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The Schuler principle : a discussion of some facts and misconceptions

Eindhoven
University of Technology
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Electrical Engineering

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and
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THE SCHULER PRINCIPLE

Summary

The well-known Schuler principle for inertial navigation has been treated in many books and articles. However, certain misconceptions centering around the so-called Schuler period and the role gravity plays in Schuler-tuned systems can be found over and again in many texts.
This report uses relatively simple explanations of the geometrical and physical situations involved, and by comparing them with the various presentations in the pertinent literature sorts out the correct and incorrect statements.
In addition, it describes a simple as well as a more sophisticated demonstration model of a Schuler-tuned system, and touches on some mechanical topics related to the Schuler principle and d'Alembert's double pendulum.

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PREFACE

As members of the Measurement and Control Group of the Department of Electrical Engineering at our University we have been studying inertial techniques. The material contained in this report had accumulated over the years, mainly as a product of our teaching activities, and we thought it right to publish it in some form. Since it is too voluminous for presentation in a periodical, but on the other hand has a different scope than textbook matter, we chose offering it in the present form, i.e. as an EUT-report.

The chapters one, two, three, five, and six were written by C.Huber, chapter four by W.J.Bogers.

Acknowledgement is due to Professor dr. C.E. Mulders for stimulating the work, for discussing with us numerous facets of the subject, for helping us to simplify and correct some mathematical presentations, and for critically reading the text.

We are grateful for the work done by various typists, and to Mr. J.A. van Dinther for the fine job done preparing all the figures.

C. Huber
W.J. Bogers
January 1983

THE SCHULER PRINCIPLE

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0. THE HISTORY OF THIS PAPER

When preparing a course on inertial techniques for measurement and control at our University Department of Electrical Engineering about as far back as 1970, we naturally wished to incorporate a chapter on inertial navigation. Studying the pertinent literature, we came across what we suspected to be inaccuracies in the various presentations of the well-known Schuler principle, inaccuracies of the kind that arise from misconceptions which are generated when authors make partial statements of a truth, which by themselves may be correct, but which other authors then use without reference to their limited applicability.

We started searching for evidence from other authors that would confirm our suppositions, but the results of our search at first remained rather meagre. So, by wise of a low priority side line of our work, we began on our own to sort things out. Gradually, however, we were finding more direct evidence of the kind we were seeking for in the literature, and the book that finally convinced us that we were right is MAGNUS, 1971.

The misconceptions we intend to point out here centre around the so-called Schuler period and the role gravity plays in Schuler-tuned systems. We should like to set before you three statements, and then elaborate on them and compare them with some quotations, hoping thereby to clarify the facts and rectify the inaccuracies. These statements are:

1. "Schuler tuning" does not necessarily imply the so-called Schuler oscillation period of 84,4 min (at the earth's surface).
2. Not all devices exhibiting the Schuler oscillation period of 84,4 min can be used as a vertical reference on a moving base.
3. The existence of a gravity field is essential for the existence of an oscillation period, but it is not an absolute requirement for the basic function of a "Schuler-tuned" vertical reference, with very few exceptions like Schuler's gyro pendulum.

The above statements were written by us a couple of years before March, 1981, when a certain chapter written by MAGNUS in 1973, in a Russian book came to our eyes. A computer search had yielded this particular reference, and we are very grateful to the author, whom we contacted, for sending us a copy of his text as the book was difficult for us to come by. He not only sent us said copy, but also copies of other pertinent articles written by him (MAGNUS, 1965 and 1966), the existence of which we had not been aware of.

These articles finally and conclusively show that our first two statements are essentially true, for on page 295 of MAGNUS, 1973 we read: "... it has been possible to show that the fixed relation, suspected by Schuler, between the 84-minute period and insensitivity to accelerations does not exist". And in the other two articles (MAGNUS, 1965 and 1966) the mathematical proof of this fact is given thoroughly and concisely.

Confronted with this circumstance we naturally questioned the relevance of finishing our treatise on the subject. However, we find in our approach a property that might appeal to a reader not so thoroughly conversant with the in-depth mathematical aspects of inertial navigation, but interested enough to be desirous of letting go oversimplified and misleading notions.

We hope we have succeeded in reducing the complex theoretical discussions to a level of plausibility by analyzing a few simple situations. We shall also describe some simple class-room demonstration set-ups which give the viewer some insight into the matter without requiring the abstraction necessary when trying to understand the principle from a full scale system demonstration.

We hope the present treatise will help to banish from the textbooks some of the often-encountered misconceptions about the scope and limits of the principle, rightly named after SCHULER, because he was the first to apply it, thereby launching the inertial type of navigation instruments into their range of usefulness.

1. BASIC ASSUMPTIONS AND DEFINITIONS

1.1 Simplified model of the earth

The Schuler principle, introduced by M. SCHULER between 1908 and 1923, is well known today. It is invariably applied in those navigational instruments which are designed to take account of the curvature of the earth's surface. In this paper we do not intend to deal with the diverse and sometimes complex details of the application of this principle, but merely with the most basic facts. To this end we shall adhere to a number of simplifications:

1. The earth shall be considered a perfect sphere with a radius of 6372 km.
2. The earth is assumed to be of homogeneous density.
3. Gravity acceleration at the earth's surface be uniform, with a value of 9,81 m/s^2.
4. The above results in a well-defined radially symmetric gravity field where $g = f(R)$ according to fig. 1.1 - 1.
5. Vehicle movements shall generally be confined to great circle trajectories.

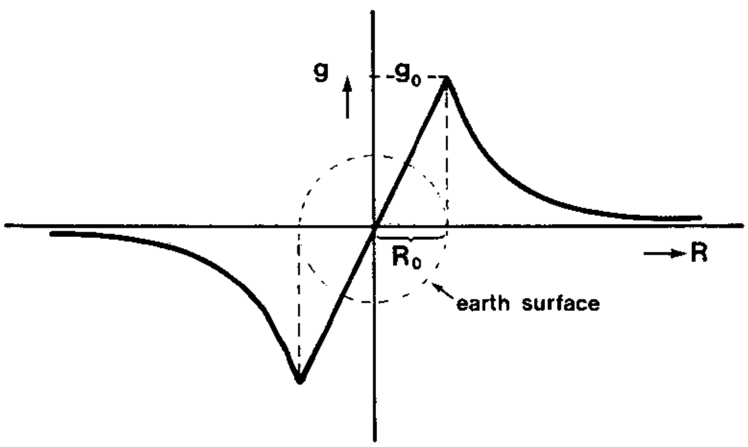


Fig. 1.1 - 1. Earth gravity g as function of distance R from centre.

$g_o = 9{,}81\ m/s^2$; $R_o = 6372$ km

for $R<R_o$ we have $\quad g = g_o(R/R_o)$ 1.1 - (1)

for $R>R_o$ we have $\quad g = g_o(R_o/R)^2$ (2)

1.2 Definition of Schuler tuning

Although we expect the reader to be familiar with the principle of the acceleration-intensitive pendulum discovered by M. SCHULER at the beginning of the twentieth century (see SCHULER, 1962, p.471), we wish to state our own definition of Schuler tuning here for clarity's sake:

An instrument member (e.g. a pendulum, or a platform), a known body axis of which points to a centre in space around which the instrument is carried by a vehicle and which keeps its said body axis pointing to said centre regardless of vehicle accelerations, is to be called Schuler-tuned.

To our taste, it should rather have been called Schuler calibrated, or Schuler adjusted, because the expression "tuned" automatically suggests the involvement of a frequency. While this expression, appearantly introduced by WRIGLEY in 1950 (compare WRIGLEY, 1977, p. 63, line 10), represents the usual practical approach to the adjustment problem, it is misleading with respect to the theoretical principle involved. To show this is one of the aims of the present paper. However, since the term "Schuler tuned" has become generally accepted we shall adhere to this custom.

1.3 Definition of the Schuler period

In many books and articles on inertial navigation the Schuler period is defined as

$$T_o = 2\,\pi\sqrt{R_o/g_o}, \qquad 1.3 - (1)$$

where R_o is the curvature radius of the earth's surface, and g_o the acceleration due to gravity at the surface of the earth. If we insert R_o = 6372 km and g_o = 9,81 m/s^2 into 1.3 - (1) we find

$$T_o \simeq 5064 \text{ s} \approx 84{,}4 \text{ min}, \qquad (2)$$

this being the approximate value of the Schuler period related to the surface of the earth.

One should be, however, more careful in stating the definition of the Schuler period.

There are two possible obvious definitions:

first, $T_s = 2\pi\sqrt{Ro/go}$, (3)

second, $T_s = 2\pi\sqrt{R\ /g\ }$, (4)

as well as a less obvious one which we will touch upon further on.

The first definition would be a logical choice in so far as Schuler himself, when developing his ideas, was concerned with earth pendulums in ships. On such vehicles — assuming the idealized earth as mentioned in our chapter 1.1 — R_0 and g_0 can be regarded as constants. So the Schuler period T_0, based on R_0 and g_0, also would be a constant, one pertaining to the earth, an earth constant thus.

A platform used in an aeroplane cannot strictly be kept tuned to T_0 after take-off, since R and g change with altitude. But it is customary to speak of Schuler-tuning also with regard to airborne systems. So we propose to use the second definition for the Schuler period, and to call the first definition the Schuler constant (for the earth).

Incidentally, this constant is the same as the smallest possible circulation time for an earth satellite. As such it had already been identified by earlier scientists (such as NEWTON and HUYGENS).
But in connection with the tuning of navigation instruments, the use of the name of Schuler is not misplaced.

The third possible definition is less obvious. It relates to the actual period of oscillation a specific Schuler-adjusted system will have when one also takes into account the gravity gradient and the mass distribution in the system, and the centrifugal forces due to the velocity of the carrying vehicle. We should like to call this the actual oscillation period.

Thus, to sum up:

(1) The Schuler constant $T_o = 2\pi\sqrt{R_o/g_o}$ 1.3 - (5)

(2) The Schuler period $T_s = 2\pi\sqrt{R/g}$ (6)

(3) The actual oscillation period: The oscillation period of a specific Schuler-adjusted system (acceleration insensitive system) under specific circumstances:

$$T = k \cdot 2\pi\sqrt{R/g} \;, \tag{7}$$

where k will always have a value between 0,5 and ∞ according to MAGNUS 1971, p.395 (see also our p.33, quotation 12).

2. INERTIAL NAVIGATION, SCHULER's PRINCIPLE, AND A SIMPLE DEMONSTRATION MODEL

In including this following chapter our intention is to state the Schuler principle in a more or less absolute form. We believe that many erroneous statements can be attributed to the inclination of authors to explain the principle along more or less historic lines of reasoning. They tend to be somewhat circuitous because they follow Schuler in always including gravity in their reasoning (compare his original article SCHULER, 1923), although gravity's role is not obligatory in all systems. Schuler, however , is justified in having done so, or can be excused for it, in that his concern was primarily with gyrocompasses which are inherently pendulous; besides that he suspected that a general law existed connecting Schuler adjustment with the Schuler period. The very title of his article carries reference to pendulums.

But we must go beyond the scope of pendulums if we want to understand the Schuler principle in a broader sense. Our intention, thus, is to explain the physical facts as clearly as we can in a matter-of-fact mode without looking back to Schuler, before we consider one of the main topics of this paper, namely the correction of some popular misstatements.

2.1 What is inertial navigation?

Navigation may be called the art of finding one's bearings. This art makes use of divers techniques, and one of those is the employment of inertial - type instruments.

To be able to navigate, you need a reference system of coordinates.

To determine your position you need to know the distance to a given point of reference (a landmark or a beacon or any other fixed point, which may be arbitrarily chosen), and directional information with respect to a given directional reference, which also may be arbitrarily chosen.

Land-based vehicles travel on a comparatively rigid medium. Distance travelled from a known starting point basically can be measured by counting the revolutions of a wheel in contact with the medium. This is demonstrated in our well-known mileage counters in automobiles. Directional reference is a problem not so easily solved, requiring a compass and/or a map and landmarks.

Sea-going vehicles travel on a fluid medium. Distance travelled is often determined by measuring the <u>speed</u> with respect to the medium and then computing its time integral. Airborne vehicles can approach the distance measuring problem in the same way.

Direction can be found in ways similar to those mentioned with the land-based vehicles.
Why this seemingly trivial discussion? It is to show an analogy with a third kind of vehicle, the "space vehicle". Its "medium" is inertial space. Distance in this medium is determined by measuring <u>acceleration</u> and then computing its double time-integral.
Directional references can be artificially created and carried on-board by way of spinning rotor gyros, laser gyros or any other form of inertial space goniostat or gonimeter.

Basically, any of the other vehicles mentioned is also a"space vehicle", since they all move in the "medium" inertial space. They can all be fitted with an inertial measurement unit to solve their navigation problem, but it is the aeroplanes that we mostly think of as using inertial navigation techiques, including the Schuler principle. Their operational modes and requirements make for the most profitable use, technically and economically, of the expensive inertial navigation instruments. But it was problems with marine instruments that initiated the discovery by M.Schuler, in 1923, of the principle carrying his name.

2.2 Determinating position by measuring acceleration

Consider a vehicle travelling parallel to the surface of a sphere on a great circle. In the vehicle there is a gimballed platform carrying an accelerometer which has its sensitive axis oriented tangentially to the great circle, i.e. parallel to the vehicle trajectory (fig. 2.2 - 1).

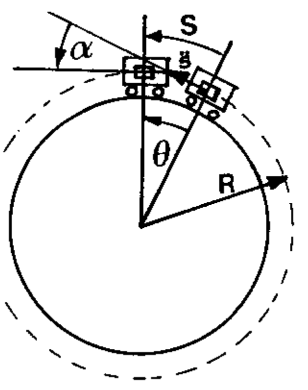


Fig. 2.2 - 1. Vehicle circling the earth along a trajectory with radius R.

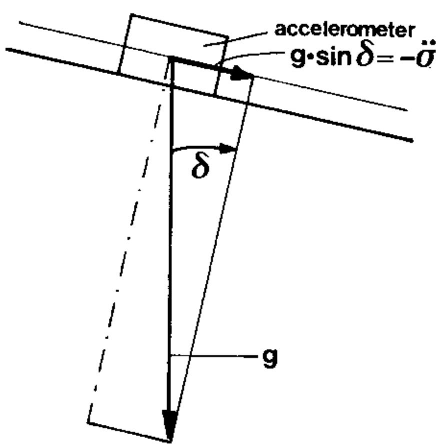
This also means that the accelerometer's sensitive axis is at right angles to the local vertical. Consequently it will sense no component of the sphere's gravity field, which is supposed to have its origin in the centre of the sphere, but only vehicle accelerations along the trajectory.

The distance s travelled by the vehicle along the trajectory can be computed from the double time-integral of the acceleration $\ddot{s}$:

$$s(t) = \iint_0^t \ddot{s}\, dt\, dt. \qquad 2.2 - (1)$$

2.3 Maintaining a vertical reference

Condition for the idea set forth in 2.2 to function properly is that the accelerometer input axis must remain horizontal all the time. A small angular deviation δ from the horizontal has two effects, (1) a relatively small loss of measuring accuracy: the measured $\ddot{s}$ will be cos δ times the true $\ddot{s}$, and (2), in the presence of gravity, the accelerometer will sense an erroneous acceleration $\ddot{\sigma}$ equal to minus sin δ times the gravitation acceleration g (see fig. 2.3 - 1).



2.3 -(1)

Fig. 2.3 - 1. Error in sensed acceleration due to gravity.

Keeping the platform with the accelerometer horizontal means causing its alignment to follow the contour of the sphere by rotating it with respect to inertial space. The rate of rotation $\dot{\alpha}$ then must equal the rate of change of the angle θ in fig. 2.2 - 1:

$$\dot{\alpha} = \dot{\theta} = \frac{\dot{s}}{R}. \qquad (2)$$

As long as 2.3 - (2) applies it is also true that

$$\ddot{\alpha} = \frac{\ddot{s}}{R}. \qquad (3)$$

We can call 2.3 - (3) the Schuler condition, and any system assuring the constant fulfillment of this condition can be called Schuler calibrated, or as remarked in par. 1.2, in commonly used terms, Schuler-tuned.

2.4 Methods for obtaining Schuler tuning

There are a number of methods to get a physical system to behave according to the Schuler condition eq. 2.3 - (3). The one we are going to explain here we chose because of its simplicity and because it leads directly to a very simple and effective demonstration model. It cannot, however, be applied when designing a "real" Schuler-tuned system for use in a terrestrial aeroplane, because its realization would require impractically large structures (e.g. SCHULER, 1923, p. 346) or impossible manufacturing accuracies (e.g. HECTOR, 1968, p. 72).
Schuler used a similar structure in discussing his discovery (SCHULER, 1923, fig. 1), but for reasons we shall state later his explanation suffers from a lack of clarity.

Let us imagine an idealized physical body resembling a pair of dumb-bells, consisting of two equal point-masses m_1, m_2, connected together by a rigid but mass-less rod (fig. 2.4 - 1), the length of which is 2r.

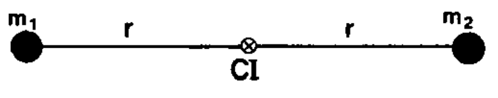


Fig. 2.4 - 1. Two point-masses connected by a mass-less rod, and floating in gravity-free space.

We know its centre of inertia CI to be half-way between the points of mass. Let this configuration be at rest in a gravity-free zone of inertial space.

If a force F_{CI} is applied to its centre of inertia, the body will be accelerated and move about without rotating. If the force is made to act on some other point of the rod, there will ensue a rotary as well as a translatory movement. Both types of movement combine to give the body a displacement of rotation around a momentary centre M, which in fig. 2.4 -2 is drawn for the

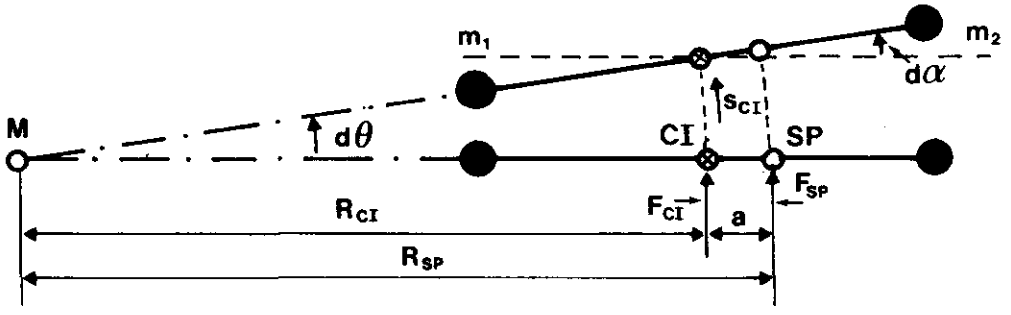


Fig. 2.4 - 2. Infinitesimal rotation dθ of the dumb-bell model around M.

case that F_{sp} is orthogonal to the connection rod.

The reason for choosing the letters SP to designate the point of attack of the force is that we want to make this point a suspension point. An imaginary vehicle is to carry the twin mass body, the latter being pivoted to the former so that it can freely rotate around SP. The vehicle is then able to exert forces on the body, the point of attack always being SP. But no torques can be transmitted to the body by rotation of the pivot.

If we now imagine the vehicle constrained to a circular trajectory around the initial M, then the suspension point SP cannot but travel along a path that always has M as its momentary centre of rotation. Thus M becomes a fixed point in space, the extended connecting line between the masses m_1 and m_2 will always pass through this fixed M, and we have created a system that obeys eq. 2.3 - (3), the Schuler condition.

Circulation around M gives rise to centrifugal forces. The vehicle, being confined to its trajectory, will counteract these forces, and for the present discussion we do not need to consider them.

2.5 The Schuler-tuned twin mass body

The twin mass body has a total mass $m = m_1 + m_2$ and a moment of inertia around its CI which amounts to $J_{CI} = mr^2$.
The force F_{sp} exerted by the vehicle gives rise to reactionary inertial forces from the body. These result in a translatory acceleration of the centre of inertia:

$$\ddot{s}_{CI} = \frac{F_{SP}}{m} = -\frac{F_{CI}}{m}, \qquad 2.5 - (1)$$

and a rotary acceleration around the centre of inertia:

$$\ddot{\alpha} = \frac{T}{J_{CI}}. \qquad (2)$$

where $\ddot{s}_{CI}$ = distance travelled by CI along trajectory
F_{sp} = force exerted by vehicle at SP
F_{CI} = reaction force generated by $\ddot{s}$ at CI
m = total mass of dumb-bell body
$\ddot{\alpha}$ = rotary acceleration of dumb-bell body
T = torque produced by F_{sp} and F_{CI}
J_{CI} = moment of inertia of dumb-bells around CI

compare fig.2.4 - 2

Now we know that

$$T = a F_{SP}, \qquad 2.5 \ - (3)$$

a being the distance between centre of inertia CI and suspension point SP (see fig. 2.4. - 2), and

$$J_{CI} = m r^2 \qquad (4)$$

with r being half the distance between m_1 and m_2. The Schuler condition given by eq. 2.3 - (3) is

$$\ddot{\alpha} = \frac{\ddot{s}}{R_{CI}},$$

where R_{CI} is the distance from centre M to CI (fig. 2.4 - 2).
Using this condition and eqs. 2.5 - (1) through (4) we can put down

$$\ddot{\alpha} = \frac{\ddot{s}}{R_{CI}} = \frac{F_{SP}}{m \cdot R_{CI}}$$

and

$$\ddot{\alpha} = \frac{T}{J_{CI}} = \frac{F_{SP} \cdot a}{m \cdot r^2}.$$

Equating these we finally get $a/r^2 = 1/R_{CI}$

or

$$\boxed{\begin{aligned} a &= \frac{r^2}{R_{CI}} \\ r &= \sqrt{a \cdot R_{CI}} \end{aligned}} \qquad (5)$$

We see that, for a given trajectory radius and a given length 2r of the twin mass body, we need only to suspend the body at a distance a from its centre of inertia to get a Schuler-tuned system.

It is worthwhile mentioning here that this result is independant of the actual magnitude of m (as long as m is not zero), and that no gravity field was needed to dertermine the design parameters.

From the second form in which eq. 2.5 - (5) is given we understand that the radius of gyration of the twin mass body has to be the geometrical mean between the suspension point excentricity a and the radius of the trajectory R_{CI}. Also the following form shows this:

$$\frac{a}{r} = \frac{r}{R_{CI}}. \qquad (6)$$

What does this amount to in terms of earth radius? Take R_{CI} = 6372 km (see par. 1.1) and r = 1m. Then

$$a = \frac{r^2}{R_{CI}} = \frac{1m^2}{6{,}372 \cdot 10^6 m} \simeq 0{,}16\,\mu m. \qquad 2.5 - (7)$$

A prohibitive requirement indeed to have to place a pivot axis within such close distance to the centre of inertia, for it would mean that the position of the centre of inertia itself would have to be known with an accuracy of a fraction of this value, say 0,01 or 0,001 μm.
Bringing the masses closer together only aggravates the difficulty since a is proportional to the square of r.

2.6 A simple table-top demonstration model

If one can do with a smaller trajectory radius R the problem gets easier. We have made a demonstration model according to fig. 2.6 - 1, the parameters of which are

Trajectory radius	R_{CI}	= 30 cm
Radius of gyration	r	= 5 cm
Suspension point excentricity	a	= 8,3 mm .

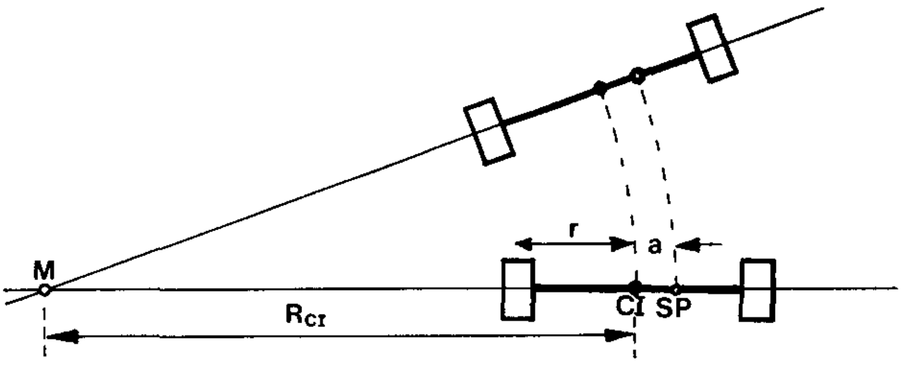


Fig. 2.6 - 1. Demonstration model of a Schuler-tuned dumb-bells body

A photograph shows the actual model which can easily be placed on top of a tabel. Demonstration proves very effective if the ball-bearings used are of a high quality instrument type and the base-plate is adjusted to be sufficiently horizontal to avoid drifting due to gravity.
Moving the carriage to and fro softly, swiftly, or abruptly, or even bumping it against the spring-loaded stops does not make the connecting rod deviate from the radial direction.

In order to show that a wrongly chosen suspension point decidedly degrades performance the connecting rod can be slid to any position between a = 8,3 mm (the Schuler tuning requirement) and a = 0 (suspension at the centre of inertia).

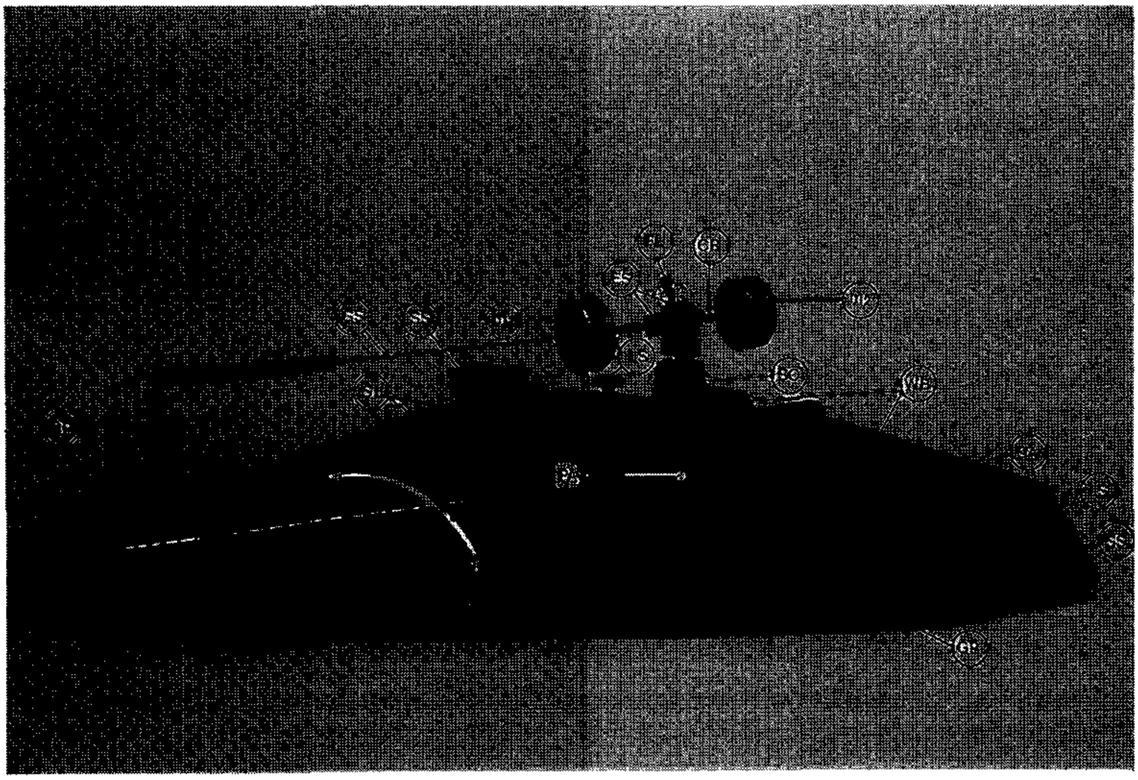


Fig. 2.6 - 2. Photograph of a demonstration model

<u>Legend to fig. 2.6 - 2</u>

BC = bearing column (aluminium)
CA = carriage (aluminium)
CP = centre pole (steel rod)
CR = connecting rod (steel)
ES = excentricity stops (brass)
FL = fixation lever
GP = ground plate (black perspex)
GS = gravity imitation spring
LS = levelling screws
m_1, m_2 = dumb-bells (brass)
PA = pivot axis
SL = spirit levels
SS = spring-loaded stops
WB = wheel boxes

The mass of the "dumb-bells" is not critical. We used brass cylinders of approx. 50 grammes each which proves sufficient to render bearing friction effects negligible. Their dimensions are 38 mm dia. and 15 mm height. Any difference between m_1 and m_2 will affect the position of the centre of inertia and the suspension point, but, after adjustement of stops ES, will not impair the proper functioning of the model.

A spring loaded gravity imitation string GS can be attached to m_1 to demonstrate gravity effects. It will be described later on.

2.7 The effects of gravity

In ch. 2.4 we assumed the twin mass system to be in a gravity-free zone of inertial space (cp. fig. 2.4 - 1). We now allow a radial gravity field to exist, the origin of which shall coincide with the centre of the trajectory (fig. 2.4 - 2). It is easy to understand why this gravity field will not enter into the Schuler-condition.
The twin mass-point body has become a pendulum in its equilibrium position, since its centre of gravity hangs beneath the suspension point.
In this position gravity can exert no torque and consequently not move the pendulum. The pendulum, designed to the Schuler condition, remains in that position whatever the movements of the suspension point may be along the circular trajectory.

