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THE SCATTERING OF RESONANCE-LINE RADIATION IN THE LIMIT OF LARGE OPTICAL DEPTH

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SUMMARY

It is shown that the essential features of the transfer of resonance-line radiation in very optically thick media of low density are described by the Poisson equation. The solution of this equation is presented in the form of an eigenfunction expansion. Simple closed expressions are given for the mean number of scatterings for escape and for the line profile at both the centre and surface of the medium. In the case of a medium with a finite probability of photon destruction upon scattering, we obtain the fraction of the photons which escape and the limiting intensity at the centre as the optical depth tends to infinity. These analytic results are shown to agree well with the available numerical solutions.

1. INTRODUCTION

The escape of resonance-line radiation from optically thick media has been considered by many authors (1)–(4). We will not discuss the history of this problem or its astrophysical applications here. We limit our attention to the behaviour of solutions for optical depths so large that the transfer problem is dominated by the redistribution of radiation in the damping wings of the line. Adams (4) recently presented numerical solutions for such cases. It is evident from his calculations that the mean number of scatterings for escape is directly proportional to the optical depth. Adams was able to offer some heurestic justification for this behaviour.

In this paper, we regard the redistribution process as a diffusion in frequency space and are thus led to solve a partial differential equation (it is, in fact, the Poisson equation). This approach owes its inspiration to a paper by Unno (5), who applied the technique to the problem of the He II Lα line. While this paper is often cited in the literature, the complexity of Unno's application and certain inadequacies of the result have obscured the potential value of this method. As we shall see, a transformation of the frequency variable allows us to obtain analytic expressions which are valid in the limit of large optical depths and which compare well with the available numerical solutions.

2. FORMULATION OF THE DIFFERENTIAL EQUATION

We consider the medium to be a uniform plane-parallel slab with no opacity other than the resonance line itself. If the density is low, the scattering will be coherent in the rest frame of the atom and we may describe the frequency redistribution by Hummer's (6) angle averaged function $R_{\mathrm{II-A}}$. We allow the possibility that a photon may be destroyed rather than scattered and denote the possibility of destruction per scattering, which we assume to be very small, by ϵ. If we introduce the Eddington approximation, we can write a second-order equation for the mean intensity in the form

$$\frac{1}{3\phi^2(x)}\frac{d^2J(\tau,x)}{d\tau^2} = J(\tau,x)-(1-\epsilon)\int_{-\infty}^{\infty} J(\tau,x')\,q(x,x')\,dx' - \frac{G(\tau)}{4\pi}. \tag{1}$$

In this equation x is the frequency in Doppler widths, $\phi(x)$ is the normalized Voigt profile, τ is the mean optical depth (i.e. $\tau_x = \phi(x)\tau$ is the monochromatic optical depth), $G(\tau)$ is the source of radiation (per unit area, per unit mean optical depth), and $q(x, x') = R_{\text{II-A}}(x, x')/\phi(x)$. Adams, Hummer & Rybicki (7) have given an expansion of the redistribution function. The leading terms are

$$R_{\text{II-A}}(x,x') = \tfrac{1}{2}\,\text{erfc}\,(|r|+|s|)+\frac{a}{\pi}\frac{1}{s^2}\;i\,\text{erfc}\,(|r|)+\ldots, \tag{2}$$

where $r = (x-x')/2$ and $s = (x+x')/2$. In the line wings the first term can be neglected and $\phi(x) \cong a/(\pi x^2)$, so

$$q(x,x') = \frac{x^2}{s^2}\,i\,\text{erfc}\,(|r|), \tag{3}$$

a form first obtained by Unno (8). We expand $J(x')$ in a Taylor's series about x and, noting that $x^2/s^2 \cong (1+2r/x)$, obtain upon integration

$$\int_{-\infty}^{\infty} J(x')\,q(x,x')\,dx' = J(x) - \frac{1}{x}\frac{dJ(x)}{dx}+\frac{1}{2}\frac{d^2J(x)}{dx^2}+\ldots \tag{4}$$

Thus, to terms of second order, the transfer equation becomes

$$\frac{\partial^2 J}{\partial\tau^2} + \frac{3}{2}\,\phi^2(1-\epsilon)\left\{\frac{\partial^2 J}{\partial x^2} - \frac{2}{x}\frac{\partial J}{\partial x}\right\} = 3\phi^2\left\{\epsilon J - \frac{G}{4\pi}\right\}. \tag{5}$$

