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THE SCHRÖDINGER OPERATOR CRITERION FOR THE STABILITY OF GALAXIES AND GAS SPHERES

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SUMMARY

A direct proof is given showing that a stellar system is stable whenever the corresponding barotropic gaseous system is secularly stable. The condition is formulated in terms of a Schrödinger equation.

It is shown that a large class of spherical steady-state stellar systems is stable to all non-spherical modes of vibration. For spherical modes the Schrödinger operator method is necessary and sufficient for the stability of barotropic spheres but for stellar systems, though correct, it is only sufficient and is not a very powerful method of proving stability. Antonov's method is harder to apply but more powerful.

The Schrödinger method is applied to some model clusters. More general and more powerful methods are needed in this field.

1. INTRODUCTION

Most spherical and elliptical galaxies show a regularity of form which leads the observer to believe that they are in steady states with the stars circulating steadily under the influence of the gravitational attraction of the whole assembly of them. There have been many discussions of such equilibria and theoretically there is such a great family of possible steady states that even the bounteous variety in the heavens is made to look as a single species.

One way in which theoreticians can be led to find too great a variety of possible steady-states is well illustrated by the pencil standing on its point. Pencils are not found like that although theory tells us such equilibria are possible. It is just conceivable that many of the theoretical models of stellar systems are likewise unstable—certainly interest should be concentrated only on the stable models (and probably only on a small subset of them). It is therefore important to discover criteria for the stability of model galaxies.

Following the pioneering work of Antonov (1), (2) one of us (D. L-B.) deduced a theorem relating the stability of stellar systems to that of corresponding gaseous systems (3). That paper and a later review (4) pay too little attention to boundary conditions which has led to misunderstandings but its results may be so interpreted as to be correct. The present paper aims firstly to show how the theorem may be proved directly* and secondly to use the resulting criterion to prove the stability of certain systems.

To allow the reader a less complicated introduction to this intricate subject only non-rotating spherical systems whose distribution functions depend on energy only are considered in the main body of the paper. Further we shall assume that the distribution function monotonically decreases with energy.

* M. Milder was the first to give a different but beautifully physical direct proof. See his thesis (5).

Generalizations to other systems are considered in appendices.

It is well known that for a general class of stellar systems there exist corresponding barotropic gaseous systems with the same equilibrium density distributions. Systems whose distribution functions depend on energy only clearly belong to such a class because if $\mathbf{v}$ is the velocity coordinate and ϵ the energy per unit mass, $(v^2/2)-\Psi$, then

$$p = \frac{1}{3}\int F(\epsilon)\, v^2 d^3v = p(\Psi)$$

while

$$\rho = \int F(\epsilon)\, d^3v = \rho(\Psi)$$

and hence

$$p = p(\rho).$$

In the above p is pressure, ρ is density, Ψ is gravitational potential and $F(\epsilon)$ is the distribution function of the steady state stellar system considered. By this we mean that

$$F\left(\frac{v^2}{2}-\Psi(\mathbf{r})\right)d^3v d^3r$$

is the total mass in a phase space box of volume d^3vd^3r about the point $\mathbf{r}$, $\mathbf{v}$ in the phase space of positions $\mathbf{r}$ and velocities $\mathbf{v}$. Jeans's equations of stellar hydrodynamics (**6**) reduce in these cases of spherical systems with isotropic pressure to the equation of hydrostatics for a barotropic gas

$$\nabla p = \rho \nabla \Psi \tag{1}$$

i.e.

$$\frac{dp}{d\Psi} = \rho. \tag{2}$$

It is these barotropic gaseous systems that we use for comparison with our stellar systems.

2. ENERGY AND ENERGY PRINCIPLES

2.1 *The total energy of a stellar system is given by*

$$W = \int f_T \frac{v^2}{2} d^6\tau - \frac{G}{2}\int\int \frac{\rho_T \rho'_T}{|\mathbf{r}-\mathbf{r}'|}\, d^3r d^3r'$$

where f_T is the total distribution function and $\rho'_T \equiv \rho_T(\mathbf{r}', t)$, $d^6\tau \equiv d^3vd^3r$ and

$$\rho_T = \int f_T d^3v.$$

We may rewrite W in the form

$$W = \int f_T \frac{v^2}{2}\, d^6\tau - \frac{G}{2}\int\int \frac{f_T f'_T}{|\mathbf{r}-\mathbf{r}'|}\, d^6\tau d^6\tau'$$

where $f'_T = f_T(\mathbf{r}', \mathbf{v}', t)$ and $d^6\tau' = d^3r' d^3v'$.

The difference in energy ΔW between the system with distribution function f_T and that with the steady-state distribution

$$F\left(\frac{v^2}{2}-\Psi\right)$$

is

$$\Delta W = \int (f_T - F)\frac{v^2}{2}\, d^6\tau - \frac{G}{2}\int\int \frac{(f_T - F)(f'_T - F')}{|\mathbf{r}-\mathbf{r}'|}\, d^6\tau d^6\tau'$$
$$-\int (f_T - F) G \int \frac{F'}{|\mathbf{r}-\mathbf{r}'|}\, d^6\tau' d^6\tau.$$

Now

$$G\int \frac{F'}{|\mathbf{r}-\mathbf{r}'|}\, d^6\tau' = G\int \frac{\rho'}{|\mathbf{r}-\mathbf{r}'|}\, d^3r' = \Psi(r)$$

the gravitational potential of the steady state. Thus writing $f = f_T - F$ for the perturbation in the distribution function we have

$$\Delta W = \int f\left(\frac{v^2}{2} - \Psi\right) d^6\tau - \frac{G}{2}\int\int \frac{ff'}{|\mathbf{r}-\mathbf{r}'|}\, d^6\tau d^6\tau'. \tag{3}$$

We shall write $\int f d^3v = \delta\rho$ and we shall deal with perturbations which do not change the mass so

$$\int f d^6\tau = \int \delta\rho d^3r = 0.$$

We now use a trick due originally to Newcombe to express the term in ΔW which is apparently linear in f as a quadratic.

The Boltzmann–Liouville equation that governs the evolution of the distribution function may be written

$$\frac{D_T f_T}{Dt} = 0. \tag{4}$$

Where

$$\frac{D_T}{Dt} \equiv \frac{\partial}{\partial t} + D \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{\partial \Psi_T}{\partial \mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{v}}$$

equation (4) tells us that f_T is convected about in phase space but its value is not changed. Consider any function $J(f_T)$ then $D_T J/Dt = 0$ so J is merely convected around and thus $\int J(f_T) d^6\tau$ is constant. Now

$$\int J(f_T) d^6\tau = \int (J(F) + fJ'(F) + \tfrac{1}{2}f^2 J''(F)) d^6\tau + O(f^3).$$

Hence since $\int J(F) d^6\tau$ is also constant in time

$$\int (fJ'(F) + \tfrac{1}{2}f^2 J''(F)) d^6\tau = \eta \text{ a constant} \tag{5}$$

η must be small since f is small. Since an unstable system is still unstable when it is started arbitrarily close to equilibrium we could choose $\eta = 0$ without spoiling our results but we shall retain it for the sake of those who (falsely) doubt that argument. We now choose $J'(F) = \epsilon$.
i.e.

$$J = \int \epsilon\, dF = \int^{\varepsilon} \epsilon \frac{dF}{d\epsilon}\, d\epsilon,$$

then

$$J''(F) = \frac{d\epsilon}{dF} = \frac{1}{\dfrac{dF}{d\epsilon}}$$

then provided that $dF/d\epsilon$ is non-zero within the cluster

$$\int f\epsilon d^6\tau = \tfrac{1}{2}\int \frac{f^2}{-\dfrac{dF}{d\epsilon}}\, d^6\tau+\eta+O(f^3) \tag{6}$$

which is the required relationship giving a quadratic form in f. The energy change ΔW is now, correct to second order in f

$$\Delta W = \frac{1}{2}\left[\int \frac{f^2}{-\dfrac{dF}{d\epsilon}}\, d^6\tau - G\iint \frac{ff'}{|\mathbf{r}-\mathbf{r}'|}\, d^6\tau d^6\tau'\right]+\eta$$

$$= \int \frac{f^2}{-2\dfrac{dF}{d\epsilon}}\, d^6\tau - \frac{G}{2}\iint \frac{\delta\rho\delta\rho'}{|\mathbf{r}-\mathbf{r}'|}\, d^3r d^3r' + \eta. \tag{7}$$

We now derive the corresponding expression for the change of energy of a barotropic gas.