If the pendulum is displaced from its equilibrium position, clearly it will exhibit an oscillatory movement around the equilibrium position. Although, as we have seen, the Schuler condition is not touched by the presence or absence (or, more generally speaking, not touched by the magnitude) of the gravity field, the oscillation period <u>is</u>.

As long as the oscillation amplitude is so small that the projection of the length of the pendulum onto the local vertical can be regarded as equal to the length of the pendulum itself ($\cos \delta \approx 1$, see fig. 2.3 - 1), the Schuler condition will not be touched by the pendulum excursions. Conversely, trajectorial vehicle accelerations will not make the pendulum oscillate nor cause it to change its oscillation mode. Both phenomena, i.e. Schuler behaviour and pendulum behaviour, are not coupled.

In a absence of a gravity field a small disturbing rotary impulse applied to the twin point mass body by a torque other than the "Schuler torque" could make the body rotate beyond limits. A gravity field keeps the excursions limited in amplitude, though it cannot prevent the oscillations from persisting over a lengthy period of time.

2.8 Including gravity effects in the demonstration model

As mentioned at the end of par. 2.6 a spring-loaded string can be attached to mass m_1 (item GS in fig. 2.6 - 2). It lends a restoring torque to the twin mass body and thus imitates the effect of a gravity field.

More properly, the point of attack of a simulated gravity force ought to be the centre of inertia, i.e. the middle of the connecting rod CR. More properly still, perhaps, a "gravity" force should have been applied to each "point mass" separately, thus enabling us to show the effects of an inhomogeneous gravity field in more detail. We would then have had to give the springs a nonlinear compliance so as to imitate the inverse square law of gravity. But we wished to keep our model as simple as possible and chose to apply a restoring force to m_1 only. In this respect the model is phenomenologic, and not quantitative.

We also devised an alternative method of introducing a restoring torque. It consists of slightly tilting the pivot axis (see fig. 2.6 - 2) towards the centre pole. In this way a predetermined component of earth gravity acts to torque the twin mass body towards its equilibrium position. That is the direction of the local radius of the trajectory, providing the ground plate is properly levelled. We have never built a model according to this idea yet, but we include a design description for the benefit of the readers (paragraph 6.4.1).

Now with the model depicted in fig. 2.6 - 2 a number of different settings of the connecting rod can be chosen. We shall list a series of them in the sequence we usually follow when giving demonstrations.

2.9 Demonstrating the Schuler principle with the model

The settings of the connecting rod CR (cp. fig. 2.6 - 2) will be indicated by prescribing the required amount of excentricity as

$a = 0$ i.e. suspension point in centre of inertia

$0 < a < 8,3$ i.e. arbitrary in-between values

$a = 8,3$ i.e. excentricity 8,3 mm which constitutes the Schuler tuning condition.

After a certain setting has been made, the demonstration consists of moving the carriage CA to and fro softly, swiftly, or abruptly, even letting it bounce back from the spring-loaded stops. This phase of the demonstration is indicated by "move".

The aim of each demonstration is to let the spectators observe the angular movements of the twin-mass body $\overline{m_1 m_2}$, either with respect to inertial space or relative to the carriage. Fig. 2.9 - 1 is given to facilitate identification of these angles.

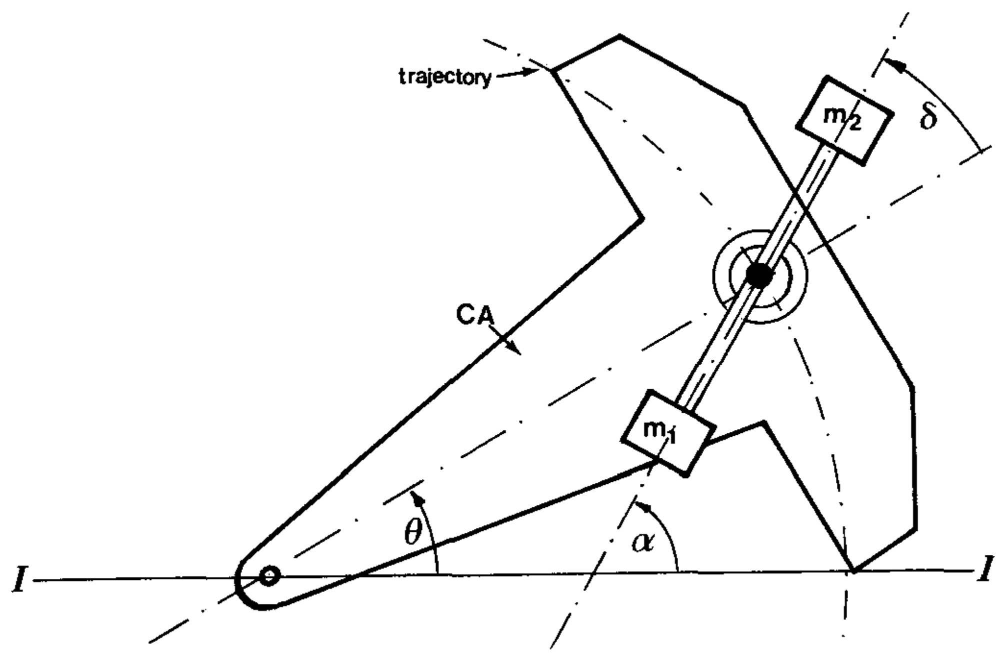


Fig. 2.9 - 1. Angles used to describe the movements of the carriage CA and the Schuler pendulum $\overline{m_1 m_2}$.

I - I = inertial directional reference

α = angle of $\overline{m_1 m_2}$ with respect to I - I

θ = angle of CA with respect to I - I

δ = angle of $\overline{m_1 m_2}$ with respect to CA

Demonstration modes

I. Gravitation simulator disconnected

A. Setting : $a = 0$

Move.

Observe : $\ddot{\alpha} = 0$; $\delta = \theta$

B. Setting : $a = 8{,}3$

Move.

Observe : $\ddot{\alpha} = \ddot{\theta}$; $\delta = 0$

C. Setting : $0 < a < 8,3$

Move.

Observe : α, $\dot{\alpha}$ and δ, $\dot{\delta}$ arbitrary

Stop.

Observe : $\dot{\alpha}$, $\dot{\delta}$ can persist

II. Gravitation simulator engaged

A. Setting : $a = 8,3$

Move.

Observe : $\ddot{\alpha} = \ddot{\theta}$; $\delta = 0$

B. Setting : $0 \leq a < 8,3$

Move.

Observe : α, δ in oscillate with arbitrary phase and amplitude

Stop.

Observe : δ-oscillation persists and induces θ-oscillations

C. Setting : $\alpha = 8,3$

Hold carriage tight, initiate δ-oscillations by hand

Release carriage

Observe : δ-oscillations persist but do not induce θ-oscillations

Then move.

Observe : δ-oscillations persist undisturbed by $\ddot{\theta}$.

2.10 Conclusions

A simple way to understand the basic nature of a Schuler-tuned system is to consider the movement of a twin-mass body that floats freely in space and is acted upon by a force, the workline of which does not pass through its centre of inertia.

Following this approach it is very easy to demonstrate the basic phenomena connected with the Schuler principle by means of a dumb-bells shaped body with a horizontal main body axis and vertical axes of rotation.

The effects of tuning and detuning can be shown. Moreover, while in an actual full-scale system for earth navigation the Schulerian behavious and the Schuler oscillation period are inseparable, the demonstration model makes it clear that those are two independant phenomena only loosely coupled because a gravity field concentric with the trajectory happens to exist.

3. REMARKS ON THE PUBLICATIONS DEALING WITH SCHULER's PRINCIPLE

In this chapter we will first examine the texts of Schuler's 1923-article, discuss their meanings and implications, and try to find out his intentions in presenting the matter as he did. Then, in the second half of this chapter, texts of a number of other authors will be given to show how their thinking has or has not been biased by Schuler's original statements.

SCHULER, in his 1923 paper, uses three different kinds of apparatuses to direct the reader to a curious phenomenon, namely the occurence of the 84 min. period in all the devices he discusses.
We shall briefly sketch their design and function before turning to Schuler's texts: -

a. The gyrocompass

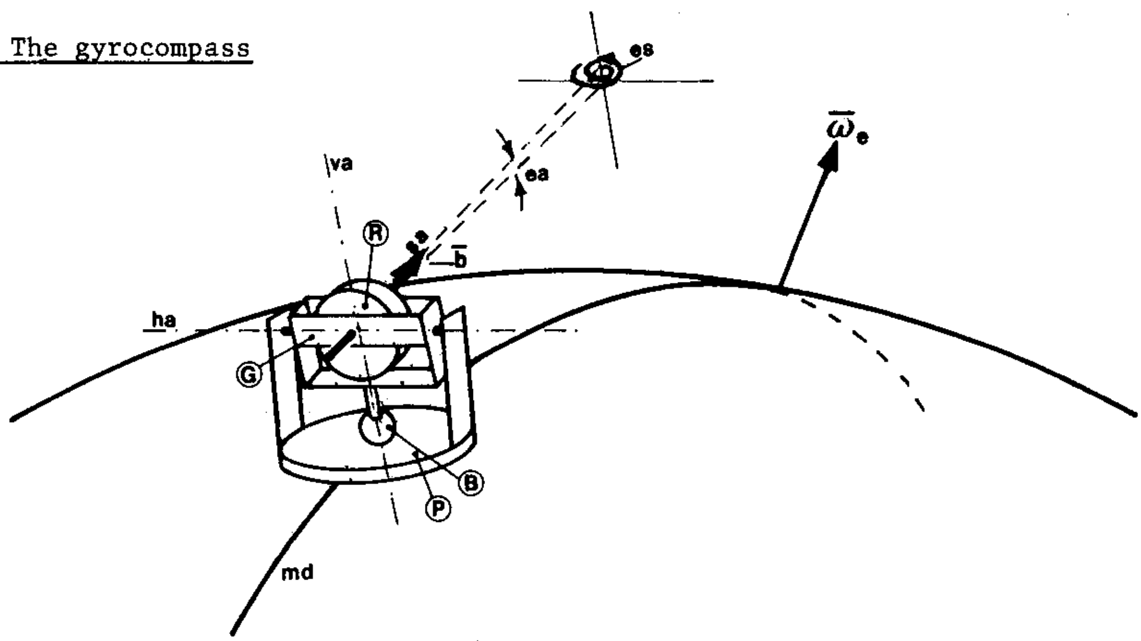


Fig. 3. - 1. Basic design of a gyrocompass

A platform P, free to rotate around the local vertical (vertical axis va) carries a gimbal G which can rotate around the platform-fixed horizontal axis ha. The pendulum bob B, attached to the gimbal G, tries to keep the spin axis sa of the gyro rotor R (which is suspended in G) at right angles to va, that is, horizontal.
Earth rotation $\bar{\omega}_e$ in general will make the spin vector $\bar{b}$ change its orientation respective to va and the meridian md.
But there is an equilibrium orientation, characterized by the elevation angle ea in the plane of the local meridian md, in which the torque produced by the bob B is exactly equal to the precession torque the gyro rotor R needs to follow the inertial rotation rate of the local meridian.

If the spin axis sa is not in that equilibrium orientation, it will swing towards it in a fashion indicated by the elliptic spiral es, which spiral is the curve traced by the projection of the spin axis sa onto a plane perpendicular to the local tangent of the meridian ma. The time required for the completion of one full round swing of sa is the period Schuler claims ought to be made 84 minutes.

b. The gyro pendulum

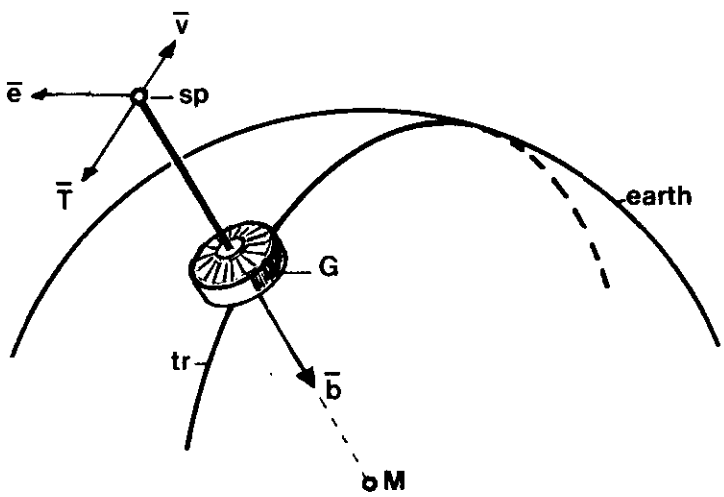


fig. 3. - 2. Gyroscopic pendulum

A gyro G, with an angular momentum vector $\bar{b}$, is suspended freely in a vehicle, its suspension point denoted by sp. The vehicle travels along a trajectory tr at a speed denoted by the vector $\bar{v}$. $\bar{e}$ is another vector in the horizontal plane, but at right angles to $\bar{v}$. The earth can be regarded as non-rotating or else as contributing to the vehicle speed resulting in the total surface speed vector $\bar{v}$ with respect to an inertial reference system. In order to maintain $\bar{b}$ in its direction to the centre M of the earth while following the curvature of the earth, the pendulum has to receive a torque $\bar{T}$ opposite to $\bar{v}$. This can be achieved, at the expense of perfect verticality, by a sideway excursion of the gyro (= "pendulum bob") G in a vertical plane in the direction of the vector $\bar{e}$. This sideway excursion can be generated by a torque parallel to $\bar{e}$, a torque that would arise during accelerations $\dot{\bar{v}}$ in the direction of $\bar{v}$. Proper tuining of the pendulum assures that $\dot{\bar{v}}$ creates just the right amount of torque parallel to $\bar{e}$ that is required to make the gyro deflect sideways by the exact amount necessary to create the precession torque $\bar{T}$, belonging to the forward velocity $\bar{v}$ resulting from $\dot{\bar{v}}$, to keep the pendulum in the vertical plane perpendicular to $\bar{v}$.

If by any cause the pendulum is displaced out of its intended direction, a conical precession movement around the intended direction will ensue. A properly designed gyroscopic pendulum will, according to Schuler, exhibit a precession rate so as to make it describe a complete cone in 84 minutes.

c. The physical pendulum

This is simply a body of arbitrary shape, freely suspended at a point above its centre of inertia. Upon horizontal acceleration of the suspension point the centre of inertia will lag behind. If the suspension point is placed at the proper distance from the centre of inertia, the body will keep its initial orientation with respect to the momentary local vertical, regardless of the suspension point acceleration (see text around our fig. 2.4 - 2). Again, according to Schuler, such a device, if disturbed, would oscillate with a period of 84 minutes.

3.1 The gyrocompass

In a review article written in 1962 by SCHULER himself we read that MARTIENSSEN had prepared a theoretical study of the behaviour of the pendulous north-seeking gyroscope when placed on a ship. (In the first footnote of SCHULER, 1923 this study can be identified as MARTIENSSEN, 1906 of our list of references). The principle of this kind of instrument had been indicated by L. Foucault a few decennia before. It consists of constraining the spin axis of a gyroscope so as to make it remain near the local horizontal plane, whereupon it will turn its spin axis into the plane of the local meridian.

Constraining the gyro as mentioned above can be comfortably done by making it pendulous. This solves the problem for a stationary north indicator, but tends to introduce disturbances when the gyro is carried on a moving base subjected to horizontal accelerations. MARTIENSSEN in his study came to the conclusion that a gyrocompass would be useless on board a ship, where it would give misreadings of dozens of degrees.
Schuler examined MARTIENSSEN's calculations and discovered that a condition could be found where the pendulosity would not make the gyro north-seeking, but would even help to keep the instrument aligned to the changing direction of the local vertical as the vehicel travelled along the surface of the earth, subjected to arbitrary accelerations.

A north-seeking gyro, when disturbed, will oscillate around the plane of the meridian. Schuler found that immunity to horizontal accelerations was concurrent with a period of this oscillation of 84,3 min.

With this type of instrument, gravity is all-important in the sense, that it is not the magnitude of gravity, but its direction, which matters. It becomes north pointing only in the presence of some sort of vertical-derived restoring torque. But its north pointing property itself essentially depends neither on the actual value of g nor on the "amount of pendulosity", i.e. the distance between suspension point and centre of gravity, and weight. The gyro will always swing into an elevated position of its spin axis by the exact amount necessary to slew the gyro to the earth rotation. (Of course we assume the gyrocompass to be sensibly dimensioned so as to keep the elevation angle within reasonably small limits).

Schuler recognized this in 1908. Although he does not state so in his famous 1923 paper, he mentions it in his article of 1962, p. 471, under "Das 84-Minuten-Prinzip ...". His reasoning (see SCHULER, 1962, p. 471) may be retold in the following way: -

Stationary on earth, the equilibrium direction of the gyro spin axis will be due north, and on a vehicle moving at constant speed it will have a known northerly steaming error independant of the amount of pendulosity. However, the amount of pendulosity will affect the period of oscillation around equilibrium direction, and it will also affect the compass' sensitivity to vehicle accelerations. Without changing its essential direction finding ability one is almost entirely free to choose the amount of pendulosity. So why not use this liberty to minimize the acceleration errors ! To his astonishment he found that not only does such a minimum exist, but that at this minimum all acceleration-induced errors become zero. The requirement for this condition seemed very simple: tune the period of swing around the equilibrium position to 84,3 minutes.
And the formula he gives (SCHULER, 1923) is

$$\frac{b}{m\ g\ a\ \Omega\ \cos\lambda} = \frac{R}{g} \qquad 3.1\ -(1)^{*)}$$

The right-hand side of this formula consists of the earth parameters:

R = radius of the earth

g = gravity acceleration at the earth's surface,

---

*) symbols Ω and λ taken from PITMAN, 1962, p. 454, instead of SCHULER's u and φ.

whereas the left side also comprises instrument parameters:

b = gyro angular momentum (SCHULER uses J)

m = equivalent point mass of unbalance

g = earth gravity acceleration

a = distance between m and suspension point

$\dot{\Omega}$ = earth rotation rate (SCHULER uses u)

λ = geographic latitude (SCHULER uses φ)

The constant R/g is a constituent of the well-known expression for the oscillation period of an "earth pendulum" (cp. 1.3 -(4)):

$$T = 2\pi\sqrt{\frac{R}{g}} = 84{,}4 \text{ min} \qquad 3.1 - (2)$$

However, as BELL (1968, p. 507) rightly notes, both sides of eq. 3. -(1) can be multiplied by g, and that shows that the instrument parameters are linked with earth parameters of geometry only, i.e. Ω, λ, and R. Whatever the value of g, within limits indicated further on, the adjusments once made on the instrument remain valid. That means, that it will not deviate from pointing in the right direction due to vehicle accelerations once it has settled, whatever the value the value of g. Only the period of oscillation following a disturbance will vary with g.
(Of course, even academically speaking, g cannot be allowed to assume just any value. The upper limit would be a technical limit, dictated by what the instrument suspension could bear. But there is a lower limit of a practical nature, beyond which settling time would be intolerably long; and of a theoretical nature, beyond which the angular excursions required for the north-keeping torque no longer permit subsituting the angles for the sine or cosine functions of the angles).

Although Schuler recognized that the north-seeking mechanism doesn't require a specific gravity torque value, he did work with the idea of a specific gravity torque value to eliminate the disturbances caused by horizontal accelerations. This, we think, is not quite the right way to state the principle, although it does not really matter for practical purposes when you deal with a fixed g-value.
It is not the gravity torque, that eliminates the acceleration sensitivity, but a specific instrument that is designed properly to be acceleration insensitive cannot but have a specific gravity torque when the g-value is given.

If the g-value changed, the gravity torque would change, but not the acceleration insentivity.

3.2 The gyro pendulum

The gyroscopic pendulum which SCHULER describes in his 1923 paper, is a different kind of instrument as far as gravity dependance is concerned. Whereas the compass utilizes only the direction of the gravity torque, but not its absolute value, to gain its north-seeking quality per se, the gyroscopic pendulum would not be insensitive to horizontal vehicle accelerations at all if there were no gravity. This pendulum needs a specific gravity torque to make it precess properly when following the earth curvature. Gravity is the "servo motor", slewing the pendulum's gyro exactly to the inertial rate of change of the local vertical.
It is not surprising, therefore, that in SCHULER's formula

$$\left(\frac{b}{m\ g\ a}\right)^2 = \frac{R}{g}, \qquad 3.2 - (1)$$

g no longer can be eliminated by multiplying both sides therewith.
g can be regarded as an instrument parameter.
If the gravity value were to change, one would either have to add an artificial torque computed from the horizontal velocity, or else to change one or more of the other instrument parameters. (Their meaning is the same as sub eq. 3.1 - (1).)

3.3 The physical pendulum

The third type of "instrument" Schuler treats in his paper of 1923 is the physical pendulum. The equation governing the relation between the pendulum's dimensions and the earth geometry were given by us in eq. 2.5.-(5) as

$$a = \frac{r^2}{R} \qquad 3.3 - (1)$$

or

$$R = \frac{r^2}{a}, \qquad (2)$$

and with eq. 2.5. - (4)

$$\frac{J}{ma} = R. \qquad (3)$$

here J = the body's moment of inertia (SCHULER uses θ)

m = the body's mass

a = the distance between the body's centre of inertia and its suspension point

If we divide both sides of 3.3 - (3) by g we get SCHULER's formula

$$\frac{J}{m\,g\,a} = \frac{R}{g} \qquad 3.3 - (4)$$

Just like with the gyro compass, so here g has nothing to do with the proper adjustment of the "instrument", but, with an instrument properly dimensioned and used in earth's gravity field, we get a specific natural frequency of the pendulum, which, if we disregard the inhomogeneity of that field, amounts to the Schuler period of 84,4 min.

3.4 Comparison of the three Schulerian instruments

The fact that the physical pendulum and the gyro pendulum act so differently with regard to gravity may seem curious at first glance.
But it can easily be explained by considering that, with the physical pendulum, the vehicle acceleration results in a pendulum excursion in the vertical plane <u>in the same direction as</u> the velocity vector, whereas with the gyro pendulum the excursion takes place in a vertical plane <u>at right angles to</u> the velocity vector. <u>Gravity pull</u> to restore the pendulum <u>only then</u> results in the pendulum's complying with the demand to follow the earth's curvature. If gravity were absent, only the lateral excursion of the gyro pendulum would build up.
For the sake of completeness, let us just briefly say, that to have at one's permanent disposal the true local vertical, of course one needs two pendulums with counterrotating rotors. (see e.g. SCHULER, 1923).
But this is a practical matter outside the scope of our present considerations.

3.5 Why Schuler did not eliminate gravity

At the end of his article of 1923, SCHULER gives a summary which shows why he introduced the 84 min. period into all the formulas describing the behaviour of the three different types of apparatus, whether the 84 min. period was relevant to the functioning of the device or not.

He thought he had found a few special cases corresponding to a general law which he tentatively formulated as follows (taken from SLATER's translation in Appendix A of PITMAN, 1962, page 453):

"An oscillatory mechanical system on whose center of gravity a central force acts will not be forced into oscillation by any arbitrary movement over a spherical surface about the center of force if its period of oscillation is equal to that of a pendulum of the length of the sphere's radius in the applied force field".

He did not say he had found the law, but said that obviously some general law lies behind it all. And he adds: "I still have to owe you, however, the general proof of the law". As we have stated in our Introduction, it is now known that there is no such law. In MAGNUS, 1973, page 295, we read: "Since then it has been possible to show that the fixed relation, suspected by Schuler, between the 84 min. period and insensivity to acceleration does not exist. There are systems with an oscillation period of 84 minutes which are not insensitive to acceleration, as well as acceleration-insensitive instruments that have other oscillation periods".

Whereas, of course, Schuler himself was aware that the fixed relationship between the 84 min. period and acceleration-insensitivity was only a supposition, other authors, endeavouring to explain the principle in a simple manner, took it for granted, or at least gave the reader the impression that it was granted. Presumably authors copied from authors, especially in the Anglo-Saxon literature area, without consulting Schuler's original work which was written in the German language. Perhaps, also, nobody really bothered to check it out for himself, since the original explanation has the beauty of simplicity, almost automatically precluding even questioning its validity.

3.6 Literature excerpts

Let us, in the light of the above-mentioned, present a few typical texts without any further comment. The reader by now will probably recognize the correct, the dubious, and the incorrect statements which we regard as typical for most handbooks and articles on the subject.

(1) "In practice, the inertial system is made to behave as if it were an 84-minute pendulum". -- KLASS, 1956, p. 7.

(2) [The condition for acceleration insensitivity, using our symbol J for the moment of inertia instead of Schuler's θ, is found to be, and we quote:]

$$"J = maR \quad \text{or} \quad \ell = \frac{J}{ma} = R \tag{31}$$

i.e. the mathematical length of the acceleration insensitive pendulum must be equal to the earth radius.
In a constant, parallel gravitation field the oscillation period of such a pendulum is

$$T_E = 2\pi\sqrt{\frac{R}{g}} = 84 \text{ minutes.} \tag{32}$$

To begin with, we find that the tuning condition (31) indeed is independant of the earth gravitation g. . . . Neither is it of consequence to the tuning condition (31) that the gravity field . . . is a central field with g decreasing with the square of the distance from earth centre. But (32) is only valid for a constant parallel gravity field. For a mathematical pendulum (31) is self-evident.
If it were possible to make such a pendulum its centre of inertia would always be at earth's centre, and one could make the suspension point travel to and fro over the earth without disturbing the pendulum's indication [of the vertical]. Also one immediately sees that formula (32) is no more valid, for earth gravity is zero at earth centre, and the oscillation period becomes infinitely large. The astonishing fact with eq. (31) however is, that the actual execution is of no importance, but only the correct tuning ratio". -- SCHULER, 1958, p. 46.

(3) "The pendulum must have an effective length equal to the earth's radius. This is Schuler tuning. The period of such a pendulum .... is 84,4 min. If by some means another device is made which oscillates with the same period, it is also Schuler-tuned". -- SAVANT, 1961, p. 19.

(4) "In Figure 1.5 [in which a spheroid physical pendulum is shown], if the angular acceleration of the pendulum about its pivot is just equal to the angular acceleration of the pivot about the earth's center due to horizontal motion, the pendulum will always remain vertical.
This condition exists if

$$\ddot{\theta} = \frac{\ddot{x}}{R} .$$

The condition expressed by [this] equation for the pendulum is called <u>Schuler tuning</u> and is the same as that for the inertial navigation system.

Furthermore, the inertial systems acts in the same way as the physical pendulum. This is true, irrespective of mechanization. ...
For example, if the pendulum is displaced from the vertical or the accelerometer displaced from the horizontal, both will detect a component of the vertical thrust acceleration. This effect will cause oscillations with a period of $1/2\pi \sqrt{R/A_R}$. If the horizontal velocity is low, $A_R \approx g$ and the period will be around 84 min for positions near the surface of the earth".
-- PITMAN, 1962, pp. 36, 37.

(5) " However, any device, which for a small perturbation angle $\delta\phi$ from the vertical undergoes a restoring acceleration $g\delta\phi$, and posseses a natural period equal to $2\pi\sqrt{a/g_0}$, where a is the earth's radius and g_0 is the magnitude of acceleration due to gravity at the earth's surface, will serve as a mechanisation of Schuler's earth-radius pendulum".
-- O'DONNELL, 1964, p. 43.

(6) "One of the essential problems in the field of vertical indication is to obtain a pendulous system with a period of 84 minutes. As was pointed out by Schuler (1923), this cannot be accomplished with a physical pendulum of reasonable size". -- ÅSTRÖM, 1965, p. 54.

(7) "With Schuler tuning, any displacement of the pendulum out of the vertical will result in an oscillation with a period of 84 min".
-- SANDRETTO, 1967, p. 6.

(8) "In inertial navigation, it is Earth that is in tune, and there is no possibility of altering the period by tinkering with the device.
I do not think that anyone can produce an inertial navigator with any other period, as e.g. the period of 'about thirty minutes' reported by Schuler in his 1923 paper, par. 31, as his best approach to an apparatus 'with full 84-minute period' ". -- BELL, 1968, p. 507.

(9) "When the pivot [of a pendulum] is part of a vehicle performing an accelerated horizontal motion the direction will deviate from the vertical. However, as we have already noted, a 'Schuler pendulum', with an oscillation time of 84.4 minutes, maintains a vertical indication, independent of the motion of the point of suspension." -- HECTOR, 1968, p.71.

(10) "If there is a gravitational field g parallel to the radius of the sphere then, if disturbed, the platform will oscillate with a period $T=2\pi\sqrt{R/g}$ ..." -- STRATTON, 1968, p. 509.

(11) "The particular virtue of Schuler, or 84-minute, tuning is simply that it eliminates transient or oscillatory errors which otherwise arise from vehicle acceleration". -- LEE, 1969, p. 268.

(12) "The practical execution of [a simple physical pendulum as an] indicator of the vertical founders when one tries to comply with the tuning condition

$$s = \frac{A}{mR} \quad {}^{*}). \tag{12.59}$$

It implies that the reduced pendulum length of the physical pendulum be equal to the earth radius R. The period of a physical pendulum thus tuned becomes

$$T = 2\pi\sqrt{\frac{R}{g}}\sqrt{\frac{A}{4A - 3C}}\,. \tag{12.60}$$

A rod-shaped pendulum with C=0 yields the value T = 42,2 minutes; for a pendulum with a spherical ellipsoid of inertia [A=C] we find T=84,4 min; and for 4A=3C we get $T \to \infty$. For a flattened out pendulum with 3C > 4A the equilibrium position z = 0 becomes instable. If one neglects in [an equation given earlier] the term that contains the gravity gradient one gets wrong results quantitatively, because then any form of the pendulum yields a period of 84,4 minutes". -- MAGNUS, 1971, p. 395.

