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The "Screening Phase Transitions" in the Two-Dimensional Coulomb Gas

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The Mayer series of a Coulomb gas with fixed ultraviolet cutoff is studied in two dimensions. In particular, we show the existence of infinitely many thresholds $T_n = (e^2/8\pi k)(1 - 1/2n)^{-1}$, k = Boltzmann's constant, e = electric charge, n = 1, 2,..., which are conjectured to reflect a sequence of transitions from pure multipole phase (the Kosterlitz–Thouless region) to a plasma phase (the Debye screening region) via an infinite number of "intermediate phases." Mathematically we prove that the Mayer series' coefficients of order up to $2n$ are finite if the temperature T is $< T_n$. For $T < T_\infty$ all the coefficients are finite and the gas can be formally interpreted as a multipole gas with multipoles with finite activity.

KEY WORDS: Coulomb gas; sine-Gordon field theory; renormalization group; multipole gas.

1. INTRODUCTION

We define the partition function for the two-dimensional Coulomb gas in a box I and with ultraviolet cutoff as

$$Z_0(I, \beta, \lambda) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^n \frac{1}{n!} \sum_{\substack{\sigma_1,\ldots,\sigma_n \\ \sum_i \sigma_i = 0}} \int_{I^n} dx_1 \cdots dx_n \times \exp\left[-\beta \sum_{i<j} \sigma_i \sigma_j W(x_i - x_j)\right] \equiv \lim_{R\to\infty} Z_0^{(-R)}(I, \beta, \lambda) \tag{1.1}$$

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where β is the "inverse temperature," $\beta = 1/kT$, λ is the "activity" of the charges, $\sigma_1, ..., \sigma_n$ are the charges of the particles at positions $x_1, ..., x_n$, W is the Coulomb potential with ultraviolet cutoff:

$$\begin{aligned} W(x, y) &= \frac{1}{(2\pi)^2} \int d^2k \, \frac{1}{k^2} \frac{e^{ik(x-y)} - 1}{k^2 + 1} \\ &= \lim_{R \to \infty} \frac{1}{(2\pi)^2} \int d^2k (e^{ik(x-y)} - 1) \left(\frac{1}{k^2 + \gamma^{-2R}} - \frac{1}{k^2 + 1} \right) \\ &\equiv \lim_{R \to \infty} W^{(-R)}(x, y) \end{aligned} \tag{1.2}$$

and $Z_0^{(-R)}(I, \beta)$ is defined as the intermediate term in (1.1) with $W^{(-R)}$ replacing W, where we have arbitrarily fixed $\gamma > 1$; $Z_0^{(-R)}$ is the partition function for the neutral gas with infrared cutoff at scale γ^R.

The name's motivation is clear if one remarks that for $|x - y| \gg 1$,

$$W(x - y) \simeq \frac{1}{2\pi} \log |x - y|^{-1} \tag{1.3}$$

If one expands $|I|^{-1} \log Z_0(I, \beta, \lambda)$ in powers of λ it is very unclear what happens to the coefficients of the resulting power series in the limit as $|I| \to \infty$.

In this paper we prove that, if $\beta > \beta_n = (8\pi/e^2)(1 - 1/2n)$, the first $2n$ coefficients have a limit as $I \nearrow R^2$. Our bounds are very poor in their n dependence.

This statement is proved by showing that important cancellations take place. The tool we use for exhibiting such cancellations is the "sine-Gordon transformation" whereby the problem is reduced to a field theory problem and the cancellations become related to a renormalizability problem of rather simple nature, which has been basically understood in Benfatto *et al.*[1] and Nicoló.[2]

2. THE SINE-GORDON TRANSFORMATION

It is convenient to observe that

$$\lim_{R \to \infty} Z_0^{(-R)}(I, \beta, \lambda) \equiv \lim_{R \to \infty} Z^{(-R)}(I, \beta, \lambda) \tag{2.1}$$

where the Z function in the right-hand side denotes a suitable nonneutral Coulomb gas partition function, i.e.,

$$Z^{(-R)}(I, \beta, \lambda) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^n \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n} \int_{I^n} dx_1 \cdots dx_n$$
$$\times \exp\left[-\frac{\beta}{2}\left(\sum_i \sigma_i\right)^2 C^{(-R,0)}(0,0)\right]$$
$$\times \exp\left[-\beta \sum_{i<j} \sigma_i \sigma_j W^{(-R)}(x_i, x_j)\right] \tag{2.2}$$

where we have introduced, for later use,

$$C^{(-R,0)}(x, y) = \frac{1}{(2\pi)^2} \int d^2k \, e^{ik(x-y)} \left(\frac{1}{k^2 + \gamma^{-2R}} - \frac{1}{k^2+1}\right) \tag{2.3}$$

so that $C^{(-R,0)}(0, 0) = (1/2\pi) \log \gamma^R$, making (2.1) obvious.

Of course there are many other weights which depress the charged configurations. The one used in (2.1) is, however, very significant as it allows us to turn the problem of studying $Z^{(-R)}(I, \beta, \lambda)$ into a field theory problem.

In fact, let $\psi^{(-R,0)}$ be the Gaussian random field on R^2 with covariance (2.3). It is easy to check (and it is well known; see Frohlich[3] and Park[4]) that if $e^2\beta = \alpha^2$,

$$Z^{(-R)}(I, \beta, \lambda) = \int P(d\psi^{(-R,0)}) \exp \lambda \int_I \cos \alpha\psi_x^{(-R,0)} \, dx$$
$$\equiv \int P(d\psi^{(-R,0)}) \exp\left[\sum_{\sigma=\pm 1} \frac{\lambda}{2} \int_I \exp(i\alpha\sigma\psi_x^{(-R,0)}) \, dx\right] \tag{2.4}$$

To prove the identity (2.4) just expand the exponential as a power series and use

$$\int P(d\psi^{(-R,0)}) \exp\left(\sum_{j=1}^{n} i\alpha\sigma_j \psi_{x_j}^{(-R,0)}\right) = \exp\left[-\frac{\alpha^2}{2} \sum_{i,j} \sigma_i \sigma_j C^{(-R,0)}(x_i - x_j)\right] \tag{2.5}$$

after some simple algebra to treat the diagonal terms in (2.5) one obtains (2.2).

3. A DIGRESSION ON THE BASIC PROPERTIES OF GAUSSIAN INTEGRALS

We shall make extensive use of the following properties of Gaussian integrals which we summarize here without proof; they are well-known consequences of definitions combined with the basic identity (2.5).

Let $\psi^{(0)}, \psi^{(1)},...$ be independently distributed Gaussian fields indexed by $x \in \Omega$, Ω being an arbitrary set of indices (R^2 in our applications).

Denote $\mathscr{E}_{(j)}$ the expectation with respect to $\psi^{(j)}$, i.e., the integral over the distribution $P(d\psi^{(j)})$ of $\psi^{(j)}$. Similarly $\mathscr{E}_{(j,p)} = \mathscr{E}_{(j)} \cdots \mathscr{E}_{(p)}$; $\mathscr{E}$ denotes the expectation with respect to all the ψs.

If $f_1,..., f_q$ are q linear functionals of $\psi^{(0)},...$ (i.e., if $f_1,..., f_q$ are q linear Gaussian random variables, one defines

$$:e^{if_k}: = e^{(\alpha^2/2)\mathscr{E}(f_k^2)} e^{i\alpha f_k} \tag{3.1}$$

"Wick ordered exponential."