2.2 *Energy of a gaseous configuration*

The internal energy of a barotropic gas is the work done in compressing it from infinite dilution. This work is

$$\int_{\frac{1}{\rho}}^{\infty} p(\rho')\, d\left(\frac{1}{\rho'}\right) = \int_0^{\rho} \frac{p(\rho')}{\rho'^2}\, d\rho' \text{ per gram.}$$

The internal energy of the configuration is therefore

$$W_{\text{int}} = \int \rho \int_0^{\rho} \frac{p(\rho')}{\rho'^2}\, d\rho' d^3r.$$

The self-gravitational energy is

$$W_{\text{grav}} = -\frac{G}{2}\iint \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r d^3r'.$$

The total energy $W_g = W_{\text{int}} + W_{\text{grav}}$.

To test the stability of a configuration we work out the difference in energy between neighbouring configurations to second order in the density difference $\delta\rho(\mathbf{r})$.

$$\Delta W_g = \Delta_1 W_g + \Delta_2 W_g$$

$$\Delta_1 W_g = \int \delta\rho \left(\int_0^{\rho} \frac{p(\rho')}{\rho'^2}\, d\rho' + \frac{p(\rho)}{\rho} - G\int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'\right) d^3r \tag{8}$$

$$\Delta_2 W_g = +\int \frac{(\delta\rho)^2}{2}\left(\frac{1}{\rho}\frac{dp}{d\rho}\right) d^3r - \frac{G}{2}\iint \frac{\delta\rho(\mathbf{r})\delta\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r d^3r'.$$

In the above ρ is the density of the unperturbed configuration $\delta\rho$ is slightly restricted since $\int \delta\rho \, d^3r = 0$. This restriction is automatically implied if we write $\delta\rho = -\operatorname{div}(\rho\boldsymbol{\xi})$ where $\boldsymbol{\xi}$ is the small displacement vector. ΔW_1 may then be rewritten

$$\Delta_1 W_g = \int \rho\boldsymbol{\xi}.\left(\frac{\nabla p}{\rho} - \nabla\Psi\right) d^3r$$

where we have used the fact that ρ vanishes at the boundary of the fluid. Evidently $\Delta_1 W_g = 0$ by reason of the equilibrium of the unperturbed fluid (equation (1)). Hence

$$\Delta W_g = \Delta_2 W_g.$$

But by reason of equation (2)

$$\frac{1}{\rho}\frac{dp}{d\rho} = \frac{d\Psi}{d\rho}$$

so we may write $\Delta_2 W_g$ in the form

$$\Delta W_g = \int \frac{(\delta\rho)^2}{2\dfrac{d\rho}{d\Psi}} d^3r - \frac{G}{2}\iint \frac{\delta\rho\delta\rho'}{|\mathbf{r}-\mathbf{r}'|} d^3r d^3r'. \tag{9}$$

2.3 *Comparison of expressions for the energy*

The second term in ΔW_g occurs in ΔW for the stellar system. The first term differs primarily in that $\delta\rho$ appears instead of f. We now work on ΔW to make it look more like ΔW_g. By Schwartz's inequality

$$\int \frac{f^2}{-\dfrac{dF}{d\epsilon}} d^3v \int -\frac{dF}{d\epsilon} d^3v \geqslant \left(\int f d^3v\right)^2 = (\delta\rho)^2.$$

The equality only holds when $f \propto -dF/d\epsilon$.

Since we are taking $dF/d\epsilon < 0$ within the cluster

$$\int \frac{f^2}{-\dfrac{dF}{d\epsilon}} d^6\tau \geqslant \int \frac{(\delta\rho)^2}{\int -\dfrac{dF}{d\epsilon} d^3v} d^3r.$$

Now

$$\int F(\epsilon)\, d^3v = \rho \text{ and } \epsilon = \frac{v^2}{2} - \Psi.$$

so

$$\int -\frac{dF}{d\epsilon} d^3v = \frac{d\rho}{d\Psi}.$$

So finally

$$\Delta W \geqslant \int \frac{(\delta\rho)^2}{2\dfrac{d\rho}{d\Psi}} d^3r - \frac{G}{2}\iint \frac{\delta\rho\delta\rho'}{|\mathbf{r}-\mathbf{r}'|} d^3r d^3r' + \eta = \Delta W_g + \eta. \tag{10}$$

Let us now *suppose* that we know that the gaseous body is secularly stable so that ΔW_g is positive for all acceptable $\delta\rho$ that do not correspond to mere displacements.

Then from equation (10)

$$\Delta W - \eta \geqslant \Delta W_g \geqslant 0.$$

For both equalities to hold both f must be proportional to $-dF/d\epsilon$ in velocity space and the $\delta\rho$ must be that of a uniform displacement. These lead to a unique $f = -\psi\, dF/d\epsilon$ where ψ is the perturbation potential corresponding to the uniform displacement. This f is the f of the mere displacement so we can deduce that at least one inequality holds except for uniform displacements of the stellar system. Had we taken $\eta = 0$ we would now have proved that $\Delta W > 0$ for all non-trivial perturbations so we could deduce that the system was stable. Since those who retain η will be of a more pedantic frame of mind we will spell out the argument for them in greater detail.

By the above inequality $\Delta W - \eta$ is positive definite for all perturbations that are not mere uniform displacements. However ΔW and η are both small initially (if f is) and are both constant.

If f ceases to be small it must do so in such a manner that $\Delta W - \eta$ still remains small. This can only be done if f is of the form corresponding to a uniform displacement of the stellar system; or rather it can only differ from such a displacement by a quantity that remains small.

Mere uniform displacements can be reduced to zero in suitable axes so f can be taken to remain small always.

Hence the system is stable.

The stellar system is therefore stable if the corresponding barotropic gaseous sphere is stable.

This theorem allows us to use our considerable knowledge of the stability of gaseous bodies to acquire a knowledge of some stable stellar-dynamical systems. However the barotropic law $p(\rho)$ is defined by the density distribution of the stellar system and not by its compressibility properties. We can never show that a stellar system is unstable by these methods because our condition is sufficient but not necessary for the stability of a stellar dynamical system.

