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THE SCATTERING OF RESONANCE-LINE RADIATION IN THE LIMIT OF LARGE OPTICAL DEPTH—II

Reflection and Transmission of Radiation Incident upon a Slab

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SUMMARY

A solution developed previously is extended to the case of a slab illuminated by monochromatic radiation. The frequency distribution of the reflected and transmitted radiation is expressed as an infinite series, and some approximate closed expressions are obtained. Application of the results to the problem of escape of photons from an expanding shell of great optical depth is discussed.

1. INTRODUCTION

In a low density medium of sufficiently great optical depth, the transfer of resonance-line radiation is dominated by the redistribution in frequency upon scattering of photons absorbed in the natural damping wings of the line. In Paper I (1) we showed that radiative transfer under such conditions can be described by a partial differential equation (the Poisson equation if we neglect photon destruction) with boundary conditions which allow an approximate solution by an eigenfunction expansion. We considered only photons introduced at the central frequency of the line by sources distributed symmetrically about the central plane of the slab. Here we extend the solution to an arbitrary space distribution of sources emitting at an arbitrary frequency. In particular, we consider photons which originate by external illumination at a given frequency. The resultant radiation field at the boundaries yields the reflection and transmission functions of the slab expressed as an infinite series. The sums of these series can in some cases be approximated by simple analytic functions. We show how these functions may be used to investigate the escape of Lα radiation from an expanding nebula.

2. THE SOLUTION FOR AN EXTERNALLY ILLUMINATED SLAB

Let radiation be incident on a plane-parallel slab at a frequency x_* measured in Doppler widths from line centre. If a distinction is made between the incident radiation before absorption and the diffuse radiation which has been scattered at least once, then the point where the incident radiation is first scattered can be considered as a source of diffuse photons. The transfer equation for the diffuse radiation is then

$$\mu \frac{dI(\tau, x, \mu)}{d\tau} = \phi(x)\, I(\tau, x, \mu) - (1-\epsilon) \times \int_{-\infty}^{\infty} J(\tau, x')\, R(x, x')\, dx' - \frac{\delta(x-x_*)}{4\pi} (1-\epsilon)\, G(\tau), \qquad (1)$$

where the source term $G(\tau)$ is the amount of incident radiation absorbed per unit area per unit mean optical depth, ϕ is the normalized Voigt profile, ϵ is the probability that a photon is destroyed rather than scattered, and $\delta(x-x_*)$ is the Dirac delta function. As in Paper I we use the Eddington approximation, expand the redistribution function $R(x, x')$ in a Taylor series, and change the frequency variable to

$$\sigma(x) = \int_0^x \{(3/2)^{1/2}\,\phi(x)\}^{-1}\,dx. \tag{2}$$

If ϵ is extremely small the only appreciable destruction of photons will be in the line core. Under this approximation, we obtain a partial differential equation for the mean intensity:

$$\frac{\partial^2 J}{\partial\tau^2}+\frac{\partial^2 J}{\partial\sigma^2} = \sqrt{6}\epsilon\delta(\sigma)\,J-\sqrt{6}\,\frac{\delta(\sigma-\sigma_*)}{4\pi}\,G, \tag{3}$$

where σ_* corresponds to the incident frequency x_*. Let the slab have a total optical thickness of $2B$ with $\tau = 0$ at the centre and $\tau = \pm B$ at the boundaries. Since there is no incident *diffuse* radiation, the boundary conditions are

$$\left(\frac{\partial J}{\partial\tau}\right)_{\pm B} = \mp\sqrt{3}\phi J(\pm B, \sigma) \tag{4}$$

and

$$\lim_{\sigma\to\pm\infty} J(\tau, \sigma) = 0. \tag{5}$$

Define the sequence of functions

$$\theta_n = N_n \begin{Bmatrix}\cos\,(z_n\tau/B)\\ \sin\,(z_n\tau/B)\end{Bmatrix} \quad\text{for}\quad \begin{Bmatrix} n = 1, 3, 5, \ldots\\ n = 2, 4, 6, \ldots\end{Bmatrix}, \tag{6}$$