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*) see our formula 2.5 - (5): $a = \frac{r^2}{R_{cr}} = \frac{mr^2}{mR_{cr}} = \frac{J}{mR}$ (=s)

4. PENDULUMS AND PERIODS

The following chapter has more detailed sub-divisions than contained in the general index in front of this paper. These sub-divisions are:-

4. PENDULUMS AND PERIODS

This chapter is intended to exhibit a number of different forms of pendulums in connection with the Schuler principle, or d'Alembert's double pendulum. In it we shall show what it means in terms of suspension point position, moment of inertia and radius of trajectory to make a pendulum insensitive to vehicle accelerations. Also the oscillation periods resulting will be shown, and the reader will see that there are pendulums with the 84 min. period which are not acceleration insensitive, as well as pendulums which are acceleration-insensitive but have a period different from 84 min. The idea is to give a survey rather than a profound discussion since the examples offered speak for themselves so to say. At any rate we would like to let the reader see for himself that the sweeping generalizations found in the textbooks and as quoted in our previous chapter can't be made to hold. There is much more to "gravity oscillators" than one would at first sight suspect. The books SCHULER, 1958, and MAGNUS, 1960 give interesting treatments. Of the many possible forms we will show only those that have some relation to the Schuler principle and the misconceptions often found in the textbooks.

4.1 Starting points

4.1.1 Conditions for acceleration-insensitive pendulums

Some two centuries ago d'ALEMBERT (1717 .... 1783) determined the conditions under which a physical pendulum will keep pointing towards a predetermined fixed centre point independant of accelerations of its suspension point. In 1923, SCHULER described the principle anew and showed how a physical pendulum -- suspended in a vehicle -- can be made to keep vertical to the earth's surface in spite of accelerations of the vehicle (see fig. 4.1.1 - 1).

If we impose the requirement $\alpha = \theta$ and $\ddot{\alpha} = \ddot{\theta}$ we arrive at the condition given by SCHULER (1923) in his eq. (5) as

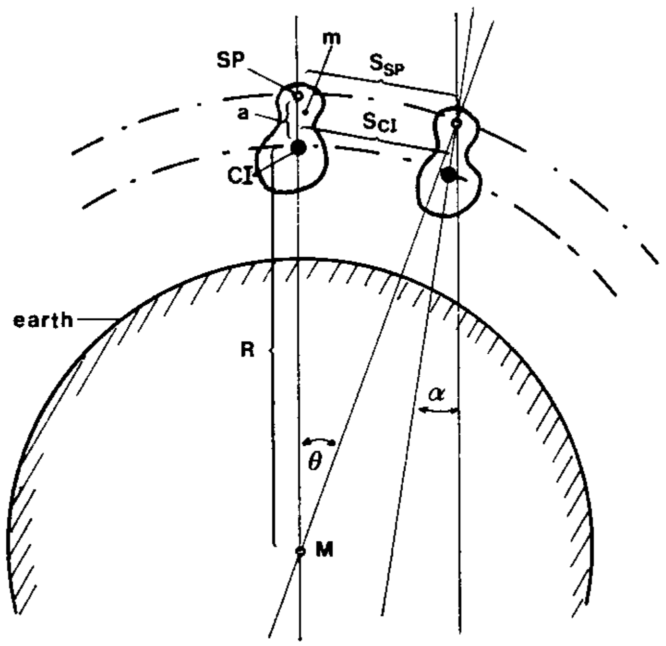
$$R = \frac{J}{ma}, \qquad 4.1.1 - (1)$$

where R = radius of trajectory

J = moment of inertia of the pendulous body

m = mass of the pendulous body

a = distance between the suspension point and the centre of inertia of the body.



M = centre of earth
CI = centre of inertia of mass m
SP = suspension point of mass m
R = radius of trajectory
θ = travel angle of SP
α = inertial rotation angle of m
S_{sp} = travel distance of SP
S_{CI} = travel distance of CI
a = distance between CI and SP

Fig. 4.1.1 - 1. Physical pendulum (arbitrarily shaped mass m) on circular trajectory around the earth.

He left it unmentioned whether J is the moment of inertia around SP ($J_{sp} = \int r^2 dm + ma$) or around CI ($J_{CI} = \int r^2 dm$), and also whether R is to be taken as the trajectory R_{sp} of the suspension point or as R_{CI} of the centre of inertia.

For a pendulum with dimensions as given in SCHULER's theoretical example (1923, p. 346, top left), namely with a radius of gyration r = 2m and a trajectory radius R $\simeq$ 6400 km, the value for the distance between SP and CI becomes a $\simeq$ 0,6 µm, a value so small that virtually $J_{sp} = J_{CI} = J$ and $R_{sp} = R_{CI} = R$.

This approximation, however, will no longer do for demonstration models with a relatively small trajectory radius R.
Because SCHULER, and later also other authors, use the rigid shaft pendulum, or even the string pendulum, to explain the functioning of acceleration insensitive pendulums, it appears to make sense to examine the significance of J_{sp}, J_{CI}, R_{sp}, R_{CI}, and the possible relationship between them.

The following types represent the range of pendulums to be considered for this purpose in ch. 4.2:

- the physical pendulum with 6 point masses
- " " " " 4 " "
- " " " " 2 " "
- the rigid shaft mathematical pendulum
- the mathematical string pendulum.

4.1.2 Pendulum periods in an inhomogeneous gravity field

Although oscillation period plays bo role whatever on the conditions making a pendulum acceleration insensitive, but is merely the consequence of placing the device in a gravity field, many authors identify correct trimming of the pendulum to make it acceleration insensitive with the Schuler-period, as is evidenced by frequent statements like "Every device with an 84-minute pendulum period is a Schuler vertical reference".
The simple, string or rigid, mathematical pendulum is usually said to have an oscillation period which is described by the formula

$$T = 2\pi\sqrt{\frac{\ell}{g}} \; . \qquad 4.1.2 - (1)$$

It is often overlooked, that this formula cannot be applied to just any mathematical pendulum at any location in the intra- or extraterrestrial gravity field. There are many situations, in which the pendulum will show a period of 84,4 min., but only one of these can be regarded as an acceleration insensitive vertical reference. So in ch. 4.3 we will give a survey of all the situations and the corresponding periods.
After that, in order to examine the relation between acceleration insensitivity and pendulum period we shall study the behaviour of the acceleration insensitive physical pendulum in the inhomogeneous gravity field. (ch.4.4).

4.2 Moment of inertia and radius of trajectory of a number of pendulums

4.2.1 General physical pendulum

Configuration

This type is represented as rigid body of arbitrary shape that is carried by a friction free universal joint at a point SP which is situated at a distance "a" from the centre of inertia CI of the body.

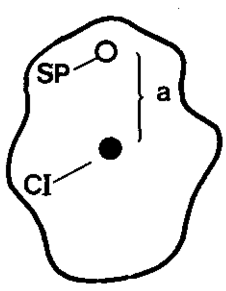


Fig. 4.2.1 - 1
Physical pendulum of arbitrary shape

Moments of inertia

With respect to CI and SP these are respectively

$$J_{CI} = \int r^2 dm \qquad 4.2.1 - (1)$$

and

$$J_{SP} = J_{CI} + ma^2 = \int r^2 dm + ma^2, \qquad (2)$$

where r = radius of gyration and m = total mass of the body.

4.2.2 Six point mass physical pendulum

Configuration

As far as inertial properties are concerned the arbitrarily shaped body of ch. 4.2.1 can always be substituted for by another body of arbitrary shape as long as the ellipsoids of inertia of both bodies are identical.
A system of six rigidly interconnected point masses is a very handy substitute.

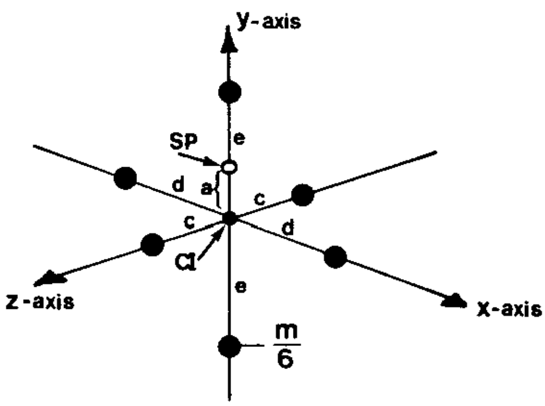


Fig. 4.2.2 - 1
Six point mass model body

The six point masses, each 1/6th of the total mass m, are positioned at the distances c, d, and e from the common centre of inertia on a system of three axes orthogonal to each other. These are then the three principal body axes. The whole system is suspended in SP at a distance "a" from CI.

Moments of inertia

The masses on the z-axis do not contribute to the moment of inertia when the body rotates around the z-axis. Thus

$$J_{CI} = 2\frac{m}{6}d^2 + 2\frac{m}{6}e^2 = \frac{1}{3}m\left(d^2+e^2\right). \qquad 4.2.2 - (1)$$

When the rotation axis passes through SP while being parallel to the z-axis, the moment of inertia will be

$$J_{SP} = J_{CI} + ma^2 = m\left[\frac{1}{3}\left(d^2+e^2\right) + a^2\right]. \qquad (2)$$

Radii of trajectory

According to th condition for Schuler tuning given in eq. 4.1.1 - (1) we try

$$R_{CI} = \frac{J_{CI}}{ma} , \qquad 4.2.2 - (3)$$

so that, with 4.2.2 - (2), we get

$$R_{CI} = \frac{1}{3}\frac{(d^2+e^2)}{a} . \qquad (4)$$

Now, from the geometry in fig. 4.1.1 - 1, we may write

$$R_{SP} = R_{CI} + a = \frac{1}{3}\frac{(d^2+e^2)}{a} + a \qquad (5)$$

Rearranging and multiplying this last expression by m/m, we see that

$$R_{SP} = \frac{m\left[\frac{1}{3}(d^2+e^2)+a^2\right]}{ma} \qquad (6)$$

which, when compared with 4.2.2 - (2), shows

$$R_{SP} = \frac{J_{SP}}{ma} . \qquad (7)$$

This means that using the moment of inertia around the suspension point results in the radius of trajectory of the suspension point, and conversely, using J_{CI} results in R_{CI}, in the formerly ambiguous expression of eq. 4.1.1 - (1).

4.2.3 Four point mass physical pendulum

Configuration

As long as we are only interested in rotations around the z-axis we can leave out the two masses lying on that axis. But to get the moment J_{sp} around the axis parallel to z right we will have to add the left-out masses to the rest. We thus arrive at a simple model with four point masses m/4 each and new, slightly reduced values of d and e.

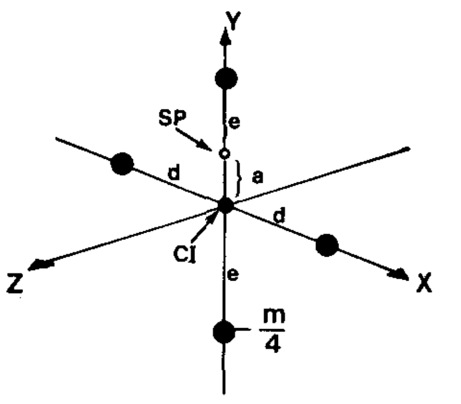


Fig. 4.2.3 - 1
Four point mass model pendulum

Again, like in fig. 4.2.2 - 1, there is a suspension point SP and the centre of inertia CI. The formulae corresponding to this model are as follows.

Moments of inertia

4.2.3 - (1)

$$J_{CI} = 2\frac{m}{4}d^2 + 2\frac{m}{4}e^2 = \frac{m}{2}(d^2+e^2)$$

(2)

$$J_{SP} = J_{CI} + ma^2 = m\left[\frac{1}{2}(d^2+e^2)+a^2\right]$$

Radii of trajectory

(3)

$$R_{CI} = \frac{J_{CI}}{ma} = \frac{1}{2}\frac{(d^2+e^2)}{a}$$

(4)

$$R_{SP} = \frac{J_{CI}}{ma} = \frac{1}{2}\frac{(d^2+e^2)}{a} + a.$$

4.2.4 Two point mass physical pendulum

Configuration

A third kind of model for a physical pendulum is that with two point masses. Many interesting situations can be studied with this simple model and it has been used in our chapter 2. Having its mass concentrated in one axis it is a substitute for a thin rod.

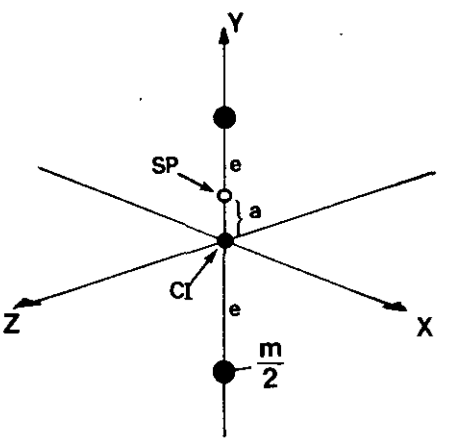


Fig. 4.2.4 - 1
Twin mass pendulum model

Just as with the previous models its SP is at a distance "a" from the CI, but its point masses now have half the total mass each.
The rotation axes to be considered ly in the xz-plane or run parallel to it.

Moments of inertia

4.2.4 - (1)

$$J_{CI} = 2\frac{m}{2}e^2 = me^2$$

$$J_{SP} = J_{CI} + ma^2 = m\left(e^2 + a^2\right) \qquad 4.2.4 - (2)$$

Radii of trajectory

$$R_{CI} = \frac{J_{CI}}{ma} = \frac{e^2}{a} \qquad (3)$$

$$R_{SP} = \frac{J_{SP}}{ma} = \frac{e^2}{a} + a \qquad (4)$$

A special case

If we choose to make a = e, i.e. to place SP at one of the point masses, we get

$$J_{SP} = 2ma^2 = 2me^2$$

$$R_{CI} = a = e$$

$$R_{SP} = 2a = 2e$$

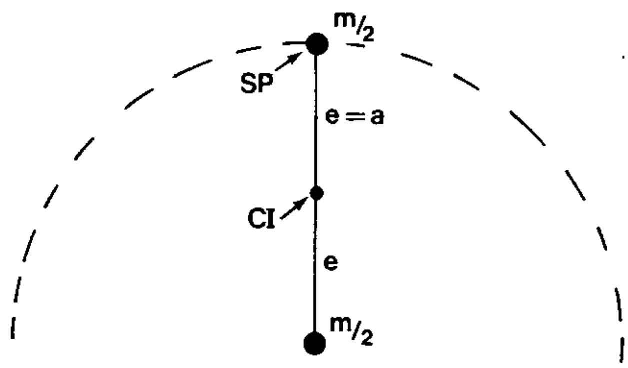


Fig. 4.2.4 - 2. "a" made equal to radius of gyration e

That means that the radius of the trajectory of the suspension point is equal to the total length of the pendulum. Thus the point mass not "captured in the vehicle" is at the centre of the trajectory and remains at rest. This situation resembles that of the rigid shaft mathematical pendulum which is to be treated in the next paragraph.
This special case also corresponds with an arbitrary physical pendulum suspended at its radius of gyration, a situation with interesting implications treated in SCHULER 1958, § 3.4.

4.2.5 Rigid shaft mathematical pendulum

Configuration

The rigid shaft mathematical pendulum is imagined to consist of a point mass (with zero radius of gyration) at the lower end of a mass-less rigid shaft, the upper end of which is hinged to the suspension SP. With this pendulum the distance "a" from its centre of inertia CI to its suspension point SP is equal to its length: $a = \ell$.

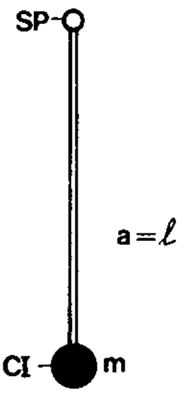


Fig. 4.2.5 - 1
Rigid shaft pendulum

Moments of inertia

Because the radius of gyration of a point mass is zero:

$$J_{CI} = mr^2 = 0 \qquad 4.2.5 - (1)$$

$$J_{SP} = ma^2 = m\ell^2 \qquad (2)$$

Radii of trajectory

Because the condition 4.1.1 - (1) and the eqs. 4.2.2 - (3) and (7) we find

$$R_{CI} = \frac{J_{CI}}{ma} = 0 \qquad (3)$$

$$R_{SP} = \frac{J_{SP}}{ma} = a = \ell. \qquad (4)$$

The radius of trajectory of the SP of such a pendulum thus is equal to its length, and the pendulum bob remains stationary at the centre of the trajectory.

4.2.6 Mathematical string pendulum

Configuration

This type of pendulum consists of a point mass with negligible radius of gyration ($r \approx 0$) hanging from a mass-less, nonrigid string that is fixed to the suspension point SP.

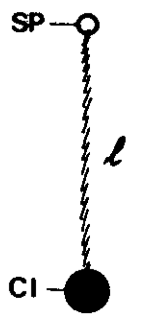


Fig. 4.2.6 - 1.
String pendulum

Moments of inertia

Again, as in 4.2.5, this pendulum has no moment of inertia with respect to CI. But, contrary to the rigid shaft type, this string pendulum neither has a useful moment of inertia around its suspension point SP. For one way of defining this moment would be as

$$J_{SP} = T_{SP} / \ddot{\alpha} \qquad 4.2.6 - (1)$$

where T_{sp} is the torque applied to the pendulum around its suspension point, and $\ddot{\alpha}$ the resulting angular acceleration of the pendulum. However, the string being flexible, no torque can be exerted via SP, and T_{sp} and $\ddot{\alpha}$ have no connection with eachother. So, in the light of what we shall be saying in par. 4.2.9 J_{sp} becomes meaningless. For the string pendulum we get

$$J_{CI} = mr^2 ; \text{ since } r = 0 \Rightarrow J_{CI} = 0 \qquad (2)$$

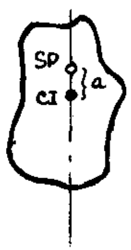
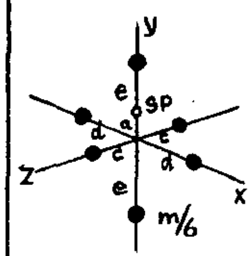
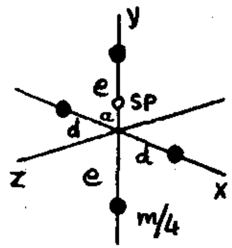
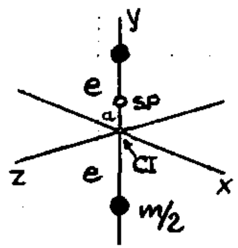
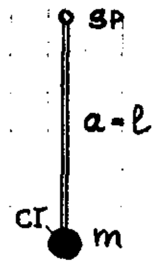
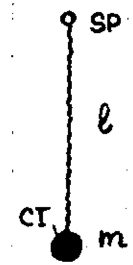
$$J_{SP} = \text{meaningless.} \qquad (3)$$

Radii of trajectory

$$R_{CI} = \frac{J_{CI}}{ma} = 0 \qquad (4)$$

$$R_{SP} = \frac{J_{SP}}{ma} = \text{meaningless.} \qquad (5)$$

4.2.7. Summary of the properties of the pendulums treated

TYPE OF PENDULUM ▷	Physical pendulum	6 point mass model	4 point mass model	2 point mass model	Rigid shaft math. pendulum	String type math. pendulum
CONFIGURATION						
REPRESENTS AN	arbitrary body	arbitrary body	thin flat disc	thin rod	ideal pendulum	ideal pendulum
MOMENTS OF INERTIA J_{CI} = J_{SP} =	$\int r^2 dm$ $\int r^2 dm + ma^2$	$\frac{1}{3} m(d^2+e^2)$ $m\left[\frac{1}{3}(d^2+e^2)+a^2\right]$	$\frac{1}{2} m(d^2+e^2)$ $m\left[\frac{1}{2}(d^2+e^2)+a^2\right]$	me^2 $m(e^2+a^2)$	zero $ma^2 = m\ell^2$	zero meaningless
RADII OF TRAJECTORY R_{CI} = R_{SP} =	J_{CI}/ma J_{SP}/ma	$\frac{1}{3}\,\frac{d^2+e^2}{a}$ $\frac{1}{3}\,\frac{d^2+e^2}{a}+a$	$\frac{1}{2}\,\frac{d^2+e^2}{a}$ $\frac{1}{2}\,\frac{d^2+e^2}{a}+a$	$\frac{e^2}{a}$ $\frac{e^2}{a}+a$	zero $a = \ell$	zero meaningless

4.2.8 Conclusions

1 Distinguishing between usable and unusable pendulums

The following types can be made to comply with the condition R = J/ma (see eq. 4.1.1 -(1)):

a) the rigid shaft mathematical pendulum
b) all forms of physical pendulum.

The string type pendulum does not physically exhibit a moment of inertia when a torque is applied through the suspension point.
As we shall see in par. 4.2.9 this renders it unusable even for model purposes.

2 Moment of inertia and radius of trajectory

As to the ambiguity mentioned just below eq. 4.1.1 -(1) the following can be said: -

a) The rigid shaft pendulum has but a moment of inertia with respect to SP. The moment of inertia around its centre of inertia is zero.
It follows that the condition R = J/ma can only mean $R_{sp} = J_{sp}/ma$.

b) With each of the other types of physical pendulum there are two possible interpretations of the condition R = J/ma:

$$R_{cI} = \frac{J_{cI}}{ma} \quad \text{and} \quad R_{sp} = \frac{J_{sp}}{ma}\,.$$

Usually one would work with R_{sp}, the radius of the suspension point's trajectory, since that point is given by the actual physical design. But working with R_{cI} can be handy at times, such as is done e.g. in ch. 6.3.

3 The relationship of R_{cI} and R_{sp} with "a" and r in a physical pendulum

Since with all physical bodies we can write $J = mr^2$ (where r is the radius of gyration), the Schuler condition R = J/ma can be written as either

$$R_{cI} = \frac{r^2}{a} \quad \Rightarrow \quad \frac{R_{cI}}{r} = \frac{1}{a/r} \qquad 4.2.8\ \text{-(1)}$$

or

$$R_{sp} = \frac{r^2}{a} + a \Rightarrow \frac{R_{sp}}{r} = \frac{1}{a/r} + \frac{a}{r}\,. \qquad (2)$$

When we draw a graph of $\frac{R}{r} = f\ (\frac{a}{r})$ for both these cases (fig. 4.2.8 - 1) we see that for every desired centre of inertia trajectory radius there is only one value of a/r, whereas every suspension point trajectory radius $R_{sp} > 2r$ can be achieved by two different values of a/r.

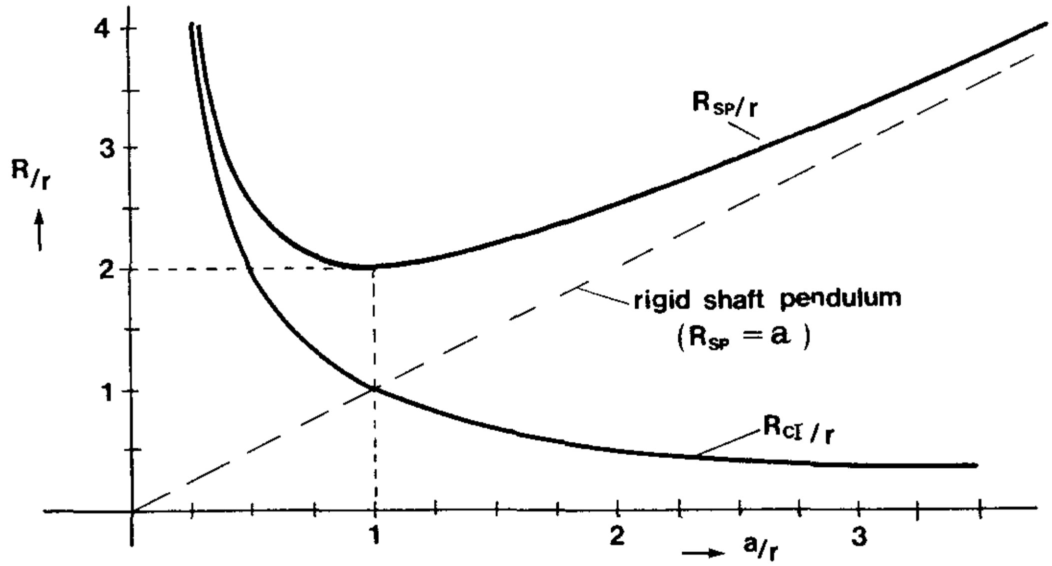


Fig. 4.2.8 - 1.
Trajectory radius R/r as a function of pendulosity a/r.
(r is the radius of gyration of the pendulum body)

We can consider two extreme ranges of a/r:

$$(1^{st})\ a \ll r \Rightarrow R_{SP} = \frac{r^2}{a} + a \simeq \frac{r^2}{a} = R_{CI}. \qquad 4.2.8 - (3)$$

This means that the suspension point SP almost coincides with the centre of inertia CI. SCHULER (1923) shows an example of this situation, and this example is mentioned right at the beginning of our ch. 5. (p.-71-).

$$(2^{nd})\ a \gg r \Rightarrow R_{SP} = \frac{r^2}{a} + a \simeq a. \qquad (4)$$

In this range the physical pendulum begins to approach the properties of the rigid shaft mathematical pendulum as can be seen from the asymptotic line (dashed) in fig. 4.2.8 - 1.

In the graph we also see that the smallest possible suspension point trajectory is $R_{SP} = 2a$. The corresponding configuration was shown in fig. 4.2.4 - 2. It coincides with the mode of suspension that yields the shortest possible period of oscillation of a given body in a given gravity field (see SCHULER 1958, §3.4.).

4 Relationship between the moments of inertia and the trajectory radii.

Combining the eqs. 4.2.2 - (3) and (7) we find that

$$\frac{J_{CI}}{J_{SP}} = \frac{R_{CI}}{R_{SP}} . \qquad 4.2.8 - (5)$$

4.2.9 Modifying trajectory radius by external torque

The situation described by eq. 4.2.5 - (4) for the rigid shaft pendulum, namely,

$$R_{SP} = \frac{J_{SP}}{ma} = a = \ell$$

means that this pendulum (provided that there are no gravitational forces acting on it) will always rotate around its centre of inertia CI (i.e. the centre of the bob) when its suspension point is subjected to an acceleration $\ddot{s}$ at right angles to the shaft. Thus the length of the pendulum is at once also the radius of trajectory of its suspension point.

If we wished to keep the pendulum parallel to its initial direction under said acceleration a torque would have to be applied according to

$$T_o = m\ell\ddot{s}, \qquad 4.2.9 - (1)$$

where m is the mass of the bob and $\ddot{s}$ the acceleration. This torque would change the radius of trajectory from $R_{sp} = \ell$ to $R_{sp} \rightarrow \infty$. By applying a smaller total torque

$$T_{tot} = m\ell\ddot{s} - T_c \qquad (2)$$

we can adjust the radius of trajectory to any desired value (see fig. 4.2.9 - 1).

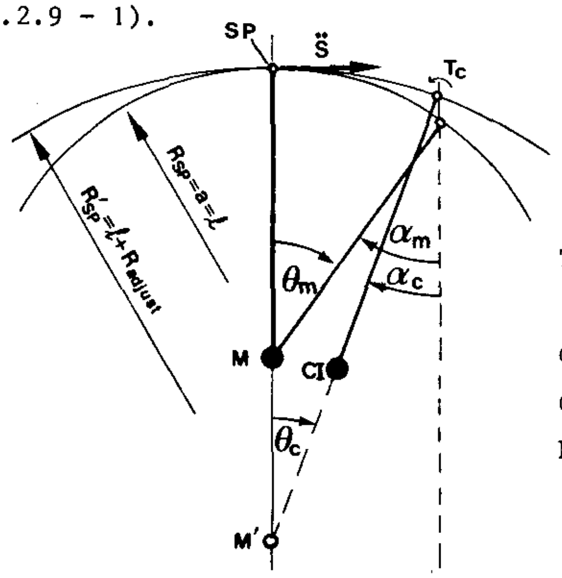


Fig. 4.2.9 - 1.
An external torque renders R_{sp} adjustable.