If $F_1,..., F_q$ are any q random variables, one defines

$$\begin{aligned}&\mathscr{E}^T(F_1,..., F_q; n_1,..., n_q)\\ &\quad = \frac{\partial^{n_1 + \cdots + n_q}}{\partial\lambda_1^{n_1} \cdots \partial\lambda_q^{n_q}} \log \mathscr{E}\left(\exp\left(\sum_{j=1}^{q} \lambda_j F_j\right)\right)\bigg|_{\lambda=0}\end{aligned} \tag{3.2}$$

and the beautiful "Wick's theorem" states (see, for instance, Gallavotti, Ref. 5, Appendix A):

$$\begin{aligned}&\mathscr{E}^T(:e^{i\alpha f_1}:,..., :e^{i\alpha f_q}:; 1, 1,..., 1)\\ &\quad = \sum_{\tau \in \mathfrak{a}} \prod_{\lambda \in \tau} (e^{-\alpha^2 C(\lambda)} - 1)\end{aligned} \tag{3.3}$$

where $\mathfrak{a}$ is the set of connected graphs joining q points so that no pair of points is joined by more than one link and $\lambda = (i, j)$ is a graph line joining (i, j) and

$$C(\lambda) = \mathscr{E}(f_i f_j) \qquad \text{if} \quad \lambda = (i, j) \tag{3.4}$$

Note that if f_1 and f_2 are independent

$$:e^{i(f_1 + f_2)}: = :e^{if_1}::e^{if_2}: \tag{3.5}$$

With the above notations, (2.4) is rewritten

$$\begin{aligned}Z^{(-R)}(I, \beta, \lambda) &= \int P(d\psi^{(-R,0)})\\ &\quad \times \left\{\exp\left[\frac{\lambda}{2} \exp\left(-\frac{\alpha^2}{2} C^{(-R,0)}(0, 0)\right) \sum_{\sigma = \pm 1} \int_I :\exp(i\alpha\sigma\psi_x^{(-R,0)}): dx\right]\right\}\end{aligned} \tag{3.6}$$

4. DUALITY TRANSFORMATION OF THE INFRARED PROBLEM INTO AN ULTRAVIOLET PROBLEM

The "infrared" problem of studying (2.4) or (3.6) involves the properties of the field $\psi^{(-R,0)}$, which is a "Yukawa field" with mass γ^{-R} and cutoff 1.

By a trivial change of scale this can be changed into a Yukawa field with mass 1 and cutoff γ^{+R}.

Simply define

$$\varphi_\xi^{(<R)} = \psi_{\gamma^R \xi}^{(-R,0)} \tag{4.1}$$

whose covariance is [see (2.3)]

$$C_{\xi\eta}^{(<R)} = \frac{1}{(2\pi)^2} \int d^2p\, e^{ip(\xi-\eta)} \left(\frac{1}{p^2+1} - \frac{1}{p^2+\gamma^{2R}} \right) \tag{4.2}$$

so that

$$Z^{(-R)}(I, \beta, \lambda) = \int P(d\varphi_\xi^{(\leq R)})$$
$$\times \exp\left[\sum_{\sigma=\pm 1} \frac{\lambda\gamma^{2R}}{2} \exp\left(-\frac{\alpha^2}{2} C^{(-R,0)}(0,0) \right) \int_{I\gamma^{-R}} :\exp(i\sigma\alpha\varphi_\xi^{(<R)}):\, d\xi \right] \tag{4.3}$$

We see that (4.3) is a "charged Yukawa gas with ultraviolet cutoff γ^R, mass 1, coupling

$$\frac{\lambda\gamma^{2R}}{2} \exp\left(-\frac{\alpha^2}{2} C^{(-R,0)}(0,0) \right) \equiv \frac{\lambda}{2} \gamma^{2R-(\alpha^2/4\pi)R}$$

and volume $\gamma^{-R}I$."

So we apply the methods for the massive Yukawa gases to study (4.3), based on the "renormalization group," developed in Benfatto *et al.*[6,1] (see Gallavotti, Ref. 5, for a review).

5. MULTISCALE DECOMPOSITION OF THE SINE-GORDON FIELD. EFFECTIVE POTENTIAL AND ITS GRAPHICAL REPRESENTATION

Let

$$C^{(0)}(\xi, \eta) = \frac{1}{(2\pi)^2} \int d^2p\, e^{ip(\xi-\eta)} \left(\frac{1}{1+p^2} - \frac{1}{\gamma^2+p^2} \right) \tag{5.1}$$

and observe the trivial identity

$$C^{(<R)}(\xi,\eta)=\sum_{k=0}^{R-1} C^{(k)}(\xi,\eta), \qquad C^{(k)}(\xi,\eta)=C^{(0)}(\gamma^k_\xi,\gamma^k_\eta) \tag{5.2}$$

This allows us to represent $\varphi^{(<R)}$ as a sum of R independent Gaussian fields $\varphi^{(0)}$, $\varphi^{(1)}$,..., $\varphi^{(R)}$,

$$\varphi^{(<R)}_\xi=\sum_{k=0}^{R-1}\varphi^{(k)}_\xi \tag{5.3}$$

with $\varphi^{(k)}$ having covariance $C^{(k)}$. It should be observed that this representation of $\varphi^{(<R)}$ has the scale-invariance property that all the fields $\varphi^{(k)}$ have identical distribution up to a trivial scaling. Namely, $\varphi^{(k)}_\xi$ has the same distribution as $\varphi^{(0)}_{\gamma^k\xi}$.

Given

$$V(\varphi^{(<R)})=\sum_{\sigma=\pm 1}\frac{\lambda}{2}\gamma^{(2-\alpha^2/4\pi)R}\int_{I\gamma^{-R}} :\exp(i\alpha\sigma\varphi^{(<R)}_\xi):\,d\xi \tag{5.4}$$

we define the "effective potential" on "scale k" by

$$\exp[V^{(k)}(\varphi^{(\leqslant k)})]=\int \exp[V(\varphi^{(<R)})\,P(d\varphi^{(k+1)})\cdots P(d\varphi^{(R-1)})] \tag{5.5}$$

and $V^{(k)}$, defined by (5.5), can be computed by a recursive application of the formula [see (3.2)],

$$\begin{aligned}\log \mathscr{E}(e^F)&=\sum_{k=1}^{\infty}\frac{\mathscr{E}^T(F;k)}{k!}=\sum_{k=1}^{\infty}\frac{\mathscr{E}^T(F,..., F;1,..., 1)}{k!}\\&=\sum_{k=1}^{\infty}\frac{\mathscr{E}^T(F,..., F)}{k!}\end{aligned} \tag{5.6}$$

where we eliminate the 1s in the truncated expectations of k variables: $\mathscr{E}^T(F,..., F;1,..., 1)\equiv\mathscr{E}^T(F,..., F)$, to simplify the notations. Note that $\mathscr{E}^T(F,..., F;1,..., 1)=\mathscr{E}^T(F;k)$.

Applying (5.6) to (5.5) we see that

$$\begin{aligned}V^{(R-2)}&=\sum_{n=1}^{\infty}\frac{\mathscr{E}^T_{(R-1)}(V;n)}{n!}\\V^{(R-3)}&=\sum_{m=1}^{\infty}\frac{1}{m!}\mathscr{E}^T_{(R-2)}\left(\sum_{n=1}^{\infty}\frac{\mathscr{E}^T(V;n)}{n!};m\right)\end{aligned} \tag{5.7}$$

and so on.

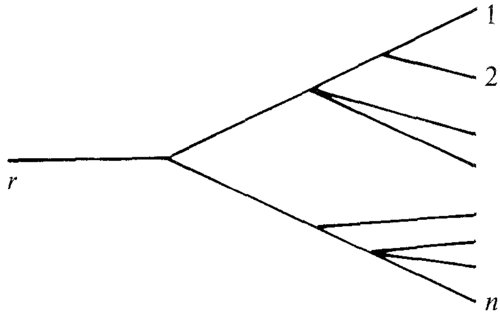

Fig. 1

The (5.7) has an obvious graphical interpretation (see Gallavotti and Nicoló[7] and Gallavotti[5]).

Let θ be a tree graph (Fig. 1) with n end points and one root r (i.e., one single branch connects the lowest tree vertex to the next vertex). The tree will be ordered from the root toward the end points. All the vertices different from the root and the end points will be called "inner vertices."

We define $n(\theta)$ as follows: If θ is a trivial tree, i.e., with no inner vertices, $\theta \equiv {}_r\!\!\longrightarrow_1$, we set $n(\theta) = 1$. If θ branches after the root into s subtrees, then let $\bar\theta_1,\dots,\bar\theta_p$ be the ones among $\theta_1,\dots,\theta_s$ which are pairwise distinct and let $q_1,\dots,q_p$ be the multiplicity that $\bar\theta_1,\dots,\bar\theta_p$ have in $\theta_1,\dots,\theta_s$. Then let $n(\theta) = q_1!(n(\bar\theta_1))^{q_1}\cdots q_p!(n(\bar\theta_p))^{q_p}$. This allows us to define $n(\theta)$ inductively.

We now interpret each end point of θ as representing V. We append to each vertex v of θ a label h_v, $-1 \leqslant h_v \leqslant R$, so that the labels strictly increase as one climbs the tree. Then we interpret each inner vertex v of the decorated tree $(\theta, \mathbf{h})$ as representing $\mathscr{E}^T_{(h_v)}$; i.e., a truncated expectation operation with respect to the field $\varphi^{(h_v)}$ and each tree branch joining a pair of vertices v', v, $v' < v$, as $\mathscr{E}_{(h_{v'}+1)}\mathscr{E}_{(h_{v'}+2)}\cdots\mathscr{E}_{(h_v-1)}$.