where

$$z_n = \frac{\pi}{2}(n-1)+\arctan\left(\frac{\sqrt{3}\phi B}{z_n}\right) \quad\text{for}\quad n = 1, 3, 5, \ldots, \tag{7}$$

$$z_n = \frac{\pi}{2}n-\arctan\left(\frac{z_n}{\sqrt{3}\phi B}\right) \quad\text{for}\quad n = 2, 4, 6, \ldots, \tag{8}$$

and the normalization constant is

$$N_n = B^{-1/2}\left(1+\frac{\sqrt{3}\phi B}{3\phi^2B^2+z_n^2}\right)^{-1/2} \simeq B^{-1/2}. \tag{9}$$

The values of z_n are always limited by $(\pi/2)(n-1)<z_n<(\pi/2)\,n$ and when the optical depth is extremely large so that $\phi B\gg 1$, z_n is well approximated by

$$z_n \simeq \frac{\pi}{2}n. \tag{10}$$

The functions θ_n are orthonormal on the interval $[-B, B]$ and they satisfy the boundary conditions (4). We expand J in terms of these functions:

$$J(\tau, \sigma) = \sum_{n=1}^{\infty} j_n(\sigma)\,\theta_n(\tau). \tag{11}$$

The fundamental assumption is that the σ dependance of θ_n (which enters because the z_n depend on σ through ϕ) is small and can be neglected. This is justifiable for $\phi B \gg 1$ in view of (10). Then equation (3) leads to

$$\frac{d^2 j_n}{d\tau^2} - \frac{z_n^2}{B^2} j_n = \sqrt{6}\epsilon\delta(\sigma) j_n - \sqrt{6}\delta(\sigma-\sigma_*) N_n Q_n \tag{12}$$

where

$$Q_n = \int_{-B}^{B} G(\tau) \begin{Bmatrix} \cos(z_n\tau/B) \\ \sin(z_n\tau/B) \end{Bmatrix} d\tau. \tag{13}$$

The solution of equation (12) then results in

$$J(\tau, \sigma) = \frac{\sqrt{6}}{8\pi} \sum_{n=1}^{\infty} \frac{Q_n}{z_n} \begin{Bmatrix} \cos(z_n\tau/B) \\ \sin(z_n\tau/B) \end{Bmatrix} \times \left[\exp(-z_n|\sigma-\sigma_*|/B) - \frac{\exp\{-z_n(|\sigma|+|\sigma_*|)/B\}}{1+2z_n/\sqrt{6}\epsilon B} \right]. \tag{14}$$

When $\sigma_* = 0$ and $G(\tau)$ is symmetrical about $\tau = 0$ (so that $Q_n = 0$ for even n) this reduces to equation (27) of Paper I.

Let a beam of radiation of unit flux πF be incident upon the $\tau = -B$ face of the slab at an angle to the normal with cosine μ. Then the radiation absorbed in a given layer is

$$G(\tau) = \frac{\phi}{\mu} \exp\{-\phi(\tau+B)/\mu\} \tag{15}$$

and the evaluation of (13) yields

$$Q_n = \left[1 + \left(\frac{\mu z_n}{\phi B}\right)^2\right]^{-1} \times [(-1)^{n-1}(\sqrt{3}\mu+1) + (\sqrt{3}\mu-1)\exp(-2\phi B/\mu)] \begin{Bmatrix} \cos z_n \\ \sin z_n \end{Bmatrix}. \tag{16}$$