T_c = ext.torque (see eq.4.2.9 - (2))
$\ddot{s}$ = suspension point acceleration
α_m , θ_m = mechanically produced angles
α_c , θ_c = angles produced with T_{tot}
M, M' = centres of trajectory

According to eq. 4.2.9 - (2) T_c must be proportional to $\ddot{s}$ if T_{tot} is to be so too. The product $m\ell\ddot{s}$ can be called a mechanically produced torque, so T_c is an artificial externally applied counter torque. To make it proportional to $\ddot{s}$ we could use the signal of an accelerometer to feed an electromechanical torquer-circuit:

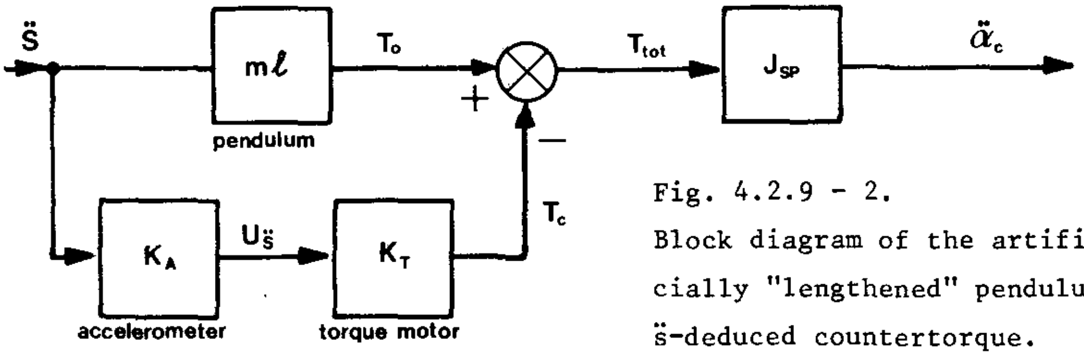


Fig. 4.2.9 - 2.
Block diagram of the artificially "lengthened" pendulum with $\ddot{s}$-deduced countertorque.

It is also possible to use an angular accelerometer for generating T_c. In the inertial navigation system described by HECTOR and ÅSTRÖM (see our fig. 5.1.2 - 1) this is done by differentiating the output signal of a rate gyro. The basic block diagram is as follows:

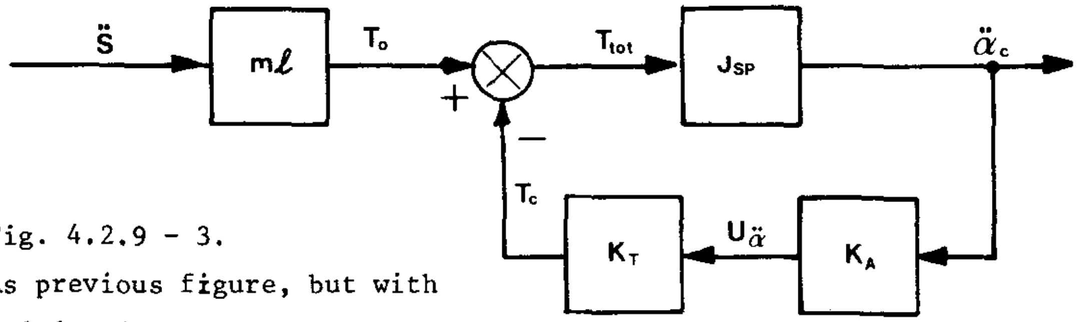


Fig. 4.2.9 - 3.
As previous figure, but with $\ddot{\alpha}$-deduced countertorque.

The abovementioned examples show the electronic solution to generating the desired countertorque. But it can also, in principle, be generated by purely mechanical means, namely e.g. by fixing a reversed pendulum to the top of the original one. The counter torque will then be automatically produced by the properly dimensioned added pendulum:

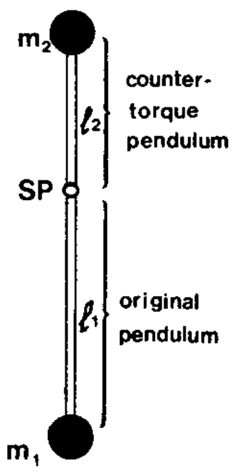


Fig. 4.2.9 - 4.
The mechanically generated
counter torque

The block diagram thereof is similar to fig. 4.2.9 - 2:

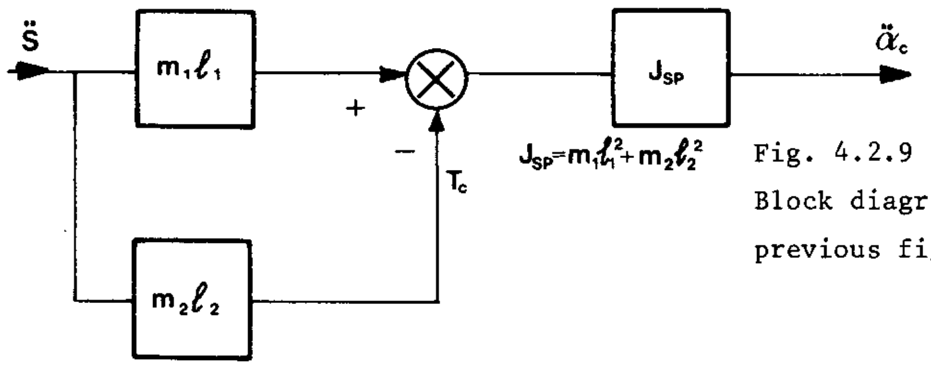


Fig. 4.2.9 - 5.
Block diagram of
previous figure

Of course this is identical with the dual point mass pendulum described in par. 4.2.4. In this context we would call it the torque reduction model, whereas fig. 4.2.9 - 3 represents a moment of inertia enhancement model. Purely mechanically it can be conceived of as a rigid shaft pendulum with added moment of inertia:

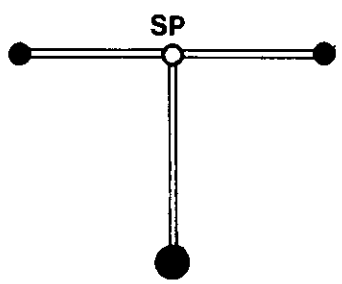


Fig. 4.2.9 - 6.
The mechanically installed additional
inertia

Its equivalent would be the four point mass body in par. 4.2.3.

4.3 Oscillation periods of various mathematical pendulums

Having discussed the conditions that lead to physical bodies being "Schuler tuned" we shall now examine the oscillation periods of such bodies in a gravity field.

4.3.1 Pendulums in a homogeneous gravity field

If we interpret the restoring torque or force as a sort of spring reaction we can calculate the oscillation periods of bodies like those compiled in 4.2 by means of the well-known formulae of inertia-spring systems:

$$T = 2\pi\sqrt{\frac{J}{\hat{S}}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{m}{S}} \;; \qquad 4.3.1 - (1)$$

Here $J = mr^2$ = moment of inertia of the body

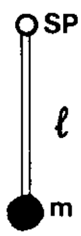
$\hat{S}$ = Torque/α = angular spring rate

m = mass of the oscillating body

S = linear spring rate involved

T = oscillation period.

For the rigid shaft mathematical pendulum with small excursion angles in a homogeneous gravity field we get



$$T = 2\pi\sqrt{\frac{J}{\hat{S}}} = 2\pi\sqrt{\frac{m\ell^2}{mg\ell}} = 2\pi\sqrt{\frac{\ell}{g}} \,. \qquad (2)$$

As long as a given pendulum is short compared with its distance to the centre of the gravity field it will comply reasonably with this formula. But with lengths comparable to the distance of its bob from said centre not only the magnitude but also the directional divergence of the field near the bob begins to play a significant role even when the amplitudes of oscillation are very small (see next two chapters).

4.3.2 Internal and external earth gravity field

The form of the gravity field of the idealized earth, which is given in our par. 1.1, shall be used in our following discussion of the periods of pendulums with which the inhomogeneity cannot be neglected.

The main characteristics of this field are repeated here:

Inside the earth : $g = f_1(R) = \frac{g_0}{R_0} \cdot R$ 4.3.2 - (1)

Outside the earth: $g = f_2(R) = g_0 R_0 \cdot R^{-2}$. (2)

Herein g_0 = gravity acceleration at earth surface = 9,81 m/s^2

R_0 = earth radius = 6372 km

R = distance from the pendulum's centre of inertia to the centre of the earth.

4.3.3 Mathematical pendulum with bob at earth surface

Situation:

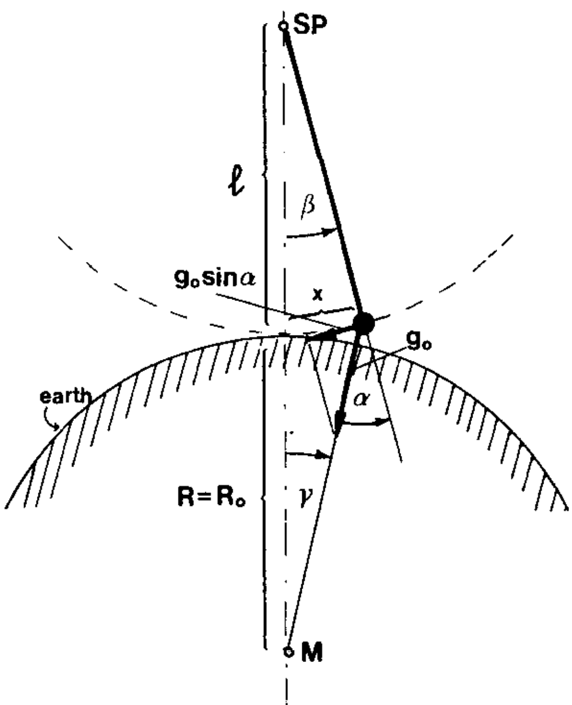


Fig. 4.3.3 - 1. Pendulum with bob at earth surface.

The pendulum has the length ℓ and is hung from its suspension point SP such that its bob grazes the earth suface. This means, that according to any of the formulas 4.3.2 -(1) or -(2) its bob experiences a gravity pull of the order g_0.

Ouside its equilibrium position the restoring force acts on the bob:

$$F = mg \cdot \sin\alpha . \qquad 4.3.3 - (1)$$

If we limit our considerations to very small excursions β we may write

$$F = mg_o\alpha. \qquad 4.3.3 - (2)$$

Now

$$\alpha = \beta + \gamma, \qquad (3)$$

and, with x as the horizontal excursion of the bob,

$$\beta = \frac{x}{\ell}; \qquad \gamma = \frac{x}{R_o} \qquad (4)$$

o that

$$\alpha = \frac{x}{\ell} + \frac{x}{R_o} = x \cdot \frac{\ell + R_o}{\ell \cdot R_o} \qquad (5)$$

and

$$F = mg_o \frac{\ell + R_o}{\ell \cdot R_o} x. \qquad (6)$$

The restoring force being proportional to the excursion, the ratio F/x can be regarded as a spring constant

$$S = \frac{F}{x} = m\frac{g_o}{R_o} \cdot \frac{\ell + R_o}{\ell}. \qquad (7)$$

The well-known expression for the oscillation period of a mechanical mass-spring system can now be applied, and we get

$$T = 2\pi\sqrt{\frac{m}{S}} = 2\pi\sqrt{\frac{R_o}{g_o}} \cdot \sqrt{\frac{\ell}{\ell + R_o}}. \qquad (8)$$

The first part of this equation is equal to the Schuler constant T_o (cf. eq. 1.3 - (1) and (5)), and the period becomes

$$T = T_o\sqrt{\frac{\ell}{\ell + R_o}}. \qquad 4.3.3 - (9)$$

For very short pendulums ($\ell << R_o$) we get the familiar formula

$$T = T_o\sqrt{\frac{\ell}{R_o}} = 2\pi\sqrt{\frac{\ell}{g_o}}. \qquad (10)$$

When the length equals the earth radius ($\ell = R_o$) the period is

$$T = T_o \cdot \sqrt{\frac{1}{2}} = \frac{84{,}4\,\text{min}}{\sqrt{2}} = 59{,}7\,\text{min}. \qquad 4.3.3. - (11)$$

When the pendulum's length finally becomes much larger than earth radius ($\ell >> R_o$) the period no longer depends on the actual length ℓ but is the Schuler constant:

$$T = T_o = 84{,}4\ \text{min}. \qquad (12)$$

Graphically we can summarize these circumstances as follows:

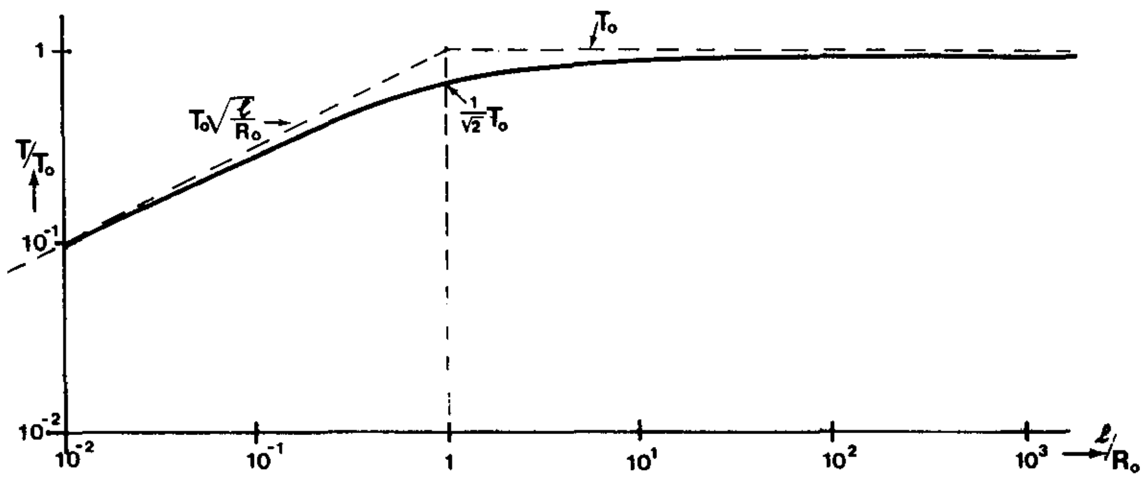


Fig. 4.3.3 - 2. Periods of earth surface pendulum
(Also valid for eq. 4.3.4 - (2) if R is substituted for R_o).

4.3.4 Mathematical pendulum with bob in the internal field of the earth

Situation:

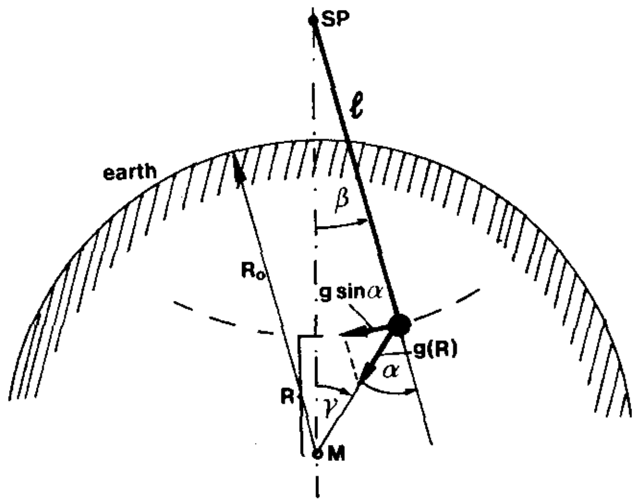


Fig. 4.3.4 - 1. Pendulum bob in the internal field

Except for the role of R_o, the geometry here is idential with that of the previous paragraph. We can copy eq. 4.3.3 - (8) substituting R, g for R_o, g_o, and the oscillation period becomes:

$$T = 2\pi \sqrt{\frac{R}{g}} \cdot \sqrt{\frac{\ell}{\ell + R}} . \qquad 4.3.4 - (1)$$

According to eq. 4.3.2 - (1) the ratio R/g is a constant inside the earth, so using the Schuler constant (cf. e.g. 4.3.3 - (12)), we have

$$T = 2\pi \sqrt{\frac{R_o}{g_o}} \cdot \sqrt{\frac{\ell}{\ell + R}} = T_o \sqrt{\frac{\ell}{\ell + R}} \qquad 4.3.4 - (2)$$

for the oscillation period of a mathematical pendulum with its bob at an arbitrary place in the internal gravity field of the earth. (Note

that it is the asymptotic value for small angular excursions β).
The discussion of this formula is the same as in the previous paragraph, and fig. 4.3.3 - 2 also is valid. It also holds for R = 0, where g = 0 and $\alpha \simeq \gamma \simeq 90^{\circ}$. This is shown in the next paragraph by using dg/dR.

4.3.5 Mathematical pendulum with bob at earth's centre.

Situation:

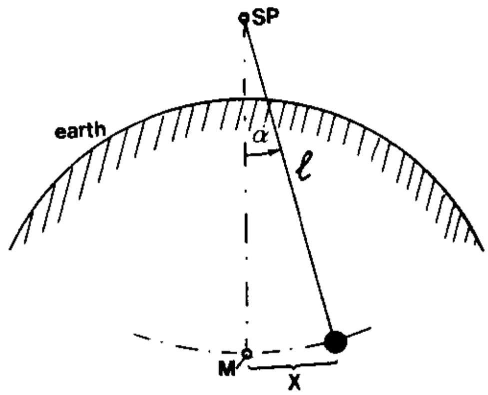


Fig. 4.3.5 - 1. Pendulum bob at earth centre

At very small excursion angles α the gravity force F trying to move the bob back to its equilibrium position following a disturbance will always act at right angles to the shaft. Its magnitude is

$$F = mg = m\frac{dg}{dx}x\,, \qquad 4.3.5 - (1)$$

where x is the linear excursion of the bob.

According to eq. 4.3.2 - (1) the field strength is proportional to the distance from the centre, so we have

$$\frac{dg}{dx} = \frac{dg}{dR} = \frac{g_0}{R_0} \qquad (2)$$

and

$$F = m\frac{g_0}{R_0}x. \qquad (3)$$

Compare eq. 4.3.3 - (7) for the idea of the spring constant, for which we now get

$$S = \frac{F}{x} = m\,\frac{g_0}{R_0}\,. \qquad 4.3.5 - (4)$$

From this the period follows as

$$T = 2\pi\sqrt{\frac{m}{S}} = 2\pi\sqrt{\frac{R_0}{g_0}} = T_0 = 84{,}4\,min. \qquad (5)$$

Note that T has become independant of the pendulum length ℓ !

Here the fallacy of comparing a Schuler-tuned system with a mathematical pendulum with bob at the centre of the earth, by equating their oscillation periods, becomes very evident. For, a Schuler-tuned system flown at a very great hight will have a period much longer than T_0, whereas the comparable pendulum will always keep its 84,4 min period.

4.3.6 Mathematical pendulum with infinite length and bob in the internal field

From the formula (2) found in par. 4.3.4 it follows that for $\ell \to \infty$ dependance on ℓ vanishes:

$$T = T_0\sqrt{\frac{\ell}{\ell+R}} \longrightarrow T_0\,. \qquad 4.3.6 - (1)$$

An infinitely long pendulum with its bob at an arbitrary distance from the earth's centre, but within the internal field appears always to have an oscillation period of 84,4 min.

4.3.7 Point mass on a straight trajectory in the internal field

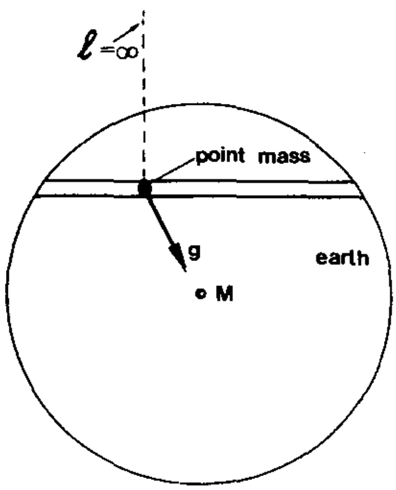


Fig. 4.3.7 - 1
Point mass in a straight "tunnel" through the earth

This situation is equivalent to having a pendulum of infinite length. The point mass thus must always oscillate with T_0. This fact, as derived from eq. 4.3.6 - (1), can also be shown by the following derivation.

We assume a point mass following a straight line through the internal field at an arbitrary distance R_{min} from the earth's centre (fig. 4.3.7 - 2). There shall be no friction.

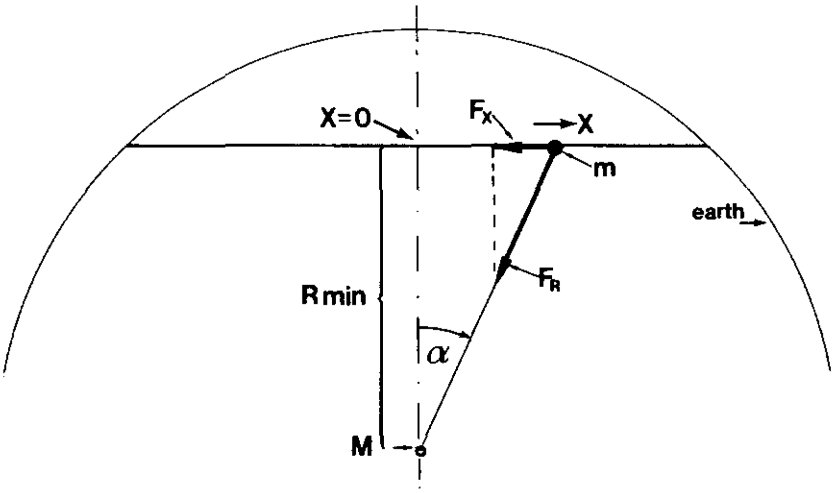


Fig. 4.3.7 - 2. Point mass on a straight trajectory

Its equilibrium position will be at the point of greatest proximity to M, the centre of the earth. We designate that point x = 0. The restoring force governing the movement of m is the x-component F_x of the gravity force F_g:

$$-F_x = F_g \sin\alpha = m g_R \sin\alpha. \qquad 4.3.7 - (1)$$

With eq. 4.3.2 - (1) we have

$$g_R = \frac{g_0}{R_0} R, \qquad (2)$$

and since sin α = x/R it follows that

$$-F_x = m \frac{g_0}{R_0} R \frac{x}{R} = m \frac{g_0}{R_0} x. \qquad (3)$$

Using, as in 4.3.3 - (7), the idea of the apparent spring constant

$$S = \frac{-F_x}{x} = m \frac{g_0}{R_0} \qquad (4)$$

we arrive at the predicted

$$T = 2\pi \sqrt{\frac{m}{S}} = 2\pi \sqrt{\frac{R_0}{g_0}} = T_0. \qquad (5)$$

Note that, apart from assuming frictionless movement of the mass, no mathematical assumptions as to small excursions or the like had to be made to find this result.

4.3.8 Orbital period of a point mass in an arbitrary plane in the internal field

The movements of the bob of a mathematical pendulum with infinite length were not restricted to a particular direction of the oscillatory movement. In general, the movement would be a Lissajons figure in a plane, and since the "stiffness" of the bob is equal in all directions, these figures will be circles or ellipses.

Because of the equivalence of a point mass on a straight line and an infinite pendulum, we can also envisage a point mass moving freely in a plane, describing circles or ellipses.
The orbital period must be equal to the oscillation period determined in the previous paragraph, namely T_o. Of course this follows directly from the linear superposition of two orthogonal oscillations, but it can also be shown using the equilibrium between the centripetal gravity force F_x and the centrifugal force F_c on a circle.

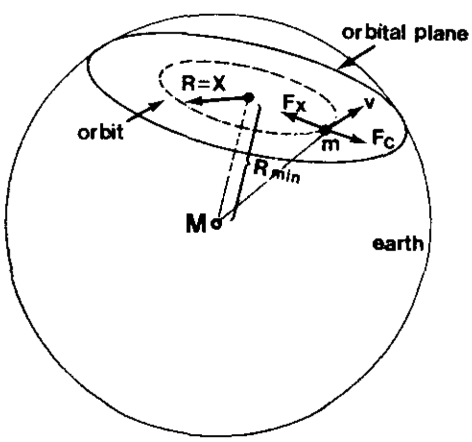


Fig. 4.3.8 - 1.
Point mass in an orbital plane

Again, like in eq. 4.3.7 - (3),

$$F_x = m \frac{g_o}{R_o} r_c , \qquad 4.3.8 - (1)$$

where now r_c is the orbital radius.

The centrifugal force is

$$F_c = m r_c \omega^2 \tag{4.3.8 - (2)}$$

with ω the arbital angular velocity. Equating F_c and F_x yields

$$\omega^2 = \frac{g_0}{R_0} = \left(\frac{2\pi}{T}\right)^2 \tag{3}$$

and thus

$$T = 2\pi\sqrt{\frac{R_0}{g_0}} = T_0 = 84{,}4\ min. \tag{4}$$

In this equation the orbital radius no longer appears, and neither does it contain the distance between the orbital plane and the earth centre M. Any infinite pendulum grazing the earth as in fig. 4.3.8 - 2 would also always have a circulation period of 84,4 min.

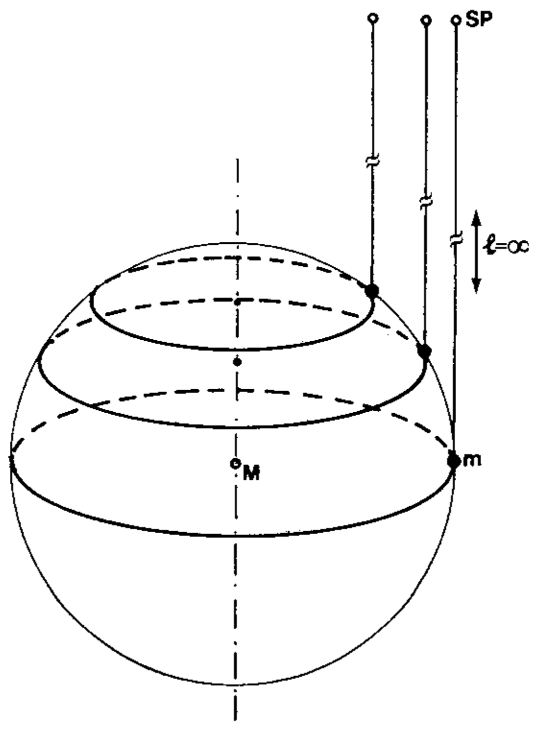


Fig. 4.3.8 - 2. Infinite pendulums grazing the earth: circulation period 84,4 min.

4.3.9 Mathematical pendulum in the external gravity field

The pendulums treated up to here moved with their bob in a gravity field in which the pull g was proportional to the distance R from the field centre M. Quite different situations arise in the external field where

$$g(R) = g_0 \frac{R_0^2}{R^2} \; . \qquad 4.3.9 - (1)$$

The general formula for the oscillation period at small excursion angles can be derived as follows (see fig. 4.3.9 - 1).

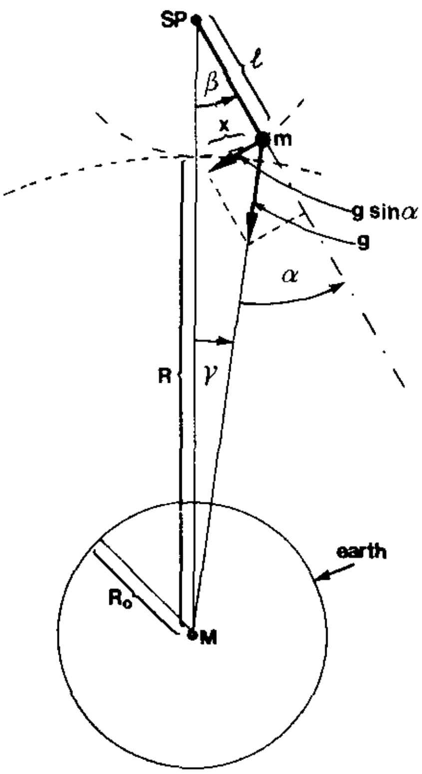


Fig. 4.3.9 - 1.
A pendulum in earth's external field

The restoring force is

$$F = mg \sin\alpha \qquad (2)$$

which for small excursions becomes

$$F = mg\alpha . \qquad (3)$$

With eq. 4.3.9 - (1) and the following geometrical relationships (fig. 4.3.9 - 1):

$$\left.\begin{array}{l} \alpha = \beta + \gamma \\ \beta = \frac{x}{\ell} \; ; \quad \gamma = \frac{x}{R} \\ \alpha = x\left(\frac{\ell + R}{\ell R}\right) \end{array}\right\} \qquad (4)$$

we get

$$F = m g_0 \frac{R_0^2}{R^2} \cdot \frac{\ell + R}{\ell R} \cdot x \,, \qquad 4.3.9 - (5)$$

$$S = \frac{F}{x} = m g_0 \frac{R_0^2}{R^2} \cdot \frac{\ell + R}{\ell R} \,. \qquad (6)$$

The oscillation period then becomes

$$T = 2\pi \sqrt{\frac{m}{S}} = 2\pi \frac{R}{R_0} \sqrt{\frac{1}{g_0} \cdot \frac{\ell R}{\ell + R}} \,. \qquad (7)$$

If we have a very short pendulum, i.e. $\ell \ll R$:

$$T = 2\pi \frac{R}{R_0} \sqrt{\frac{\ell}{g_0}} \,, \qquad (8)$$

which is the same formula as found for the short earth surface pendulum, eq. 4.3.3 - (10), multiplied by the distance ratio R/R_0.

If we have a relatively long pendulum ($\ell \gg R$) there remains

$$T = 2\pi \frac{R}{R_0} \sqrt{\frac{R}{g_0}} \,. \qquad (9)$$

Here the period is still a function of the distance R, but its dependance on the pendulum length ℓ has vanished.

Of course, inserting $R = R_0$ in all the above formulas yields the familiar expressions for the earth surface pendulum (ch. 4.3.3).

If we take a very long pendulum, the only possibility to tune it to T_0 = 84,4 min. in the external field is to choose $R = R_0$, that is to make it an earth surface pendulum. For any R of larger value the period only becomes longer. But a short pendulum (eq. 4.3.9 - (8)) can be tuned to 84,4 min. at any place in the outer field by making

$$\frac{R}{R_0} \sqrt{\frac{\ell}{g_0}} = \sqrt{\frac{R_0}{g_0}}$$

$$\ell = \frac{R_0^3}{R^2} \,. \qquad (10)$$

To give a numerical example let us take $R = 10\, R_0$.