3. THE SCHRÖDINGER OPERATOR CRITERION

ΔW_g may be re-expressed in terms of the perturbation in the potential

$$\psi = G \int \frac{\delta\rho'}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'.$$

$$\nabla^2\psi = -4\pi G\delta\rho$$

Then

$$\Delta W_g = \int \frac{\delta\rho}{2\dfrac{d\rho}{d\Psi}} \left(\frac{\nabla^2\psi}{-4\pi G} - \frac{d\rho}{d\Psi}\,\psi\right) d^3r.$$

We now define the Schrödinger operator S by

$$S\psi = \left(-\frac{\nabla^2}{4\pi G} - U\right)\psi$$

where the analogy to the quantum mechanical Hamiltonian H is obvious where U is the negative of the potential

$$H\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 - U\right)\psi.$$

In the Schrödinger operator we put

$$U = \frac{d\rho}{d\Psi} > 0.$$

We may now write ΔW_g in the following ways.

$$\Delta W_g = \int \frac{\delta\rho}{2U} S\psi d^3r = \int \frac{-\nabla^2\psi}{8\pi G U} S\psi d^3r = \frac{1}{2}\int \left(\frac{S\psi}{U}+\psi\right) S\psi d^3r$$
$$= \frac{1}{2}\int \left(\frac{(S\psi)^2}{U}+\psi S\psi\right) d^3r. \tag{11}$$

Now ψ is subject to the boundary conditions for a perturbation potential. That is $\nabla^2\psi = 0$ outside the cluster and ψ is $O(1/r^2)$ at ∞ since perturbations do not involve change of mass. Also

$$\frac{S\psi}{U} = \frac{\delta\rho}{\rho}\frac{dp}{d\rho} - \psi$$

must be finite where U vanishes on the edge of the cluster and ψ and $\nabla\psi$ must be continuous over that surface. If $\Delta W_g > 0$ for all such non-zero ψ which are not merely uniform displacements then the system is stable. Mere uniform displacement will give $\Delta W_g = 0$ but this will not affect the stability.

We see from expression (11) that if $\int \psi S\psi d^3r > 0$ for all ψ that satisfy the boundary conditions and that do not correspond to mere displacements, then $\Delta W_g > 0$ and the system is stable. This leads us to consider the minimizing of

$$\frac{\int \psi S\psi d^3r}{\int \psi U\psi d^3r}$$

subject to the boundary conditions that $\psi = O(1/r^2)$ at ∞ etc. Stationary values of this ratio occur when

$$S\psi = \lambda\, U\psi \tag{12}$$

which reduces to $\nabla^2\psi = 0$ outside the body and ensures $S\psi/U$ finite so the boundary conditions are fulfilled if ψ is $O(1/r^2)$.

For eigenfunctions

$$\Delta W_g = \frac{\lambda(\lambda+1)}{2}\int \psi U\psi d^3r$$

which is positive for $\lambda > 0$ and negative for $\lambda < 0$.*

Now S is a Hermitian operator over all space for functions that die like $1/r$ or more at ∞ and U is positive definite over all ψ satisfying our boundary conditions. We therefore deduce that the eigenfunctions ψ_λ of equation (12) form a complete set.

It therefore would seem as if the condition $\Delta W_g > 0$ were equivalent to the

* $S+U = -\dfrac{\nabla^2}{4\pi G}$ which is positive definite over all space. Hence $\lambda+1>0$ even when $\lambda<0$.

statement that S must be positive definite, i.e. that every eigenvalue must be positive. (We must of course allow $\lambda = 0$ but only for uniform displacements.) However difficulties with this chain of argument occur as soon as we pay careful attention to the boundary conditions. Equation (12) gives a complete set of eigenfunctions for our problem under the boundary condition that ψ is $O(1/r)$ at ∞, not $O(1/r^2)$. Thus not all the eigenfunctions are acceptable as possible ψ because some of them correspond to a change in the mass of the system. For spherically symmetrical modes our $O(1/r^2)$ boundary condition leads to $\psi \equiv 0$ outside the cluster. Solving equation (12) under the boundary condition that $\psi = 0$ and $d\psi/dr = 0$ at the edge gives the unique solution $\psi \equiv 0$. Thus no spherical eigenfunction can obey our boundary conditions. However there is no difficulty in finding linear combinations of pairs of spherical eigenfunctions with different eigenvalues that do satisfy them. We would be utterly wrong to delete all spherical modes as possible causes of instability. We note that this difficulty is occurring only with the spherical modes because the solutions outside the body behave like $A/r+B$ both of which terms are forbidden by the boundary condition. The dying solutions of all non-spherical modes are at least $O(1/r^2)$ so non-spherical modes cause us no difficulty.

Let us first prove that any disturbance of a spherical star cluster can be split as usual into its spherical and non-spherical parts, each of which are possible disturbances.

If $f(\mathbf{r}, \mathbf{v}, t)$ is a solution to our problem so is the rotated solution $f(\mathscr{R}(\mathbf{r}), \mathscr{R}(\mathbf{v}), t)$ when $\mathscr{R}$ is any rotation operator. Further as the equations are linear any linear combination of solutions is a solution. Now define

$$\bar{f}(\mathbf{r}, \mathbf{v}, t) = \text{Average over all rotations } \mathscr{R} \text{ of } f(\mathscr{R}(\mathbf{r}), \mathscr{R}(\mathbf{v}), t).$$

Clearly $\bar{f}$ is spherically symmetrical and $f-\bar{f}$ describes a possible disturbance of the star cluster which has no spherically symmetrical part. In particular the potential corresponding to $f-\bar{f}$ will be expressible as a combination of the spherical harmonics $Y_l^m(\theta, \phi)$ starting not with $l = 0$ but with $l = 1$. The only modes left out in such an analysis are those for which $\bar{f} = f$, $\bar{\psi} = \psi$, i.e. those which are spherically symmetrical. These we shall consider presently. Now we shall prove our theorem on the aspherical modes by which phrase we shall mean those with no spherically symmetric part.

4. ASPHERICAL MODES

Theorem

Any spherical star cluster whose distribution function is a decreasing function of energy alone is stable to all aspherical perturbations.

All the aspherical eigenfunctions of the equation

$$S\psi = \lambda U\psi$$

satisfy the required boundary condition $\psi = O(1/r^2)$. Furthermore S is Hermitian and U is positive definite. Thus the aspherical eigenfunctions will form a complete set and each individual eigenfunction is an acceptable perturbation potential. We now show that all aspherical eigenfunctions that do not correspond to mere displacements have positive eigenvalues.

Consider the mere displacement caused by an infinitesimal uniform displacement $\mathbf{p}$. The perturbation potential is then $\psi = \mathbf{p}.\nabla\Psi$. Taking S of this ψ

$$S\psi = -\frac{\nabla^2}{4\pi G}\left(\mathbf{p}.\frac{\partial}{\partial\mathbf{r}}\Psi\right) - \frac{d\rho}{d\Psi}\mathbf{p}.\frac{\partial}{\partial\mathbf{r}}\Psi$$

i.e.

$$S\psi = \mathbf{p}.\frac{\partial}{\partial\mathbf{r}}\rho - \frac{d\rho}{d\Psi}\mathbf{p}.\frac{\partial}{\partial\mathbf{r}}\Psi = 0$$

since ρ is a function of Ψ alone.

Thus the uniform displacements give eigenvalues of zero as we should expect. Since Ψ decreases monotonically these eigenfunctions have no spherical node. Furthermore the three independent infinitesimal uniform displacements together make up the complete set of $2l+1$ eigenfunctions that have $l = 1$ in their spherical harmonic representations and give the same radial dependence (with no spherical node). Thus all other independent eigenfunctions either have $l > 1$ or at least one spherical node.

But by a well known theorem (Morse & Feshbach, p. 719) the eigenvalues for given l increase as the number of spherical nodes increases. Therefore all $l = 1$ eigenvalues that do not correspond to mere uniform displacements are positive.