For the remainder of this discussion we will assume that the incident radiation can be represented by a beam entering at $\mu = 1/\sqrt{3}$. Since the Eddington approximation is equivalent to a discrete ordinate formulation with $\mu = 1/\sqrt{3}$, this choice is in some sense consistent. Making use of some relations which follow from the boundary conditions,

$$\begin{Bmatrix} \cos^2(z_n) \\ \sin^2(z_n) \end{Bmatrix} = \left[1 + \left(\frac{\sqrt{3}\phi B}{z_n}\right)^2\right]^{-1} \quad \text{for} \quad \begin{Bmatrix} n = 1, 3, 5, \ldots \\ n = 2, 4, 6, \ldots \end{Bmatrix}, \tag{17}$$

to evaluate equation (14) at the boundaries $\tau = \pm B$, we obtain

$$J(\mp B, \sigma) = \frac{\sqrt{6}}{4\pi} \frac{1}{3\phi^2 B^2} \sum_{n=1}^{\infty} (\pm 1)^{n-1} z_n \left[1 + \left(\frac{z_n}{\sqrt{3}\phi B}\right)^2\right]^{-1} \times \left\{ \exp(-z_n|\sigma-\sigma_*|/B) - \frac{\exp\{-z_n(|\sigma|+|\sigma_*|)/B\}}{1+2z_n/\sqrt{6}\epsilon B} \right\}. \tag{18}$$

Let us define a reflection coefficient $\mathbf{R}(\sigma, \sigma_*)$ as the diffuse flux which is scattered back at $\tau = -B$ at the frequency σ as a result of illumination with a unit beam at frequency σ_*. Similarly, the transmission function $\mathbf{T}(\sigma, \sigma_*)$ is the flux scattered out of the $\tau = B$ face. Since $\pi F = (4\pi/\sqrt{3}) J$, upon setting $z_n = (\pi/2) n$ and defining the total optical thickness $T = 2B$, we find

$$\begin{Bmatrix} \mathbf{R}(\sigma, \sigma_*) \\ \mathbf{T}(\sigma, \sigma_*) \end{Bmatrix} = \frac{4\pi}{3\sqrt{2}} \frac{1}{\phi^2 T^2} \sum_{n=1}^{\infty} \frac{(\pm 1)^{n-1}}{(1+pn^2)^2} n \left[\exp(-\alpha n) - \frac{\exp(-\beta n)}{1+sn} \right], \tag{19}$$

where $\alpha = \pi|\sigma - \sigma_*|/T$, $\beta = \pi(|\sigma| + |\sigma_*|)/T$, $p = \pi^2/3\phi^2T^2$ and $s = 2\pi/\sqrt{6}\epsilon T$.

3. EVALUATION OF THE REFLECTION AND TRANSMISSION FUNCTIONS

While it is sometimes feasible to sum equation (19) numerically, in cases of extreme optical depth, α, β and p can become very small, so that an excessive number of terms is required. We can, however, obtain some useful approximations for such conditions.

We first examine the case where there is no photon destruction. Then $\epsilon = 0$ and the second exponential term in the sum, $\exp(-\beta n)/(1+sn)$, vanishes. Consider the transmitted radiation. For $\phi T \gg 1$, the incident beam is scattered near $\tau = -B$, so that the source term $G(\tau)$ is appreciable only near this boundary. In the expression for $T(\sigma, \sigma_*)$ the low n terms will be the important ones since they correspond to the long wavelength components which can propagate to $\tau = B$, while the high n terms will effectively cancel one another. Then, since $(1+pn^2)^2 \simeq 1$ until n becomes very large

$$\mathbf{T}(\sigma, \sigma_*) \simeq \frac{4\pi}{3\sqrt{2}} \frac{1}{\phi^2 T^2} \sum_{n=1}^{\infty} (-1)^{n-1} n \exp(-\alpha n). \tag{20}$$

Recalling that in the wings of the line $\phi \simeq (a/\pi x^2)$ and $\sigma \simeq (2/3)^{1/2}(\pi/a)(x^3/3)$, we sum the series to obtain

$$\mathbf{T}(x, x_*) \simeq \frac{\pi^3}{3\sqrt{2}} \frac{x^4}{(aT)^2} \operatorname{sech}^2 \left[\frac{\pi^2}{6} \left(\frac{2}{3}\right)^{1/2} \frac{|x^3 - x_*{}^3|}{aT} \right]. \tag{21}$$

Because of the x^4 factor, the radiation is not symmetric about x_*, but is skewed away from line centre to frequencies where the opacity is smaller.