Then

$$\ell = \frac{1}{100} R_0 \,.$$

This means, that a pendulum of approx. 64 km length will swing with the 84,4 min. period if its bob is at a distance of some 64 000 km from the earth centre. But by no means this one can be regarded as an acceleration insensitive vertical reference with respect to the earth.

4.4 Oscillation periods of Schuler-tuned physical pendulums

Whereas a mathematical pendulum by itself can only be made acceleration insensitive by placing its bob at the trajectory centre, it can be tuned to 84,4 min. under a variety of conditions. Contrariwise, a physical pendulum can, at least in principle, always be made acceleration insensitive; however, the ensuing oscillation period will only be equal to the Schuler period under exceptional conditions.
MAGNUS, 1971, gives the comprehensive formula (eq. 12.60 on p. 395) for this property of the physical pendulum*). In our treatise we will give a simple derivation for the twin mass body only.

4.4.1 Twin point mass body in a homogeneous gravity field

The configuration is the same as in our ch. 2, namely two point masses connected to each other by means of a rigid mass-free rod, pivoted at its suspension point SP at a distance "a" above its centre of inertia CI.

The total torque acting on this body is (cf. fig. 4.4.1 - 1):

$$M = M_1 + M_2 = F_1 r_1 \sin\alpha_o + F_2(-r_2)\sin\alpha_o . \qquad 4.4.1 - (1)$$

With

$$\left.\begin{array}{l} F_1 = F_2 = \frac{m}{2} g_o \\ r_1 = r - a \quad \text{and} \quad r_2 = r + a \\ \alpha_o \ll 1 \end{array}\right\} \qquad (2)$$

we get

$$M = \frac{m}{2} g_o \left[r - a - (r+a)\right]\alpha_o = -m g_o a \alpha_o . \qquad (3)$$

---

*) See translation of this passage at the end of our ch. 3.6 under number (12).

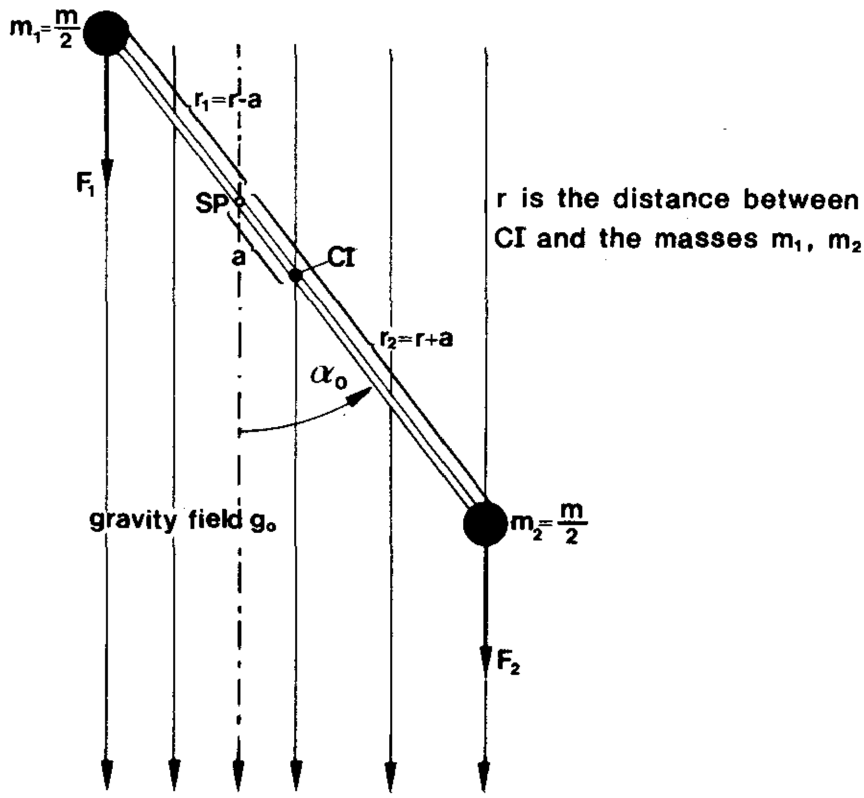


Fig. 4.4.1 - 1. Twin mass body in homogeneous gravity field

The well-known formula(see 4.3.1 - (1)) for the oscillation period of a spring-mass-system is

$$T = 2\pi\sqrt{\frac{J}{\hat{S}}} \, , \qquad 4.4.1 - (4)$$

and for the given pendulum

$$\left.\begin{aligned} J = J_{SP} &= mr^2 + ma^2 \\ \hat{S} = \frac{M}{\alpha_0} &= mg_0 a \, , \end{aligned}\right\} \qquad (5)$$

so that

$$T = 2\pi\sqrt{\frac{1}{g_0}\left(\frac{r^2}{a} + a\right)} \, . \qquad (6)$$

The pendulum was assumed to be acceleration insensitive, thus from eq. 4.2.4 - (4)

$$R_{SP} = \frac{r^2}{a} + a \qquad (7)$$

and

$$T = 2\pi\sqrt{\frac{R_{SP}}{g_0}} \, . \qquad (8)$$

If we choose for the trajectory radius R_{sp} the earth radius R_o we get the Schuler constant

$$T = 2\pi \sqrt{\frac{R_o}{g_o}} = T_o = 84{,}4 \text{ min.} \qquad 4.4.1 - (9)$$

Note that there is incongruity in the fact that we assumed a homogeneous gravity field while introducing a finite trajectory radius with eq. 4.4.1 - (7). But exactly this is the effect of the common practice of neglecting the gravity gradient when explaining the Schuler principle by means of a physical pendulum. MAGNUS, 1971, mentions this fact explicitly (see under number (12) at the end of our ch. 3.6).

4.4.2 Twin point mass body near earth surface

Inthe homogeneous gravity field absolute dimensions of the pendulum do not matter. But in the inhomogeneous field certain terms can or cannot be neglected according as the pendulum dimensions compare with the trajactory radius. In order to be able to give a simple calculation of the oscillation period we shall assume the trajectory radius to be earth radius, and the radius of gyration of the pendulum (which is half the distance between the two point masses) to be one meter. From this there follows a theoretical value of the suspension point distance "a" that is impractically small (see introduction to our ch. 5), but for the sake of our calculation example we must just accept it. To summarize

$$\left.\begin{aligned} R &= R_o = 6372 \text{ km} \\ r &= 1 \text{ m} \\ a &= r^2/R_o = 0{,}16 \cdot 10^{-6} \text{ m} \end{aligned}\right\} \qquad 4.4.2 - (1)$$

The calculation then proceeds as follows.

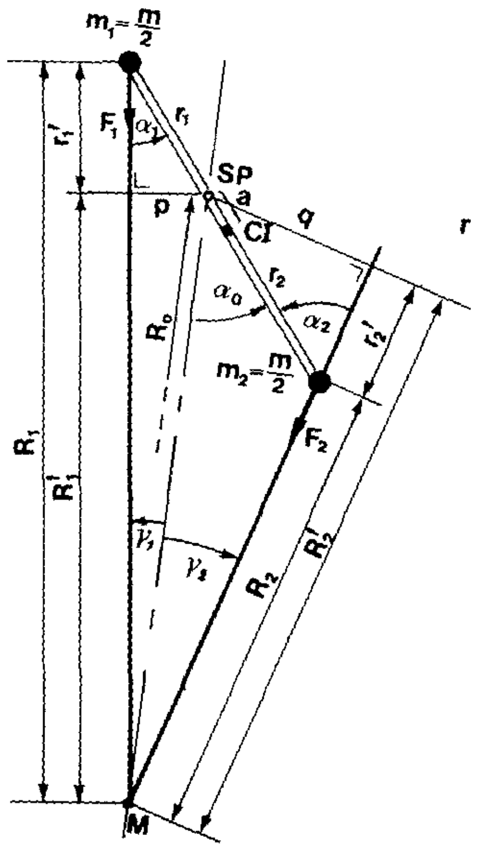


Fig. 4.4.2 - 1.
The pendulum in the inhomogeneous gravity field

From the fig. we can see

$$
\left.
\begin{aligned}
&R_1 = R_1' + r_1' &\qquad& R_2 = R_2' - r_2' \\
&R_1' = R_0 \cos\gamma_1 && R_2' = R_0 \cos\gamma_2 \\
&r_1' = r_1 \cos\alpha_1 && r_2' = r_2 \cos\alpha_2 \\
&r_1 = r - a && r_2 = r + a \\
&p = r_1 \sin\alpha_1 && q = -r_2 \sin\alpha_2 \\
&\text{and for small excursion angles } \alpha_0 \ll 1: && \\
&\gamma_1 = \frac{r_1}{R_0}\alpha_1 && \gamma_2 = \frac{r_2}{R_0}\alpha_2 \\
&\alpha_1 = \frac{1}{1 + r_1/R_0}\alpha_0 && \alpha_2 = \frac{1}{1 - r_2/R_0}\alpha_0 \\
&p = r_1\alpha_1 = r_1 \frac{1}{1 + r_1/R_0}\alpha_0 && q = -r_2\alpha_2 = -r_2 \frac{\alpha_0}{1 - r_2/R_0}
\end{aligned}
\right\} \quad 4.4.2 - (2)
$$

With the dimensional details 4.4.2 - (1) and with $\alpha_0 \ll 1$ the neglections now can be introduced:

$$\left.\begin{aligned} R_1' &= R_0 \cos\left(\frac{r_1}{R_0}\,\frac{\alpha_0}{1+r_1/R_0}\right) \approx R_0 \\ R_2' &= R_0 \cos\left(\frac{r_2}{R_0}\,\frac{\alpha_0}{1-r_2/R_0}\right) \approx R_0 \\ r_1' &= r_1 \cos\frac{\alpha_0}{1+r_1/R_0} \approx r_1 \\ r_2' &= r_2 \cos\frac{\alpha_0}{1-r_2/R_0} \approx r_2 \end{aligned}\right\} \quad 4.4.2 - (3)$$

As in par. 4.4.1 we determine the restoring spring constant S by first deriving the total torque M acting on the body:

$$M = M_1 + M_2 = F_1 p + F_2 q \tag{4}$$

where p and q are to be taken from 4.4.2 - (2) and the forces are

$$F_1 = \frac{m}{2} g_0 \frac{R_0^2}{R_1^2} \; ; \qquad F_2 = \frac{m}{2} g_0 \frac{R_0^2}{R_2^2} \, . \tag{5}$$

Making use of 4.4.2 - (2) and (3) we finally find the torque acting on the body as composed of a number of separately indentifiable contributions, namely

$$M = \left(\underbrace{\frac{m}{2} g_0 r_1}_{A} \cdot \underbrace{\frac{R_0^2}{(R_0+r_1)^2}}_{B} \cdot \underbrace{\frac{1}{1+r_1/R_0}}_{C} - \underbrace{\frac{m}{2} g_0 r_2}_{A} \cdot \underbrace{\frac{R_0^2}{(R_0-r_2)^2}}_{B} \cdot \underbrace{\frac{1}{1-r_2/R_0}}_{C} \right) \alpha_0 \tag{6}$$

A: torque in a homogeneous field; these terms also give the dimension of torque to the equation

B: dimensionless corrective terms introducing the difference in the <u>value</u> of g at m_1 and m_2

C: dimensionless corrective terms dealing with the difference in the in the <u>direction</u> of g at m_1 and m_2.

Due to the torsional stiffness being

$$\hat{S} = \frac{M}{\alpha_o} \qquad 4.4.2 - (7)$$

the bracketed part of 4.4.2 - (6) is the expression we are seeking. Rearranging and simplifying it we find:

$$\left.\begin{aligned} \hat{S} &= \frac{m}{2} g_o \left[\left(\frac{1}{1 + r_1/R_o} \right)^3 r_1 - \left(\frac{1}{1 - r_2/R_o} \right)^3 r_2 \right] \\ &= \frac{m}{2} g_o \frac{(1 - r_2/R_o)^3 r_1 - (1 + r_1/R_o)^3 r_2}{(1 + r_1/R_o)^3 \cdot (1 - r_2/R_o)^3} \, . \end{aligned} \right\} \qquad (8)$$

Because of 4.4.2 - (1) the denominator of this expression becomes practically unity:

$$\left[(1 + r_1/R_o)\,(1 - r_2/R_o) \right]^3 \approx 1 - 3 \cdot 10^{-13}, \qquad (9)$$

and we are left with

$$\hat{S} = \frac{m}{2} g_o \left[\left(1 - \frac{r_2}{R_o}\right)^3 r_1 - \left(1 + \frac{r_1}{R_o}\right)^3 r_2 \right] . \qquad (10)$$

Rewriting this by converting to the appropiate series and neglecting higher order terms (which may be done due to 4.4.2 - (1))we arrive at

$$\left.\begin{aligned} \hat{S} &= \frac{m}{2} g_o \left[\left(1 - 3\frac{r_2}{R_o}\right) r_1 - \left(1 + 3\frac{r_1}{R_o}\right) r_2 \right] \\ &= \frac{m}{2} g_o \left(r_1 - r_2 - 6\frac{r_1 r_2}{R_o} \right) . \end{aligned} \right\} \qquad (11)$$

Using $r_1 = r-a$ and $r_2 = r+a$ this becomes

$$\left.\begin{aligned} \hat{S} &= \frac{m}{2} g_o \left(r - a - (r+a) - 6\frac{(r-a)(r+a)}{R_o} \right) \\ &= \frac{m}{2} g_o \left(-2a - 6\frac{r^2}{R_o} + 6\frac{a^2}{R_o} \right) . \end{aligned} \right\} \qquad (12)$$

To make the pendulum acceleration-insensitive we must demand that

$$a = \frac{r^2}{R_o} . \qquad \text{(see e.g. 4.2.4 - (3))}$$

This yields

$$\hat{S} = \frac{m}{2} g_o \left(-2a - 6a + 6\frac{a^2}{R_o}\right) = -4mag_o\left(1 - \frac{6}{8}\frac{a}{R_o}\right) \qquad 4.4.2 - (13)$$

In our example 4.4.2 - (1)

$$\frac{a}{R_o} \approx 2{,}5 \cdot 10^{-14} \qquad (14)$$

so that effectively

$$\hat{S} \approx 4mag_o . \qquad (15)$$

Again, according to eq. 4.3.1 - (1), the oscillation period is

$$T = 2\pi\sqrt{\frac{J_{SP}}{\hat{S}}} = 2\pi\sqrt{\frac{mr^2 + ma^2}{4mag_o}} = \pi\sqrt{\frac{1}{g_o}\left(\frac{r^2}{a} + a\right)} . \qquad (16)$$

Since according to 4.2.4 - (4) we have

$$\frac{r^2}{a} + a = R_{SP} ,$$

and R_{sp} was assumed by us to be R_o, we finally arrive at

$$T = \pi\sqrt{\frac{R_{SP}}{g_o}} = \pi\sqrt{\frac{R_o}{g_o}} = \frac{1}{2} T_o = 42{,}2 \text{ min.} \qquad (17)$$

This result was predicted from the formula given by MAGNUS, 1971, p. 395, where he states that a thin rod (the equivalent of which is our twin point mass body) will oscillate with said period $1/2\ T_o$.

4.4.3 Twin mass body at a more general distance

Trying to derive the general expression for the oscillation period along the line we used in the previous paragraph leads to very complicated equations. If we lift the restriction of small pendulum dimensions but keep it as to small excursion angles, we can rewrite eq. 4.4.2 - (6) in a more general fashion:

$$M = \frac{m}{2} g_o r_1 \frac{\sin\frac{\alpha_o}{1 + r_1/R_o}}{\left[\cos\left(\frac{r_1}{R_o}\frac{\alpha_o}{1 + r_1/R_o}\right) + \frac{r_1}{R_o}\cos\left(\frac{\alpha_o}{1 + r_1/R_o}\right)\right]^2} - \frac{m}{2} g_o r_2 \frac{\sin\frac{\alpha_o}{1 - r_2/R_o}}{\left[\cos\left(\frac{r_2}{R_o}\frac{\alpha_o}{1 - r_2/R_o}\right) - \frac{r_2}{R_o}\cos\left(\frac{\alpha_o}{1 - r_2/R_o}\right)\right]^2}$$

4.4.3 - (1)

If we introduce the approximations 4.4.2 - (3) we arrive at the same result as in par. 4.4.2. But if we retain some higher order terms in working out 4.4.2 - (8) we finally arrive at

$$|\hat{S}| = 4mag_o\left[1 - \frac{5}{4}\frac{a}{R_o} + \frac{3}{4}\left(\frac{a}{R_o}\right)^2 - \frac{1}{4}\left(\frac{a}{R_o}\right)^3\right], \tag{2}$$

an expression that compares with 4.4.2 - (13).
With the same pendulum and trajectory dimensions as in the preceding paragraph the factor between the straight brackets appears to become slightly smaller than 1, namely approximately $1-10^{-13}$. The resulting oscillation period thus again turns out to be $1/2\ T_o$.

5. ACTIVE SYSTEM SCHULER REFERENCES

This chapter has the following additional sub-divisions:

5. ACTIVE SYSTEM SCHULER REFERENCES

Mechanizations like those shown in chapter 4 are impractical when one wants to design an acceleration-insensitive vertical reference for use on earth-bound vehicles. This is due to the large radius of curvature of the earth's surface which, in consequence of equation 2.5 - (5), either requires very large instrument dimensions (represented by the radius of gyration r in said equation) or very small displacements of the suspension point relative to the centre of inertia of the pendulum ("a" in eq. 2.5. - (5)). In his article of 1923 SCHULER of course mentions this fact, writing on page 346: "Such a physical pendulum, however, is completely impractical; for even with 20,000 kilograms and 4-meter radius of gyration for the mass, the separation of point of support and center of gravity is only 0.6 micron".*)
Incidentally, it is interesting to note that SCHULER mentions a mass of 20,000 kg. It is difficult for us to believe that he didn't realize that the actual value of the mass involved doesn't really matter (see our chapter 2.5, second paragraph following eq. 2.5 - (5)). It may be due to the fact that he was thinking in terms of gravity torques (see our chapter 3.1, last paragraph), inter alia for reasons like those we give in our ch. 3.5.

Also the same reasoning error we find in HECTOR, 1968, p. 72. Here the author uses the formula

$$J = mar$$

to determine the distance "a" between the suspension point and the centre of inertia. He then assumes values for both J and m, which is not necessary if we rewrite the formula:

$$a = \frac{J}{mR} = \frac{mr^2}{mR} = \frac{r^2}{R} \, .$$

For use in vehicles one has to resort to systems incorporating active measuring and controlling components like accelerometers, integrators, torques, gyros. The reasons for discussing the simple physical pendulum in SCHULER's original article, and in most textbooks since, are purely didactical or academic.

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*) Taken from SLATER's translation in PITMAN, 1962. The value of r = 4m is due to a translation inaccuracy. It ought to read r = 2m (cp. STRATTON, 1968, p. 508, lines 30 ... 32).

Schuler didn't have at his disposal the sensors, electronic circuits, and actuators we know now. In his gyroscopic pendulum he uses the torque produced when accelerating the system horizontally as a sensor to measure the acceleration and at the same time as an actuator to swing the pendulum sideways in a controlled manner. Together with gravity this sideway swing represents another actuator to precess the gyro according to the inertial angular velocity of the local vertical travelling along with the carrying vehicle.

This instrument, showing a cross relation between acceleration induced torque and gravity torque, cannot be designed so as to keep absolutely to the local vertical even in theory, as MAGNUS (1971) explains on page 397. Also, because of this cross relation, it does not lend itself to an explanation by means of a simple block diagram as the two kinds of instruments to be treated below. We should just like to add, quoting MAGNUS, 1971, that provided the angular momentum of the gyro employed is sufficiently large to allow neglection of the relevant moments of inertia of the entire instrument, the period of precession due to disturbances will practically amount to the Schuler period, i.e. 84,4 min at the surface of the earth, regardless of the gravity gradient.

The same holds with regard to the oscillation period of all electronically assisted gyroscopic Schuler references, as MAGNUS, 1971, shows in chapter 16.4.

The reasons for our including these instruments in our treatise is, (1) to complete the treatment of pendulums in ch. 4 with the electronically assisted kind, (2) to give block diagrams of these instruments in a manner we have not yet encountered elsewhere, (3) to show with these block diagrams the decoupling, or rather the absence of coupling, between the gravity-generated pendulosity and the condition of Schuler tuning, and (4) to give a background to the description of our classroom demonstration model.

5.1 The electronically assisted physical pendulum

In chapter 4, one of the pendulums discussed is the rigid shaft mathematical pendulum. For it to be an acceleration insensitive vertical indicator it would either have to have its bob resting at the centre of the trajectory, which of course is impractical for terrestrial applications, or it will have to be provided with compensating torques acting around its suspension point.
Inertial navigation systems using this latter kind of implementation of the Schuler principle have been built and tested (ÅSTRÖM, 1965, HECTOR, 1968). For use in our lecture series we set up a simple block diagram distinctly showing the independant natures of (1) the parameter adjustment to achieve acceleration insensitivity and (2) the resulting oscillation period.

5.1.1 The geometry of the vertical indicating system

Fig. 5.1.1 - 1 shows a rigid shaft mathematical pendulum suspended in a carriage above the surface of the earth (SP = suspension point).
It is to be moved from position (1) ($\theta = 0$) to position (2) (local vertical advanced by the angle θ). The earth is to be considerd non rotating for simplicity's sake, since earth rotation and movement of a vehicle can be combined to a resultant total inertial movement. So θ is at the same time an earthbound angle and an angle in inertial space.

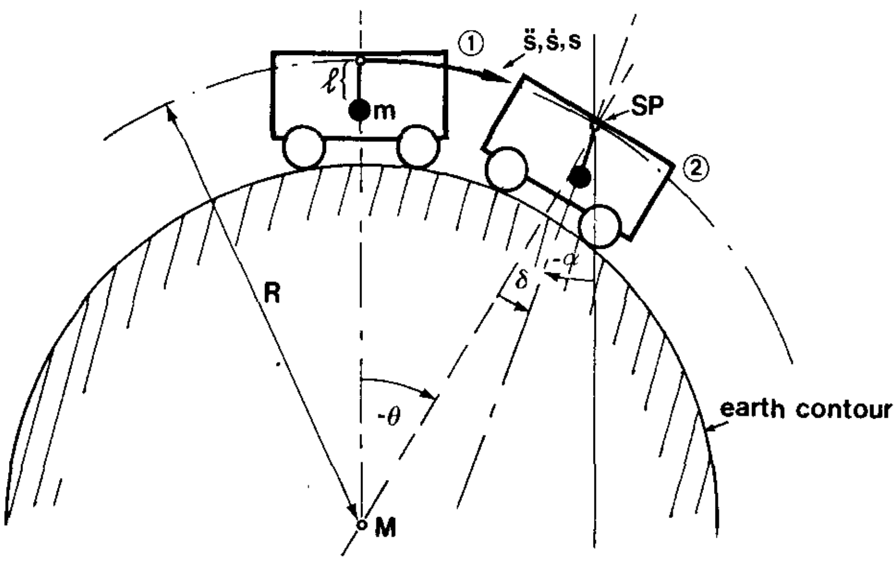


Fig. 5.1.1 - 1. Pendulum in carriage.
(Movements to the right and CCW angles and torques counted positive).

The objective is to make the pendulum follow the directional changes of the local vertical when the carriage moves. In terms of the angles shown in fig. 5.1.1 - 1 this means that at all times

$$\alpha = \theta. \qquad 5.1.1 - (1)$$

The torque that has to be applied to the pendulum around its suspension point to archieve this is proportional to $\ddot{\theta}$, and because of the geometric relationship

$$s = -R\theta \qquad (2)$$

it is also proportional to $\ddot{s}$. From this it follows that, by integration, the distance s travelled by the carriage can be computed from the time function of the torque.

5.1.2 The design of the pendulum

If no torque at all is applied to the pendulum the bob will start lagging behind upon horizontal acceleration of the suspension point. In terms of the angles of fig. 5.1.1 - 1 this means that at the start of the carriage (initial values $\theta = \alpha = \dot{\theta} = \dot{\alpha} = \delta = 0$) we get

$$\ddot{\alpha} > \ddot{\theta}. \qquad 5.1.2 - (1)$$

This leads to undesirable transient oscillations rendering the pendulum useless as a vertical reference.

If, on the other hand, we apply a torque so as to keep the pendulum shaft more parallel to itself ($\ddot{\alpha} = \dot{\alpha} = \alpha \rightarrow 0$), the pendulum mass would no longer lag behind, but advance in front of the movement of the carriage. In terms of the angles involved this means

$$\ddot{\alpha} \leqslant \ddot{\theta}. \qquad (2)$$

The torque necessary to achieve $\ddot{\alpha} = 0$ would have to be

$$T = m\ell\ddot{s}, \qquad (3)$$

where

m = pendulum bob mass

ℓ = pendulum length

$\ddot{s}$ = suspension point acceleration.

This arrangement corresponds with a symmetric dumb-bell body, like in chs. 2.4 and 4.2.6, suspended in its centre of inertia, except that the counter-torque produced by the "upper half" of the dumb-bell now has been replaced by the artificial torque produced by some feedback arrangement.

In order to achieve the desired relation

$$\ddot{\alpha} = \ddot{\theta} \qquad 5.1.2 - (4)$$

which is equivalent to 5.1.1 - (1), the torque given by 5.1.2 - (3) will have to be reduced by a specific amount. We can write

$$T_{total} = T_o + T_c , \qquad (5)$$

where T_c is the artificial countertorque. The angular acceleration of the pendulum with its moment of inertia mr^2 is

$$-\ddot{\alpha} = \frac{T_{tot}}{mr^2} = \frac{T_o + T_c}{mr^2} . \qquad (6)$$

According to 5.1.1 - (1) and (2) this becomes

$$-\ddot{\alpha} = \frac{T_o + T_c}{mr^2} = \frac{\ddot{s}}{R} . \qquad (7)$$

By means of 5.1.2 - (3) we substitute for $\ddot{s}$:

$$\frac{T_o + T_c}{mr^2} = \frac{T_o}{m\ell R} \qquad (8)$$

or

$$T_c = -T_o\left(1 - \frac{r^2}{R\ell}\right). \qquad (9)$$

For a rigid mathematical or point-mass pendulum, where $r = \ell$, we finally get

$$T_c = -T_o\left(1 - \ell/R\right). \qquad (10)$$

This means that the counter-torque to be added to make such a pendulum obey the prescription $\alpha = \theta$ must be less than the mechanical acceleration torque $T_o = m\ell\ddot{s}$ by an amount dictated by the ratio of the pendulum length and the earth radius.

One way to achieve this is given by ÅSTRÖM (1965) and HECTOR (1968). It consists of measuring $\dot{\alpha}$ by means of a rate gyro fixed to the pendulum and feeding the differenciated gyro output to a torquer affecting the pendulum swing (fig. 5.1.2 - 1).

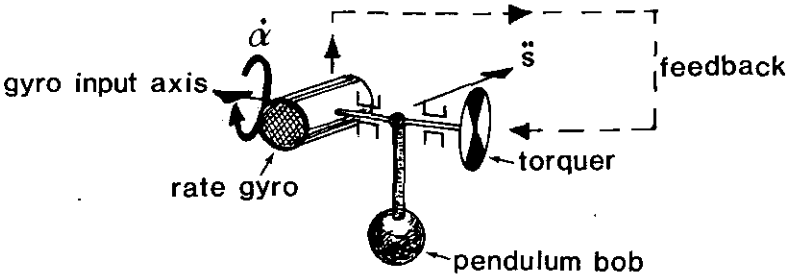


Fig. 5.1.2 - 1. The pendulum fitted with rate gyro and torquer.

We can derive the required instrument data in the following way: -
We insert 5.1.2 - (3) into 5.1.2 - (9) and get

$$T_c = -\ddot{s}m\ell\left(1 - \frac{r^2}{R\ell}\right) \qquad 5.1.2 - (11)$$

which, with 5.1.1 - (1) and (2), gives

$$T_c = \ddot{\alpha}m\left(R\ell - r^2\right). \qquad (12)$$

Now the feedback loop of fig. 5.1.2 - 1 has the following character:

$$\begin{aligned} T_{torquer} &= \tau\frac{d}{dt}\left(\dot{\alpha}K_G K_T\right) \\ &= \ddot{\alpha}\tau K_G K_T. \end{aligned} \qquad (13)$$

The torquer is to deliver the counter-torque, so we equate 5.1.2 - (12) and (13):

$$\tau K_G K_T = m\left(R\ell - r^2\right). \qquad (14)$$

Here the left side of the equation has the electromechanical conversion factors K_G of the rate gyro and K_T of the torquer, and the time constant τ of an electronic integrator, whereas the right hand side contains the purely mechanical magnitudes bob mass m, pendulum length ℓ, and pendulum radius of gyration r, and the earth radius R. (Compare also fig. 5.1.5 - 1).

5.1.3 Discussion of the pendulum design

There are two ways of looking at the nature of the torque feedback of the pendulum.