Now by another similar general theorem on eigenvalues (Morse & Feshbach, p. 754) the $l = 2$ eigenfunctions with a given number of spherical nodes lie above the $l = 1$ eigenfunctions with the same number of spherical nodes. Thus all $l = 2$ eigenvalues are positive. Similarly using the general theorem that the $l = L+1$ eigenvalues lie above the corresponding $l = L$ eigenvalues we may now apply mathematical induction to prove that all aspherical eigenvalues are positive unless they are zero and correspond to mere uniform displacements. Any aspherical function ψ satisfying our boundary conditions may be expanded in the form

$$\psi = \sum_i a_i\psi_i$$

where the ψ_i are a complete set of eigenfunctions orthogonalized and normalized by the condition

$$\int \psi_i U \psi_j \, d^3r = \delta_{ij}.$$

Then

$$\begin{aligned}\Delta W_g &= \int \left\{\frac{[S(\Sigma a_i\psi_i)]^2}{U} + \Sigma a_j\psi_j U \Sigma a_i\psi_i\right\} d^3r \\ &= \Sigma\Sigma\, \lambda_i(\lambda_j+1)\, a_i a_j \int \psi_i U \psi_j d^3r \\ &= \Sigma\Sigma\, \lambda_i(\lambda_j+1) a_i a_j \delta_{ij} = \Sigma\lambda_i(\lambda_i+1) a_i^2.\end{aligned}$$

Since $\lambda_i(\lambda_i+1) \geqslant 0$ for all i, $\Delta W_g \geqslant 0$. Equality only occurs if every non-zero a_i has $\lambda_i = 0$ in which case as we have seen the ψ is that corresponding to a linear combination of uniform displacements; i.e. it corresponds to a uniform displacement. This completes the proof of the theorem that all aspherical modes are stable.

We remark that the same theorem that proves that all other aspherical eigenfunctions have eigenvalues greater than the $l = 1$ displacement eigenvalue of zero, shows us that the unacceptable $l = 0$, no spherical node eigenvalue must always be negative.

5. SPHERICAL MODES

The boundary conditions on the edge of the stellar system now reduce to $\psi = 0$ and $d\psi/dr = 0$ and $S\psi/U$ must not be infinite. We now minimize not $\int \psi S\psi d^3r$ but rather the full energy ΔW_g. Since we are interested in its passage through zero we may normalize by any positive definite quadratic integral and it proves convenient to choose

$$\int \psi \frac{-\nabla^2}{4\pi G} \psi d^3r = \int \frac{(\nabla\psi)^2}{4\pi G} d^3r$$

which is positive definite under these boundary conditions.

Note

$$S+U = \frac{-\nabla^2}{4\pi G}.$$

Our problem is therefore to minimize the ratio R

$$R = \frac{\int \left[\frac{(S\psi)^2}{U} + \psi S\psi\right] d^3r}{\int \psi(S+U)\, \psi d^3r} = \frac{\Delta W_g}{\int \frac{(\nabla\psi)^2}{4\pi G} d^3r}.$$

If $R_{\min} > 0$ for all spherical perturbations the system is stable. The corresponding eigenvalue equation is

$$S\frac{S\psi_\lambda}{U} + S\psi_\lambda - \lambda(S+U)\psi_\lambda = 0. \tag{13}$$

that is

$$(S+U)\left[\frac{1}{U}(S-\lambda U)\, \psi_\lambda\right] = 0.$$

Since in spherical symmetry

$$(S+U)Q = \frac{-1}{4\pi G}\frac{1}{r}\frac{d^2}{dr^2} rQ$$

we may solve this equation to obtain

$$\frac{1}{U}(S-\lambda U)\, \psi_\lambda = c'_\lambda + \frac{D_\lambda}{r}. \tag{14}$$

But $S\psi_\lambda/U$ and $\lambda\psi_\lambda$ must be finite at $r = 0$ so $D_\lambda = 0$. Hence $(S-\lambda U)\psi_\lambda = c'_\lambda U$. By a normalization of ψ_λ we may take $c'_\lambda = (\lambda+1)c_\lambda$. To know that this can be done we must know that none of the $\lambda+1$'s are equal to zero. To prove that $\lambda+1 \neq 0$ we merely rewrite equation (13) in the form

$$(S+U)\left[\frac{1}{U}(S+U)\psi_\lambda\right] = (\lambda+1)(S+U)\psi_\lambda$$

and notice that the operator on each side is positive definite. Multiplying by ψ_λ and integrating with the use of the boundary conditions:

$$\lambda+1 = \frac{\int \frac{[(S+U)\, \psi_\lambda]^2}{U} d^3r}{\int \psi_\lambda(S+U)\psi_\lambda d^3r} > 0.$$

Hence equation (14) has been reduced to

$$\left(\frac{-\nabla^2}{4\pi G}-(\lambda+1)U\right)\psi_\lambda = (\lambda+1)Uc_\lambda.$$

That is

$$\left(\frac{-\nabla^2}{4\pi G}-U-\lambda U\right)(\psi_\lambda+c_\lambda) = 0.$$

i.e.

$$S\chi_\lambda = \lambda\chi_\lambda \tag{15}$$

where

$$\chi_\lambda = \psi_\lambda+c_\lambda.$$

We have again reduced our problem to the properties of the Schrödinger operator but now for the spherical modes the boundary conditions are $\chi_\lambda = c_\lambda$, $d\chi_\lambda/dr = 0$ on the edge of the system.

We now show that the necessary and sufficient condition that $R \propto \Delta W_g$ is positive definite is that no spherical eigenfunction have a negative or zero eigenvalue λ. Since $S+U$ is positive definite under our boundary conditions and $(S+U)U^{-1}S$ is Hermitian, the solutions of equation (13) for ψ_λ will form a complete set. The ψ_λ corresponding to different λ are $S+U$—orthogonal for multiply equation (13) by $\psi_\lambda{}'$ and integrate over the volume of the cluster. After subtraction from the same equation with λ' and λ exchanged one obtains

$$-\lambda\int\psi_\lambda{}'(S+U)\psi_\lambda d^3r = \lambda'\int\psi_\lambda(S+U)\psi_\lambda{}'d^3r = 0.$$

But $S+U$ is Hermitian over the volume of the cluster for ψ_λ obeying our boundary conditions so

$$(\lambda-\lambda')\int\psi_\lambda{}'(S+U)\psi_\lambda d^3r = 0.$$

Hence if $\lambda\neq\lambda'$, $\psi_\lambda{}'$ and ψ_λ are orthogonal. Actually as it will appear in the next section there is no degeneracy so we have a complete set of spherical $S+U$ orthogonal functions ψ_λ.

Any ψ satisfying the boundary conditions may be expanded in the form

$$\psi = \Sigma a_\lambda\psi_\lambda.$$

So

$$R = \frac{\sum_\lambda\sum_\mu \mu a_\lambda a_\mu\int\psi_\lambda(S+U)\psi_\mu d^3r}{\sum_\lambda\sum_\mu a_\lambda a_\mu\int\psi_\lambda(S+U)\psi_\mu d^3r}$$

$$= \frac{\sum_\lambda \lambda a_\lambda{}^2\int\psi_\lambda(S+U)\psi_\lambda d^3r}{\sum_\lambda a_\lambda{}^2\int\psi_\lambda(S+U)\psi_\lambda d^3r}.$$

Notice that since $S+U$ is positive definite R is always positive if all the eigenvalues λ are positive. Further if one or more eigenvalue is negative (or zero) then the corresponding eigenfunction yields

$$R = \lambda<0 \text{ (or zero)}.$$

Thus the necessary and sufficient condition for ΔW_g to be positive definite is that the eigenvalues λ should be positive. Using our spherical symmetry the eigenvalue equation (15) takes the form

$$-\frac{1}{4\pi G}\frac{1}{r}\frac{d^2}{dr^2}(r\chi_\lambda)-(\lambda+1)U\chi_\lambda = 0$$

i.e.

$$\frac{d^2\phi_\lambda}{dr^2}+(\lambda+1)4\pi GU\phi_\lambda = 0, \tag{16}$$

where $\phi_\lambda = r\chi_\lambda$ is subject to the conditions $\phi_\lambda(0) = 0$ and $d\phi_\lambda/dr = c_\lambda$ at $r = r_b$ the boundary of the system and $1/r\ \phi_\lambda = c_\lambda$ at $r = r_b$.