Then, given $(\theta, \mathbf{h})$ we can define $V(\theta, \mathbf{h})$ as follows:

$$V(\theta, \mathbf{h}) = \mathscr{E}_{h_r+1}\cdots\mathscr{E}_{h_{v_0}-1}\mathscr{E}^T_{h_{v_0}}(V(\theta_1;\mathbf{h}_1),\dots, V(\theta_s;\mathbf{h}_s)) \tag{5.8}$$

if v_0 is the first inner vertex of θ out of which bifurcate s trees $\theta_1,\dots,\theta_s$-with the labels $\mathbf{h}_1,\dots,\mathbf{h}_s$ inherited from $(\theta, \mathbf{h})$. Equation (5.8) allows us to define $V(\theta, \mathbf{h})$ inductively in terms of $V(\theta, \mathbf{h})$ for the trivial tree ${}^{h_r}_{r}\!\!\longrightarrow^{R}$ which, of course, will be defined to be $\mathscr{E}_{h_r+1}\cdots\mathscr{E}_{R-1}(V)$. Hence $V(\theta, \mathbf{h})$ has a meaning for all the tree's decorations $\mathbf{h}$ in which the end points are given scale index R: "admissible trees." Since we only consider here such trees we shall not explicitly write in the pictures the end points' scale index.

It is easy to check by induction that (5.7) can be written as

$$V^{(k)}=\sum_{\substack{(\theta,\mathbf{h})\\ h_r=k}}\frac{V(\theta,\mathbf{h})}{n(\theta)} \tag{5.8}$$

Clearly $V(\theta, \mathbf{h})$ is an integral over n points if θ has n end points and a sum over n charges $\sigma_1,..., \sigma_n=\pm 1$ as follows from (5.4).

Therefore, it is convenient to have an expression for the integral's argument.

For this purpose we simply add to the tree θ other decorations: namely, the ith endpoint will be given indices x_i, σ_i and the interpretation of the ${}^{h}\!\!-\!\!\!-\!\!\!-\!\!\!-_{(\xi,\sigma)}$ branches will be (if $\theta_0\equiv$ trivial tree)

$$\bar{V}(\theta_0, h, \sigma, \xi)=\frac{\lambda}{2}\gamma^{(2-\alpha^2/4\pi)R}{:}\exp(i\alpha\sigma\varphi_\xi^{(\leqslant h)}){:} \tag{5.9}$$

If we denote $\gamma=(\theta, \mathbf{h}, \boldsymbol{\sigma}, \boldsymbol{\xi})$ a decorated tree of this new type

$$V^{(k)}=\sum_{n=1}^{\infty}\int d\xi_1\cdots d\xi_n\sum_{\sigma_1\cdots\sigma_n}\sum_{\substack{\theta\\ N(\theta)=n}}\sum_{\mathbf{h};h_r=k}\frac{\bar{V}(\gamma)}{n(\theta)} \tag{5.10}$$

where $N(\theta)$ is the number of end points of θ and $\bar{V}(\gamma)$ is defined as in (5.8) recursively:

$$\bar{V}(\gamma)=\mathscr{E}_{h_r+1}\cdots\mathscr{E}_{h_{v_0}-1}\mathscr{E}^T_{h_{v_0}}(\bar{V}(\gamma_1),..., \bar{V}(\gamma_s)) \tag{5.11}$$

with obvious notations.

6. THE COULOMB GAS AS A MULTIPOLE GAS

We find now an explicit form for (5.11) and give a possible interpretation to it.

Given a decorated tree γ and vertex $v>r\equiv$ root of γ, we define v' as the vertex of γ immediate predecessor of v, $v^{(1)},..., v^{(s_v)}$ the immediate successors of v, γ_v the subtree of γ consisting of v', v and the successors of v, and

$$\varphi^{(\cdot)}(\gamma_v)=\sum_{i\in v}\sigma_i\varphi_{\xi_i}^{(\cdot)} \tag{6.1}$$

where $i\in v$ means that the end point ξ_i is an end point of γ_v; one could also write $i\geqslant v$, $i\equiv$ end point.

If $(\theta, \mathbf{h}, \boldsymbol{\sigma}, \boldsymbol{\xi}) = \gamma$ we call $S(\gamma) =$ "shape of γ" $= (\theta, \boldsymbol{\sigma})$ and put $n(\gamma) \equiv n(\theta)$.

A tree γ, actually its shape, is allowed to identify a family of hierarchically arranged clusters of end points' positions. We call v also the cluster $(\xi_i)_{i \in v}$ of end point positions of γ_v and

$$d(v) \equiv \text{diameter of the cluster } v \tag{6.2}$$

We define, given γ and two vertices v, $w \in \gamma$,

$$C^{(\cdot)}(v, w) \equiv \mathscr{E}(\varphi^{(\cdot)}(\gamma_v)\, \varphi^{(\cdot)}(\gamma_w)) \equiv \sum_{\substack{i \in v \\ j \in w}} C^{(\cdot)}(\xi_i, \xi_j)\, \sigma_i \sigma_j \tag{6.3}$$

where $(\cdot)$ denotes any index [e.g., (h) or $(<h)$ or others that will be introduced later] and $C^{(\cdot)}$ denotes the appropriate covariance. So (6.3) is some kind of electrostatic energy between the clusters v and w. If $v = w$ it is a "self-energy."

Then one guesses that

$$\bar{V}(\gamma) = \tilde{V}(\gamma) {:}\exp[i\alpha\varphi^{(\leqslant h_r)}(\gamma)]{:} \tag{6.4}$$

where $\tilde{V}(\gamma)$ is a suitable function of γ and one easily proves inductively a recursion relation on $\tilde{V}(\gamma)$. Namely,

$$\tilde{V}(\gamma) = \left[\prod_{i=1}^{s_v} \tilde{V}(\gamma_i)\right] \exp\left[-\alpha^2 \sum_{i<j}^{1, s_v} C^{(<h_v)}(v^{(i)}, v^{(j)})\right]$$
$$\times \left(\sum_{\tau \in \mathfrak{a}_v} \prod_{(i,j) \in \tau} \{\exp[-\alpha^2 C^{(h_v)}(v^{(i)}, v^{(j)})] - 1\}\right) \tag{6.5}$$

which is an immediate consequence of (5.11) combined with the ansatz (6.4) and Wick's theorem (3.3), if v denotes the first inner vertex of γ and $\mathfrak{a}_v$ are the graphs described in (3.3) joining s_v abstract points (which can be concretely identified with the clusters $v^{(1)}, \ldots, v^{(s_v)}$ which are enclosed in v thought as a cluster).

Equation (6.5) reduces, inductively, the proof of the ansatz (6.4) to the case in which γ has no inner vertex: in this case it is, however, obvious as (5.9) says.

Equation (6.5) allows us to write

$$\log Z^{(-R)}(\lambda, I, \beta) = \sum_{n=1}^{\infty} \sum_{\substack{\theta; N(\theta) = n \\ \mathbf{h}, h_r = -1}} \sum_{\sigma_1, \ldots, \sigma_n} \int_{\gamma^{-R} I} \frac{\tilde{V}(\gamma)}{n(\gamma)}\, d\xi_1 \cdots d\xi_n \tag{6.6}$$

which is, of course, the Mayer series for our Coulomb gas. Bounds on the nth order term in (6.6), obviously proportional to λ^n, will be discussed in following sections.

Here we make a single remark which permits the interpretation of the Coulomb gas partition function as the partition function of a set of multipoles with *short-range* forces.

For this purpose we go back to (3.6) and define the fields ψ and $\psi^{(-R,-1)}$ with covariances given, respectively, by

$$C_{xy}=\frac{1}{(2\pi)^2}\int dp\, e^{ip(x-y)}\left(\frac{1}{p^2+\gamma^{-2}}-\frac{1}{p^2+1}\right)\equiv C(x-y)$$
$$C_{xy}^{(-R,-1)}=\frac{1}{(2\pi)^2}\int dp\, e^{ip(x-y)}\left(\frac{1}{p^2+\gamma^{-2R}}-\frac{1}{p^2+\gamma^{-2}}\right) \tag{6.7}$$

so that $\psi+\psi^{(-R,-1)}$ has the same distribution as $\psi^{(-R,0)}$ [see (2.3)].

