_k\|_{\boldsymbol{L}^2_{\delta/4-1/2,0}(\mathbb{R}^2)} \leqslant C\|f\|_{L^2_{1/2,(\delta/2-1/2)(s')}}. \tag{4.22}$$

We get then Estimate (4.13) and we verify easily that u is a solution of Equation 3.1. The uniqueness of u follows from the fact that the space $K^2_{\delta/2-1,0}(\mathbb{R}^2)$ contains no polynomials. $\square$

5. Behaviour of u_λ when $\lambda \to 0$

Assume $1 < p < 2$, $f \in L^p(\mathbb{R}^2)$, and, for $\lambda > 0$, consider the equation

$$-\Delta u_\lambda + \lambda \frac{\partial u_\lambda}{\partial x_1} = f \quad \text{in } \mathbb{R}^2. \tag{5.1}$$

If we set

$$\boldsymbol{y} = \lambda \boldsymbol{x}, \quad u_\lambda(\boldsymbol{x}) = v(\boldsymbol{y}), \quad \text{and} \quad f(\boldsymbol{x}) = \lambda^2 g(\boldsymbol{y}),$$

then v satisfies the equation

$$-\Delta v(\boldsymbol{y}) + \frac{\partial v}{\partial y_1}(\boldsymbol{y}) = g(\boldsymbol{y}) \quad \text{in } \mathbb{R}^2, \tag{5.2}$$

where, clearly, $g \in L^p(\mathbb{R}^2)$. We know by Theorem 3.9 that, if $1 < p < 2$, Equation (5.2) has a solution v such that, in particular, $\nabla v \in \boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)$, $\nabla^2 v \in (L^p(\mathbb{R}^2))^{2\times 2}$, $\partial v/\partial x_1 \in L^p(\mathbb{R}^2)$ and

$$\|\nabla v\|_{\boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)} + \|\nabla^2 v\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leqslant C\|g\|_{L^p(\mathbb{R}^2)}. \tag{5.3}$$

By a simple calculation we obtain from Inequality (5.3) the estimate

$$\|\nabla u_\lambda\|_{\boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)} + \|\nabla^2 u_\lambda\|_{L^p(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}, \tag{5.4}$$

where C does not depend on λ. We deduce that the sequences ∇u_λ and $\nabla^2 u_\lambda$ remain bounded in $\boldsymbol{L}^p(\mathbb{R}^2)$ and $(L^{p^*}(\mathbb{R}^2))^{2\times 2}$, with $p^* = 2p/(2-p)$, respectively. Now, setting

$$-\Delta u_\lambda = f_\lambda \quad \text{in } \mathbb{R}^2, \tag{5.5}$$

then the sequence f_λ is bounded in $L^p(\mathbb{R}^2) \cap W_0^{-1,p^*}(\mathbb{R}^2)$. These spaces are reflexive, so extracting a subsequence if necessary, also denoted f_λ, we have

$$f_\lambda \rightharpoonup f \text{ in } L^p(\mathbb{R}^2) \quad \text{and} \quad f_\lambda \rightharpoonup f \text{ in } W_0^{-1,p^*}(\mathbb{R}^2).$$

Further, note that $p^* > 2$, so there exist $z \in W_0^{1,p^*}(\mathbb{R}^2)$ and $w \in W_0^{2,p}(\mathbb{R}^2)$ such that

$$-\Delta z = -\Delta w = f \quad \text{in } \mathbb{R}^2.$$

Since $\nabla z \in \boldsymbol{L}^{p^*}(\mathbb{R}^2)$, $\nabla w \in \boldsymbol{L}^{p^*}(\mathbb{R}^2)$ by Sobolev embedding and $\nabla z - \nabla w$ is harmonic, it follows that $\nabla z - \nabla w = 0$ in $\mathbb{R}^2$. Hence there exists $k \in \mathbb{R} \subset W_0^{2,p}(\mathbb{R}^2)$ such that $z = w + k$, thus $z \in W_0^{2,p}(\mathbb{R}^2) \cap W_0^{1,p^*}(\mathbb{R}^2)$. Now, since the norm in $W_0^{2,p}(\mathbb{R}^2)/\mathbb{R}$

is equivalent to its semi-norm, we deduce from the inequality (5.4) that there exist $k_\lambda \in \mathbb{R}$ and $u \in W_0^{2,p}(\mathbb{R}^2) \cap W_0^{1,p^*}(\mathbb{R}^2)$ such that

$$u_\lambda + k_\lambda \rightharpoonup u \text{ in } W_0^{2,p}(\mathbb{R}^2) \text{ and in } W_0^{1,p^*}(\mathbb{R}^2).$$

Since $-\Delta u = f$ in $\mathbb{R}^2$, there exists $k \in \mathbb{R}$ such that $z = u+k$. We have thus recovered the result obtained by Amrouche, Girault, and Giroire in [1] for $f \in L^p(\mathbb{R}^2)$. The following proposition is thus acquired.