Note that our potential perturbation $\psi_\lambda = (\phi_\lambda/r)-c_\lambda$. We shall compare the solutions of equation (16) for ϕ_λ with ϕ the solution of the equation obtained by putting $\lambda = 0$

$$\frac{d^2\phi}{dr^2}+4\pi GU\phi = 0. \tag{17}$$

This follows Jacobi's standard discussion of the variational problem. We imagine ourselves as integrating these equations inwards from r_b and we apply the boundary conditions $d\phi/dr = 1$, $\phi/r = 1$ at $r = r_b$ (taking $c_\lambda = 1$ normalization).

We shall show that the necessary and sufficient condition for the existence of negative or zero eigenvalues λ in our problem is that ϕ should have at least two zeros in the range $0 \leqslant r < r_b$. We postpone the formal proof to get the idea first. Suppose $\lambda < 0$. At r_b, ϕ_λ and ϕ are tangent to the line through the origin that passes through 1 at r_b. Equations (16) and (17) show us that near r_b both ϕ_λ and ϕ have negative acceleration, $(d^2\phi/dr^2)$, with the acceleration of ϕ having the larger magnitude (as $\lambda < 0$). Thus just inside r_b, ϕ_λ will be greater than ϕ. Both accelerations remain negative until their respective ϕ's become negative. Thus as r decreases both ϕ graphs curve downwards away from the straight line (Fig. 1). We show presently that ϕ becomes negative first. If ϕ has only one zero then it remains negative down to and including $r = 0$. Under these circumstances we shall show that ϕ_λ also has just one zero down to and including $r = 0$ (always assuming $\lambda < 0$). However since ϕ_λ has a negative acceleration down to its first zero that zero is not at $r = 0$. We have therefore a contradiction with the condition that $\phi_\lambda = r(\psi_\lambda+1)$ must be zero at $r = 0$. Hence our supposition that $\lambda < 0$ can not hold when ϕ has only one zero in the interval $[0, r_b]$. We now show that when ϕ has two or more zeros in the range $[0, r_b]$ then there is a negative or zero eigenvalue λ. (Of course if there is an eigenvalue with $\lambda = 0$ then $\phi(0) = 0$ and ϕ is the eigenfunction itself.) Integrate equation (16) with λ put equal to $-1+\delta$ and δ assumed small. We obtain using our boundary conditions at $r = r_b$,

$$\phi_{-1+\delta} = r-\delta\int_r^{r_b}\int_r^{r_b} 4\pi GUr\,dr+O(\delta^2).$$

This has only one zero for δ sufficiently small. Now ϕ has at least two zeros, so if we integrate the ϕ_λ equation with $\lambda = -1+\delta$ there is one and if we integrate it with $\lambda = 0$ there are at least two. Hence there is a least λ, λ_1 which has two zeros and $-1 < \lambda_1 \leqslant 0$. If the zero of ϕ_{λ_1} closest to $r = 0$ is not at $r = 0$ it is simple to show that $\phi_{\lambda_1-\delta}$ would have two zeros which is impossible since λ_1 is the least value of λ with two zeros. Hence $\phi_{\lambda_1}(0) = 0$. Thus λ_1 is a negative (or zero) eigenvalue of our problem and $\psi_{\lambda 1} = (\phi_{\lambda 1} - r)/r$ is the spherical eigenfunction of equation (13). Two zeros of ϕ are thus sufficient to show that there is a negative (or zero) eigenvalue λ.

Note that zero eigenvalues only occur when ϕ itself actually passes through the origin.

We still need a formal proof of the necessity.

5.1 *Proof*

Suppose $\lambda < 0$. Multiply equation (16) by ϕ and equation (17) by ϕ_λ, subtract and integrate from r to r'.

$$\left[\phi \frac{d\phi_\lambda}{dr} - \phi_\lambda \frac{d\phi}{dr}\right]_r^{r'} = -\lambda \int_r^{r'} \phi \, 4\pi G U \phi_\lambda \, dr. \tag{18}$$

Let r be the first zero of $\phi_\lambda - \phi$ and $r' = r_b$ the boundary. Equation (18) reduces to

$$-\phi \frac{d}{dr}(\phi_\lambda - \phi)\bigg|_r = -\lambda \int_r^{r'} \phi \, 4\pi G U \phi_\lambda \, dr.$$

Since $\phi_\lambda - \phi$ is positive near r_b hence $d/dr\,(\phi_\lambda - \phi)$ must be positive at its first zero as we proceed inwards from r_b. If ϕ had not already achieved its first zero it would be negative in which case the l.h.s. of equation (18) would be negative and the r.h.s. positive, which is impossible. Thus ϕ achieves its first zero before $\phi_\lambda - \phi$, i.e. ϕ achieves its first zero before ϕ_λ. ϕ is negative in the range from the first zero of ϕ_λ down to and including $r = 0$ because ϕ does not have a second zero by hypothesis. If ϕ_λ is to be an eigenfunction it must have at least a second zero because it must be zero at $r = 0$. Now apply equation (18) with r the second zero of ϕ_λ and r' the first zero. $d\phi_\lambda/dr > 0$ at the first zero r' and < 0 at the second zero r. $\phi < 0$ at both. Equation (17) reduces to

$$\left[\phi \frac{d\phi_\lambda}{dr}\right]_r^{r'} = -\lambda \int_r^{r'} \phi \, 4\pi G U \phi_\lambda \, dr,$$

and the l.h.s. is negative.

But with λ assumed < 0 the r.h.s. is positive for ϕ_λ and ϕ are also negative in the range between ϕ_λ's first and second zeros. Hence contradictions.

Thus if ϕ has only one zero there is no negative eigenvalue. If ϕ has only one zero there is no zero eigenvalue either since ϕ is the solution of the equation with $\lambda = 0$ and it can not pass through the origin.

This completes the proof that the necessary and sufficient condition for ΔW_g to be positive definite is that ϕ should not have more than one zero in the range $0 \leqslant r < r_b$.

The proofs of the general theorems on the increase of eigenvalues with l and with a number of radial nodes quoted earlier are very similar to the above argument.

6. APPLICATIONS OF THE METHOD

Since we proved a stellar system to be stable if the corresponding gaseous system was secularly stable we can already deduce that certain stellar systems are stable from our knowledge of the secular stability of polytropes. It is known that for non-rotating barotropes secular and ordinary stability are the same (7) and that when $\gamma = 1+(1/n)$ the polytropes are stable for $n<3$. The corresponding stellar systems (**8**) have distribution functions of the form

$$F \propto (\epsilon_0 - \epsilon)^{n-3/2} \qquad \epsilon < \epsilon_0 < 0$$
$$0 \qquad \epsilon > \epsilon_0$$

which decrease with energy for $n > 3/2$. Thus for $3 > n > 3/2$ the polytropic stellar systems are stable. Antonov was the first to derive this result, but his second more powerful method enabled him to prove stability for the range $n \geqslant 3$ also (**2**). This is some measure of the crudity of the Schrödinger operator criterion. We note however that for barotropic gaseous spheres our method is necessary and sufficient for stability (**7**) and the restriction of decreasing distribution function does not arise. To show how the method is applied in practice we begin with the illustrative example of Plummer's model.