We now introduce the variable σ defined by

$$\sigma(x) = \int_0^x \{(3/2)^{1/2}\,\phi(x)\}^{-1}\,dx, \tag{6}$$

so that

$$\frac{dx}{d\sigma} = \left(\frac{3}{2}\right)^{1/2}\phi. \tag{7}$$

We then have

$$\frac{3}{2}\phi^2\left\{\frac{\partial^2 J}{\partial x^2} - \frac{2}{x}\frac{\partial J}{\partial x}\right\} = \frac{\partial^2 J}{\partial\sigma^2} - \left\{\frac{1}{\phi}\frac{\partial\phi}{\partial\sigma} + \frac{\sqrt{6}}{x\,\phi}\right\}\frac{\partial J}{\partial\sigma}. \tag{8}$$

Now for large values of x we can easily show that

$$\sigma(x) \simeq \left(\frac{2}{3}\right)^{1/2}\frac{\pi}{a}\frac{x^3}{3} \tag{9}$$

so that in the line wings where $\phi \simeq a/(\pi x^2)$ we find

$$\frac{1}{\phi}\frac{\partial\phi}{\partial\sigma} + \frac{\sqrt{6}}{x}\phi \simeq -\frac{\sqrt{6}}{\pi}\frac{a}{x^3} + \frac{\sqrt{6}}{x}\phi = 0. \tag{10}$$

Thus, since $(1-\epsilon) \cong 1$, we obtain

$$\frac{\partial^2 J}{\partial \tau^2}+\frac{\partial^2 J}{\partial \sigma^2} = 3\phi^2\left\{\epsilon J-\frac{G}{4\pi}\right\}. \tag{11}$$

In terms of the overall width of the line, ϕ^2 is very sharply peaked at $\sigma = 0$, so we approximate it with a delta function. To preserve the normalization, we note that

$$\int_{-\infty}^{\infty} 3\phi^2\, d\sigma = \int_{-\infty}^{\infty} 3\phi^2 \frac{d\sigma}{dx}\, dx = \sqrt{6}, \tag{12}$$

by the normalization of ϕ. Thus we can represent the transfer problem by the following partial differential equation:

$$\frac{\partial^2 J}{\partial \tau^2}+\frac{\partial^2 J}{\partial \sigma^2} = \sqrt{6}\left\{\epsilon J-\frac{G}{4\pi}\right\}\delta(\sigma). \tag{13}$$

We consider the medium to have a total optical thickness $2B$, with $\tau = 0$ at the centre and boundaries at $\pm B$. The boundary conditions can then be written as

$$\left(\frac{\partial J}{\partial \tau}\right)_{\pm B} = \mp \frac{3}{2}\phi J(B, \sigma) \tag{14}$$

and

$$\lim_{\sigma\to\pm\infty} J(\tau, \sigma) = 0. \tag{15}$$

3. SOLUTION OF THE EQUATION

Following Unno (5), we employ an eigenfunction expansion. Consider the homogeneous equation

$$\frac{d^2\theta}{d\tau^2}+\lambda^2\theta = 0. \tag{16}$$

We can easily verify that solutions of the form*

$$\theta_n = A\cos(\lambda_n\tau); \quad n = 1, 2, \ldots, \tag{17}$$

will satisfy the boundary conditions (14) as long as λ_n satisfies the equation

$$\lambda_n \tan(\lambda_n B) = \tfrac{3}{2}\phi. \tag{18}$$

If we now define $z_n = \lambda_n B$, this equation is equivalent to

$$z_n = \pi(n-1)+\tan^{-1}\left\{\frac{3}{2}\frac{\phi B}{z_n}\right\}; \quad n = 1, 2, \ldots, \tag{19}$$

where the principal value of the arctangent is understood. Thus we see that $\pi(n-1) \leq z_n \leq \pi(n-1/2)$. Now, for problems where the optical depth is large, $\phi B \gg 1$ out to any frequency where there is appreciable radiation (i.e. the photons escape before they diffuse to frequencies where the whole slab is optically thin). It then follows that

$$z_n \simeq \pi(n-1/2) \tag{20}$$

to within a term of the order $\pi n/\phi B$. Thus, while the λ_n depend on σ through ϕ

