PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

THE SCHWARZIAN DERIVATIVE AND UNIVALENT FUNCTIONS

SHIGEO OZAKI AND MAMORU NUNOKAWA

ABSTRACT. In this paper we prove under certain conditions the function $w=f(z)$ is univalent in $|z|<1$.

It is customary to formulate the inequalities of the "Verzerrungssatz" type for analytic functions $w=f(z)$, schlicht in the unit circle, with reference to a specific normalization. The two normalizations mainly used are: (a) $f(z)$ is finite in $|z|<1$, $f(0)=0$, $f'(0)=1$; (b) $f(z)$ has a pole at $z=0$ with the residue 1. If we want to obtain inequalities which are independent of any particular normalization, we have to use quantities which are invariant with regard to an arbitrary linear transformation of the z-plane. The simplest quantity of this type is the Schwarzian differential parameter

$$\{w, z\} = (w''/w')' - \tfrac{1}{2}(w''/w')^2,$$

also called the Schwarzian derivative of w with regard to z.

It is easy to obtain an upper bound for $\{w, z\}$ by a simple transformation of the classical inequality $|a_1|\leqq 1$ valid for functions $w=f(z)=z^{-1}+a_0+a_1z+\cdots$ schlicht in the unit circle. Indeed, applying this inequality to the coefficient of z in the expansion of the schlicht function

$$\begin{aligned} g(z) &= \frac{f'(x)(1-|x|^2)}{f((z+x)/(1+\bar{x}z)) - f(x)} \\ &= \frac{1}{z} + \bar{x} - \frac{1}{2}\frac{f''(x)}{f'(x)}(1-|x|^2) \\ &\quad - \frac{1}{6}(1-|x|^2)^2\left[\left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right]z + \cdots, \quad |x|<1, \end{aligned}$$

we obtain $|\{w, z\}|\leqq 6/(1-|z|^2)^2$ [3, p. 226].

It is known that by replacing the number 6 in this inequality by 2, this necessary condition for the univalence of $f(z)$ in $|z|<1$ becomes sufficient ([4], [2], [5], [1]).

Received by the editors March 8, 1971.
AMS 1969 *subject classifications*. Primary 3040, 3042.
Key words and phrases. Univalent functions, Schwarzian derivative.

Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular in $|z|<1$ and

$$g(z) = \frac{f'(x)(1-|x|^2)}{f((z+x)/(1+\bar{x}z)) - f(x)} = \frac{1}{z} + h(z, x), \qquad |x| < 1.$$

Then $f(z)$ is univalent in $|z|<1$ if and only if $g(z)$ is univalent in $|z|<1$. In this case, the above theorem can be stated as follows:

THEOREM (NEHARI [1]). *In order that the function $w=f(z)$ be univalent in $|z|<1$, it is necessary that*

$$|h'(0, x)| = |\tfrac{1}{6}(1-|x|^2)^2\{f, x\}| \leqq 1 \quad \text{for } |x| < 1$$

and sufficient that

$$|h'(0, x)| \leqq \tfrac{1}{3} \quad \text{for } |x| < 1.$$

We can now prove the following similar theorem for a sufficient condition that the function $w=f(z)$ be univalent in $|z|<1$.

THEOREM 1. *In order that the function $w=f(z)$ be univalent in $|z|<1$, it is sufficient that*

$$|h'(z, x)| \leqq 1 \quad \text{for } |z| < 1$$

where x is in the unit disk.

PROOF. Evidently, $g(z)-1/z=h(z, x)$ is regular in $|z|<1$ and

$$(g(z_2) - 1/z_2) - (g(z_1) - 1/z_1) = h(z_2, x) - h(z_1, x) = \int_{z_1}^{z_2} h'(z, x)\, dz$$

where the integral is taken on the line segment z_1z_2, $z_1 \neq z_2$, $|z_1|<1$ and $|z_2|<1$. Putting

$$z = z_1 + t(z_2 - z_1), \qquad 0 \leqq t \leqq 1,$$

and

$$dz = (z_2 - z_1)\, dt$$

the above integral can be written in the form

$$\int_0^1 (z_2 - z_1) h'(z, x)\, dt.$$

Multiplying $|-z_1z_2/(z_2-z_1)|$ on both sides we have

$$\left| \frac{(g(z_1) - g(z_2))z_1z_2}{z_2 - z_1} - 1 \right| = \left| z_1z_2 \int_0^1 h'(z, x)\, dt \right|$$
$$< \int_0^1 |h'(z, x)|\, dt \leqq \int_0^1 dt = 1.$$

This shows that $g(z_2)-g(z_1)\neq 0$ for $z_1\neq z_2$ and therefore $f(z)$ is univalent in $|z|<1$.

THEOREM 2. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular in $|z|<1$ and*

$$\mathrm{Re}(f^2(z)/z^2f'(z)) \geqq \tfrac{1}{2} \quad \textit{in } |z| < 1.$$

Then $f(z)$ is univalent in $|z|<1$.

PROOF. Applying Theorem 1 to $f(z)$, $f(z)$ is univalent in $|z|<1$ if $|h'(z, 0)|\leqq 1$ in $|z|<1$. By the maximum principle, $|z^2h'(z, 0)|\leqq 1$, in $|z|<1$, implies $|h'(z, 0)|\leqq 1$ in $|z|<1$. On the other hand, it is easily confirmed that

$$|-z^2h'(z, 0)| = |z^2f'(z)/f^2(z) - 1|.$$

Therefore if we suppose that

$$|z^2f'(z)/f^2(z) - 1| \leqq 1 \quad \text{in } |z| < 1$$

or

$$\mathrm{Re}(f^2(z)/z^2f'(z)) \geqq \tfrac{1}{2} \quad \text{in } |z| < 1,$$

then $f(z)$ is univalent in $|z|<1$.

REFERENCES

1. S. Friedland and Z. Nehari, *Univalence conditions and Sturm-Liouville eigenvalues*, Proc. Amer. Math. Soc. **24** (1970), 595–603. MR **40** #4435.
2. E. Hille, *Remarks on a paper by Zeev Nehari*, Bull. Amer. Math. Soc. **55** (1949), 552–553. MR **10**, 697.
3. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR **13**, 640.
4. ———, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551. MR **10**, 696.
5. V. V. Pokornyĭ, *On some sufficient conditions for univalence*, Dokl. Akad. Nauk SSSR **79** (1951), 743–746. (Russian) MR **13**, 222.

DEPARTMENT OF MATHEMATICS, SCIENCE UNIVERSITY OF TOKYO, TOKYO, JAPAN

DEPARTMENT OF MATHEMATICS, GUNMA UNIVERSITY, MAEBASHI, JAPAN