(a) *Plummer's model*

Potential $$\Psi = \frac{GM}{(r^2+a^2)^{1/2}}$$

Density $$\rho = \frac{3a^2M}{4\pi}\left(\frac{\Psi}{GM}\right)^5$$

Distribution function $$f = \frac{48}{7\sqrt{2}}\frac{a^2}{\pi^3}\frac{M}{(GM)^5}(-\epsilon)^{7/2}.$$

This is the polytropic model with $n = 5$, we should not therefore expect stability. Our method tells us to form

$$\frac{d\rho}{d\Psi} = \frac{3a^2}{4\pi G}\left(\frac{\Psi}{GM}\right)^4$$

and then to integrate the equation

$$\frac{d^2\phi}{dr^2} + 4\pi G\frac{d\rho}{d\Psi}\phi = 0 \quad \text{i.e.} \quad \frac{d^2\phi}{dr^2} + \frac{3a^2}{(r^2+a^2)^2}\phi = 0$$

starting with $\phi = r$, $\phi' = 1$ at large r and integrating inwards. The result of such an integration is shown in Fig. 1 curve (5). Evidently ϕ has two zeros so the gaseous system is secularly (and ordinarily) unstable. We can therefore prove nothing about the stability of the stellar system from our criterion.

(b) *The $n = 3$ polytrope*. We need not perform the integration in this case for it is simple to show that the system is neutrally stable to a uniform contraction. We can use this fact to determine the form that the function ϕ would take had we performed the integration. In a uniform contraction $\rho_r((1-\delta)r) = \rho(r)(1-\delta)^{-3}$ and $\Psi_r((1-\delta)r) = \Psi(r)(1-\delta)^{-1}$ so $\delta\rho \propto 3\rho + r(d\rho/dr)$ and $\psi \propto \Psi + r(d\Psi/dr)$. We now demonstrate that this ψ does satisfy the eigenvalue equation (14).

$$S\psi = \delta\rho - \frac{d\rho}{d\Psi}\psi = 3\rho + r\frac{d\rho}{dr} - \frac{d\rho}{d\Psi}\Psi - r\frac{d\rho}{d\Psi}\frac{d\Psi}{dr} = 3\rho - \frac{d\rho}{d\Psi}\Psi.$$

But for $n = 3$ polytropes $\rho \propto (\Psi - \Psi_0)^3$ and therefore

$$\Psi\frac{d\rho}{d\Psi} = 3\rho + \Psi_0\frac{d\rho}{d\Psi}$$

where Ψ_0 is constant. Hence

$$S\psi = -\Psi_0 \frac{d\rho}{d\Psi} = cU.$$

So ψ satisfies equation (14). Since Ψ is $O(1/r)$ at $\infty \psi = d/dr\,(r\Psi)$ is $O(1/r^2)$ and satisfies our boundary condition. Since $d^2/dr^2\,(r\Psi)$ is $-4\pi G\rho r$ which is negative within and zero outside the boby, we deduce that $d/dr\,(r\Psi)$ is positive within (since it is zero outside). Thus ψ has no zeros and is the fundamental mode. ψ may be calculated from the $n = 3$ Emden solution given by Eddington (9). The ϕ corresponding to this ψ suitable normalized is plotted as curve (3) of Fig. 1. Notice that it just has two zeros one of which is at the origin.

(c) As our example of a system whose stability may be shown by the Schrödinger operator method we take the $n = 1$ polytrope. The equations for ϕ integrates to give simply a sinusoidal curve. This has but one zero in the range so the system

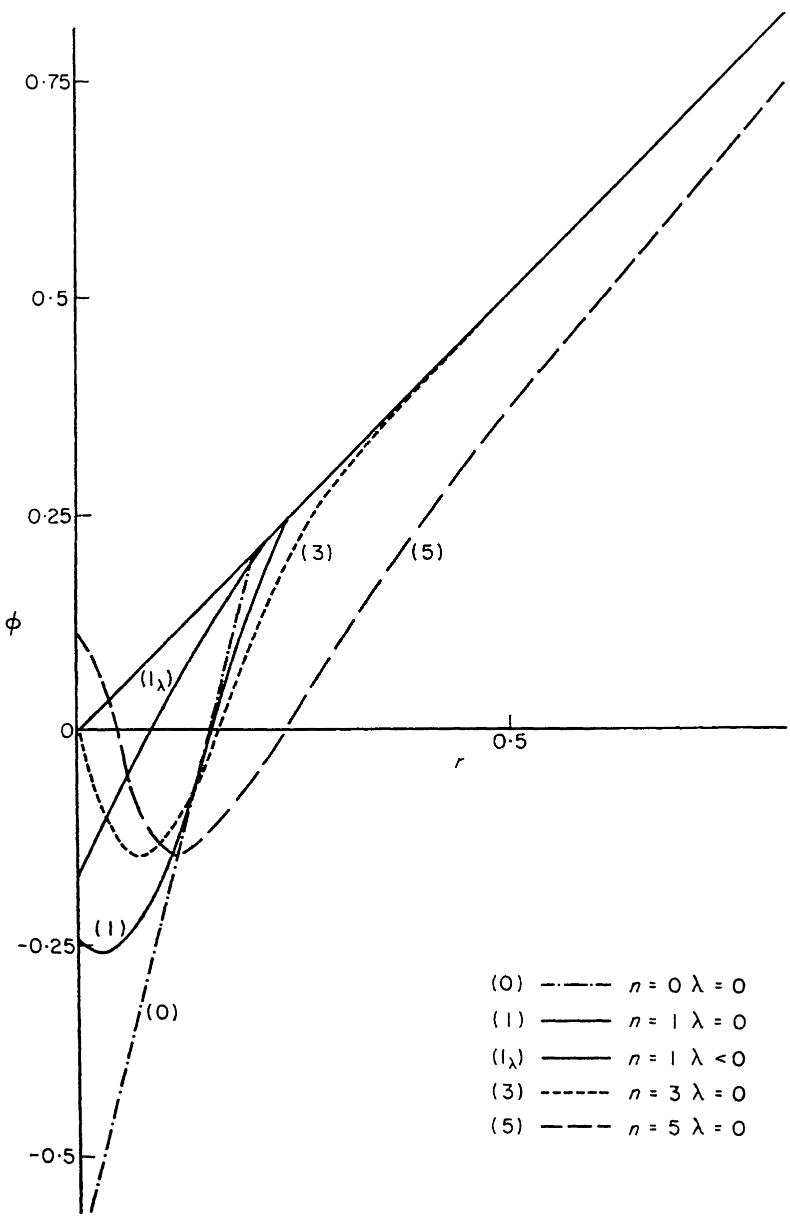

FIG. 1

is stable. The ϕ graph for the stable $n = 0$ liquid sphere has been added to Fig. 1 for greater completeness.

(d) We could have deduced the results (a), (b), (c) from our knowledge of gaseous polytropes; however in dealing with general stellar systems the integration for ϕ is necessary. As our last example we take the isochrone model cluster discussed by Henon and used by Eggen, Lynden-Bell & Sandage (**12**). This example is important for in performing the integration we found most confusing results which only made sense when we realized that for systems whose densities fall like r^{-4} or less rapidly it is incorrect to apply the limit of our boundary conditions as the boundary is let tend to infinity. Rather the correct operation is to find the asymptotic form of the solutions apply the boundary conditions to these as some large finite value of r_b and only then let r_b tend to infinity. It is a curious and interesting mathematical quirk that these procedures are inequivalent, the first one giving no solutions at all to our problem. We shall not enlarge on this example of the non-reversibility of limit processes but will simply apply the correct procedure.