* We will only consider media with source terms symmetric about the central plane. The solutions must then be even functions of τ.

in equation (18), this dependence is very slight. Now, the θ_n are orthogonal on the interval $[-B, B]$ and

$$\int_{-B}^{B} \cos^2(\lambda_n\tau)\,d\tau = B + \frac{(3/2)\phi}{(9/4)\phi^2+\lambda_n^2} \simeq B, \tag{21}$$

so that the functions $B^{-1/2}\cos(\lambda_n\tau)$ form an orthonormal set and we may represent the mean intensity as

$$J(\tau,\sigma) = \sum_{k=1}^{\infty} B^{-1/2}\cos(\lambda_k\tau)\,j_k(\sigma). \tag{22}$$

We introduce this series into equation (13), and upon multiplying by $B^{-1/2}\cos(\lambda_n\tau)$ and integrating over τ, obtain

$$\frac{d^2 j_n}{d\sigma^2} - \lambda_n^2 j_n = \sqrt{6}\left\{\epsilon j_n - B^{-1/2}\frac{Q_n}{4\pi}\right\}\delta(\sigma), \tag{23}$$

where

$$Q_n = \int_{-B}^{B} G(\tau)\cos(\lambda_n\tau)\,d\tau. \tag{24}$$

Away from $\sigma = 0$, the solution of equation (23) which satisfies the boundary condition (15) is

$$j_n(\sigma) = C\exp(-\lambda_n|\sigma|). \tag{25}$$

Because of the delta function, the derivative of $j_n(\sigma)$ must have a jump at $\sigma = 0$ which satisfies

$$\left(\frac{dj_n}{d\sigma}\right)_{\sigma\to+0} - \left(\frac{dj_n}{d\sigma}\right)_{\sigma\to-0} = -2\lambda_n C = \sqrt{6}\left\{\epsilon C - B^{-1/2}\frac{Q_n}{4\pi}\right\}, \tag{26}$$

and this determines the constant C. Thus we obtain the solution of equation (13):

$$J(\tau,\sigma) = \frac{\sqrt{6}}{8\pi}\sum_{n=1}^{\infty}\frac{Q_n\cos(z_nB^{-1}\tau)\exp(-z_nB^{-1}|\sigma|)}{z_n+\sqrt{6}\epsilon B/2}. \tag{27}$$

4. EVALUATION FOR A CENTRAL SOURCE

In this and the following section we will consider $\epsilon = 0$. The simplest case is that of a source of unit strength located at the central plane of the slab. Then

$$Q_n = \int_{-B}^{B}\delta(\tau)\cos(\lambda_n\tau)\,d\tau = 1 \tag{28}$$

and

$$J(\tau,\sigma) = \frac{\sqrt{6}}{8\pi}\sum_{n=1}^{\infty} z_n^{-1}\cos(z_nB^{-1}\tau)\exp(-z_nB^{-1}|\sigma|). \tag{29}$$

We may evaluate this expression at the centre of the slab with the help of approximation (20) and the series expansion of $\tanh^{-1}$:

$$J(0,\sigma) = \frac{\sqrt{6}}{8\pi}\sum_{n=1}^{\infty}\frac{\exp(-\pi(n-1/2)B^{-1}|\sigma|)}{\pi(n-1/2)} = \frac{\sqrt{6}}{4\pi^2}\tanh^{-1}\{\exp(-\pi|\sigma|/2B)\}. \tag{30}$$

Using the approximate value of σ given by (9), this becomes

$$J(0, x) = \frac{\sqrt{6}}{4\pi^2} \tanh^{-1} \left(\exp \left\{ -\frac{\pi^2}{6} \left(\frac{2}{3}\right)^{1/2} \frac{|x^3|}{aB} \right\} \right). \tag{31}$$

While this function has a logarithmic singularity at $x = 0$ which is characteristic of a solution of Poisson's equation for a point source in the $\sigma\tau$ plane, away from the origin it agrees well with numerical solutions. If we integrate the series term by term and sum we obtain the integrated intensity at the centre:

$$\langle J \rangle_0 = \int_{-\infty}^{\infty} J(0, \sigma)\, dx = \frac{\sqrt{6}\Gamma(1/3)}{12\pi^2} (2^{4/3} - 1)\, \zeta(4/3) \times \left\{ \frac{3}{\pi^2} \left(\frac{3}{2}\right)^{1/2} aB \right\}^{1/3} = 0{\cdot}218136\{aB\}^{1/3}, \tag{32}$$

where $\zeta(\ )$ is the Riemann zeta function.