Turning to the reflection coefficient, we can make the same approximation if $|x - x_*|$ is large enough for the exponential term in equation (19) to choke off the series before $(1+pn^2)^2$ differs much from unity. The result is

$$\mathbf{R}(x, x_*) \simeq \frac{\pi^3}{3\sqrt{2}} \frac{x^4}{(aT)^2} \operatorname{csch}^2 \left[\frac{\pi^2}{6} \left(\frac{2}{3}\right)^{1/2} \frac{|x^3 - x_*{}^3|}{aT} \right]. \tag{22}$$

Since $\operatorname{csch}^2(x) = 1/x^2 - \frac{1}{3} + (7/90)x^2 - \ldots$, this approximation blows up as $x \to x_*$. However, if α and p are both small, we can consider replacing the summation by an integration:

$$\sum_{n=1}^{\infty} \frac{n \exp(-\alpha n)}{(1+pn^2)^2} \simeq \int_0^{\infty} \frac{n \exp(-\alpha n)}{(1+pn^2)^2} dn = \frac{1}{p}\{1 - \nu f(\nu)\}, \tag{23}$$

where

$$\nu = \sqrt{3}\phi|\sigma - \sigma_*| = \frac{\sqrt{2}}{3} \frac{|x^3 - x_*{}^3|}{x^2} \tag{24}$$

and

$$f(\nu) = \int_0^{\infty} \frac{\exp(-\nu y)\, dy}{1+y^2} = Ci(\nu)\sin(\nu) + \left\{\frac{\pi}{2} - Si(\nu)\right\}\cos(\nu) \tag{25}$$

expresses the integral in terms of the cosine integral function Ci and the sine integral function Si (2). Thus we obtain

$$\mathbf{R}(x, x_*) = \frac{\sqrt{2}}{\pi}\{1-\nu f(\nu)\}. \tag{26}$$

Fig. 1 shows this expression as a function of ν. Note that the optical thickness T does not appear. As $T \to \infty$ the integral representation (23) becomes exact. Thus equation (26) is actually the proper result for the reflection from a semi-infinite slab. At $x = x_*$, $\mathbf{R}(x, x_*) = \sqrt{2}/\pi$, while the asymptotic expansion of $f(\nu)$ is

$$f(\nu) \sim \frac{1}{\nu}\left\{1-\frac{2!}{\nu^2}+\frac{4!}{\nu^4}-\dots\right\}, \tag{27}$$

so that away from x_* equation (26) tends toward

$$\mathbf{R}(x, x_*) \sim \frac{\sqrt{2}}{\pi}\frac{2}{\nu^2} = \frac{9\sqrt{2}}{\pi}\frac{x^4}{(x^3-x_*{}^3)^2}. \tag{28}$$

Furthermore, if $|x_*| \gg |x-x_*|$, $\nu \simeq \sqrt{2}|x-x_*|$ and (28) becomes

$$\mathbf{R}(x, x_*) \sim \frac{\sqrt{2}}{\pi}\frac{1}{(x-x_*)^2}. \tag{29}$$

Thus, the back scattering from a semi-infinite slab produces, in the wing of the line, a sort of Lorentzian profile about the input frequency. Those photons which