Then $\psi^{(-R,-1)}$ has the same distribution up to scaling as $\varphi^{(<R-1)}$ [see (4.1)] and using [see (3.5)],

$$\begin{aligned}&:\exp[i\alpha\sigma(\psi_{\gamma^R\xi}+\psi_{\gamma^R\xi}^{(-R,-1)})]:\\&\quad=\exp\left(\frac{\alpha^2}{2}C_{00}\right)\exp(i\alpha\sigma\psi_{\gamma^R\xi}):\exp(i\sigma\alpha\varphi_\xi^{(<R-1)}):\end{aligned} \tag{6.8}$$

we see, from (4.3), that

$$\begin{aligned}Z^{(-R)}(\lambda,I,\beta)=\int P(d\psi)\Bigg\{&\exp\sum_{n=1}^{\infty}\sum_{\boldsymbol{\sigma}}\sum_{\substack{\theta,N(\theta)=n\\ \mathbf{h},h_r=-1}}^{*}\int_{\gamma^{-R}I}\\&\times d\xi_1\cdots d\xi_n\cdot\left[:\prod_{j=1}^{n}\left(\frac{\lambda}{2}\gamma^{2R}\gamma^{-(\alpha^2/4\pi)R}\exp(i\alpha\sigma_j\psi_{\gamma^R\xi_j})\right):\right]\\&\times\hat{V}(\theta,\mathbf{h},\boldsymbol{\sigma},\boldsymbol{\xi})\exp\left[-\alpha^2\sum_{i<j}\sigma_i\sigma_jC(\gamma^R(\xi_i-\xi_j))\right]\Bigg\}\end{aligned} \tag{6.9}$$

where $(1/2\pi)\log\gamma^{(R-1)}$ is the covariance at 0 (i.e. $C_{00}^{(-R,-1)}$) and the * recalls that the trees considered in the above formula differ from the ones so far considered because the end point frequency is $R-1$ instead of R and the $\hat{}$ on V reminds one that in the inductive definition (5.11) one starts from (5.9) *without* the factor $(\lambda/2)\,\gamma^{2R}\gamma^{(-\alpha^2/4\pi)R}$ and with $\varphi^{(<R-1)}$ instead of $\varphi^{(<R)}$.

So, for suitably defined Λ_{σ} [see (9.1)], (6.9) implies that $Z^{(-R)}$ can be written as

$$Z^{(-R)}(\lambda, I, \beta) = \int P(d\psi) \exp \sum_{n=1}^{\infty} \sum_{\sigma_1 \cdots \sigma_n} \int_{I^n} \Lambda_{\sigma}(x_1 \cdots x_n)$$
$$\times :\exp\left(i\alpha \sum_j \sigma_j \psi_{x_j}\right): dx_1 \cdots dx_n \tag{6.10}$$

If we develop the exponential in powers and perform the (trivial) Gaussian integrals proceeding in the same way used to prove the identity between (2.2) and (2.4), we realize that (6.10) can be interpreted as the partition function of a gas of multipoles. There will be multipoles localized in $dx_1\, dx_2 \cdots dx_n$ with charge σ_i in dx_i. Multipoles differing by a translation will be considered "of the same species"; their activity is

$$\Lambda_{\sigma_1 \cdots \sigma_n}(x_1, ..., x_n)\, dx_2 \cdots dx_n \tag{6.11}$$

and two multipoles interact with energy

$$\alpha^2 \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_i \sigma'_j C_{x_i y_j} \tag{6.12}$$

with C given by (6.7). So the interaction is a short-range "electrostatic" interaction with positive Fourier transform and ultraviolet cutoff too.

Of course, if one wants to go beyond a formal interpretation, one has to check that the total activity of each species is well defined even as $R \to \infty$.

The following seems to us a reasonable definition of the "activity of the multipole Q": Consider the Gibbs factor associated with N multipoles $(X_1, \boldsymbol{\sigma}_1), ..., (X_N, \boldsymbol{\sigma}_N)$ of positions $X_1 = (x_1^{(1)}, ..., x_{n_1}^{(1)}), ..., X_N = (x_1^{(N)}, ..., x_{n_N}^{(N)})$ and charges $Q^{(i)}$:

$$\sum_{\substack{\boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_N \\ n_1, ..., n_N \text{fixed}}} \Lambda_{\boldsymbol{\sigma}_1}(X_1) \cdots \Lambda_{\boldsymbol{\sigma}_N}(X_N) \exp[-U_{\boldsymbol{\sigma}_1 \cdots \boldsymbol{\sigma}_N}(X_1, ..., X_N)] \tag{6.13}$$

and suppose that the integral of this expression over $X_1, ..., X_N$, with one point fixed in each X_j, can be bounded by

$$\prod_{j=1}^{N} \zeta(n_i) \tag{6.14}$$

Then we say that the activity of the multipole of charge $Q^{(i)}$ is bounded by $\zeta(n_i)$, n_i being the number of charges in the multipole.

It will be a corollary of this paper that the bound (6.14) holds with an R-independent $\zeta(n)$ if $n \leqslant 2p$ and if $\beta > \beta_p$; furthermore $\zeta(n) \to_{R\to\infty} 0$ if n is odd.

In other words, the multipoles have finite activity in the above sense if the Mayer coefficients of the same order are finite as $R \to \infty$.

A better definition of the activities of the multipoles would be the following: Consider the integral of (6.13) over all the X_i keeping for each of them one point fixed and this quantity should be positive. If so, one would consider its logarithm and write it as

$$\tilde{\zeta}(Q_1^{(1)}) \cdots \tilde{\zeta}(Q^{(N)}) \exp[\tilde{U}_{Q_1,\ldots,Q_N}(\xi_1,\ldots,\xi_N)] \tag{6.15}$$

Then, if $\tilde{U}_{Q_1\cdots Q_N}(\xi_1,\ldots,\xi_N)$ can be interpreted as a short-range (many-body) potential and if the potentials of $\tilde{U}$ and the $\tilde{\zeta}$s admit a limit as $R \to \infty$, one would say that the multipoles have finite activity.

This second deeper sense will not be investigated here.

7. THE MAIN BOUND ON THE COEFFICIENTS OF MAYER SERIES

The results stated in the abstract follow easily from the following estimate:

Let θ be a tree and let $Q_v \geqslant 0$ be defined for each vertex v of θ.

Let $\boldsymbol{\sigma}$ be a $\mathbf{Q}$-admissible charge configuration on the end points of θ, i.e., a charge configuration such that the charge $Q(v)$ of the cluster v of end points ($Q(v) = \sum_{i\in v} \sigma_i$) verifies $|Q(v)| = Q_v$.

Then

$$\begin{aligned}
&\left| \sum_{\boldsymbol{\sigma}, |Q(v)| = Q_v} V(\theta, \mathbf{h}, \boldsymbol{\sigma}, \boldsymbol{\xi}) \right| \\
&\quad \leqslant K_n \left(\prod_{i=1}^{n} \gamma^{(\alpha^2/4\pi) h_{v_i}} \right) \cdot \gamma^{-(\alpha^2/4\pi) Q_{v_0}^2 h_{v_0}} \\
&\quad \times \left(\prod_{\substack{v > v_0 \\ v\,\text{inner}}} \gamma^{-(\alpha^2/4\pi) Q_v^2 (h_v - h_{v'})} \right) \left(\prod_{\substack{v > v_0 \\ Q_v = 0}} \gamma^{-2(1-\varepsilon)(h_v - h_{v'})} \right) \\
&\quad \times \left\{ \exp\left[-\bar{\kappa} \sum_{v > r} \gamma^{h_v} d(v) \right] \right\} (\lambda \gamma^{(2-\alpha^2/4\pi) R})^n
\end{aligned} \tag{7.1}$$

where K_n is a suitable constant $\bar{\kappa} > 0$, h_{v_i} is the frequency of the vertex to which the ith end point of θ is attached, v_0 is the first inner vertex of the tree θ, and n is the number of end points.

We postpone to the next section checking (7.1). Here we proceed to show that it easily implies that all the Mayer coefficients of order $\leqslant 2p$ are finite uniformly in R if $\alpha^2 > \beta_{2p}$, and tend to zero as $R \to \infty$ if they are of odd order $\leqslant 2p$.

The contribution of the trees with given θ to the virial coefficient of order n is, from (6.6), simply estimated by the integral of the right-hand side of (7.1) over all the ξs except one, times γ^{-2R}.[3]

It is easy to check that

$$\int \exp\left[-\bar{\kappa} \sum_{v>r} \gamma^{h_v} d(v)\right] d\xi_2 \cdots d\xi_n \leqslant K^n \prod_{v\,\text{inner}} \gamma^{-2(s_v-1)h_v} \tag{7.2}$$

where K is a suitable constant and s_v is the number of vertices in θ which follow v immediately. This will be left to the reader [see Gallavotti[5], for a proof, in Appendix A].