The isochrone model has the potential

$$\Psi = \frac{GM}{b+\sqrt{(r^2+b^2)}}$$

and density

$$\rho = \frac{M}{4\pi b^3}\frac{2u-1}{u^2(u-1)^3}$$

where

$$u = 1+\left\{\left(\frac{r}{b}\right)^2+1\right\}^{1/2}.$$

Thus

$$\frac{d\rho}{d\Psi}4\pi G = \frac{8r^2+9(r^2+1)^{1/2}+11}{(r^2+1)^2(1+(r^2+1)^{1/2})} = 4\pi GU$$

say where we have taken units so that $b = 1$.

We have to solve

$$\frac{d^2}{dr^2}\phi+4\pi GU\phi = 0.$$

Now for large r,

$$4\pi GU = \frac{8}{r^3}$$

$\phi''+8r^{-3}\phi = 0$ has asymptotic solutions

$$\phi = A\left(1-\frac{4}{r}+O\left(\frac{1}{r^2}\right)\right)+B\left(r-16\log\frac{2\sqrt{2}}{r^{1/2}}+o\left(\frac{\log r}{r^{1/2}}\right)\right).$$

Taking the boundary conditions $\phi = cr_b$ at $r = r_b$

$$\frac{\partial\phi}{\partial r} = c \text{ at } r = r_b$$

we find

$$A\left(1-\frac{4}{r_b}\right)+B\left(r_b+8\log\frac{r_b}{8}\right) = cr_b+O\left(\frac{\log r_b}{r_b^{1/2}}\right)$$

$$A4/r_b^2+B(1+8/r_b) = c$$

$$\therefore \qquad A\left(1-\frac{8}{r_b}\right)+8B(\log r_b/8-1) = O\left(\frac{\log r_b}{r_b^{1/2}}\right).$$

Hence for large r_b

$$-\frac{A}{B} = 8\log r_b.$$

Hence it is the A solution that satisfies the boundary conditions for large r_b. To get on this solution at some finite r at which we wish to start integrating $r = 100$ say we put $\phi = -4/100$ and $d\phi/dr = +4/10^4$ and integrate inwards.

The result is given in Fig. 2 which shows that ϕ has two zeros so we have not been able to prove that the isochrone model is stable (or unstable).

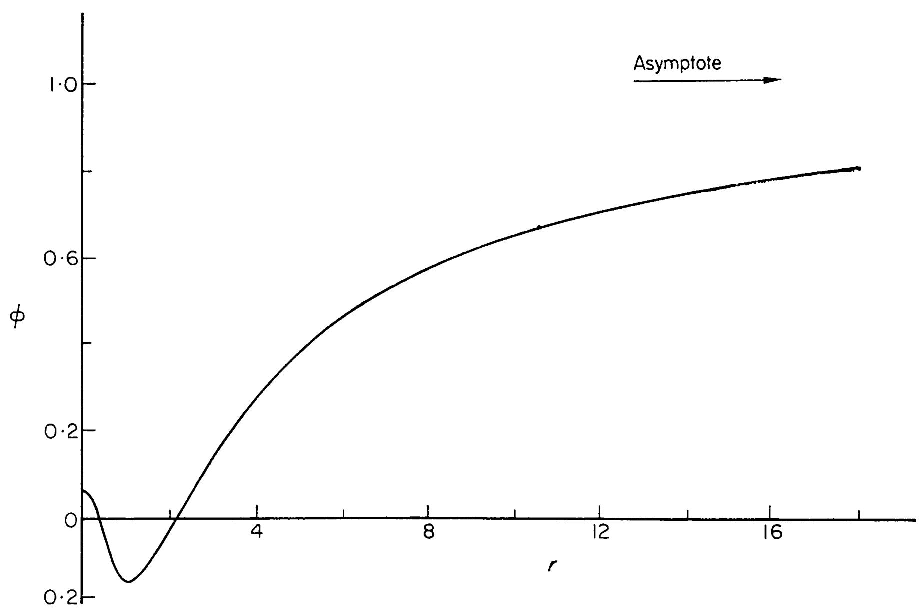

FIG. 2

It was our original aim to find out whether the truncated isothermal sequences of Woolley (**13**) and of Michie (**14**) and King (**15**) were stable. However it appears that the Schrödinger equation method is so feeble for spherical modes that it would only be capable of proving the very heavily truncated members of these sequences to be stable. This is of little interest so we have postponed that attempt until we have finished developing more powerful necessary and sufficient criteria for stability.

However the method outlined here is probably the best for determining the stability of Barotropic spheres. In particular the differential equation to be solved only has one functional coefficient U defining the system whereas most other methods require two coefficients.

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APPENDIX I

Generalizations of the method

There are three ways in which the methods given here can be extended to a wider range of problems:

(1) Extension to uniformly rotating stellar systems with $F = F(\epsilon)$.

(2) Extension to cover the spherically symmetrical modes only of spherical systems whose distribution functions are of the form $F(\epsilon, h^2)$ where $\mathbf{h} = \mathbf{r}\times\mathbf{v}$.

(3) Extension to cover the axially symmetrical modes only of axially symmetrical systems whose distribution functions are of the form $F(\epsilon, h_z)$.

These extensions were discussed in an earlier paper (3) and a review (4) and Milder gave a direct proof of the first extension (5). However there is some advantage in seeing how the Schrödinger criterion extends when we take care with our boundary conditions. By contrast to the earlier development we use at each time axes rotating with the angular velocity H/I_T where H is the total angular momentum which we assume fixed and equal to that of the equilibrium and I_T is the moment of inertia of the system about the axis of **H**. The total energy of the system is then

$$W_T = \frac{1}{2}\int f_T\left[\left(v_\phi+\frac{H}{I_T}R\right)^2+v_R^2+v_z^2\right]d^6\tau-\frac{G}{2}\int\frac{f_Tf'_T}{|\mathbf{r}-\mathbf{r}'|}d^6\tau d^6\tau'$$

where

$$I_T = \int f_T R^2 d^6\tau$$

and

$$H = \int f_T R\left(v_\phi+\frac{H}{I_T}R\right)d^6\tau, \quad \text{i.e.} \quad \int f_T R v_\phi d^6\tau = 0. \tag{A.1}$$

We subtract from W_T the energy of the unperturbed system and for its expression we use coordinates rotating with angular velocity $\Omega = H/I$ where I is the unperturbed moment of inertia.

$$W = \int F\left\{\frac{1}{2}[(v_\phi+\Omega R)^2+v_R^2+v_z^2]\right\}d^6\tau-\frac{G}{2}\int\int\frac{FF'}{|\mathbf{r}-\mathbf{r}'|}d^6\tau d^6\tau'.$$

$$H = \int FR(v_\phi+\Omega R)d^6\tau = \int FRv_\phi d^6\tau+I\Omega,$$

thus

$$\int FRv_\phi d^6\tau = 0. \tag{A.2}$$

Note that in the above v_R, v_ϕ, v_z are with respect to different axes than those used for W_T. Nevertheless v_ϕ, v_R, v_z etc. are dummy variables integrated over in both expressions. Hence we may subtract the two expressions and obtain a correct expression for ΔW incorporating the integrands under one integral.

$$\Delta W = W_T-W = \int f\left\{\frac{1}{2}[(v_\phi+\Omega R)^2+v_R^2+v_z^2]-\Psi\right\}d^6\tau$$
$$-\frac{G}{2}\int\frac{ff'}{|\mathbf{r}-\mathbf{r}'|}d^6\tau d^6\tau'+\int f_T R\Delta\left(v_\phi\frac{H}{I_T}+\frac{H^2}{2I_T^2}R\right)d^6\tau$$

where the first two terms are obtained analogously to equation (3) and $f = f_T-F$.