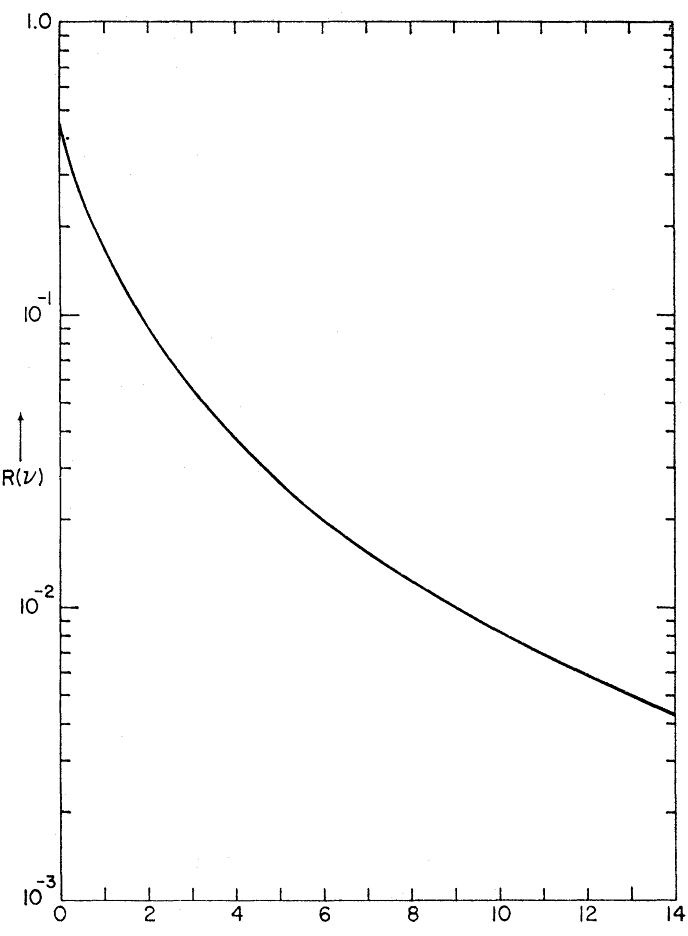

FIG. 1. *The reflection function for a semi-infinite slab as given by equation* (26).

stray far from x_* do so by diffusing far into the slab before working their way out again. If the slab is of finite thickness, then for some sufficiently large $|x-x_*|$ the photons will be more likely to leak out the far side than to continue to diffuse in frequency. Thus, approximation (29) will go over to approximation (22), which falls off exponentially for large $|x-x_*|$. If $aT \gg 1$, then the two approximations will merge smoothly. This can be clearly seen by noting that if we insert the approximation $\operatorname{csch}^2(x) \simeq 1/x^2$ into equation (22) we obtain exactly equation (28).

It is possible to write down various expressions which reduce to equation (26) near x_* but tend towards equation (22) for large $|x-x_*|$. For example,

$$\mathbf{R}(x, x_*) \simeq \frac{\sqrt{2}}{\pi}\{1-\nu f(\nu)\}\{\gamma^2 \operatorname{csch}^2(\gamma)\}, \tag{30}$$

where

$$\gamma = \frac{\pi^2}{6}\left(\frac{2}{3}\right)^{1/2}\frac{|x^3-x_*{}^3|}{aT}$$

has been found to provide a good representation of $\mathbf{R}$ for all values of x in a number of cases where aT is extremely large.

Turning to the case of a finite probability ϵ of photon destruction, we see that the reflection and transmission functions consist of the $\epsilon = 0$ forms discussed above, minus loss terms which we will denote $\mathbf{R}_l$ and $\mathbf{T}_l$, respectively. If we again assume that we can set $(1+pn^2)^2 \simeq 1$, then we must evaluate

$$\begin{Bmatrix}\mathbf{R}_l(\sigma, \sigma_*)\\ \mathbf{T}_l(\sigma, \sigma_*)\end{Bmatrix} \simeq \frac{4\pi}{3\sqrt{2}}\frac{1}{\phi^2 T^2}\sum_{n=1}^{\infty}\frac{(\pm 1)^{n-1}\, n \exp(-\beta n)}{1+sn}, \tag{31}$$

where

$$\beta = \frac{\pi}{T}(|\sigma|+|\sigma_*|) = \frac{\pi^2}{3}\left(\frac{2}{3}\right)^{1/2}\frac{|x^3|+|x_*{}^3|}{aT}.$$