Substituting (7.2) into the integral of (7.1), one obtains the following bound on the contribution of θ to the nth-order Mayer series coefficient:

$$\begin{aligned}\sum_{\mathbf{h}} K_n K^n &\left(\prod_{i=1}^{n} \gamma^{(\alpha^2/4\pi)h_{v_i}}\right) \gamma^{-(\alpha^2/4\pi)Q_{v_0}^2 h_{v_0}} \\ &\times \left(\prod_{\substack{v\,\text{inner}\\ v>v_0}} \gamma^{-[(\alpha^2/4\pi)Q_v^2 + 2(1-\varepsilon)\delta_{Q_v 0}](h_v - h_{v'})}\right) \\ &\times \left(\prod_{v\,\text{inner}} \gamma^{-2(s_v-1)h_v}\right) \cdot \gamma^{(2-\alpha^2/4\pi)Rn}\gamma^{-2R}\end{aligned} \tag{7.3}$$

having used $C^{(-R,0)}(0,0) = (1/2\pi)R\log\gamma$ and $C^{(0)}(0,0) = (1/2\pi)\log\gamma$.

Using $\sum_i h_{v_i} = \sum_{v\,\text{inner}} n_v(h_v - h_{v'})$, $\sum_{v\,\text{inner}} h_v(s_v - 1) \equiv \sum_{v\,\text{inner}} (n_v - 1)(h_v - h_{v'})$ if $n_v \equiv$ number of points in the cluster then (7.3) becomes, if $a = \alpha^2/4\pi - 2$,

$$\begin{aligned}\tilde{K}_n \gamma^{-2R}\gamma^{-naR} \sum_{\mathbf{h}} &\left(\prod_{\substack{v>v_0\\ v\,\text{inner}}} \gamma^{[an_v + 2 - (\alpha^2/4\pi)Q_v^2 - 2(1-\varepsilon)\delta_{Q_v 0}](h_v - h_{v'})}\right) \\ &\times \gamma^{[an_{v_0} + 2 - (\alpha^2/4\pi)Q_{v_0}^2]h_{v_0}}\end{aligned} \tag{7.4}$$

So we have reduced the problem to the easy discussion of a bunch of geometric sums.

The analysis of (7.4) is as follows:

Consider first the case $\alpha^2 \geqslant 8\pi$.

[3] The γ^{-2R} comes from (6.6) which has to be divided by $|I|$ rather than $|I|\,\gamma^{-2R}$, while the integral over ξ_1 only gives $|I|\,\gamma^{-2R}$.

In this case $a \geqslant 0$ and (7.4) can be bounded by

$$\gamma^{-2R}\gamma^{-naR}\sum_{\{h_v\}}\gamma^{[an_{v_0}-(\alpha^2/4\pi)Q_{v_0}^2+2]h_{v_0}} \times \prod_{\substack{v>v_0\\ v\,\text{inner}}}\gamma^{(an_v+2\varepsilon)(h_v-h_{v'})}$$

Since $\sum_{v>v_0}(h_v-h_{v'}) \leqslant n(R-h_{v_0})$ this expression can be bounded by

$$\begin{aligned}
&\text{if } Q_{v_0}=0: \quad \sum_{\{h_v\}}\gamma^{-(2-2\varepsilon n)(R-h_{v_0})}\gamma^{-naR}\prod_{\substack{v>v_0\\ v\,\text{inner}}}\gamma^{an_v(h_v-h_{v'})}\\
&\text{if } Q_{v_0}\neq 0: \quad \gamma^{-(2-2\varepsilon n)R}\left[\gamma^{-aRn}\sum_{\{h_v\}}\prod_{\substack{v>v_0\\ v\,\text{inner}}}\gamma^{an_v(h_v-h_{v'})}\right]
\end{aligned} \tag{7.6}$$

Leaving the easier case $\alpha^2=8\pi$, $a=0$, to the reader, suppose $a>0$ and choose $\varepsilon<1/2n$. Then (7.6), (7.7) imply uniform boundedness of the order n Mayer coefficient if we show that

$$J=\sum_{\mathbf{h}}\gamma^{-aRn}\prod_{v\,\text{inner}}\gamma^{an_v(h_v-h_{v'})} \tag{7.7}$$

is uniformly bounded in R, which also implies that the odd coefficients vanish necessarily (because $Q_{v_0}\neq 0$).

To bound J observe that θ must contain a vertex w out of which bifurcate only branches ending in end points.
Summing over h_w, between $h_{w'}$ and R, alone yields

$$\begin{aligned}
J &\leqslant \gamma^{-aRn}\sum_{\{h_v\}_{v\neq w}}\frac{\gamma^{a(R-h_{w'})n_w}}{1-\gamma^{-a}}\prod_{\substack{v\,\text{inner}\\ v\neq w}}\gamma^{an_v(h_v-h_{v'})}\\
&=\frac{\gamma^{-aR(n-n_w)}}{(1-\gamma^{-a})}\sum_{\{h_v\}_{v\neq w}}\prod_{\substack{v\,\text{inner}\\ v\neq w}}\gamma^{an'_v(h_v-h_{v'})}\equiv\frac{J'}{(1-\gamma^{-a})}
\end{aligned} \tag{7.8}$$

where n'_v is the number of end points that can be reached from v *not* counting the n_w ones that can be reached from w.

Of course J' is very similar to J but "pertains to a simpler tree" with one vertex and n_w end points less. Hence repeating the argument

$$J \leqslant (1-\gamma^{-a})^{-n} \tag{7.9}$$

Suppose now that $\alpha^2 < 8\pi$, $\alpha^2 > \alpha_{2p}^2 = 8\pi(1-1/2p)$. In this case $a<0$ and by taking ε small enough the summations over h_v, v inner and $v > v_0$, in (7.4) do not cause any problem so that (7.4) is bounded by

$$\begin{aligned}
&\hat{K}_n \gamma^{-2R} \gamma^{-anR} \sum_{h_{v_0}=0}^{R} \gamma^{[an+2-(\alpha^2/4\pi)Q_{v_0}^2]h_{v_0}} \\
&\leqslant \begin{cases} \hat{K}_n \gamma^{R[-2-an+an+2-(\alpha^2/4\pi)Q_{v_0}^2]2,5]} = \hat{K}_n \gamma^{-(\alpha^2/4\pi)Q_{v_0}^2 R} \text{ or} \\ \hat{K}_n \gamma^{-2R} \gamma^{-naR} R \end{cases}
\end{aligned} \tag{7.10}$$

where the first bound holds if the series over h_{v_0} geometrically diverges if extended up to $h_{v_0} = +\infty$, i.e., if $an + 2 - (\alpha^2/4\pi)\, Q_{v_0}^2 > 0$, and the second holds if the series converges or diverges linearly. Clearly this is exactly the bound that we want, because $na + 2 > 0$ for $n \leqslant 2p$.

8. CANCELLATIONS

It remains to prove the bound (7.1).

The recursion relation (6.5) allows us to represent the quantity to bound in the left-hand side of (7.1) as the factor $[(\lambda/2)\,\gamma^{(2-\alpha^2/4\pi)R}]^n$ multiplied by the factor

$$\begin{aligned}
&\sum_{\substack{\sigma \\ |Q(v)| = Q_v}} \prod_{v \text{ inner}} \left\{ \exp\left[-\alpha^2 \sum_{i<j} C^{(<h_v)}(v^{(i)}, v^{(j)}) \right] \right. \\
&\qquad\qquad \left. \times \prod_{(i,j) \in \tau_v} (\exp[-\alpha^2 C^{(h_v)}(v^{(i)}, v^{(j)})] - 1) \right\}
\end{aligned} \tag{8.1}$$

where $v^{(1)},\ldots, v^{(s_v)}$ are the s_v immediate followers of v. The expression that we really have to bound is obtained by summing (8.1) over the selections of the graphs τ_v, per each v. Recall that τ_v is a connected graph connecting the clusters $v^{(1)},\ldots, v^{(s_v)}$.