Now $\int f_T R\Delta(v_\phi H/I_T)d^6\tau = \int fRv_\phi d^6\tau\Delta(H/I_T) = 0$ by equations (A.1) and (A.2). Further

$$\int f_T R\Delta\left(\frac{H^2}{2I_T^2}R\right)d^6\tau = \frac{I_T H^2}{2}\Delta\left(\frac{1}{I_T^2}\right)$$

$$= \frac{(I+\Delta I_T)H^2}{2}\left[-\frac{2\Delta I_T}{I^3}+\frac{3(\Delta I_T)^2}{I^4}\right]+O(\Delta I_T)^3$$

$$= -\Omega^2\int fR^2d^6\tau+\frac{\Omega^2}{2}\frac{(\Delta I_T)^2}{I}+O(f^3)$$

incorporating these results to simplify ΔW and noting from (A.1) and (A.2) that $\int fRv_\phi d^6\tau = 0$ we have

$$\Delta W = \int f\left[\frac{1}{2}(v_\phi{}^2+v_R{}^2+v_z{}^2)-\left(\Psi+\frac{1}{2}\Omega^2R^2\right)\right]-\frac{G}{2}\iint\frac{ff'}{|\mathbf{r}-\mathbf{r}'|}d^6\tau d^6\tau' +\frac{\Omega^2}{2}\frac{(\Delta I_T)^2}{I}.$$

This appears to be in contradiction to the result derived in references (3) and (4). However such an idea is illusory because f has a different meaning. Referring our distribution functions to absolute space rather than rotating axes then f in this section is given by

$$f(\mathbf{r},\mathbf{v}+\boldsymbol{\Omega}\times\mathbf{r},t) = f_T\left(\mathbf{r},\mathbf{v}+\frac{\mathbf{H}}{I_T}\times\mathbf{r},t\right)-F(\mathbf{r},\mathbf{v}+\boldsymbol{\Omega}\times\mathbf{r},t)$$

whereas the f of Refs (3) and (4) is

$$f(\mathbf{r},\mathbf{v}+\boldsymbol{\Omega}\times\mathbf{r},t) = f_T(\mathbf{r},\mathbf{v}+\boldsymbol{\Omega}\times\mathbf{v},t)-F(\mathbf{r},\mathbf{v}+\boldsymbol{\Omega}\times\mathbf{v},t).$$

We now use the dodge of equations (4)–(6) to re-express the first term in ΔW and obtain

$$\Delta W = \frac{1}{2}\int\frac{f^2}{-\dfrac{dF}{\partial\epsilon}}d^6\tau-\frac{G}{2}\iint\frac{ff'}{|\mathbf{r}-\mathbf{r}'|}d^6\tau d^6\tau'+\frac{\Omega^2}{2}\frac{(\Delta I_T)^2}{I}$$

where

$$\epsilon = \tfrac{1}{2}(v_\phi{}^2+v_R{}^2+v_z{}^2)-(\Psi+\tfrac{1}{2}\Omega^2R^2).$$

Proceeding as before

$$\Delta W \geqslant \frac{1}{2}\int\frac{(\Delta\rho)^2}{\dfrac{d\rho}{d\Phi}}d^6\tau-\frac{G}{2}\int\frac{\Delta\rho\Delta\rho'}{|\mathbf{r}-\mathbf{r}'|}d^6\tau d^6\tau'+\frac{\Omega^2}{2}\frac{(\Delta I_T)^2}{I}+\eta \qquad (A.3)$$

where $\Phi = \Psi+\frac{1}{2}\Omega^2R^2$.

However expression (A.3) less the constant η is precisely the configuration energy of the uniformly rotating gaseous system as we now show. Define W_{int_T} as the total pressure plus gravitational energy.

$$W_{g_T} = W_{\mathrm{int}_T}+\frac{H^2}{2I_T}.$$

Then

$$\Delta W_g = \Delta_1 W_{\mathrm{int}}+\Delta_2 W_{\mathrm{int}}-\frac{H^2}{2I^2}\Delta I_T+\frac{1}{2}\frac{H^2}{I^3}(\Delta I_T)^2.$$

Now $\Delta_1(W_{\mathrm{int}_T}-\frac{1}{2}\Omega^2 I_T) = 0$ because the unperturbed state is an equilibrium. Hence

$$\Delta W_g = \Delta_2 W_{\mathrm{int}}+\frac{1}{2}\frac{\Omega^2}{I}(\Delta I_T)^2. \qquad (A.4)$$

It is clear from our earlier expressions (8) and (9) for $\Delta_2 W_{\mathrm{int}}$ that in the present case it gives the first two terms of expression (A.3). Hence from (A.3) and (A.4)

$$\Delta W \geqslant \Delta W_g+\eta.$$

Following our argument of Section 2.3 we deduce that the stellar system is secularly stable if the gaseous system is. We can of course again formulate the first two terms of expression (A.3) using the Schrödinger operator. This is equivalent to secular stability of a gaseous system forced to rotate with angular velocity Ω for then the relevant energy to be minimized for the gaseous configuration is $W_{g_T} = W_{\mathrm{int}_T} - \frac{1}{2}\Omega^2 I_T$ where the last term is the centrifugal potential energy. On variation we then get

$$\Delta W_g = \Delta_2 W_{\mathrm{int}}.$$

It is just the $\Delta_2 W_{\mathrm{int}}$ that can be written in terms of the Schrödinger operator. Thus the criterion given by the Schrödinger operator is the same as secular stability of the gas when the elements of it are forced to rotate with their equilibrium angular velocities. Many stable systems such as the Earth in its orbit do not satisfy such a stability criterion. However the Maclaunin spheroids* satisfy this criterion up to the point of bifurication with the Jacobi ellipsoids and only up to there. The Jacobi ellipsoids do not satisfy it.

Spherical systems whose distribution functions are not functions of energy only can not be treated generally by the Schrödinger operator method. However as shown previously (3), (4) their spherical modes are stable if ΔW_g is positive definite where the $dF/d\epsilon$ of equation (7) is replaced by $\partial F/\partial \epsilon$ with angular momenta held fixed. This destroys the relationship of $U = \int -(\partial F/\partial \epsilon) d^3v$ with $d\rho/d\Psi$. Otherwise application of the criterion is exactly the same as that described in Sections 5 and 6.

Only axially symmetrical modes of axially symmetrical (non-spherical) stellar systems with distribution functions $F = F(\epsilon, h_z)$ can be treated by the Schrödinger operator method. The method given in Refs (3) and (4) can be shown to be again equivalent to the stability of the gaseous system when every ring is forced to rotate with its equilibrium angular velocity. Again care must be taken to use trial functions in the energy

$$\Delta W_g = \frac{1}{2} \int \left(\frac{(S\psi)^2}{\int -\frac{\partial F}{\partial \epsilon}} d^3v \right) d^3r + \frac{1}{2} \int \psi S \psi d^3r$$

which gives $\psi = O(1/r^2)$ at infinity, $\nabla^2 \psi = 0$ outside the equilibrium body and $S\psi/U$ finite within and on the edge of that equilibrium figure. Again

$$U = \int -\frac{\partial F}{\partial \epsilon} d^3v.$$

It is worth remarking that the uniform displacement along the axis of rotation is a neutral mode which gives $S\psi = 0$ and is antisymmetric with respect to the galactic plane. An application of the method of Section 4 then proves that all antisymmetric modes are stable.

Note added in proof

We remark that it is still true that the cluster is stable if S is positive definite in the following sense. We demand as boundary condition that $\nabla\psi$ is $O(1/r^3)$ at infinity, but from any trial function ψ we subtract $\psi(\infty)$ before forming the integral of $\psi S \psi$.

* There are no stellar systems rigorously corresponding to these liquid configurations.