We also obtain closed expressions at the boundaries. Here we have

$$J(B, \sigma) = \frac{\sqrt{6}}{8\pi} \sum_{n=1}^{\infty} z_n^{-1} \cos(z_n) \exp(-z_n B^{-1} |\sigma|). \tag{33}$$

From equation (18) we see that

$$\frac{\cos(z_n)}{z_n} = \frac{\sin(z_n)}{(3/2)\phi B} \simeq \frac{(-1)^{n-1}}{(3/2)\phi B}, \tag{34}$$

so that (33) reduces to

$$J(B, x) = \frac{\sqrt{6}}{24} \frac{x^2}{aB} \operatorname{sech} \left\{ \frac{\pi^2}{6} \left(\frac{2}{3}\right)^{1/2} \frac{|x^3|}{aB} \right\}. \tag{35}$$

This expression can be integrated to show that the flux emerging from both boundaries is

$$2\pi\langle F \rangle_B = 4\pi \langle J \rangle_B = 4\pi \int_{-\infty}^{\infty} J(B, x)\, dx = 1, \tag{36}$$

in agreement with the unit source. Setting the derivative of $J(B, x)$ to zero we obtain a transendental equation for the maximum of the emergent profile. The resulting solution is

$$x_m = \pm 0{\cdot}88119\{aB\}^{1/3}. \tag{37}$$

Finally, we consider the mean number of scatterings for escape. This is just the number of photons absorbed (scattered) per unit time over the whole medium divided by the generation rate. Thus

$$\langle N \rangle = \int_{-B}^{B} \int_{-\infty}^{\infty} 4\pi J(\tau, \sigma)\, \phi\, dx\, d\tau. \tag{38}$$

Integration over τ yields

$$\langle N \rangle = \sqrt{6}\, B \sum_{n=1}^{\infty} (-1)^{n-1} \pi^{-2} (n - 1/2)^{-2} \int_{-\infty}^{\infty} \exp(-z_n B^{-1} |\sigma|)\, \phi\, dx. \tag{39}$$

When the optical depth is very large, we can easily verify that over the core of the line $z_n B^{-1}|\sigma| \ll 1$ up to large values of n. This simply establishes that at the line centre where most of the scattering occurs, the intensity *averaged over the whole medium* has a flat maximum. Setting the exponential to unity and recalling that ϕ is normalized with respect to x, we have

$$\langle N \rangle = \frac{\sqrt{6}}{\pi^2} B \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1/2)^2} = \frac{4\sqrt{6}}{\pi^2} u_2 B = 0{\cdot}909316B, \tag{40}$$

where $u_2 = 1-3^{-2}+5^{-2}-7^{-2}+\ldots = 0{\cdot}9159656$.

5. EVALUATION FOR A UNIFORM SOURCE

Another case of interest is that of a uniform source distribution in the slab. If we again normalize to unit generation in a column of unit area through the slab, we must set $G = (2B)^{-1}$. Then we find that

$$Q_n = \frac{\sin(z_n)}{z_n} \simeq \frac{(-1)^{n-1}}{z_n} \tag{41}$$

and

$$J(\tau, \sigma) = \frac{\sqrt{6}}{8\pi} \sum_{n=1}^{\infty} (-1)^{n-1} z_n^{-2} \cos(z_n B^{-1}\tau) \exp(-z_n B^{-1}|\sigma|). \tag{42}$$

Proceeding analogously to the preceding section, we find

$$J(0, x) = \frac{\sqrt{6}}{2\pi^3} S\left(\exp\left\{-\frac{\pi^2}{6}\left(\frac{2}{3}\right)^{1/2}\frac{|x^3|}{aB}\right\}\right), \tag{43}$$

where the function S is defined as

$$S(z) = z - \frac{z^3}{3^2} + \frac{z^5}{5^2} - \frac{z^7}{7^2} + \ldots = \int_0^z \operatorname{arccot}(\theta^{-1})\, \theta^{-1}\, d\theta. \tag{44}$$