With some simple algebra one rewrites (8.1) as

$$\begin{aligned}
&\sum_{\sigma : |Q(v)| = Q_v} \left(\prod_{i=1}^{n} \exp\left[\frac{\alpha^2}{2} C^{(<h_{v_i})}(\xi_i, \xi_i) \right] \right) \\
&\quad \times \prod_{v \text{ inner}} \left\{ \exp\left[-\frac{\alpha^2}{2} \sum_{i,j} C^{(h_{v'}, h_v)}(v^{(i)}, v^{(j)}) \right] \right. \\
&\quad \times \prod_{(i,j) \in \tau_v} (\exp[-\alpha^2 C^{(h_v)}(v^{(i)}, v^{(j)})] - 1)
\end{aligned} \tag{8.2}$$

where h_{v_i} is the frequency of the inner vertex which is directly connected to the ith end point ξ_i, and

$$C^{(p,q)} \equiv \sum_{p \leqslant h < q} C^{(h)} = C^{(<q)} - C^{(<p)} \tag{8.3}$$

Let

$$\delta C^{(h_v)}(\xi, \eta) = C^{(h_{v'}, h_v)}(\xi, \eta) - C^{(h_{v'}, h_v)}(0, 0) \tag{8.4}$$

and observe that

$$\sum_{i,j} C^{(h_{v'}, h_v)}(v^{(i)}, v^{(j)}) = Q_v^2 C^{(h_{v'}, h_v)}(0, 0) + \sum_{i,j} \delta C^{(h_v)}(v^{(i)}, v^{(j)}) \tag{8.5}$$

so that (8.1) becomes the product of

$$\gamma^{(\alpha^2/4\pi)(\sum_{i=1}^n h_{v_i} - \sum_{v \text{inner}} (h_v - h_{v'}) Q_v^2)} \tag{8.6}$$

times

$$G = \sum_{\sigma: |Q(v)| = Q_v} \prod_{v \text{inner}} \left\{ \exp\left[-\frac{\alpha^2}{2} \sum_{i,j} \delta C^{(h_v)}(v^{(i)}, v^{(j)}) \right] \right.$$
$$\left. \times \prod_{(i,j) \in \tau_v} (\exp[-\alpha^2 C^{(h_v)}(v^{(i)}, v^{(j)})] - 1) \right\} \tag{8.7}$$

having used $C^{(0)}(0, 0) = (1/2\pi) \log \gamma$, $C^{(h_{v'}, h_v)}(0, 0) = (h_v - h_{v'})\, C^{(0)}(0, 0)$.

Hence it remains only to prove that (8.7) can be bounded by

$$\left\{ \exp\left[-\bar{\kappa} \sum_{v > v_0} \gamma^{h_v} d(v) \right] \right\} \prod_{\substack{v > v_0 \\ v \text{inner}}} \gamma^{-2(h_v - h_{v'})(1-\varepsilon)} \tag{8.8}$$

The following simple inequalities, for some $K > 0$,

$$|C^{(h_v)}(w, t)| \leqslant K[(\gamma^{h_v} d(w))^{(1-\varepsilon)\delta_{Q_w 0}} (\gamma^{h_v} d(t))^{(1-\varepsilon)\delta_{Q_t 0}}] \exp[-\kappa \gamma^{h_v} d(w, t)]$$
$$|\delta C^{(h_v)}(w, t)| \leqslant K\{[\gamma^{h_v}(d(w))]^{(1-\varepsilon)\delta_{Q_w 0}} [\gamma^{h_t} d(t)]^{(1-\varepsilon)\delta_{Q_t 0}}\}$$
$$\times\ [\gamma^{h_v} d(w \cup t)]^{(1-\varepsilon)(1-\delta_{Q_w 0})(1-\delta_{Q_t 0})} \tag{8.9}$$

(see Appendix A) imply immediately, from (8.7), that the sum in (8.7) is termwise bounded by

$$\text{const} \left(\prod_{\substack{v > r \\ v \text{inner}}} e^{-\bar{\kappa} \gamma^{h_v} d(v)} \right) \left(\prod_{\substack{v > v_0 \\ Q_v = 0}} \gamma^{-(h_v - h_{v'})(1-\varepsilon)} \right) \tag{8.10}$$

In fact it is clear that each term if (8.7) has the bound

$$\text{const}\left\{\left(\prod_{v\,\text{inner}} e^{-\kappa\gamma^{h_v}d^*(v)}\right)\left[\prod_{\substack{v>v_0\\ Q_v=0}} \gamma^{h_{v'}}\, d(v)\right]^{(1-\varepsilon)}\right\} e^{K\Sigma_v[\gamma^{h_v}d(v)]^{1-\varepsilon}} \tag{8.11}$$

where the first factor comes from bounding the $\prod_{(i,j)}$ in (8.7) using its exponential decay in (8.9) and observing that each neutral vertex v appears as a member of at least one of the pairs $(v^{(i)}, v^{(j)}) \in \tau_{v'}$ except v_0 itself, i.e., except $v'=r$, and the remaining part from bounding the other factor of (8.7) using the estimates (8.9). $d^*(v)$ is the length of the shortest path connecting the clusters $v^{(1)},\ldots, v^{(s_v)}$ immediately inside v. The path itself need not be connected, being connected only modulo $v^{(1)},\ldots, v^{(s_v)}$, i.e., only if the points in $v^{(1)},\ldots, v^{(s_v)}$ are regarded connected. Hence $d^*(v)$ may be substantially smaller than $d(v)$.

Nevertheless the inequality $\gamma^h \geqslant (1-\gamma^{-1})(\gamma^h+\gamma^{h-1}+\cdots+1)$ and the connectedness of each τ_v immediately implies

$$\sum_{v>r} \gamma^{h_v}\, d^*(v) \geqslant (1-\gamma^{-1}) \sum_{v>r} \gamma^{h_v}\, d(v) \tag{8.12}$$

so that (8.11) can be bounded by

$$\text{const}\left\{\prod_{\substack{v>r\\ v\,\text{inner}}} \exp[K[\gamma^{h_v}\, d(v)]^{1-\varepsilon} - \kappa(1-\gamma^{-1})\,\gamma^{h_v}]\right\}\left\{\prod_{\substack{v>v_0\\ Q_v=0}} [\gamma^{h_{v'}}\, d(v)]^{1-\varepsilon}\right\} \tag{8.13}$$

Setting $\tilde{\kappa}=\kappa(1-\gamma^{-1})/2$ and using

$$\left\{\exp\left[K[\gamma^{h_v}\, d(v)]^{1-\varepsilon} - \frac{\kappa}{2}(1-\gamma^{-1})\gamma^{h_v}\, d(v)\right]\right\} d(v)^{1-\varepsilon} \leqslant \text{const}\, \gamma^{-(1-\varepsilon)h_v} \tag{8.14}$$

one gets (8.10).

To see that the second factor in (8.10) can be replaced by its square requires showing that the sum over the charges leads to cancellations in the bound to G. The rest of the section is devoted to this problem.

We first write the admissible configurations $\boldsymbol{\sigma}$ in a convenient way. Namely, we associate to each vertex v a number $\mu_v = +1$ if $Q_v > 0$ and $\mu_v = \pm 1$ if $Q_v = 0$. Then we fix one admissible configuration $\bar{\boldsymbol{\sigma}}$ and define

$$\sigma_i = \bar{\sigma}_i \prod_{i\in v} \mu_v, \qquad i=1,\ldots, n \tag{8.15}$$

where $v \ni i$ means that the ith end point of θ is inside v.

Then it is clear that it will be sufficient to prove the bound (8.8) for the partial sum of (8.1) made over the $\boldsymbol{\sigma}$s of the form (8.15).

The existence of the cancellations is easily done in an abstract context.

We call $w_1,..., w_q$ the neutral vertices following v_0. They are supposedly labeled so that first come the minimals, then the minimals among the ones which follow each of the minimals, etc., hierarchically.

We imagine that the above sum is partially performed by summing over $\mu_1,..., \mu_p$ at fixed $\mu_{p+1},..., \mu_q$ with $\mu_i \equiv \mu_{w_i}$ and we guess after a while its structure.

It is clear that such a sum will have to be a function of

$$\bar{C}^{(h_v)}_{w\bar{v}}, \qquad \bar{C}^{(h_v)}_{w\tilde{w}}, \qquad \bar{C}^{(h_v)}_{ab}, \qquad \bar{C}^{(h_v)}_{v_1v_2} \tag{8.16}$$

where $\bar{C}$ denote covariances of type $C^{(h)}$ or $\delta C^{(h)}$ evaluated in a charge configuration with $\mu_1 = \mu_2 = \cdots = \mu_p = 1$ and $\mu_{p+1},..., \mu_q$ fixed; the arguments of the covariances are written as subscripts and a, b are any end points of the tree that can be reached climbing it from v without meeting any of the $w_{p+1},..., w_q$ vertices; w, $\tilde{w}$ are among the $w_1,..., w_q$; v_1, v_2 are not comparable to any among $w_{p+1},..., w_q$ or follow the "same" among them; $\bar{v}$ is an arbitrary vertex and such is also v in h_v.