The first way has been shown in 5.1.2 and regards the torque produced by horizontally accelerating the pendulum mass ($T_0 = \ddot{s}m\ell$) as being partially counteracted by the artificially created torque. This leaves a net torque T_{tot} which is much more feeble than T_0, and thus the moment of inertia of the pendulum Jmech = mr^2 cannot be accelerated by more than the required $\ddot{\alpha}$. In formula:

$$\ddot{\alpha} = \frac{T_{tot}}{J_{mech}} = \frac{T_0 + T_c}{J_{mech}} \quad . \qquad 5.1.3 - (1)$$

This can be called the torque reduction model.

The other way of viewing the effect of the torque feedback loop is to regard it as introducing an extra moment of inertia. In this view only the acceleration torque T_0 acts on the pendulum, but it has to accelerate its greatly enlarged moment of inertia. In formula:

$$\ddot{\alpha} = \frac{T_0}{J_{tot}} \quad . \qquad (2)$$

In this expression the total moment of inertia turns out to be as follows: from eq. 5.1.3 - (1) we get

$$\frac{T_0 + T_c}{J_{mech}} = T_0 \frac{1 + T_c/T_0}{J_{mech}} = \frac{T_0}{J_{tot}} \quad , \qquad (3)$$

so that

$$J_{tot} = \frac{J_{mech}}{1 + T_c/T_0} \quad . \qquad (4)$$

Using 5.1.2 - (10) we can write

$$\frac{T_c}{T_0} = -1 + \frac{\ell}{R} \quad .$$

Inserting this into 5.1.3 - (4) finally yields

$$J_{tot} = J_{mech} \, R/\ell \qquad (5)$$

or, with $J_{mech} = mr^2$

$$J_{tot} = mr^2 \, R/\ell \, . \qquad (6)$$

If the pendulum is of the rigid shaft point mass type; where $r = \ell$, then

$$J_{tot} = m\ell R. \qquad 5.1.3 - (7)$$

We can call this the inertia enhancement model.

5.1.4 Interaction between torque feedback and gravity

Let the rigid mathematical pendulum have an angular displacement δ with respect to the vertical:

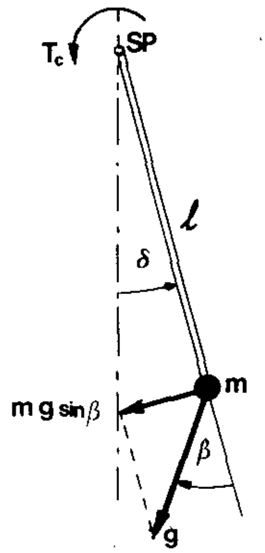


Fig. 5.1.4 - 1. The pendulum in the (inhomogeneous) gravity field

According to eq. 4.3.3 - (6) the gravity torque trying to restore the pendulum to its equilibrium position is

$$T_g = -m\ell^2 g\left(\frac{1}{\ell} + \frac{1}{R}\right)\delta \qquad 5.1.4 - (1)$$

in which the expression for T_g/δ or

$$|\hat{S}| = m\ell^2 g\left(\frac{1}{\ell} + \frac{1}{R}\right) \qquad (2)$$

can be recognized as having the properties of a torsional spring stiffness.

Together with the apparent moment of inertia given in 5.1.3 - (7) this will form an undamped oscillator. For its period of oscillation we find

$$T = 2\pi\sqrt{\frac{J}{\hat{S}}} = 2\pi\sqrt{\frac{m\ell R}{m\ell^2 g\left(\frac{1}{\ell}+\frac{1}{R}\right)}}$$

$$T = 2\pi\sqrt{\frac{R}{g}\cdot\frac{1}{1+\ell/R}} \quad . \qquad 5.1.4 - (3)$$

If the inhomogeneity of the gravitation field may be neglected, i.e. if $\ell/R \ll 1$, there remains the Schuler period:

$$T = 2\pi\sqrt{\frac{R}{g}} \quad . \qquad (4)$$

With a pendulum in which e.g. $\ell = 6$ cm (that is the order of magnitude indicated in ÄSTRÖM, 1965, p. 58), intended for earth surface use,

$$\frac{\ell}{R} = \frac{6\,cm}{6\cdot 10^8\,cm} = 10^{-8}, \qquad (5)$$

so that we can safely neglect the inhomogeneity effect.

If we use, not a single point mass pendulum but a pendulum with mechanically enlarged moment of inertia as depicted in fig. 5.1.4 - 2, the following will result.

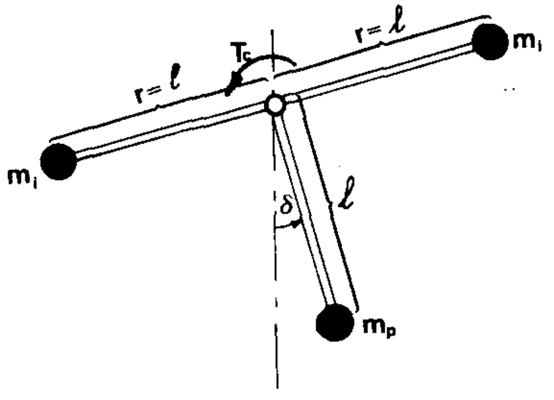


Fig. 5.1.4 - 2. Pendulum with enlarged moment of inertia

In a homogeneous gravitation field the masses m_i do not contribute any gravity torque. So the restoring "spring" constant, taken from 5.1.4 - (2) with $R = \infty$, will be

$$\hat{S} = m_p \ell g. \qquad 5.1.4 - (6)$$

To obtain the expression for the total moment of inertia we have to go back to 5.1.2 - (8), which we rewrite:

$$\frac{T_o + T_c}{(2m_i + m_p)r^2} = \frac{T_o}{m_p \ell R} \quad . \qquad (7)$$

The left side represents the ratio of the total (reduced) torque and the mechanical moment of inertia, whereas the right side shows that of the "mechanical" torque and the total apparent moment of inertia. This latter we need for calculating the period of oscillation:

$$T = 2\pi\sqrt{\frac{J}{\hat{S}}} = 2\pi\sqrt{\frac{m_p \ell R}{m_p \ell g}} = 2\pi\sqrt{\frac{R}{g}} \,. \qquad (8)$$

This means that in the homogeneous gravity field it makes no difference to the period to what extent the necessary moment of inertia is produced mechanically or artificially (cp. 5.1.4 - (4).

In the inhomogeneous field we do find a difference between the single point mass pendulum and the pendulum with the additional masses m_i. Besides the extra term ℓ/R in 5.1.4 - (3), which represents a slightly enlarged "stiffness", there will be a reduction of "stiffness" due to the fact that the masses m_i in fig. 5.1.4 - 2 deliver a negative torque because of their swinging around a position of unstable equilibrium. But, since the ratio of the total to the mechanical moments of inertia is very large for terrestrial applications (cp. 5.1.3 - (5)), the influence of this negative "spring" torque will not be noticeable in practical systems. So, for practical considerations, a physical pendulum will be as good as the theoretical rigid point mass pendulum.

5.1.5 The block diagram

Following the foregoing discussions it is now not difficult to draw a general block diagram of an electronically assisted vertical reference pendulum (fig. 5.1.5 - 1).

The transfer function is to be derived as follows (s representing the Laplace operator):

$$\begin{aligned}\alpha &= \frac{1}{s^2}\ddot{\alpha} = \frac{1}{s^2}\cdot\frac{1}{mr^2}T_{tot} = \frac{1}{s^2 mr^2}\left(T_o + T_c\right)\\ &= \frac{1}{s^2}\cdot\frac{1}{mr^2}\left(a_{in}\,m\ell + \dot{\alpha}K_G K_T \tau s\right)\\ &= \frac{1}{s^2}\cdot\frac{1}{mr^2}\left[(\ddot{s}+\ddot{\sigma})m\ell + \ddot{\alpha}K_G K_T\tau\right]\\ &= \frac{1}{s^2}\cdot\frac{1}{mr^2}\left(\ddot{s}m\ell + \ddot{\sigma}m\ell - \ddot{s}\frac{K_G K_T \tau}{R}\right).\end{aligned} \qquad 5.1.5 - (1)$$

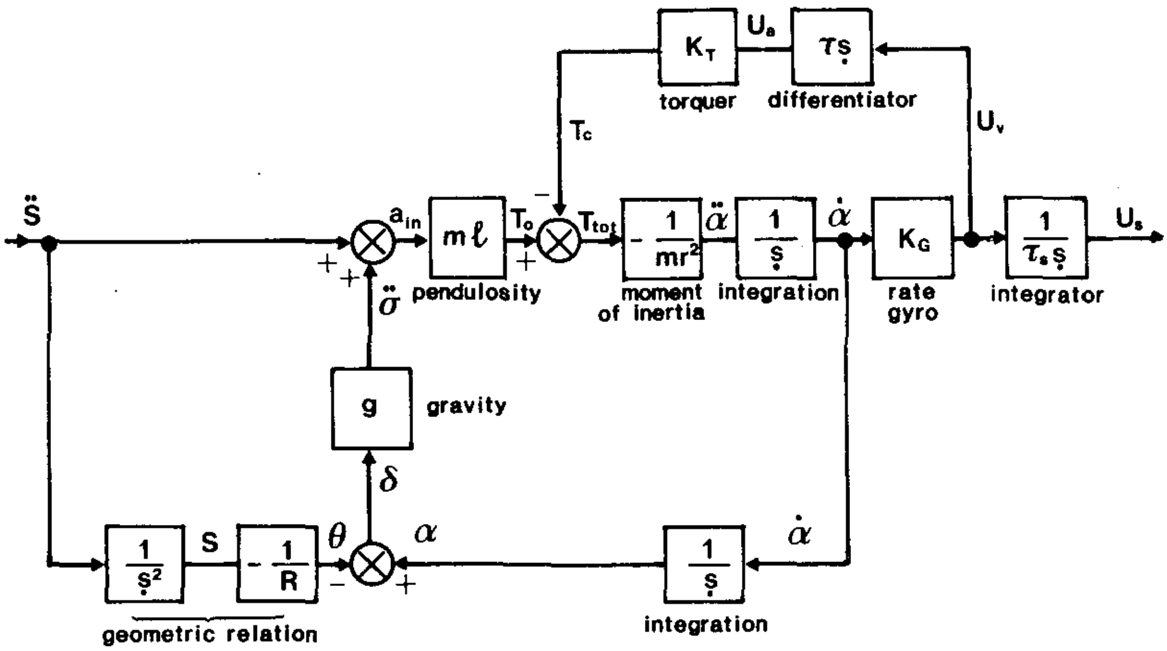


Fig. 5.1.5 - 1. Block diagram of an electronically assisted rigid mathematical pendulum.

$\ddot{s}$ = vehicle acceleration

a_{in} = input to acceleration sensitive system

T_o = mechanical torque

T_{tot} = *total torque*

$\ddot{\alpha}$ = angular accel. of pendulum

$\dot{\alpha}$ = angular velocity of pendul.

α = angular excursion of pendul.

U_v = voltage proportional to $\dot{s}$

U_s = voltage proportional to s

$\dot{s}$ = vehicle velocity

s = vehicle travel

U_a = voltage proportional to $\ddot{s}$

T_c = compensation torque

Θ = direction of local vertical

δ = deviation of pendulum from local vertical

$\ddot{\delta}$ = error acceleration

m = mass of pendulum bob

ℓ = length of pendulum (= R)

r = radius of gyration of pendul.

s = Laplace operator

K_G = gyro scale factor

τ_s = integrator time constant

τ = differentiator time constant

K_T = torquer scale factor

R = radius of vehicle trajectory (or earth radius)

g = gravity acceleration

What we want to achieve is $\delta = 0$ (see fig. 5.1.1 - 1). That means that

$$\ddot{\sigma} = g\delta = 0. \qquad 5.1.5 - (2)$$

In that case we can equate the angular excursion of the pendulum and the change θ of the local vertical:

$$\alpha = \theta. \qquad (3)$$

Relation 5.1.1 - (2) tells us that

$$\theta = -\frac{s}{R}, \qquad (4)$$

so that we can use 5.1.5 - (1) to get

$$-\frac{s}{R} = \frac{s}{mr^2}\left(m\ell - \frac{K_G K_T \tau}{R}\right)$$

and

$$-\frac{mr^2}{R} = m\ell - \frac{K_G K_T \tau}{R}$$

so that

$$K_G K_T \tau = m\left(R\ell - r^2\right), \qquad (5)$$

which is the same formula as 5.1.2 - (14), derived there using the geometric considerations and the required torques.

From the block diagram we can also derive the pendulum properties with $\ddot{s} = 0$ and $\ddot{\sigma} \neq 0$, and so check the correctness of the diagram.

With $s = \dot{s} = \ddot{s} = 0$ the pendulum excursion α becomes equal to δ, and the input acceleration $a_{in} = \ddot{\sigma}$. From the second line in 5.1.5 - (1) we get accordingly

$$\delta = \frac{1}{s^2}\cdot\frac{-1}{mr^2}\cdot\left(\ddot{\sigma}m\ell + \dot{\delta}K_G K_T \tau s\right) \qquad (6)$$

and, with $g\delta = \ddot{\sigma}$, and rearranged

$$s^2 \delta m r^2 = -\left(g\delta m \ell + \delta K_G K_T \tau s^2\right)$$

$$g\delta m \ell = -s^2 \delta\left(m r^2 + K_G K_T \tau\right)$$

$$s^2 = -\frac{g m \ell}{m r^2 + K_G K_T \tau} \quad . \qquad 5.1.5 - (7)$$

With eq. 5.1.5 - (5) this becomes

$$s^2 = -\frac{g m \ell}{m r^2 + m(R\ell - r^2)} = -\frac{g}{R} . \qquad (8)$$

For a harmonic oscillation we thus find

$$\omega = \sqrt{\frac{g}{R}} \qquad (9)$$

just like in eq. 5.1.4 - (4).

5.2 A classroom demonstration model

In order to study the electronically assisted pendulum system according to the ideas of ÄSTRÖM (1965) and HECTOR (1968) we designed and built a single axis model that can be wheeled along the floor and needs no more space than will be readily found in a lecture room. Its photograph is shown below, and side and top views in the next two drawings.

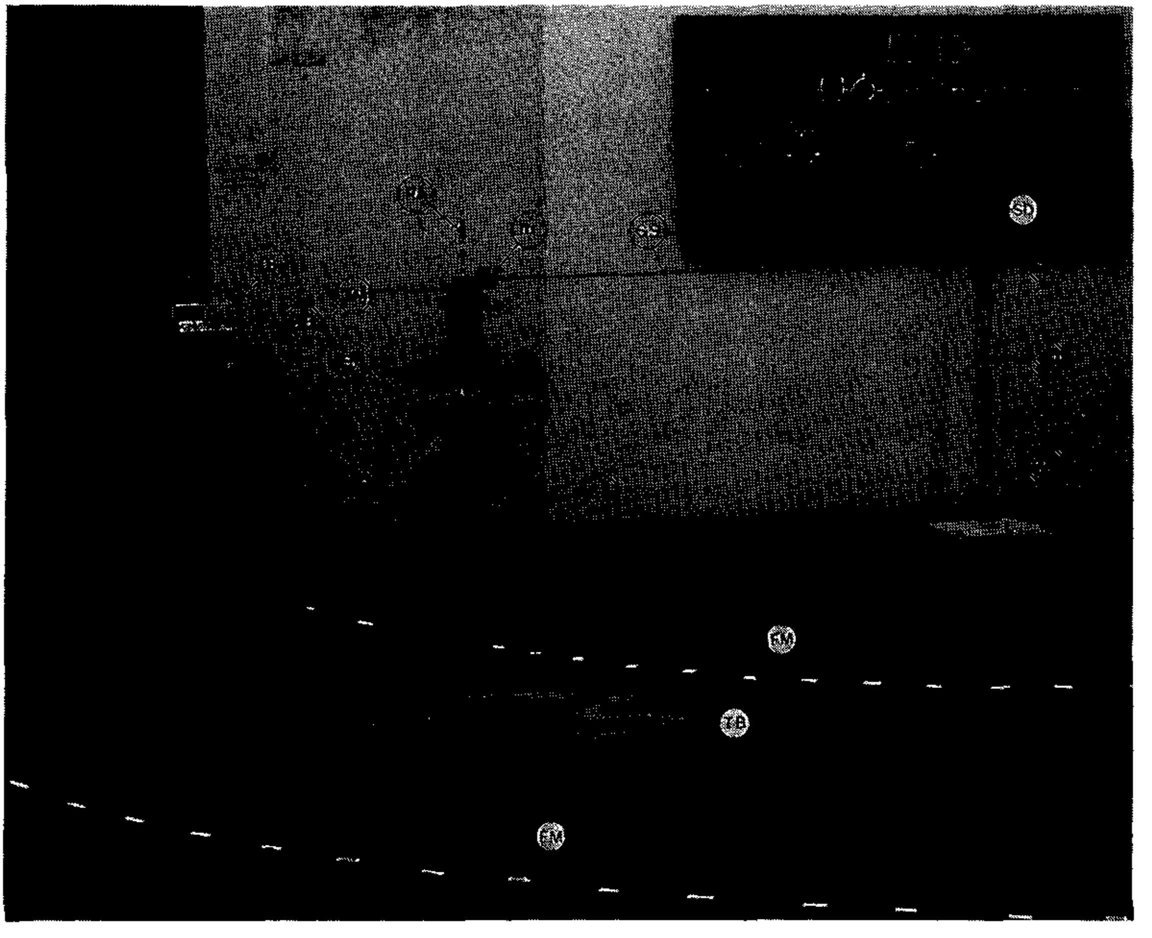


Fig. 5.2. - 1. View of the demonstration set-up.

<u>Legend to figs. 5.2 - 1, - 2, and - 3.</u>

CA = carriage
CB = control box
CP = centre pole
EB = electronics box
FM = floor markings
GS = gravity imitation string
IB = indicator box
m = pendulum bob mass
PA = pivot axis
PC = potentiometer cable
PM = ten-turn potentiometer
SD = string drum
SL = spirit level
TB = triangular bogies
TM = torque motor
WT = weight

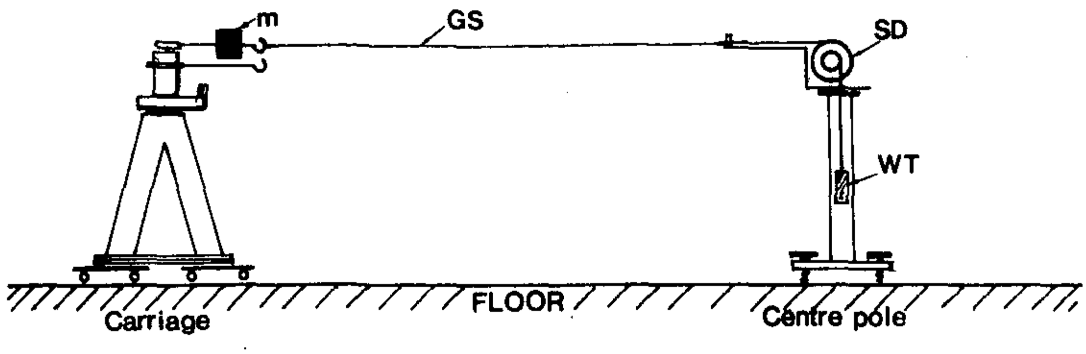


Fig. 5.2 - 2. Side view showing the way the carriage and the centre pole are connected up.

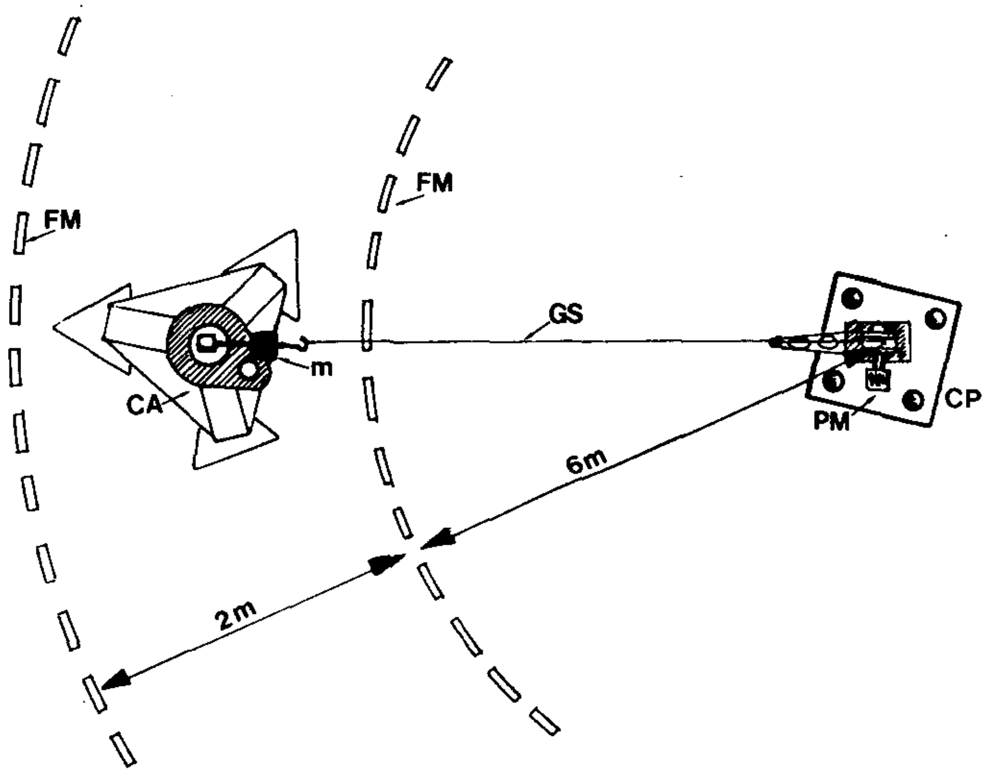


Fig. 5.2 - 3. Top view of the set-up with the carriage CA, the centre pole CP, and the floor markings FM.

(see legend page -85-)

It works similarily to the table-top passive model depicted in fig. 2.6 - 2. Only the "point mass" m_1 is present (called m in fig. 5.2. - 1), the torque exerted by mass m_2 of fig. 2.6 - 2 now being generated by the torque motor TM.

We see the carriage CA, which simulates the navigating vehicle. It is carried by a triple set of triangular bogies TB to ensure keeping the pivot axis PA sufficiently vertical despite minor floor irregularities.

The pendulum mass m is "attracted" towards the centre of the trajectory by means of the gravity imitation string GS which is attached to a string drum SD on the centre pole CP. This string drum is torque loaded by a weight WT which acts on SD by a reduction gear 1 : 4. Thus the string GS can be extended or released by an amount of $\pm$ 1 meter with respect to its nominal length of 7 meters, making the weight WT travel up and down the perspex tube of the centre pole CP over a range of $\pm$ 25 cm from its middle position. The centre pole is positioned so as to mark the origin of curvature of the two concentric floor markings FM which indicate the navigable area.

The axle of the string drum SD carries an electric ten-turn potentiometer PM. Its position corresponds with the extension length of the gravity string GS and is sensed by the electronic circuitry in the electronics box EB, to which it is connected by the potentiometer cable PC. In this way information about the radius of trajectory R can be used to keep the scale factor $K_G K_T \tau$ of eq. 5.1.5 - (5) "tuned" to the momentary R.

The electronic box EB of course contains the circuitry required to operate the system. It is described in the (Dutch language) student thesis reports HAGENBERG 1969, VAN OORT 1971, JANSON 1973, KLEIKAMP 1981. There is also an indicator box IB which can be connected either to the carriage CA (to simulate "on board navigation") or to the electronics box EB (to simulate a telemetry situation). The panel of the indicator box IB is copied in fig. 5.2 - 4.

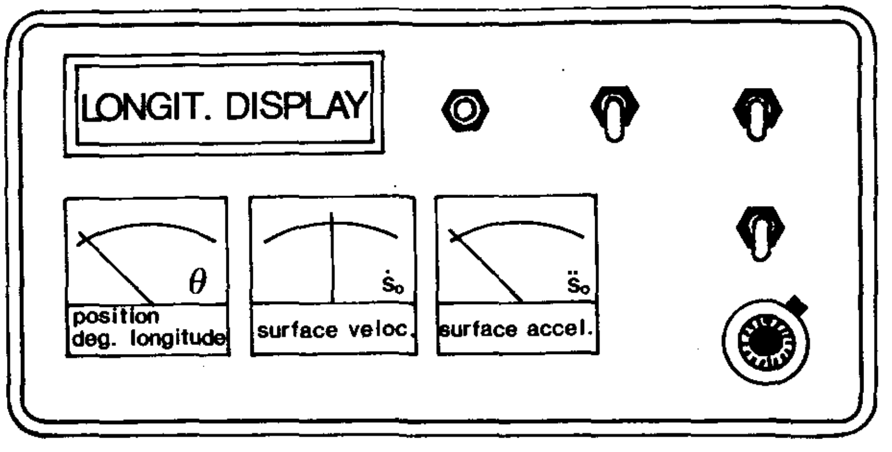


Fig. 5.2 - 4. The indicator box panel
(Switches and potentiometer serve for integrator adjustment)

The torque motor TM is placed on top of a turret that can be rotated around the pivot axis PA (see cross-sectional drawing 5.2 - 5). A syncro-resolver between the axle of the pendulum and this turret feeds a signal to a servo-motor, thereby forming a servo-loop that causes the turret to keep rotationally aligned with the pendulum. This was arranged in order to give the carriage complete rotational freedom around PA in spite of the torque motor TM having only a limited working range of $\pm$ 60 deg. There is a hook attached to the turret to allow the demonstrator to disengage the "gravity string" from the pendulum bob and fix it to the turret, thereby cancelling the gravity effect but retaining the trajectory radius information at the ten turn potentiometer PM.

Inside the turret, but carried by the pendulum axle, one can see the rate gyro necessary to establish the torque feedback loop (compare figs. 5.1.2 - 1 and 5.1.5 - 1). All the required electrical connections to rotating parts have been made by installing slip-rings.
Working a demonstration with the system is easy and straightforward, provided the floor in the navigable area is level to within 2 milliradians, and flat to within a few millimeters.

The standard setup is as shown in fig. 5.2 - 1. Arbitrary movements of the carriage within the navigationable area will not cause the pendulum to desert its orientation towards the centre pole. Acceleration $\ddot{s}$,

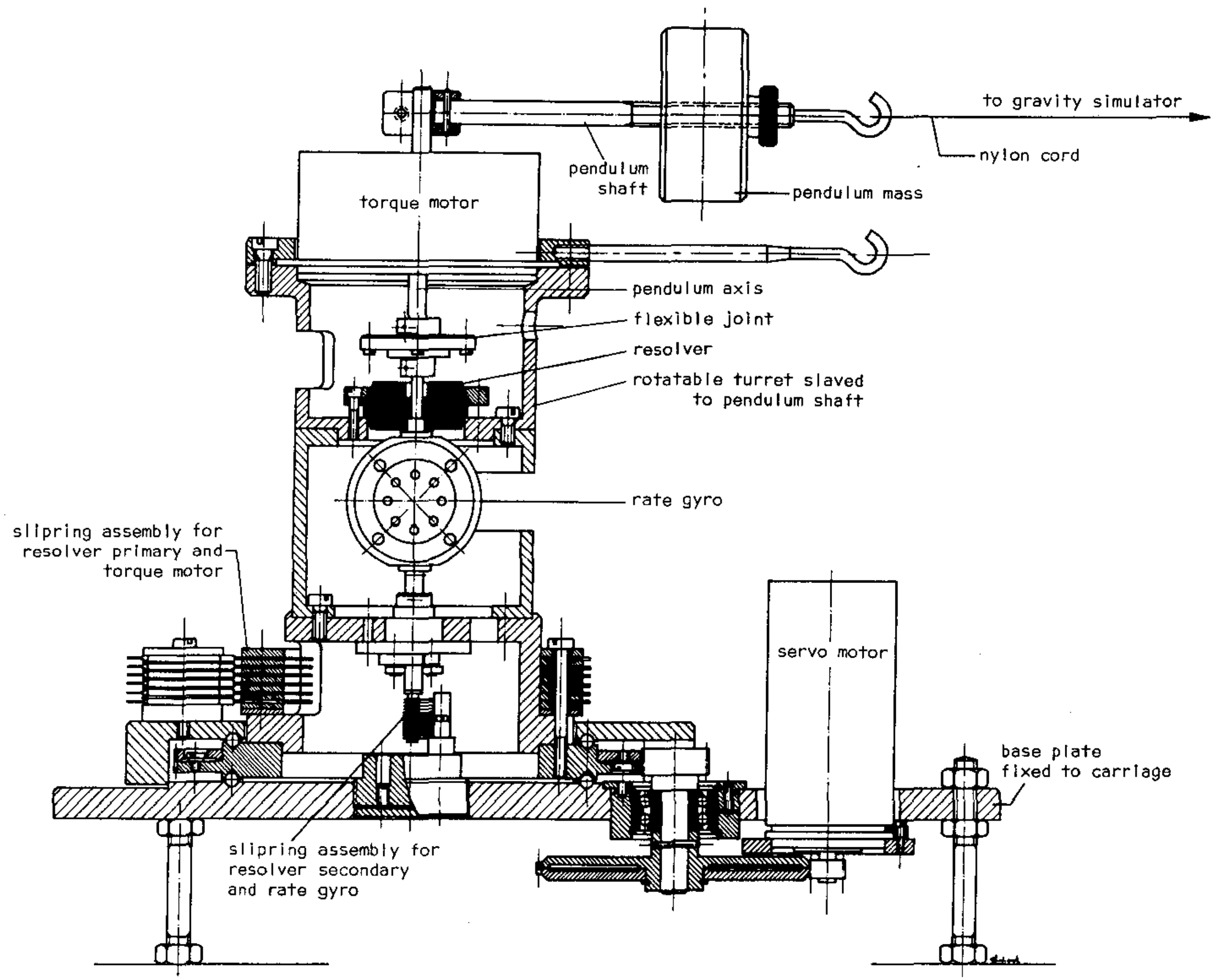


Fig. 5.2 - 5. Cross-sectional drawing of the turret assembly

momentary velocity $\dot{s}$, and relative position θ can be read from the pointer instruments on the panel (fig. 5.2 - 4). There is on it also a digital display to indicate θ. (For the meaning of the symbols $\ddot{s}$, $\dot{s}$, θ and also α, see fig. 5.1.1 - 1).