Now

$$\sum_{n=1}^{\infty}\frac{(\pm 1)^{n-1}\, n \exp(-\beta n)}{1+sn} = \frac{1}{s}\left\{\frac{1}{\exp(\beta)\mp 1} - \sum_{n=1}^{\infty}\frac{(\pm 1)^{n-1}\{\exp(-\beta)\}^n}{1+sn}\right\}, \tag{32}$$

and the sum may be expressed as an integral so that equation (31) becomes

$$\begin{Bmatrix}\mathbf{R}_l\\ \mathbf{T}_l\end{Bmatrix} = \frac{2\pi^2}{\sqrt{3}}\frac{x^4}{aT}\left(\frac{\epsilon}{a}\right)\left\{\frac{1}{\exp(\beta)\mp 1} - \frac{\exp(\beta/s)}{s}\int_0^{\exp(-\beta)}\frac{\eta^{(1/s)}\, d\eta}{1\mp\eta}\right\}. \tag{33}$$

Repeated integration by parts then yields

$$\begin{Bmatrix}\mathbf{R}_l\\ \mathbf{T}_l\end{Bmatrix} = \frac{4\pi^3}{3\sqrt{2}}\frac{x^4}{(aT)^2}\left\{\frac{w}{1+s} \pm \frac{w^2}{(1+s)(1+2s)}\right.$$
$$\left. -\frac{1}{s^2}\left[\frac{2w^3}{(1+s^{-1})(2+s^{-1})(3+s^{-1})} + \dots + \frac{(n-1)!\, w^n}{(1+s^{-1})\dots(n+s^{-1})} + \dots\right]\right\}, \tag{34}$$

where $w = \{\exp(\beta)\mp 1\}^{-1}$. This series will converge if $w < 1$. In evaluating $\mathbf{T}_l$, $w = \{\exp(\beta)+1\}^{-1} < \frac{1}{2}$ for all β and the convergence will be rapid. But in evaluating $\mathbf{R}_l$, $w = \{\exp(\beta)-1\}^{-1} < 1$ only if $\beta > \ln 2$. Thus some other form will be necessary for small β (although if $s \ll 1$ the series will be a useful asymptotic expansion).

One representation of the digamma function $\psi(u+1)$ is (2)

$$\int_0^1 \frac{1-\eta^u}{1-\eta}\,d\eta = \psi(u+1)+\gamma, \tag{35}$$

where $\gamma = 0{\cdot}577215\ldots$ is Euler's constant. We can use this relation to express the integral in equation (33) as an integration over the interval $[\exp(-\beta), 1]$. We then expand the numerator in a binomial series to obtain

$$\mathbf{R}_l(x, x_*) = \frac{2\pi^2}{\sqrt{3}}\frac{x^4}{aT}\left(\frac{\epsilon}{a}\right)\left\{\frac{1}{\exp(\beta)-1}+\frac{\exp(\beta/s)}{s}\left[\ln s+\psi(1+1/s)+\gamma\right.\right.$$
$$\left.\left.+\ln v - v + \frac{(1-s)}{2{\cdot}2!}v^2 - \frac{(1-s)(1-2s)}{3{\cdot}3!}v^3+\ldots\right]\right\}, \tag{36}$$

where $v = \{1-\exp(-\beta)\}/s$. While this series will converge for any v, it will not be useful if v is large and s small, since there will be severe cancellation in such cases.

Another approach is to make use of the continued fraction expansion (3)

$$\sum_{n=1}^{\infty}\frac{\{\exp(-\beta)\}^n}{1+sn} = \frac{\exp(-\beta)}{\{1-\exp(-\beta)\}+}\,\frac{s}{1+}\,\frac{s\exp(-\beta)}{\{1-\exp(-\beta)\}+}$$
$$\frac{2s}{1+}\,\frac{2s\exp(-\beta)}{\{1-\exp(-\beta)\}+}\,\frac{3s}{1+}\cdots. \tag{37}$$