We make the following ansatz: The result of the summation over $\mu_1,..., \mu_p$ is a sum of expressions like

$$\left[\prod \bar{C}^{(h_v)}_{wa} \prod \bar{C}^{(h_v)}_{w\tilde{w}} \prod \bar{C}^{(h_v)}_{w\hat{w}}\right]\left[\prod \bar{C}^{(h_{w'})}_{w\bar{v}} \prod \bar{C}^{(h_{w'})}_{w\tilde{w}}\right]$$
$$\times \Phi(\{\bar{C}^{(h_v)}_{w\bar{v}}\}, \{\bar{C}^{(h_v)}_{w\tilde{w}}\}, \{\bar{C}^{(h_v)}_{ab}\}, \{\bar{C}^{(h_v)}_{v_1v_2}\}) \tag{8.17}$$

where Φ is a suitable analytic function, the w, $\tilde{w}$ in the first bracket are among $w_1,..., w_p$, while $\hat{w}$ is among $w_{p+1},..., w_q$; and the w, $\tilde{w}$ in the second bracket are all among the $w_{p+1},..., w_q$. Φ is assumed bounded by the exponential of the sum of its arguments and the following two properties are supposed to hold:

(1) The set of the w, $\tilde{w}$s appearing in the second bracket together with the $\hat{w}$s in the first product covers $w_{+1},..., w_q$.

(2) Consider the "intervals" $(v, w]$ or $(v, \tilde{w}]$ relative to pairs v, w and v, $\tilde{w}$ appearing inside the same covariance in the first product: then every one among $w_1,..., w_p$ is inside at least two possibly identical of such "tree intervals."

The ansatz is valid for $p = 0$, as is proved by using $e^x - 1 = xE(x)$, where $E(x)$ is analytic of exponential growth, in the factors of the second product of (8.7) and absorbing the Es and the factors in the first product into the function Φ. If it is assumed valid for $p=1,\ldots,\bar{p}$ then it easily follows for $p=\bar{p}+1$, if one makes use of the analyticity of Φ and of the following relations, valid if $\tilde{\ddot{C}}^{(\cdot)} \equiv \bar{\ddot{C}}^{(\cdot)}$ with $\mu_{\bar{p}+1}=1$:

$$\begin{aligned}
&\bar{C}^{(h_v)}_{ww_{\bar{p}+1}}=\mu_{\bar{p}+1}\tilde{C}^{(h_v)}_{ww_{\bar{p}+1}}, \quad \bar{C}^{(h_v)}_{w_{\bar{p}+1}\bar{v}}=\mu_{\bar{p}+1}\tilde{C}_{w_{\bar{p}+1}\bar{v}} && \text{for} \quad w\neq w_{\bar{p}+1},\ \bar{v}\neq w_{\bar{p}+1}\\
&\bar{C}^{(h_v)}_{wa}=\mu_{\bar{p}+1}\tilde{C}^{(h_v)}_{w_{\bar{p}+1}a}+(\tilde{C}^{(h_v)}_{wa}-\tilde{C}^{(h_v)}_{w_{\bar{p}+1}a}) && \text{if} \quad w_{\bar{p}+1}>w\\
&\bar{C}^{(h_v)}_{w\tilde{w}}=\mu_{\bar{p}+1}\tilde{C}^{(h_v)}_{w_{\bar{p}+1}\tilde{w}}+(\tilde{C}^{(h_v)}_{w\tilde{w}}-\tilde{C}^{(h_v)}_{w_{\bar{p}+1}\tilde{w}}) && \text{if} \quad w_{\bar{p}+1}>\tilde{w}\\
&\bar{C}^{(h_v)}_{v_1v_2}=\tilde{C}^{(h_v)}_{v_1v_2}
\end{aligned} \tag{8.18}$$

and then performs the sum over $\mu_{\bar{p}+1}=\pm 1$.

Hence property (1) above implies that for $p=q$ the summation over the $\boldsymbol{\mu}$s and hence the whole G [see Eq. (8.7)] can be bounded as a sum of terms like

$$\text{const}\exp\left\{K\sum_{v}[\gamma^{h_v}d(v)]^{1-\varepsilon}\right\}\prod_{w>v}[\gamma^{h_v}d(w)]^{1-\varepsilon} \tag{8.19}$$

where the product ranges over a suitable set of pairs v, w and the union of the tree intervals $(v, w]$ covers each neutral vertex $w_1>v_0$ at least twice, i.e.,

$$\sum_{w>v}(h_w-h_v)\geqslant 2\sum_{\substack{v>v_0\\ Q_v=0}}(h_v-h_{v'}) \tag{8.20}$$

Moreover from (8.10) G can also be bounded by

$$\text{const}\left(\prod_{\substack{v>r\\ v\,\text{inner}}}e^{-\bar{K}\gamma^{h_v}d(v)}\right) \tag{8.21}$$

and using the bounds (8.19), (8.21) together the proof of the bound (7.1) is completed.

9. THE ACTIVITY SERIES

It follows from (6.9) that

$$\begin{aligned}
A_{\boldsymbol{\sigma}}(x_1,\ldots,x_n)=&\sum_{\theta:N(\theta)=n}\ \sum_{\mathbf{h};h_r=-1}^{*}\left(\frac{\lambda}{2}\gamma^{(2-\alpha^2/4\pi)R}\right)^n\hat{V}(\theta,\mathbf{h},\boldsymbol{\sigma},\gamma^{-R}x)\\
&\times\exp\left[-\sum_{i<j}\alpha^2\sigma_i\sigma_jC(x_i-x_j)\right]
\end{aligned} \tag{9.1}$$

where the * reminds us that the end points' frequencies of θ are now $R-1$ rather than R, and where the last term arises because

$$\prod_{j=1}^{n} :\exp(i\alpha\sigma_j\psi_{\xi_j\gamma^R}):$$
$$=\exp\left[-\sum_{i<j}\alpha^2 C(x_i,x_j)\,\sigma_i\sigma_j\right] :\exp\left(i\alpha\sum_j \sigma_j\psi_{\xi_j\gamma^R}\right): \tag{9.2}$$

Consider an expression like (6.13), integrated over all the points in $X_1,..., X_N$ except one in each of them.

The result will be, setting $\xi_j = x_j\gamma^{-R}$,

$$\int d^*\Xi_1\cdots d^*\Xi_N\left[\prod_{j=1}^{N}\gamma^{(n_j-1)2R}\Lambda_{\sigma_j}(\gamma^R\Xi_j)\right]\exp[-\alpha^2 U_{\sigma_1\cdots\sigma_N}(\Xi_1,...)] \tag{9.3}$$

where $\Xi_1,..., \Xi_N$ are simply $\gamma^{-R}X_1$, $\gamma^{-R}X_2$,... and d^* means that one integrates only over the components $\xi_2^{(j)},..., \xi_{n_j}^{(j)}$ of the multipole $\Xi_j = (\xi_1^{(j)},..., \xi_{n_j}^{(j)})$.

To study the bound (6.14) we treat first the case $N=1$. The others reduce to it quite easily.

So we study

$$\sum_{\substack{\sigma\\ |Q(v)|=Q_v}}\int d\xi_2\cdots d\xi_n\,\gamma^{-2R}\left(\frac{\lambda}{2}\gamma^{(2-\alpha^2/4\pi)R}\right)^n \exp[-\alpha^2 U_\sigma(\gamma^R\xi)]$$
$$\times\prod_{v\,\text{inner}}\left(\exp\left[-\alpha^2\sum_{i<j} C^{(<h_v)}(v^{(i)},v^{(j)})\right]\right.$$
$$\left.\times\prod_{(i,j)\in\tau_v}\{\exp[-\alpha^2 C^{(h_v)}(v^{(i)},v^{(j)})]-1\}\right) \tag{9.4}$$

because this is the expression for $\sum_\sigma\int\Lambda_\sigma(\xi_1,...,\xi_n)\,d\xi_2\cdots d\xi_n$ that is obtained from (9.1) proceeding in the same way as done to obtain (8.1).

The situation is now very simple: (9.4) is very similar to (8.1) except for a factor which we write as

$$\exp[-\alpha^2 U_\sigma(\gamma^R\xi)] = 1-\alpha^2 U_\sigma(\gamma^R\xi)\,E(U_\sigma(\gamma^R\xi)) \tag{9.5}$$

where $E(x)=(e^x-1)/x$ is a function, analytic in x, bounded by e^x.