Over the range $0 < z < 1$, $S(z) \cong z$. Thus the intensity at the centre displays a flat maximum with steep sides. The integrated intensity at the centre is

$$\langle J \rangle_0 = \frac{\sqrt{6}\Gamma(1/3)}{12\pi^3} 2^{7/3} u_{7/3} \left(\frac{3}{\pi^2}\left(\frac{3}{2}\right)^{1/2} aB\right)^{1/3} = 0{\cdot}060063\{aB\}^{1/3}, \tag{45}$$

where $u_{7/3} = 1-3^{-7/3}+5^{-7/3}-\ldots = 0{\cdot}9393807$.
The intensity at the surface is

$$J(B, x) = \frac{1}{\sqrt{6}\pi} \frac{x^2}{aB} \tanh^{-1}\left(\exp\left\{-\frac{\pi^2}{6}\left(\frac{2}{3}\right)^{1/2}\frac{|x^3|}{aB}\right\}\right), \tag{46}$$

which attains a maximum at

$$x_m = \pm 0{\cdot}71025(aB)^{1/3}. \tag{47}$$

Finally, the mean number of scatterings for escape is

$$\langle N \rangle = \frac{7\sqrt{6}}{\pi^3} \zeta(3) B = 0{\cdot}664736B. \tag{48}$$

6. EVALUATION FOR A MEDIUM WITH A FINITE DESTRUCTION PROBABILITY

We next consider the case of a non-zero photon destruction probability, but still demand that this effect be small enough that the radiation spreads far out into the damping wings. The principal example is the H I Lα line in a static dust-free H° region, where conversion to the two-photon continuum occurs at a rate of $\epsilon = 4{\cdot}4 . 10^{-9}$ for sufficiently high densities (3). We consider a source at the centre of the medium. Then the intensity at the central plane is given by

$$J(0, \sigma) = \frac{\sqrt{6}}{8\pi} \sum_{n=1}^{\infty} \frac{\exp\left(-\pi(n-1/2)B^{-1}|\sigma|\right)}{\pi(n-1/2)+\sqrt{6}\epsilon B/2}. \tag{49}$$

An important quantity is the integrated intensity. If we define $p = \sqrt{6}\epsilon/2\pi$, we obtain

$$\langle J \rangle_0 = \frac{\sqrt{6}\Gamma(1/3)}{12\pi^2} \left(\left(\frac{3}{2}\right)^{1/2} \frac{3}{\pi^2} aB\right)^{1/3} \sum_{n=1}^{\infty} \frac{1}{(n-1/2)^{1/3}\{(n-1/2)+pB\}}. \tag{50}$$

Let us consider optical depths so large that most photons are destroyed rather than escape. Then $pB \gg 1$, and the terms of the series will at first decrease slowly. Beyond the first few terms, however, we can approximate the sum by an integral over $(n-1/2)$. With some manipulation we obtain

$$\langle J \rangle_0 = \begin{pmatrix}\text{The first } M \text{ terms} \\ \text{of equation (50).}\end{pmatrix} + \frac{\sqrt{6}\Gamma(1/3)}{4\pi^2} \left\{\frac{3}{\pi}\frac{a}{\epsilon}\right\}^{1/3} \int_{w_0}^{\infty} \frac{w\,dw}{1+w^3}, \tag{51}$$

where $w_0 = (M/pB)^{1/3}$. The integral is elementary. In the limit as $B \to \infty$, $w_0 \to 0$ and the value of the integral is $2\pi/(3\sqrt{3})$, so that

$$\lim_{B\to\infty} \langle J \rangle_0 = \frac{\Gamma(1/3)\sqrt{2}}{6\pi} \left\{\frac{3}{\pi}\frac{a}{\epsilon}\right\}^{1/3} = 0{\cdot}1979(a/\epsilon)^{1/3}. \tag{52}$$

This is the greatest intensity which can be maintained in a medium with a given photon destruction probability. The temperature dependence of the result is very weak: $\langle J \rangle_0 \propto a^{1/3} \propto T^{-1/6}$.