The nth convergent of a continued fraction can be expressed in the form

$$C_n = \rho_1+\rho_1\rho_2+\rho_1\rho_2\rho_3+\ldots+\rho_1\ldots\rho_n. \tag{38}$$

Using this form and incorporating ρ_1 and ρ_2 explicitly, we obtain

$$\mathbf{R}_l(x, x_*) = \frac{4\pi^3}{3\sqrt{2}}\frac{x^4}{(aT)^2}\frac{\exp(-\beta)}{\{1-\exp(-\beta)\}\{1-\exp(-\beta)+s\}}\{1+\rho_3+\rho_3\rho_4+\ldots\}, \tag{39}$$

where $\rho_2 = -s/\{1-\exp(-\beta)+s\}$, $\rho_n+1 = \{1+r_n(1+\rho_{n-1})\}^{-1}$ for $n>2$, $r_n = (n/2)\,s/\{1-\exp(-\beta)\}$ for n even, and $r_n = \{(n-1)/2\}\,s\exp(-\beta)/\{1-\exp(-\beta)\}$ for n odd. This form is especially useful if s is very small.

We note that the quantity β in equation (19), unlike α, does not vanish as $x\to x_*$, but only as both $x\to 0$ and $x_*\to 0$. The approximations found above will fail due to the omission of the $(1+pn^2)^2$ factor only when $p/\beta^2 = 4{\cdot}5x^4/(|x|^3+|x_*|^3)^2$ is not very small, and thus, only when both x and x_* are, say, $\gtrsim 10$. But near the line core the approximation to the redistribution function fundamental to this whole discussion is invalid. We may, however, desire an expression for the solution as defined by equation (19) in the region of the line centre for the sake of consistency in numerical applications. We can obtain such a result if we note that (ϵ/a) is extremely small in most cases of interest (e.g. $(\epsilon/a)\sim 10^{-6}$ for losses by conversion to the two-photon continuum in the H I Lα line) and hence, to a good approximation

$$\sum_{n=1}^{\infty}\frac{n\exp(-\beta n)}{(1+pn^2)^2(1+sn)} \simeq \frac{1}{s}\int_0^{\infty}\frac{\exp(-\beta n)}{(1+pn^2)^2}\,dn \tag{40}$$

for x and x_* small. This leads to the expression

$$\mathbf{R}_l(x, x_*) \simeq (\epsilon/a)\, x^2\{f(u)+ug(u)\}, \tag{41}$$

where $u = \beta/\sqrt{p} = (\sqrt{2}/3)(|x|^3+|x_*|^3)/x^2$ and the function $g(u)$ is similar to $f(u)$ introduced previously:

$$g(u) = \int_0^\infty \frac{y \exp(-uy)\, dy}{1+y^2} = -Ci(u)\cos(u)+\left\{\frac{\pi}{2}-Si(u)\right\}\sin(u). \tag{42}$$

This expression has the property that as $u \to 0$, $\mathbf{R}_l \sim (\pi/2)(\epsilon/a)\, x^2$.

We have now obtained a set of approximations which are suitable for the rapid numerical evaluation of the reflection and transmission functions for extremely large optical depths.

4. AN APPLICATION TO THE ESCAPE OF RADIATION FROM AN EXPANDING NEBULA

The analysis presented in this paper was motivated by the following problem: What is the intensity of H I Lα radiation trapped in an expanding shell of neutral hydrogen? This problem arises in connection with the earliest stages in the development of planetary nebulae (4). Ionized hydrogen in the interior provides the source of Lα photons, but the neutral shell, due to its much greater optical depth, dominates the transfer problem. It is reasonable to consider the shell to be thin and without appreciable internal velocity gradients. The problem is then equivalent to that of photons introduced at the central plane between two plane-parallel slabs.