Actually these instruments respond to $\ddot{\alpha}$, $\dot{\alpha}$ and α. But the "navigator", i.e. the demonstrator and his audience, will be more interested in the trajectory acceleration and velocity. Since, according to eqs. 5.1.1 - 1 and 5.1.1 - 2, the system is designed to make $\alpha = \theta$, and the trajectory geometry links s to $-R\theta$, we have calibrated two scales to read $\ddot{s}$ in m/sec^2, and $\dot{s}$ in m/sec equivalent "surface speed", that is speed projected unto the inner circle of floor marks FM, the third scale and the digital display unit to indicate "longitude travelled" in degrees.

On a small control box CB, fixed to the carriage CA, three switches are mounted to enable the demonstrator to close down different parts of the electronic system (see fig. 5.2 - 6).

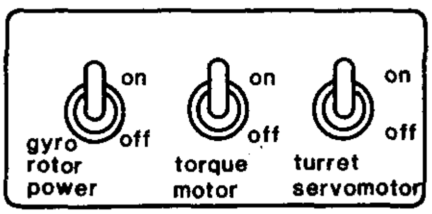


Fig. 5.2 - 6. The three switches on the control box CB.

When the feedback loop is interrupted, either by shutting down gyro running power or by disconnecting the torquer signal, it can be demonstrated that the pendulum's oscillation period sharply decreases from a previous 5s to perhaps 0,5s. Besides that, of course, without feedback the pendulum is very sensitive to vehicle accelerations.

By unhooking the gravity string GS from the pendulum (and engaging it with the fixed hook on the turret) one can show that absence of the gravity pull does not affect the tuning, and that disturbances no longer result in oscillations but in a permanent drift rate. By giving the gravity string an arbitrary length one will find that the pendulum fixes itself upon a new centre of trajectory.

5.3 The electronically controlled horizontal platform

In analogy to the electronically assisted physical pendulum we can draw a block diagram of a horizontal platform, in which the independant natures of the parameter adjustment and the oscillation period can also be clearly seen.

5.3.1 The geometrical situation

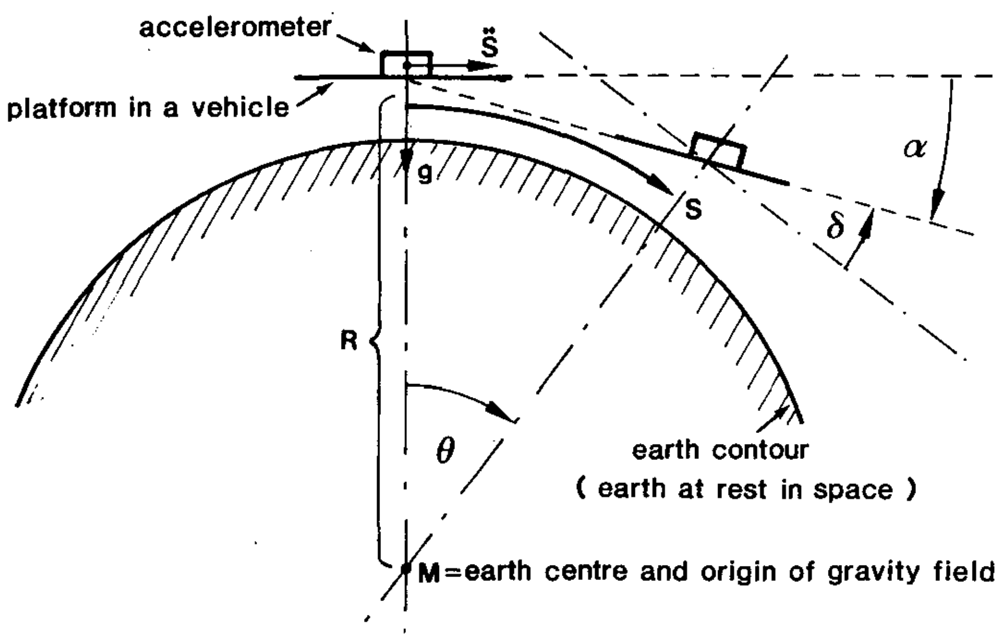


Fig. 5.3.1 - 1. A platform at the surface of the earth.

A short explanation of the situation may be desired: The platform, initially horizontal and at rest, is accelerated horizontally ($\ddot{s}$). The trajectory magnitudes, $\ddot{s}$ and s, although schematically entered at different distances from the earth centre M, all refer to the same trajectory radius R, which is also by approxmation the earth radius. Thus it is, as in ch. 5.1.1, that

$$s = -R\theta . \qquad 5.3.1 - (1)$$

(We reckon translatory movement to the right and rotatory movement CCW as positive).

After having travelled a distance corresponding to the "longitude" angle θ the platform is assumed to have, by whatever reason, turned by an angle α with respect to its initial alignment. It will be off the local horizontal in the new position by an angle δ. Now

$$\theta = \alpha - \delta. \qquad 5.3.1 - (2)$$

The objective is to determine s by double integration of $\ddot{s}$. For this reason δ must be kept zero, to avoid a component of gravity acceleration to enter the integration proces. It becomes the task of the platform alignment system to keep

$$\alpha = \theta . \qquad (3)$$

5.3.2 The error input to the accelerometer

The occurrence of any error angle

$$\delta = \alpha - \theta \qquad 5.3.2 - (1)$$

(compare figs. 5.3.1 - 1 and 5.3.2 - 1) will give rise to an input into the accelerometer that we shall call the acceleration error

$$\ddot{\sigma} = g\delta . \qquad (2)$$

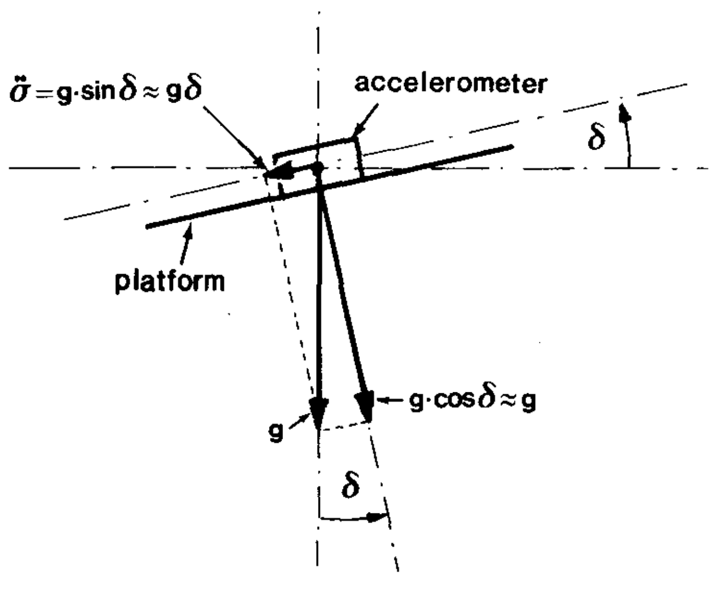


Fig. 5.3.2 - 1. Accelerometer on a platform in a gravity field.

A positive angle δ entailes a gravity component to the left, which the accelerometer interprets as an acceleration to the right (positive).

The total input signal to the accelerometer now becomes

$$a_{in} = \ddot{s} + \ddot{\sigma} . \qquad 5.3.2 - (3)$$

With the scale factor K_A there appears an electrical signal, e.g. a voltage, at the output of the accelerometer:

$$U_A = K_A (\ddot{s} + \ddot{\sigma}) . \qquad (4)$$

The desired output, however, is only $K_A\ddot{s}$, so, again, the inertial navigation system has to be designed to keep $\ddot{\sigma} = 0$, and thus to ensure $\delta = 0$.

5.3.3 The block diagram

There are two ways to go about explaining the Schuler tuned platform: (1) describe the design principle an deduce from it the functional block diagram; (2) give first the block diagram, and then explain how it is realized. We choose for the latter, so here the next figure, 5.3.3 - 1, shows the block diagram.

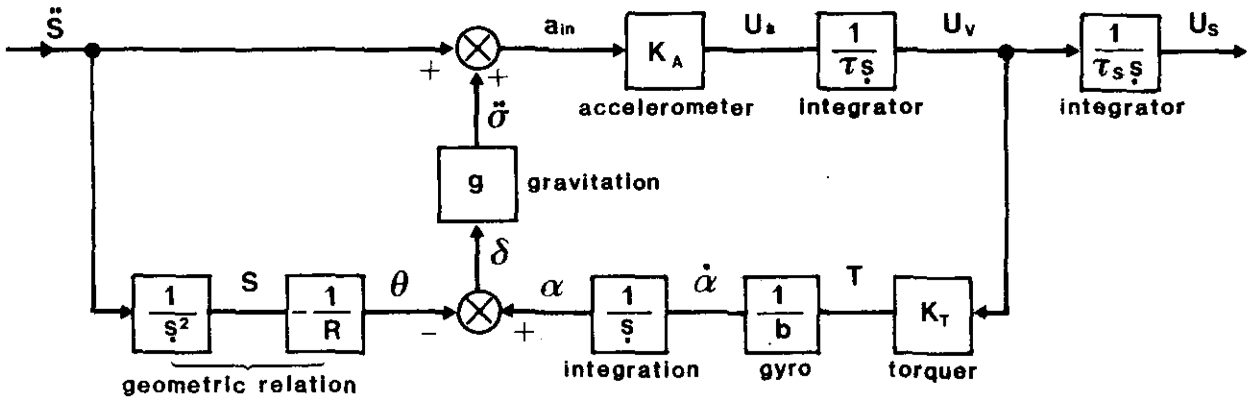


Fig. 5.3.3 - 1. Block diagram of an electronically controlled horizontal platform.

$\ddot{s}$ = vehicle acceleration	θ = vehicle angle of travel
a_{in} = input accelerometer	s = vehicle distance of travel
U_a = voltage proportional to a_{in}	K_A = accelerometer scale factor
U_v = voltage indicating $\dot{s}$ = v	τ = integrator time constant
U_s = voltage indicating s	$\underset{\cdot}{s}$ = Laplace operator
T = torque applied to gyro	K_T = torquer scale factor
$\dot{\alpha}$ = platform angular velocity	b = angular momentum of gyro
δ = error angle of platform	g = gravity acceleration
$\ddot{\sigma}$ = error acceleration	R = earth radius

5.3.4 The transfer function

The accelerometer produces an electrical signal, in our case a voltage

$$U_A = K_A a_{in} = K_A (\ddot{s} + \ddot{\sigma}). \qquad 5.3.4 - (1)$$

Two integrators transform this signal into a voltage

$$U_s = \frac{K_A}{\tau \tau_s s^2} (\ddot{s} + \ddot{\sigma}). \qquad (2)$$

If the error term $\ddot{\sigma}$ were zero, this voltage U_s would be proportional to the travel s, and that is what is desired:

$$U_s = \frac{K_A}{\tau \tau_s s^2} = \frac{K_A}{\tau \tau_s} \cdot S \quad . \qquad (3)$$

5.3.5 Elimination of the error term

Making $\ddot{\sigma}$ zero means [cf. eq. 5.3.2 - (1) and (2)]:

$$\ddot{\sigma} = g\delta = g(\alpha - \theta) = 0, \qquad 5.3.5 - (1)$$

or

$$\alpha = \theta. \qquad (2)$$

From the block diagram we can read

$$\alpha(\ddot{s}) = \frac{K_A K_T}{\tau b s^2} (\ddot{s} + \ddot{\sigma}) \qquad (3)$$

and

$$\theta(\ddot{s}) = -\frac{1}{R s^2} \ddot{s} \; . \qquad (4)$$

Using the requirement $\ddot{\sigma} = 0$ and equating eqs. 5.3.5 - (3) and (4), we get

$$\frac{K_A K_T}{\tau b} = -\frac{1}{R} \qquad (5)$$

as condition for tuning the system. For, left of the equation mark we find only instrument parameters, and right of it the only geometric trajectory parameter there is, namely R.

Neither the vehicle acceleration $\ddot{s}$ not the gravity acceleration g appear in 5.3.5 - (5). So the tuning is valid for all $\ddot{s}$, and independant of g.

5.3.6 The system realisation

The reader will probably be familiar with the fact, that in a gyroscopically controlled horizontal Schuler tuned platform system the platform can be governed by a servo-loop slaving it to the gyro. Schematically we can picture it as follows:

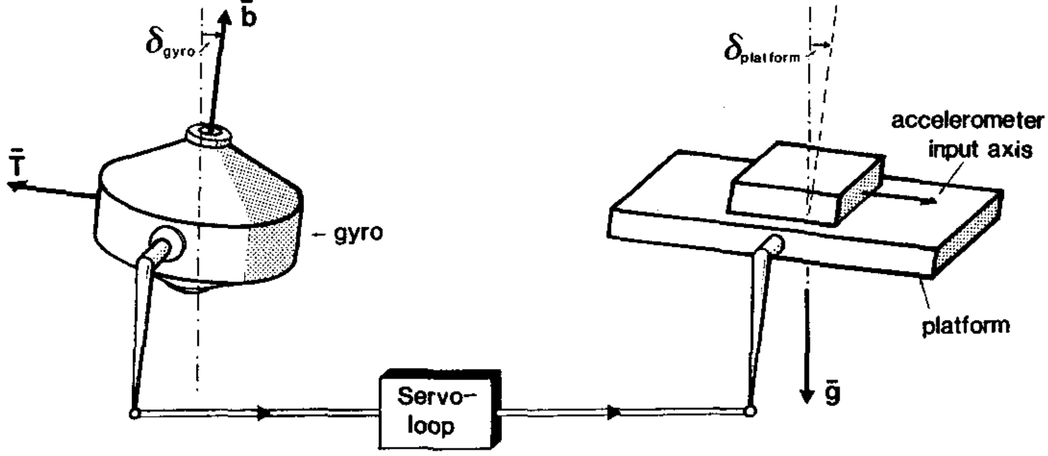


Fig. 5.3.6 - 1. The platform slaved to the gyro

As a consequence always

$$\alpha_{platform} = \alpha_{gyro} = \alpha \quad \text{and} \quad \delta_{platform} = \delta_{gyro} = \delta$$

The precession rate $\dot{\alpha}$ of the gyro (and thus also of the platform) is governed by the torque T_G that is exerted on the gyro by the torquer according to the block diagram, and this results in an angular rotation α. In formula:

$$\dot{\alpha} = \frac{\dot{b}}{b} = \frac{T_G}{b} \qquad 5.3.6 - (1)$$

$$T_G = K_T U_V \qquad (2)$$

$$\alpha = \frac{\dot{\alpha}}{s} = \frac{K_T U_V}{s b} \; . \qquad (3)$$

This last formula explains the corresponding branch of the loop in the block diagram. This loop is a feed-forward loop as far as the generation of α is concerned, but a feed-back loop for δ! This feed-back loop is undamped and enables the system to sustain oscillations once they have been excited by disturbances other than the vehicle accelerations $\ddot{s}$.

The whole feed-forward loop goes according to

$$\alpha = \frac{K_A K_T}{\tau b s^2} \cdot a_{in} \qquad 5.3.6 - (4)$$

which was already given in 5.3.5 - (3).

The feed-back loop contains no damping because it is impossible instrumentally to distinguish between $\ddot{s}$ and $\ddot{\sigma}$ unless one of them is known. Damping would introduce unwanted terms in the measurement of $\ddot{s}$. In situations where $\ddot{s}$ is known to be zero any oscillations the feed-back loop may have built up can be damped out by temporarily introducing the proper signals somewhere in the loop.

5.3.7 The diffential equation of the feedback loop

Take a situation where $\ddot{s} = \dot{s} = 0$, and let us call that position of the vehicle $\theta = 0$. In this situation eq. 5.3.1 - (2) yields

$$\delta = \alpha \qquad 5.3.7 - (1)$$

and, because of 5.3.2 - (2) and (3)

$$a_{in} = \ddot{e} = g\alpha . \qquad (2)$$

Substituting this into the loop function 5.3.6 - (4) results in

$$\alpha = g \frac{K_A K_T}{\tau b} \cdot \frac{\alpha}{s^2} \qquad (3)$$

$$s^2 = g \frac{K_A K_T}{\tau b} . \qquad (4)$$

We transform into the frequency domain and find

$$\omega = \sqrt{-g \frac{K_A K_T}{\tau b}} , \qquad (5)$$

this being the frequency at which the loop would oscillate if disturbed.

In 5.3.5 - (5) we gave the instrument parameter condition for elimination of the acceleration sensitivity of the horizontal reference platform as

$$\frac{K_A K_T}{\tau b} = -\frac{1}{R} .$$

Inserting this into 5.3.7 - (5) we finally find

$$\omega = \sqrt{\frac{g}{R}} \qquad 5.3.7 - (6)$$

or

$$T = 2\pi\sqrt{\frac{R}{g}} \qquad (7)$$

as the oscillation period of the disturbed Schuler tuned platform.

Since the accelerometer, unaffected by the inhomogeneity of the gravity field, only reacts to the local value of g, the system will indeed have a period of 84,4 min for a position near the surface of the earth, where $R = R_o$ and $g = g_o$.

This circumstance is discussed for various systems in MAGNUS 1971, in ch. 16.4.

5.4 Conclusions

In this chapter the electronically tuned Schuler systems are treated as a logical extension of the purely mechanical models of par. 4.2.9.

The block diagrams (figs. 5.1.5-1 and 5.3.3-1) give the impression of control systems with feed-back loops. But a closer inspection shows that there are two feed-forward loops — one consisting of the geometrical relationships and the other of the instrument parameters — of which the outputs are compared by means of the influence of the radially symmetric gravity field. The comparison result is used as a kind of feedback which unavoidably leads to an undamped oscillatory system. The oscillation period thereof depends on the "tuning" of the instrument parameters <u>and</u> gravity, but the correct instrument parameters are <u>not</u> a function of gravity.

With the first system described (i.e. the pendulum) the gravity gradient can be neglected in practical earth surface applications because of the low ratio of the mechanically real to the electronically simulated moments of inertia.
With the second system (i.e. the platform) the gravity gradient is irrelevant as long as the accelerometers are unaffected by the gradient. Thus all practical electronically tuned Schuler systems will exhibit the 84 min. period.

By understanding that gravity plays only a secondary role in Schuler tuned systems, it is easy to design and build class-room models not only of a purely mechanical system but also of electronically tuned systems.

Whereas a purely mechanical system, as our twin mass demonstration model of par. 2.6, could theoretically be operated floating freely in space, all the electronic systems need a carrier vehicle that is able to exert the necessary reaction torques: with the pendulum to generate the reduction torque, and with the platform to apply the gyro control torque.

6. MISCELLANEOUS TOPICS

The principle Schuler discovered when endeavouring to find a mode of using the gyrocompass on board of moving ships involves an array of aspects: geometry, translational dynamics, rotational dynamics, pendulum theory, gyro theory, gravitation, measurement, and control. It is not surprising then that there are more situations than those created by navigation problems where derivatives of his principle can be discerned. It is not proper exclusively to couple Schuler's name with those phenomena, since e.g. NEWTON, D'ALAMBERT and RITTER appear to have mused over certain situations, facts, and conditions that are related to Schuler's principle but have nothing or very little to do with navigation problems as such.

Our purpose in presenting a choice of various related topics is not to introduce novel systems, bur merely to show perhaps unexpected relationships, and to underline our statement no. 3 in our chapter 0, namely that "The existence of a gravity field ... is not an absolute requirement for the basic function of a "Schuler-tuned vertical refererence ....".

6.1 Satellite orbital period

In our chapter 4.3.8 we showed that a pendulum of infinite length, the bob of which is launched to perform circular or elliptical trajectories inside or at the surface of the earth, will complete a round in the Schuler period of 84, 4 min. A great circle trajectory along the surface of the earth belongs to this set of movements, and this is what a satellite ideally would do. But it is remarkable, that also "impossible" satellites, like those following a minor circle (e.g. at constant latitude other than 0 deg.), would have the same orbit period. (See fig. 4.3.8 - 2).

6.2 Gravity trains

The type of free movements the bob of an infinitely long pendulum can make include those that a mass would perform in a straight frictionless tunnel through the earth. It follows, that the distance between any two places on (the idealized) earth could be covered in 42,2 min. by a train propelled by gravity only, through a straight tunnel between the two places (provided of course there were no frictional or other losses). It would start and arrive with zero velocity (see e.g. MISNER, 1971).

Quicker "gravity trains" are possible with curved tunnels on short distances. They can be compared with the pendulums treated in ch. 4.3.4, although the restriction to small angles of excursion β would have to be abandoned. But to the antipodes the shortest time is 42,2 min.

6.3 "Schuler-tuned" pounders and doors

Problem: You are setting up a fence, and to do this you have to drive in the posts. Lacking a heavy hammer or axe you use a surplus pole as a pounder (fig. 6.3 - 1).

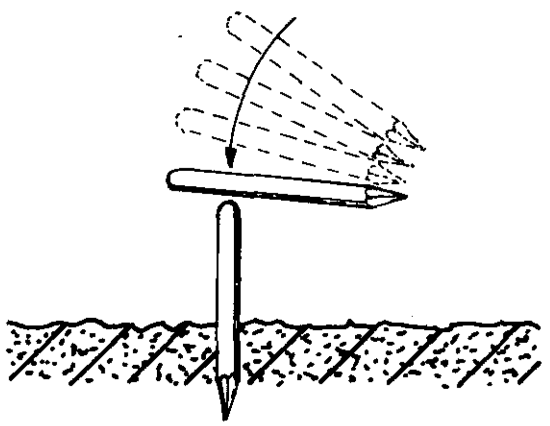


Fig. 6.3 - 1.
Using a fence post for pounding

You quickly notice that your hands get severe shocks if you choose the wrong place along the pounder-pole to strike the fence post.

It appears that the right place with which to strike depends on the radius of curvature of the movement of the descending pole. If you let the pounder descend without imparting rotational energy to it, i.e. in a parallel movement, you will have to let the centre of that pole touch the head of the fence post. The impact will not lead to the build-up of rotational energy, and the pole will come to rest immediately after impact (see fig. 6.3 - 2a).

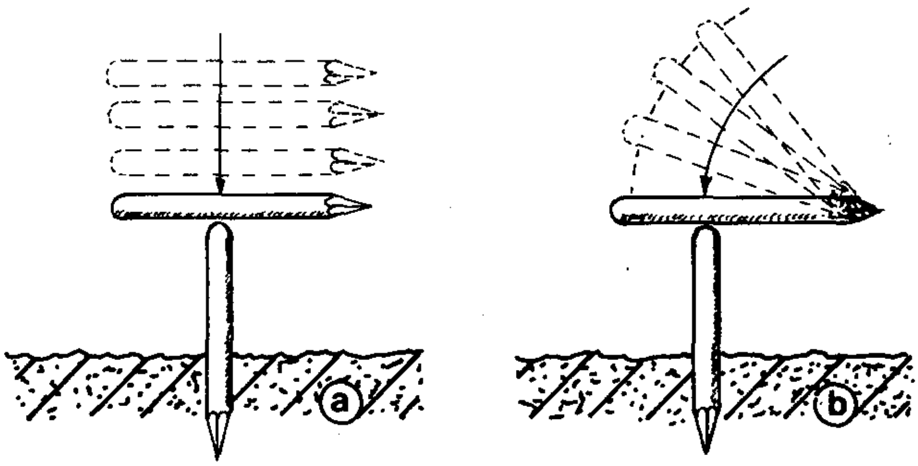


Fig. 6.3 - 2. Two situations discussed in the text.

This movement of the pole can be described in terms of a trajectory with an infinite radius of curvature.

But if one holds the pole at one end and lets it strike the head of the other post by letting it fall in the fashion of fig. 6.3 - 2b, the centre of the trajectory lies at the end of the pole where the hand holds it.

When we look at ch. 4.2.4, where the relationship between radius of trajectory, radius of gyration, and position of suspension point is given, we understand that any acceleration we impart to the suspension point will never displace the centre of the trajectory. So if we select the point of impact along the pounder so as to make it coincide with the suspension point belonging to the end of the pole as centre of trajectory, the hand holding that end of the pole will feel no jerk upon hitting the fence post.

The relationship given in 4.2.4 - (3)

$$R_{cI} = \frac{J_{cI}}{ma} = \frac{mr^2}{ma} = \frac{r^2}{a} \qquad 6.3 - (1)$$

must be translated to the circumstances of the pole. We introduce the parameters according to fig. 6.3 - 4.

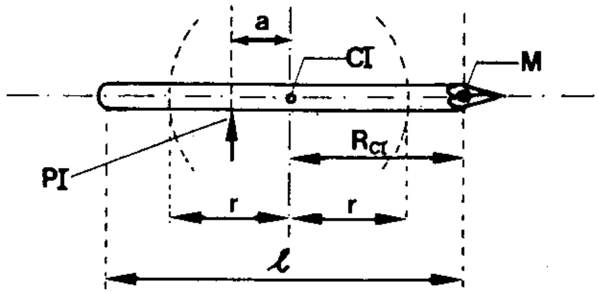


Fig. 6.3 - 4. Parameters descriptive of the pole

ℓ = effective length of pole
r = radius of gyration of pole
R_{CI} = radius of trajectory of the centre of inertia
M = centre of trajectory
CI = centre of inertia of pole
PI = point of impact
a = distance between CI and PI

The radius of gyration of a rod is known to be

$$r = \frac{1}{\sqrt{3}} \cdot \frac{\ell}{2} \quad . \qquad 6.3 - (2)$$

With this and

$$R_{CI} = \frac{\ell}{2} \qquad (3)$$

the formula 6.3 - (1) becomes

$$\left.\begin{aligned} \frac{\ell}{2} &= \frac{1}{3a}\left(\frac{\ell}{2}\right)^2 \\ 3a &= \frac{\ell}{2} \\ a &= \frac{\ell}{6} \ . \end{aligned}\right\} \qquad (4)$$

Because the radius of trajectory of the suspension point always must be

$$R_{SP} = R_{CI} + a \qquad 6.3 - (5)$$

we find for the suspension point alias point of impact of the pounder pole held at one end

$$R_{SP} = \frac{\ell}{2} + \frac{\ell}{6} = \frac{2}{3}\ell \;. \qquad (6)$$

The recipe for avoiding painful reaction jerks to your hand when using a pole to pound with would run something like this:
Take hold of one extreme end of the pole, lift it up high and let it descend in a manner as shown in fig. 6.3 - 1. Aim at hitting the object with one third of the pole extending beyond the object. See to it that the end your hand is holding comes level with the head of the object to be pounded before the pounder strikes that object. Then arrest the end you are holding at that level position and let the pole complete the travel by hinging around the position of your hand. In this wise your hand has become the centre of the last part of the total trajectory and will not experience reaction forces if the "Schuler-condition" is satisfied by the point of impact at the moment of impact.

In fig. 6.3 - 5 such a movent is qualitatively depicted.

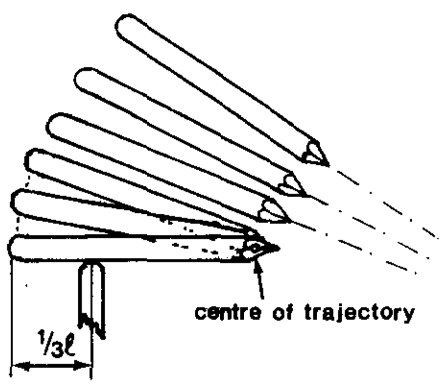


Fig. 6.3 - 5. The last phase of the pounder trajectory

The same technique can be applied to doorbuffers to avoid damage to the hinges and their mounting base.

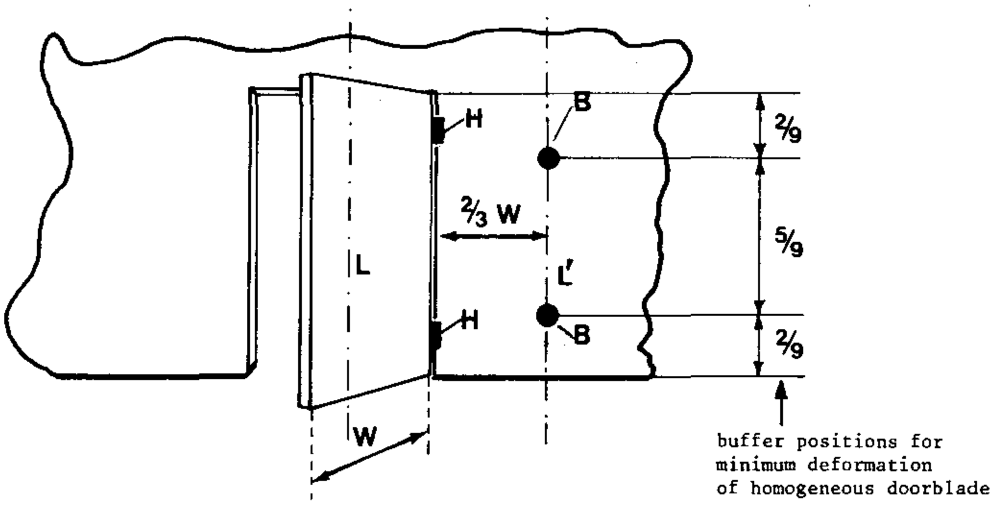


Fig. 6.3 - 6. "Schuler-tuned" doorbuffers

B = buffers
H = hinges
L = line of impact on door
L' = line of impact on wall
W = width of door

If the stops are placed symmetrically along the line of impact at a distance of 2/3 the width of the door from the line of the hinges, even a heavy door violently flung open will not pull out or deform the hinges.