We can also obtain the amount of radiation which escapes the medium. We evaluate equation (27) at $\tau = B$ with the aid of equation (34) and integrate over x to obtain

$$\langle J \rangle_B = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\pi(n-1/2)+\sqrt{6}\epsilon B/2}. \tag{53}$$

This sum can be expressed exactly as an integral. Since we have a unit source, the fraction of the original photons which finally escape the slab is $f_e = 4\pi\langle J \rangle_B$. Thus we find

$$f_e = \frac{2}{\pi} \int_0^1 \eta^{r-1}(1+\eta)^{-1}\,d\eta, \tag{54}$$

where $r = \frac{1}{2}+\sqrt{6}\epsilon B/2\pi$. Note that for $\epsilon = 0$, $r = \frac{1}{2}$ and we recover (36). A more useful expansion than the series in (53) follows upon integration by parts:

$$f_e = \frac{1}{\pi}\left(\frac{1}{r}+\frac{1}{2r(r+1)}+\ldots+\frac{n!}{2^n r(r+1)\ldots(r+n)}+\ldots\right). \tag{55}$$

7. COMPARISON WITH NUMERICAL CALCULATIONS

Adams (4) has presented results of numerical calculations for line centre optical depths up to 10^8. These results show that the number of scatterings for escape is proportional to the optical thickness of the medium. Photons in the medium are constantly changing frequency as they scatter, taking steps of about one Doppler width. They originate in the line core and are scattered very frequently there, but it is only on their excursions into the wings that they can travel an appreciable distance from their point of origin. Since they are executing a sort of random walk in frequency, they may return many times to the core. By introducing *a priori* the assumption that only the *longest single excursion* into the wings is effective in moving a photon the distance needed for escape, Adams showed that the photons should escape at a frequency of about $0{\cdot}68\,(aB)^{1/3}$ Doppler widths and that the mean number of scatterings for escape should be about $1{\cdot}5B$. Aside from the numerical constants, these are our equations (37) and (40). Examination of Adams' Fig. 1 and Fig. 5 shows that the constants we have obtained fit the numerical results as closely as can be judged from diagrams on this scale.

In a discussion of the dense early stages of planetary nebulae we have given numerical results for the Lα radiation field in an H^+ region surrounded by a thick neutral shell (9). The solutions were obtained by the Feautrier method; the redistribution function was handled in a manner similar to that described by Adams *et al.* (7). The parameters of the model were as follows: (a) the temperature of the

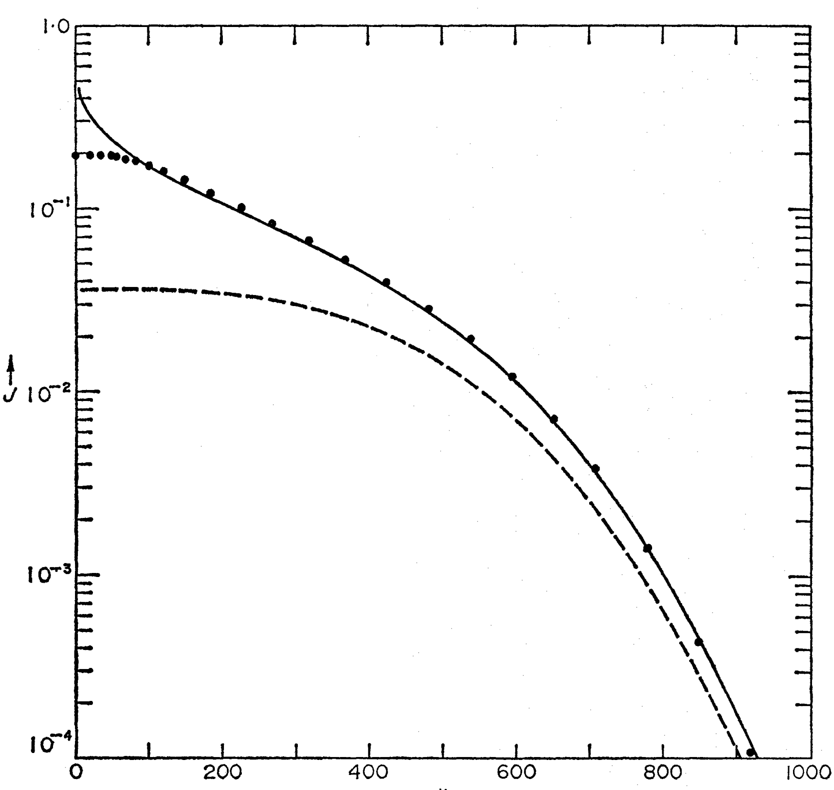

FIG. 1. *The intensity at the centre of a slab with an optical half-thickness of* $B = 5\,.\,10^{10}$ *and a damping parameter of* $a = 3{\cdot}33\,.\,10^{-3}$. *The solid curve is the analytic expression for a source concentrated in the central plane as given by equation* (31); *the dashed curve is the corresponding expression for a uniform source distribution given by equation* (43). *The filled circles are the results of a numerical solution described in the text.*