Proposition 5.1. *Assume that $1 < p < 2$ and let $f \in L^p(\mathbb{R}^2)$. Then Equation (5.1) has at least a solution u_λ of the form (3.41) such that $\nabla u_\lambda \in \boldsymbol{L}^{3p/(3-p)}(\mathbb{R}^2) \cap \boldsymbol{L}^{2p/(2-p)}(\mathbb{R}^2)$, $\nabla^2 u_\lambda \in (L^p(\mathbb{R}^2))^{2\times 2}$, and $\partial u_\lambda/\partial x_1 \in L^p(\mathbb{R}^2)$. Moreover, if $1 < p < \frac{3}{2}$, then $u_\lambda \in L^{3p/(3-2p)}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Furthermore, there exists $k_\lambda \in \mathbb{R}$ such that, when $\lambda \to 0$,*

$$u_\lambda + k_\lambda \rightharpoonup u \text{ in } W_0^{2,p}(\mathbb{R}^2) \text{ and in } W_0^{1,p^*}(\mathbb{R}^2),$$

where u is the unique solution of Poisson's Equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^2, \tag{5.6}$$

with the estimate

$$\|\nabla u\|_{\boldsymbol{L}^{p^*}(\mathbb{R}^2)} + \|\nabla^2 u\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leqslant C\|f\|_{L^p(\mathbb{R}^2)}. \tag{5.7}$$

For $f \in W_0^{-1,p}(\mathbb{R}^2)$ we have the following result.

Proposition 5.2. *Assume $1 < p < 2$ and let $f \in W_0^{-1,p}(\mathbb{R}^2)$ satisfy the compatibility condition*

$$\langle f, 1\rangle_{W_0^{-1,p}(\mathbb{R}^2)\times W_0^{1,p'}(\mathbb{R}^2)} = 0. \tag{5.8}$$

Then Equation (5.1) has a unique solution $u_\lambda \in L^{3p/(3-p)}(\mathbb{R}^2) \cap L^{p^*}(\mathbb{R}^2)$ such that $\nabla u_\lambda \in \boldsymbol{L}^p(\mathbb{R}^2)$ and $\partial u_\lambda/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$. Moreover,

$$u_\lambda \rightharpoonup u \text{ in } W_0^{1,p}(\mathbb{R}^2) \text{ as } \lambda \to 0,$$

where u is the unique solution of Poisson's Equation

$$-\Delta u = f \quad \text{in } \mathbb{R}^2, \tag{5.9}$$

and the following estimate holds

$$\|u\|_{L^{p^*}(\mathbb{R}^2)} + \|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{5.10}$$

P r o o f. By Isomorphism (2.7), there exists $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^2)$ such that $f = \operatorname{div} \boldsymbol{F}$ and

$$\|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leqslant C\|f\|_{W_0^{-1,p}(\mathbb{R}^2)}. \tag{5.11}$$

Setting

$$\boldsymbol{y} = \lambda \boldsymbol{x}, \quad u_\lambda(\boldsymbol{x}) = v(\boldsymbol{y}), \quad \boldsymbol{F}(\boldsymbol{x}) = \lambda \boldsymbol{G}(\boldsymbol{y}), \quad \text{and} \quad g = \operatorname{div} \boldsymbol{G},$$

v satisfies Equation (5.2) where $g \in W_0^{-1,p}(\mathbb{R}^2) \perp \mathbb{R}$. By Theorem 3.11, this equation has a unique solution $v \in L^{3p/(3-p)}(\mathbb{R}^2) \cap L^{p^*}(\mathbb{R}^2)$ such that $\nabla v \in \boldsymbol{L}^p(\mathbb{R}^2)$ and $\partial v/\partial x_1 \in W_0^{-1,p}(\mathbb{R}^2)$, with the estimate

$$\|v\|_{L^{p^*}(\mathbb{R}^2)} + \|\nabla v\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leqslant C\|g\|_{W_0^{-1,p}(\mathbb{R}^2)} \leqslant C\|\boldsymbol{G}\|_{\boldsymbol{L}^p(\mathbb{R}^2)}. \tag{5.12}$$

As previously, we get the estimate

$$\|u_\lambda\|_{L^{p^*}(\mathbb{R}^2)} + \|\nabla u_\lambda\|_{\boldsymbol{L}^p(\mathbb{R}^2)} \leqslant C\|\boldsymbol{F}\|_{\boldsymbol{L}^p(\mathbb{R}^2)}. \tag{5.13}$$