6.4 Schuler tuning and small circle movements

In his 1923 article, SCHULER says that a properly tuned pendulum will remain pointing towards the centre of the earth "no matter how the vehicle moves" (p. 346). This is affirmed in a paragraph on p. 395 of MAGNUS 71, where of a physical pendulum it is said that it would stay vertical "completely independant of the movements the suspension point may make along the surface of the earth".
Most authors explain the Schuler principle by analyzing great circle movements of the pendulum carrier, and then assume that arbitrary movements

can be regarded as a linear superposition of great circle components, not affecting the desired behaviour of the reference system adversely.

This view, however, is challenged by Bell in his commentary called "The Schuler Pendulum's Fatal Flaw" (BELL, 1969). He contends that "clearly there are many motions of [the suspension point] on a sphere of radius R for which a pendulum having [length] L ≡ R does not point the 'vertical', amongst which ... motions in small circles ..."

There is a simple analysis of a two point mass physical pendulum moving on a small circle that we will give here and which does not support BELL's contention. We do not assume that our treatment gives a conclusive proof of the matter, but we offer it for the benefit of readers that might like to study this problem more deeply.

Imagine a vehicle travelling along a line of latitude λ. A Schuler tuned twin mass pendulum carried by it will, if properly aligned, describe a conical ring in space (fig. 6.4 - 1).

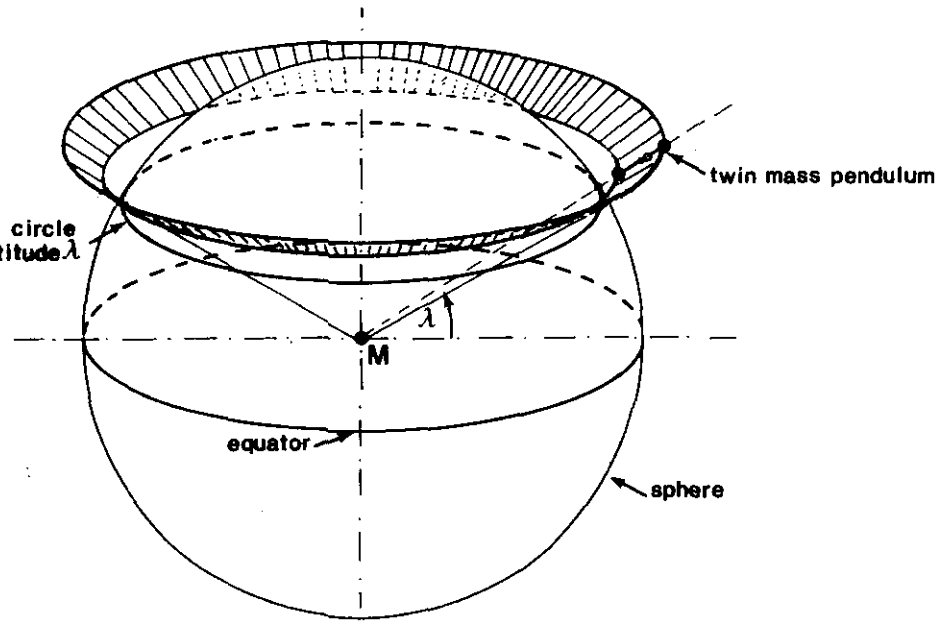


Fig. 6.4 - 1. A twin mass pendulum on a small circle trajectory

Its centre of inertia then follows a circle, the radius of which we shall call R'. Having understood that Schuler tuning has nothing to do with the presence or absence of a central gravity field, we can detach this circular suspension point trajectory from the sphere and study it separately in space.

By this we arrive at the next figure, showing the geometrical relationships.

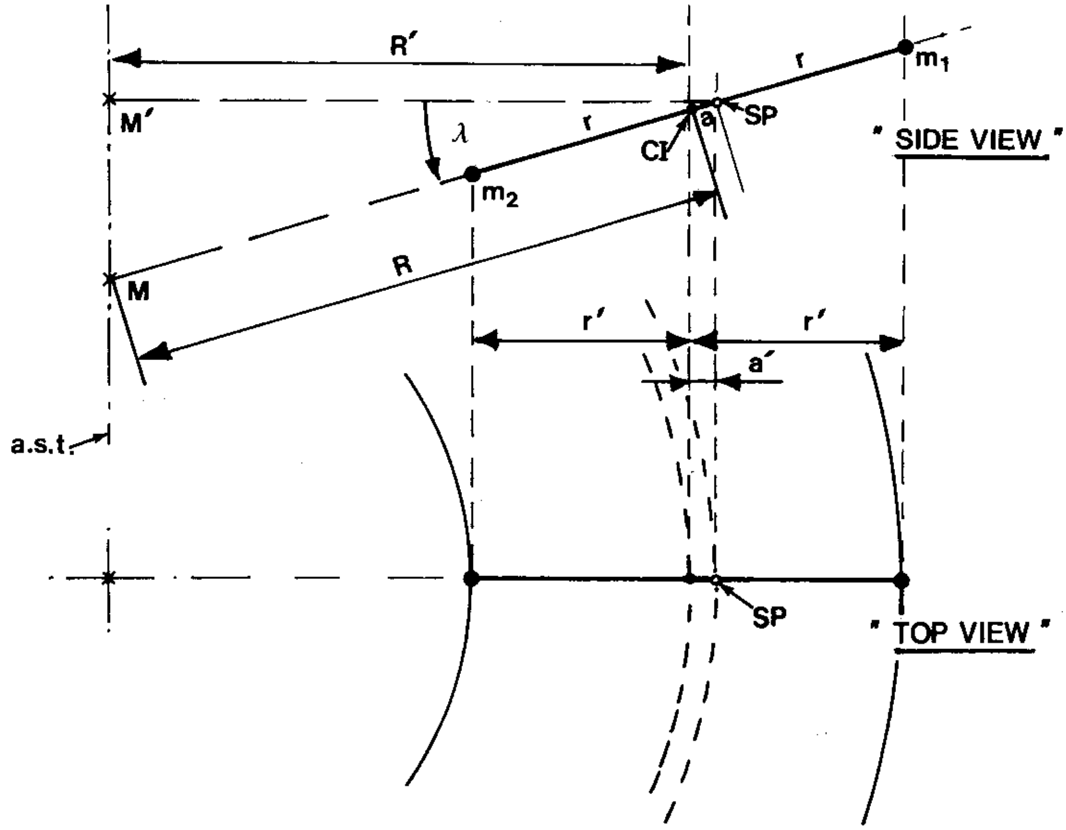


Fig. 6.4 - 2. Equivalent of a small circle trajectory

CI	= centre of inertia	R'	= radius of trajectory
M	= centre of the sphere	a.s.t.	= axis of sphere and trajectory
M'	= centre of trajectory	$m_{1,2}$	= mass points
Sp	= suspension point	λ	= angle of latitude
R	= radius of sphere	a', r'	= projections of a, r

The acceleration forces acting on the pendulum are those exerted on SP by the accelerations of the vehicle tangential to the trajectory, and by the centrifugal forces acting on the two mass points.

With zero trájectory velocity we have only the tangential acceleration forces. Even though the vehicle travels around M', the momentary centre of the pendulum movement must be M, since no torques are transmitted by the vehicle to the pendulum via SP. The momentary rotational acceleration $\dot{\omega}$ around this centre can be vectorially split into a component $\dot{\omega}'$ at right angles to the trajectory plane, and another, $\dot{\omega}''$, along the radius of the trajectory:

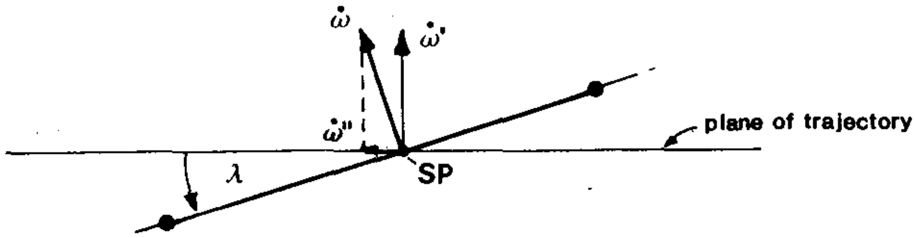


Fig. 6.4 - 3. The components of the momentary angular acceleration of the twin mass pendulum

Now it appears that the Schuler condition

$$R = \frac{r^2}{a} \qquad \text{cf. eq. 2.5 - (5)}$$

is also valid for the separate rotation acceleration components. Thus for $\dot{\omega}'$ we have to take the "TOP VIEW" projections:

$$R' = \frac{(r')^2}{a'} \Rightarrow R\cdot\cos\lambda \Rightarrow \frac{(r\cdot\cos\lambda)^2}{a\cdot\cos\lambda} \Rightarrow R = \frac{r^2}{a} \qquad 6.4 - (1)$$

and for the front view (not drawn in fig. 6.4 - 2, but easily deduced from it):

$$R'' = \frac{(r'')^2}{a''} \Rightarrow R\cdot\sin\lambda \Rightarrow \frac{(r\cdot\sin\lambda)^2}{a\cdot\sin\lambda} \Rightarrow R = \frac{r^2}{a}\,. \qquad (2)$$

We conclude that, certainly as long as no centrifugal forces are to be taken into account, the Schuler tuned twin mass pendulum will also work on small circle trajectories.

But what about the centrifugal forces? The vehicle following the trajectory exerts an inward pull f_v at the SP, whereas m_1 and m_2 generate the outward forces f_{c1} and f_{c2} (see fig. 6.4 - 4).

Equilibrium would be established if

$$f_{c1}\,(r-a)\sin\lambda - f_{c2}\,(r+a)\sin\lambda = 0\,. \qquad (3)$$

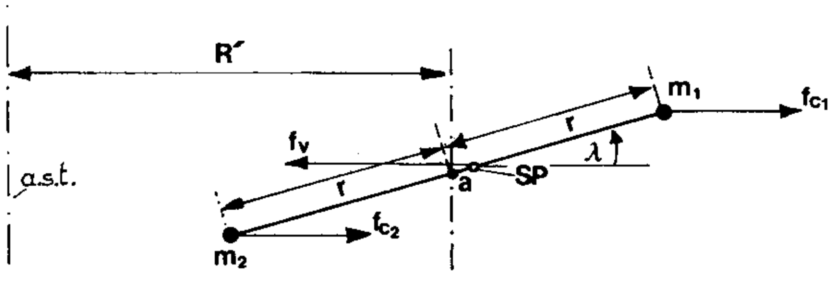


Fig. 6.4 - 4. Centrifugal forces acting on the pendulum

With ω as rate of rotation of the vehicle with respect to the centre M' of the trajectory:

$$m_1\omega^2(R+r)\cos\lambda\cdot(r-a) - m_2\omega^2(R-r)\cos\lambda\cdot(r+a) = 0. \qquad 6.4 - (4)$$

Since $m_1 = m_2$,

$$(R+r)(r-a) - (R-r)(r+a) = 0$$
$$Rr + r^2 - Ra - ra - (Rr - r^2 + Ra - ra) = 0$$
$$2(r^2 - Ra) = 0 \qquad (5)$$
$$R = \frac{r^2}{a},$$

which again is exactly the condition for Schuler-tuning (cf. eq. 2.5 - (5)). This means that if the twin mass pendulum is Schuler tuned it will be at the same time unaffected by the centrifugal forces, whether the trajectory is a great circle or a small circle.

It must be added, however, that this immunity only exists when the pendulum is in its nominal position, i.e. pointing towards the centre of the sphere. This is easily seen from eq. 6.4 - (5) in combination with fig. 6.4 - 2; for, if the angle designated λ in this figure changes, while R' and r remain the same, R would change and demand a new adjusment of "a".

Whereas the existence of a central gravity field makes the nominal pendulum position a stable equilibrium, the effect of the centrifugal forces tends to destabilize the pendulum.

6.4.1 A demonstration model

Using the facts described above a modified version of the table-top model of fig. 2.6 - 2 can be designed. The idea is to replace the gravity imitation string GS of that model by a component of earth gravity by tilting the pivot axis PA. In order to be able to show the behaviour of the pendulum with and without gravity restoring torque the pendulum must keep tuned in both situations. For this the hinge point and tilt angle cannot be chosen arbitrarily, but must satisfy the condition to be given here:-

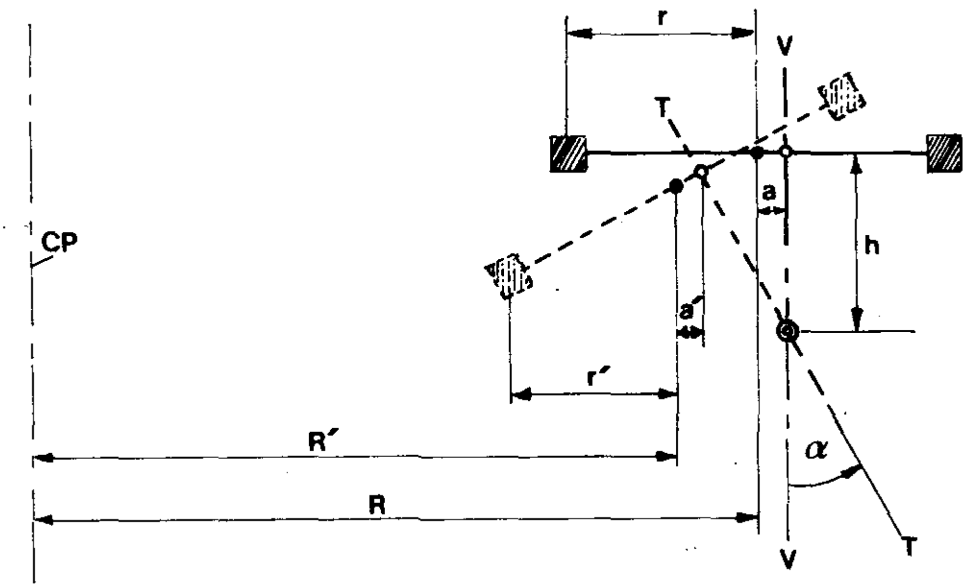


Fig. 6.4.1 - 1. Tiltable pivot axis geometry.
This schematic drawing represents a cross section through the model of fig. 2.6 - 2 in a plane passing through PA and CP.

CP = centre pole axis (cf. fig. 2.6 - 2)
R = trajectory radius with vertical pivot axis
R' = trajectory radius with tilted pivot axis
V-V = PA in vertical position
T-T = PA in tilted position
a,r as in fig. 2.6 - 1
a', r' idem but with tilted pivot axis
α = tilt angle
h = height of suspension point above hinge point
● = centre of inertia of the dumb-bells
o = "suspension point"
◎ = hinge point

The Schuler condition for the vertical PA is:

$$R = \frac{r^2}{a} \,,$$

and for the tilted PA:

$$R' = \frac{(r')^2}{a'} = \frac{r^2 \cos^2\alpha}{a\cdot\cos\alpha} = R\cdot\cos\alpha \,. \qquad 6.4.1 - (1)$$

We select α so as to achieve a desired restoring torque. Then h follows from

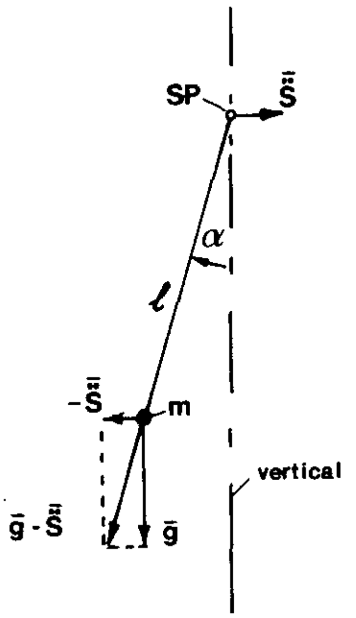
$$h = \frac{R - R'}{\sin\alpha} = R\cdot\frac{1-\cos\alpha}{\sin\alpha} = R\cdot\tan\frac{\alpha}{2} \,. \qquad (2)$$

As example let us choose $\alpha = 0{,}1$ radian, thereby making the pendulum "feel" one tenth of gravity. Tan 0,05 is approx. 0,05, so that h=0,05R. In our model (see ch. 2.6) R was 30 cm. The hinge point thus would have to be 1,5 cm below the connection rod axis. Then, in both situations ($\alpha = 0$ and $\alpha = 0{,}1$ rad) the dumb-bells model would be Schuler tuned, simulating zero gravity in the first case, and a central gravity field in the second.

As already stated in the third paragraph of ch. 2.8, we have not built such a model. We would be grateful to hear of the experience of any reader who has.

6.5 A Schuler tuned liquid level

A mathematical pendulum can be regarded as an acceleration ratio indicator. Hanging in its equilibrium position in indicates zero horizontal acceleration. When the suspension point is subjected to a steady state horizontal acceleration $\ddot{s}$, after transients have subsided the pendulum shaft will be deflected from the vertical by an amount α as shown in the following figure: -



$$\alpha = \arctan \frac{-\ddot{s}}{g} \qquad 6.5 - (1)$$

or, for small ratio's,

$$\alpha = -\frac{1}{g}\,\ddot{s}\,. \qquad (2)$$

Fig. 6.5 - 1. The pendulum is an acceleration ratio indicator.

Gravitational acceleration g can be regarded as a bias acceleration, or a reference acceleration determining the scale factor of the pendulum as an accelerometer.It replaces the spring constant of a spring restrained pendulous (seismic type) accelerometer.

As long as this pendulum is considered to be in a homogeneous gravity field its scale factor does not depend on its length ℓ. In a radial symmetric gravity field matters become more complicated. We shall not analyze the circumstances here but just state that the scale factor decreases with growing ℓ until it disappears with $\ell = R$, R being the radius of the circular trajectory along which $\ddot{s}$ is defined. This means, that Schuler tuning of such a pendulum, i.e. making α disappear for all and any $\ddot{s}$, cannot be achieved except for $\ell = R$.

Now a liquid level also is a kind of gravity biased acceleration ratio meter. Are there conditions under which it can be made to indicate the local level irrespective of horizontal accelerations?

To examine this question we take not a conventional bubble type spirit level, but an open ended tube as in fig. 6.5 - 2, with constant cross sectional area A, filled up to height h with a liquid of density ρ. When subjected to horizontal acceleration $\ddot{s}$ the level of the liquid in one vertical part will rise and in the other it will fall due to the force of inertia F acting on the horizontal part of the tube. Gravity exerts a restoring force G due to the level difference 2Δh. The ratio of these is

$$\frac{F}{G} = \frac{\ddot{s} \cdot \rho \cdot bA}{-g \cdot \rho \cdot 2\Delta h A} = -\frac{\ddot{s} \cdot b}{g \cdot 2\Delta h} . \qquad 6.5 - (3)$$

Equilibrium is reached when F = G, so that we get

$$2\Delta h = -\frac{b}{g}\ddot{s}. \qquad (4)$$

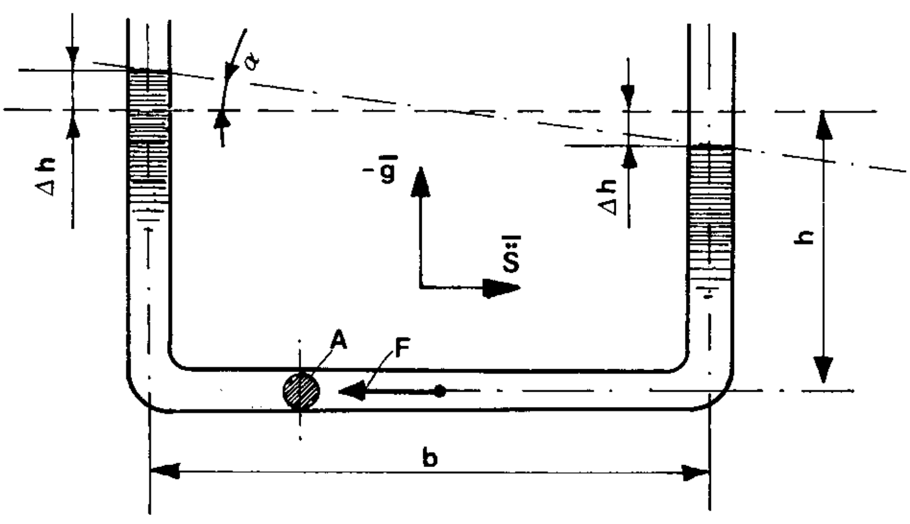


Fig. 6.5 - 2. Open-ended water level

We can draw a connecting line between the two surfaces of the liquid columns. In the steady state this line will run at an angle α with respect to the local level the size of which is given by

$$\tan\alpha = \frac{2\Delta h}{b} = -\frac{\ddot{s}}{g} , \qquad (5)$$

just like in the case of the mathematical pendulum (cf. eq. 6.5 - (2)). In contrast, however, the liquid level can be Schuler tuned theoretically: -

In a gravity-free zone in space we imagine the system of fig. 6.5 - 2 to be accelerated by $\ddot{s}$ parallel to the "horizontal" section with length b. The liquid mass in that section would (disregarding friction) remain in its original place with regard to inertial space were it not that the two "vertical" columns resisted being accelerated vertically to give place. The pressure p_v they generate on being accelerated is

$$p_v = m_v \Delta\ddot{h}/A = 2h\rho\Delta\ddot{h}. \qquad 6.5 - (6)$$

The pressure p_h of the horizontal mass is due to its acceleration $\ddot{x}$ with respect to inertial space. This $\ddot{x}$ is equal to the difference in system acceleration $\ddot{s}$ and internal acceleration of the liquid with respect to the tube, which is $\Delta\ddot{h}$:

$$\ddot{x} = \ddot{s} - \Delta\ddot{h} \qquad (7)$$

Thus

$$p_h = m_h\ddot{x}/A = b\rho(\ddot{s} - \Delta\ddot{h}). \qquad (8)$$

The pressures p_v and p_h must be equated, so we get

$$\begin{aligned} 2h\Delta\ddot{h} &= b(\ddot{s} - \Delta\ddot{h}) = b\ddot{s} - b\Delta\ddot{h} \\ (2h + b)\Delta\ddot{h} &= b\ddot{s} \end{aligned} \qquad (9)$$

and, since 2h + b is the total length of the liquid column:

$$\frac{\Delta\ddot{h}}{b} = \frac{\ddot{s}}{\ell} . \qquad (10)$$

We defined the tilt angle α in 6.5 - (5), so

$$\ddot{\alpha} = \frac{2\Delta\ddot{h}}{b} = \frac{2\ddot{s}}{\ell} . \qquad (11)$$

Our object in making the system behave Schulerian is to move it along a trajectory with radius R and make it follow the local radius vector by making its tilt angle α equal to the trajectory travel angle θ, or

$$\ddot{\alpha} = \ddot{\theta} = \frac{\ddot{s}}{R}$$

⟨see e.g. eq. 5.1.1 - (1) through 5.1.2 - (4)⟩.Using 6.5 - (11) in this relation results in

$$2\frac{\ddot{s}}{\ell} = \frac{\ddot{s}}{R}$$

$$\ell = 2R. \qquad 6.5 - (12)$$

The recipe for making a Schuler tuned openended liquid level, i.e. a liquid level unaffected by horizontal accelerations thus must be:
Take a length of tube equal to twice the radius of the trajectory, fill it with the liquid, and affix the ends of the tube vertically to the vehicle at an arbitrary distance b from each other. Do what you like with the rest of the length: coil it, fold it up, zigzag it or stow it away in any other manner, carry it within the vehicle or leave it partially at rest outside. Provided the two ends of the tube are not moved up or down independantly of each other, the liquid surfaces will then always remain in the same horizontal plane, irrespective of horizontal accelerations of the vehicle.

Once Schuler-tuned, the system can be placed back in a central gravity field. Of course this field is necessary to keep the liquid in its place in a "real" experiment. A "real" experiment need not (and practically could not) be performed with an earth radius trajectory, but a demonstration model similar to the one described in ch. 2.6 could be executed by using a small horizontal trajectory but bending the utmost ends of the tube upward.

Why is it, that the length 2R is required? A hint to this can be given by the following configuration:

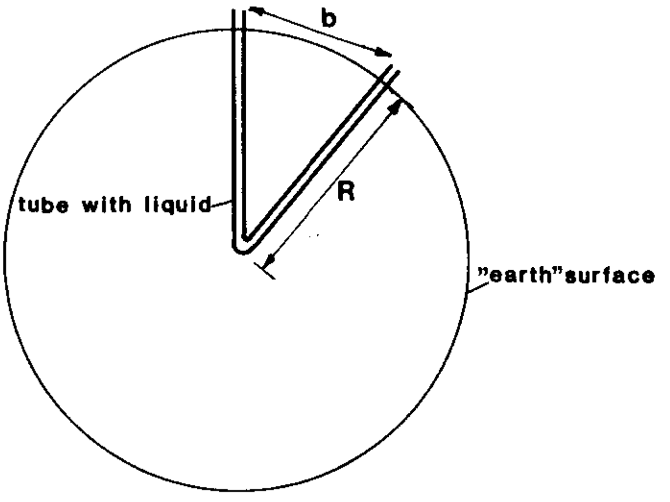


Fig. 6.5 - 3. A configuration to explain the requirement ℓ = 2R.

In it the system consists only of vertical columns which are insensitive to horizontal accelerations.

According to MAGNUS 61 (p. 29), a liquid column in a U-shaped tube has a resonance frequency of

$$\omega^2 = \frac{2g}{L} .$$

Rewriting this expression to conformity with our rendering (e.g. in eq. 1.3 - (6)) and substituting L = 2R according to our eq. 6.5 - (12), we get

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2g}{L}}$$

$$T = 2\pi\sqrt{\frac{L}{2g}} = 2\pi\sqrt{\frac{2R}{2g}}$$

which is the Schuler period

$$T_s = 2\pi\sqrt{R/g} .$$

No wonder Schuler expected T_s to reflect something like a universal law, and that many people have stated that any acceleration insensitive pendulous device will exhibit the Schuler period, when this period turns up in the most unexpected ways!

7. LIST OF MAIN SYMBOLS

The symbols used very specifically in certain figures only, as well as symbols occurring in quotations from other authors, are not included in this list. The numbers at the right indicate the chapter in which the symbol is first mentioned.

a	distance between SP and CI	2.5
b	angular momentum ($\bar{b}$ for the vector)	3.
$\mathfrak{b}$	base length of water level	6.5
g	gravity acceleration	1.1
g_o	g at earth surface (=9,81 m/s^2)	1.1
h	height of liquid column	6.5
ℓ	length of a (mathematical) pendulum	4.1.2
ℓ	length of liquid column	6.5
m	mass	2.4
p	pressure	6.5
r	radius of gyration	2.4
s	distance of travel along trajectory	2.2
$\underset{\bullet}{s}$	Laplace operator (also $\mathcal{s}$ is used)	5.1.5
t	time coordinate	2.2
x	horizontal displacement	4.3.3

A	cross-sectional area of water column	6.5
CI	centre of inertia	2.4
F	force	2.4
G	weight	6.5
J	moment of inertia	2.5
K_A	accelerometer scale factor	4.2.9
K_G	rate gyro scale factor	5.1.2
K_T	torque motor scale factor	4.2.9
M	centre of trajectory	2.4
M	torque	4.4.1
R	radius of trajectory	1.1
R	earth surface (trajectory) radius = 6372 km	1.1
$\hat{S}$	angular spring constant	4.3.1
SP	suspension point	2.4

T	oscillation period	1.2
T	torque	2.5
T_o	"Schuler constant" $2\pi\sqrt{R_o/g_o}$ = 84,4 min	1.2
T_S	Schuler period	1.2
U	voltage	5.3.2

α	pendulum or platform inertial rotation angle (except in par. 4.3.3/4.3.4/4.3.9/6.4.1)	2.2
δ	pendulum (or platform) deviation from local vertical (or horizontal)	2.3
λ	geographic latitude	3.1
ρ	density of liquid	6.5
$\ddot{\sigma}$	gravity component sensed by accelerometer	2.3
τ	time constant	5.1.2
ω	angular frequency of oscillation	5.1.5
ω	angular velocity	6.4
ω_e	earth rotation rate = 73 µrad/s	3.
θ	angle of travel along circular trajectory	2.2
Ω	same as ω_e	3.1

8. REFERENCES

The list has been arranged chronologically.

An asterisk * preceding a year of publication marks references not consulted by ourselves. They are somewhat less easily accessible, and we believe that to make our point the additional information they possibly could offer is not strictly required. But we found them cited by other authors, and we include them to more or less complete the chain of articles dealing with the subject.

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