ionized hydrogen was 10 000 K and the mean optical depth of this zone in the line was 10^6, (b) the temperature of the neutral shell was 200 K ($a = 3{\cdot}33 . 10^{-3}$) with a line optical depth of $5 . 10^{10}$. Because of the great optical depth of the shell, the H^+ region, in spite of the higher temperature, has only minor influence on the solution and acts mainly as a photon source.

In Fig. 1 we show the numerical solution at the H^+–H° boundary in units of the Doppler width in the neutral shell. The solid curve is the analytic solution for the intensity at the centre of a slab with a central source as given by equation (31), evaluated for $B = 5 . 10^{10}$ and $a = 3{\cdot}33 . 10^{-3}$. The broken curve is the corresponding solution for a uniform source given by equation (43). The plateau at the centre of the numerical solution is due mainly to the fact that the source is not concentrated but extends from $\tau = 0$ to 10^6 and has a greater Doppler width due to the higher temperature. The failure of the analytic solution due to the use of a delta function in frequency in the source term would only show up a few Doppler widths from line centre. (See Fig. 3 in the paper by Adams (4) for a true central source, where the intensity at $\tau = 0$ continues to increase up to the core-wing transition.)

In Fig. 2 we show the numerical solution at the surface of the H° shell compared to equations (35) and (46). The emergent intensity is in excellent agreement with the analytic solution for a central source.

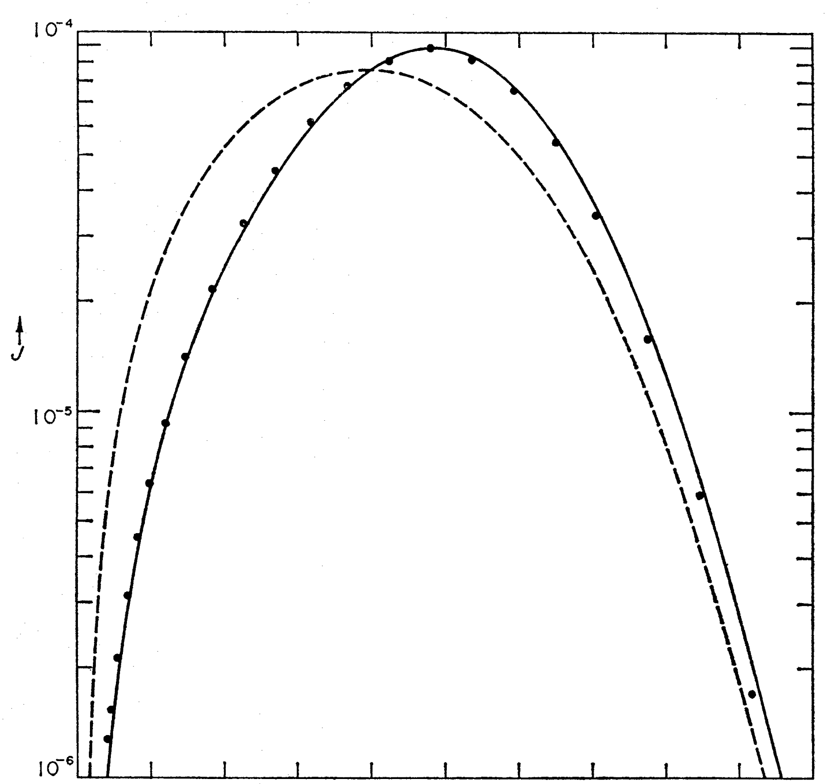

FIG. 2. *The surface intensity for a slab with the same parameters as Fig. 1. The solid curve is the expression for a central source given by equation (35); the dashed curve is the corresponding expression for a uniform source given by equation (46). The filled circles are a numerical solution.*

Thus it appears that the expressions derived in this paper not only clarify the asymptotic behaviour of this type of radiative transfer problem, but also provide solutions of sufficient accuracy to be of practical application.

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