The results of Paper I provide the solution for the static case. If the slabs are moving away from the central plane with velocity V, then a photon which leaves one slab experiences a Doppler shift of

$$\Delta x = \frac{2\mu V}{(2kT/M)^{1/2}} \tag{43}$$

Doppler widths in crossing the interior, relative to a reference frame moving with the second slab. Here, μ is the cosine of the angle to the perpendicular, T the gas temperature, and M the mass of the H atom. Thus radiation scattered between the inner faces of the slabs suffers a systematic redshift which transports photons out the wing of the line until they can finally escape. The flux which enters one side is the sum of photons created in the centre and photons reflected from the opposite side. Because of the symmetry involved we see that

$$F(x+\Delta x) = S(x+\Delta x/2)+\int_{-\infty}^{\infty} F(x_*)\,\mathbf{R}(x, x_*)\, dx_*, \tag{44}$$

where $S(x)$ is the source at the central plane (confined to the line centre) and $\mathbf{R}(x, x_*)$ is the reflection function developed in the previous section.

This equation provides the basis for a numerical solution of the problem. We introduce a set of discreet frequencies x_i ($i = 1, 2, \ldots, N$) such that $x_{i+1} = x_i+\Delta x$. The first frequency is chosen far enough to the blue that $F(x_1) \simeq 0$, while the last frequency x_N must be far enough to the red that essentially all the flux escapes or is destroyed before diffusing that far. The flux between frequency points is represented by some interpolation formula. If we chose linear interpolation for simplicity, equation (44) becomes

$$F_{i+1} = S_{i+1/2}+\sum_{k=1}^{N-1} F_k \int_{x_k}^{x_{k+1}} \left\{\frac{x_{k+1}-x_*}{x_{k+1}-x_k}\right\} \mathbf{R}(x_i, x_*)\, dx_*$$

$$+\sum_{k=1}^{N-1} F_{k+1} \int_{x_k}^{x_{k+1}} \left\{\frac{x_*-x_k}{x_{k+1}-x_k}\right\} \mathbf{R}(x_i, x_*)\, dx_*; \quad i = 1, 2, \ldots, N-1. \quad (45)$$

The integrals are just constants which can be evaluated numerically once the x_i are chosen. Thus equation (45) has the form

$$F_{i+1} = S_{i+1/2}+\sum_{k=1}^{N} F_k C_{ik}. \quad i = 1, 2, \ldots, N-1. \quad (46)$$

The nature of this problem is such that upon each reflection the fraction of radiation which escapes or is destroyed is very small, but this small fraction determines the solution. Thus the reflection matrix C_{ik} must be carefully normalized. Now if no photons are destroyed,

$$\int_{-\infty}^{\infty} \mathbf{R}_{\epsilon=0}(x, x_*)\, dx+\int_{-\infty}^{\infty} \mathbf{T}_{\epsilon=0}(x, x_*)\, dx+\exp\{-\phi(x_*)\, T/\mu\} = 1, \quad (47)$$

where the last term is the radiation which passes through the slab without absorption. Thus the proper normalization is

$$\sum_{i=1}^{N} C_{ik} = \int_{-\infty}^{\infty} \mathbf{R}(x, x_*)\, dx = 1-\exp\{-\phi(x_*)\, T/\mu\}$$

$$-\int_{-\infty}^{\infty} T_{\epsilon=0}(x, x_*)\, dx-\int_{-\infty}^{\infty} \mathbf{R}_l(x, x_*)\, dx. \quad (48)$$

While the approximations of Section 3 for $\mathbf{R}(x, x_*)$ are not accurate enough to ensure proper normalization, the right-hand side of (48) is quite adequate for this purpose. Once F_i has been obtained from the solution of equation (46), the profile of the emergent flux follows immediately from the transmission function.

At a temperature of 200° and a velocity of $V = 10\ \mathrm{km\ s^{-1}}$, the value of Δx is 5·4 Doppler widths. Since the x_i must extend many hundreds of Doppler widths from line centre, the system of equations can become quite large, though not prohibitively so. We have, in fact, obtained solutions for several cases with optical depths of $T = 10^{10}$ by this method. It is beyond the scope of this paper to discuss the physical basis for such models and the details of solutions for various expansion velocities; we hope to present this material elsewhere.

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