The sequences u_λ and ∇u_λ remain bounded in $L^{p^*}(\mathbb{R}^2)$ and $\boldsymbol{L}^p(\mathbb{R}^2)$, respectively. These spaces are reflexive, so there exists $u \in L^{p^*}(\mathbb{R}^2)$ such that $u_\lambda \rightharpoonup u$ in $L^{p^*}(\mathbb{R}^2)$ and $\nabla u_\lambda \rightharpoonup \nabla u$ in $\boldsymbol{L}^p(\mathbb{R}^2)$. We easily verify that u is a solution of Poisson's Equation (5.9) and satisfies Estimate (5.10). The uniqueness of u follows by the fact that the space $L^{p^*}(\mathbb{R}^2)$ contains no polynomials. We deduce that $u \in W_0^{1,p}(\mathbb{R}^2)$ and we have also recovered the result obtained in [1] for $f \in W_0^{-1,p}(\mathbb{R}^2)$. $\square$

References

[1] *C. Amrouche, V. Girault, J. Giroire*: Weighted Sobolev spaces for Laplace's equation in $\mathbb{R}^n$. J. Math. Pures Appl., IX. Sér. *73* (1994), 579–606. zbl

[2] *C. Amrouche, U. Razafison*: Weighted Sobolev spaces for a scalar model of the stationary Oseen equation in $\mathbb{R}^3$. J. Math. Fluids Mech. To appear.

[3] *R. Farwig*: A variational approach in weighted Sobolev spaces to the operator $-\Delta + \partial/\partial x_1$ in exterior domains of $\mathbb{R}^3$. Math. Z. *210* (1992), 449–464. zbl

[4] *R. Farwig*: The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces. Math. Z. *211* (1992), 409–448. zbl

[5] *R. Farwig, H. Sohr*: Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains. Proc. 3rd International Conference on the Navier-Stokes

Equations: Theory and Numerical Methods, Oberwolfach, Germany, June 5–11, 1994 (J. G. Heywood, ed.). World Scientific, Ser. Adv. Math. Appl. Sci. Vol. 47, Singapore, 1998, pp. 11–30. zbl
[6] *R. Finn*: On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. Arch. Ration. Mech. Anal. *19* (1965), 363–406. zbl
[7] *R. Finn*: Estimates at infinity for stationary solutions of the Navier-Stokes equations. Bull. Math. Soc. Sci. Math. Phys. R. P. R. *51* (1960), 387–418. zbl
[8] *G. P. Galdi*: An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I: Linearized steady problems. Springer Tracts in Natural Philosophy, Vol. 38. Springer, New York, 1994. zbl
[9] *B. Hanouzet*: Espaces de Sobolev avec poids application au problème de Dirichlet dans un demi espace. Rend. Sem. Mat. Univ. Padova *46* (1972), 227–272. zbl
[10] *S. Kračmar, A. Novotný, M. Pokorný*: Estimates of Oseen kernels in weighted L^p spaces. J. Math. Soc. Japan *53* (2001), 59–111. zbl
[11] *A. Kufner*: Weighted Sobolev spaces. Wiley-Interscience, New York, 1985. zbl
[12] *P. I. Lizorkin*: (L^p, L^q)-multipliers of Fourier integrals. Dokl. Akad. Nauk SSSR *152* (1963), 808–811. zbl
[13] *C. W. Oseen*: Über die Stokessche Formel und über eine verwandte Aufgabe in der Hydrodynamik. Arkiv fór Mat. Astron. och Fys. *7* (1911), 1–36. zbl
[14] *C. W. Oseen*: Neuere Methoden und Ergebnisse in der Hydrodynamik. Akadem. Verlagsgesellschaft, Leipzig, 1927. zbl
[15] *C. Pérez*: Two weighted norm inequalities for Riesz potentials and uniform L^p- weighted Sobolev inequalities. Indiana Univ. Math. J. *39* (1990), 31–44. zbl
[16] *M. Reed, B. Simon*: Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-adjointness. Academic Press, New York-San Francisco-London, 1975. zbl
[17] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. University Press, Princeton, 1970. zbl

Authors' addresses: *C. Amrouche*, Laboratoire de Mathématiques Appliquées, UMR-CNRS 5142, Université de Pau et des Pays de l'Adour, France, e-mail: `cherif.amrouche@univ-pau.fr`; *H. Bouzit*, Département de Mathématiques, Université de Mostaganem, Algérie, e-mail: `hamidbouzit@yahoo.fr`.