Inserting (9.5) into (9.4) gives a first term, from the 1, which is exactly the one studied in Sections 7 and 8 and a second term proportional to

$$\sum_{i<j} C(\gamma^R(\xi_i-\xi_j))\,\sigma_i\sigma_j \tag{9.6}$$

This is a charge-dependent term which can affect the cancellation argument of Section 8, hence the bounds of Section 7.

Without going through the argument of Section 8, it should be clear that the contribution to (9.4) from the second term of (9.5), proportional to (9.6), will be bounded in the same way as described in Section 8 except that some w may now be counted less than two times because the corresponding $\bar{C}$s are replaced by $C(\gamma^R(\xi-\xi'))$ coming from (9.6) with $\xi, \xi' \in w'$.

Clearly this spoils part of the bounds in Section 8. However, C has range 1. Hence, the presence of a term like $C(\gamma^R(\xi-\xi'))$ leads to an essential improvement. In fact, our bound will now become a sum of terms like

$$\left[\exp\left(-\bar{\kappa}\sum_{v>r}\gamma^{h_v}d(v)\right)\right]\prod_{\substack{Q_v=0\\ v\in B}}\gamma^{2(1-\varepsilon)(h_v-h_{v'})}$$
$$\times\prod_{\substack{Q_v=0\\ v\notin B}}|C(\gamma^R(\xi_{1,v}-\xi_{2,v}))\,\gamma^{(1-\varepsilon)(h_v-h_{v'})}| \tag{9.7}$$

where B is a subset of the set of neutral vertices and $\xi_{1,v}$, $\xi_{2,v}$ are in two different clusters immediately following v', one of which is v itself.

The C factors together with the exponential factor integrated over $\xi_2,..., \xi_n$ improve the bound (7.2) by a factor

$$\prod_{\substack{v\notin B\\ Q_v=0}}\gamma^{-2(R-h_{v'})} \tag{9.8}$$

because each C forces $\xi_{2,v}$ to stay within γ^{-R} of $\xi_{1,v}$ while in the bound (7.2) it is forced to stay only within $\gamma^{-h_{v'}}$ of it so that one can multiply (7.2) by the factor (9.8) to obtain the bound on the relevant integral.

Hence, since $2(R-h_v)\geqslant 2(h_v-h_{v'})$ the improvement (9.8) on the integral compensates for the fact that in (9.7) some neutral vertices contribute only $\gamma^{(1-\varepsilon)(h_v-h_{v'})}$ and not $\gamma^{2(1-\varepsilon)(h_v-h_{v'})}$. So we find that (9.4) is uniformly bounded in R and vanishes if $Q_{v0}\neq 0$ for $R\to\infty$.

The same argument applies to the case $N>1$. One gets exactly the same bound on $\zeta(n)$ multiplied by a suitable N-independent factor.

Remarks. (i) In principle the bound (7.4) to the contribution of θ to the nth-order Mayer series coefficient is also an upper bound and it could happen that the coefficient of order k of the Mayer series stays finite when $\alpha^2>\alpha_{2n}^2$ also for $k<2n$ although its upper bound diverges. To rule out this highly unlikely possibility one should prove a lower bound for the coefficients of the same type but with different values of the constants. Simple investigations on some examples seem to show that this is true unless some very special cancellations occur.

(ii) It is remarkable that the thresholds α_{2n}^2 above which the Coulomb gas (with u.v. cutoff) "generates molecules" made by $p \leqslant 2n$ are precisely the same thresholds at which the Yukawa gas collapses into neutral clusters of $p \leqslant 2n$ particles in the ultraviolet limit (Benfatto *et al.*[1]). This shows a "duality" between the infrared properties of the Coulomb gas and the ultraviolet properties of the Yukawa gas in the interval $[4\pi, 8\pi)$.

(iii) If we write

$$\begin{aligned} P &= \frac{1}{|I|} \log Z^{(-R)}(\lambda, I, \beta) \\ &= \sum_{p \leqslant k} \mathscr{A}_p(\beta, I, R)\, \lambda^p + \mathscr{R}_{k+1}(\beta, \lambda, I, R)\, \lambda^{k+1} \end{aligned} \tag{9.9}$$

we have shown that the coefficients $\mathscr{A}_p(\beta, I, R)$ admit limits as $R \to \infty$ which are uniformly bounded in I: in fact it is clear that they also converge to a limit as $I \to R^2$. It is therefore natural to conjecture that, $\forall k$, if $\alpha^2 > 8\pi$ and for $k \leqslant 2n$ if $\alpha^2 > \alpha_{2n}^2$ it is

$$\mathscr{R}_{k+1}(\beta, \lambda, I, R) \leqslant \text{const} \qquad \forall I, R \tag{9.10}$$

if λ is small enough. If this is true it should be reasonable to think that the pressure is in fact $C^{(n)}$ in λ if $\alpha^2 > \alpha_{2n}^2$ and λ small enough. In fact the pressure might even be analytic in λ for α^2 large and λ small (private communication by T. Spencer). This conjecture suggests that while the temperature decreases the Coulomb gas presents an infinite sequence of phase transitions passing from the plasma phase, $\alpha^2 < 4\pi$, with the Debye screening phenomena, to the multipole phases, $4\pi < \alpha^2 < 8\pi$, where one can conjecture that some partial screening effects are left which prevent the formation of too large multipoles. The "Kosterlitz Thouless" regime, $\alpha^2 > 8\pi$, would be the last stage of this sequence of phase transitions in which bound states of any size are possible in thermal equilibrium.

(iv) The analysis of the conjecture on $\mathscr{R}_{k+1}$ made after Eq. (9.10), for any k if $\alpha^2 > 8\pi$ and for $k \leqslant 2n$ if $\alpha^2 > \bar{\alpha}_{2n}^2$, with $\bar{\alpha}_{2n}^2 < 8\pi$ but $\bar{\alpha}_{2n}^2 > \alpha_{2n}^2$, is in progress and hopefully will be published in a forthcoming paper.

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APPENDIX A: THE BOUNDS (8.9)

For the first bound it suffices to take $h_v = 0$ by scale invariance.

Then, if $w = (x_1,..., x_{n_1})$, $t = (y_1,..., y_{n_2})$ with respective charges $\sigma_1,..., \sigma_{n_1}$, $\mu_1,..., \mu_{n_2}$, it is

$$C^{(0)}(w, t) = \frac{1}{(2\pi)^2} \int \frac{(\sum_{i=1}^{n_1} \sigma_i e^{-px_i})(\sum_{j=1}^{n_2} \mu_j e^{py_j})}{(1+p^2)(\gamma^2+p^2)} dp \tag{A1}$$

so that if x_0, y_0 are the centers of mass of w, t and using

$$\sum_{i=1}^{n_1} \sigma_i e^{ipx_i} = e^{ipx_0} \left[Q_w + \sum_{i=1}^{n} \sigma_i (e^{ip(x_i - x_0)} - 1) \right] \tag{A2}$$

and the similar identity for t:

$$\begin{aligned} C^{(0)}(w, t) = {} & Q_w Q'_t \frac{1}{(2\pi)^2} \frac{e^{-ip(x_0 - y_0)}}{(1+p^2)(\gamma^2+p^2)} dp \\ & + \frac{Q_w}{(2\pi)^2} \int e^{-ip(x_0 - y_0)} \frac{\sum_j \mu_j (e^{ip(y_i - y_0)} - 1)}{(1+p^2)(\gamma^2+p^2)} dp \\ & + \frac{Q'_t}{(2\pi)^2} \int e^{-ip(x_0 - y_0)} \frac{\sum_j \sigma_j (e^{-ip(x_j - x_0)} - 1)}{(1+p^2)(\gamma^2+p^2)} dp \\ & + \frac{1}{(2\pi)^2} \int e^{-ip(x_0 - y_0)} \frac{(\sum_j \sigma_j e^{-ip(x_j - x_0)} - 1)(\sum_j \mu_i e^{ip(y_j - y_0)} - 1)}{(1+p^2)(\gamma^2+p^2)} dp \end{aligned} \tag{A3}$$

implying the first bound in (8.9) via

$$\left| \sum_j \sigma_j (e^{-ip(x_j - x_0)} - 1) \right| \leqslant 2 \sum_j |p(x_j - x_0)|^{1-\varepsilon} \tag{A4}$$

and the independently known exponential decay of $C^{(0)}(x-y)$ as $|x-y| \to \infty$.

If at least one among Q_w, Q_t is zero, the second of (8.9) follows from the first written for h instead of h_v and then summing it over h between $h_{v'}$ and h_v, forgetting the exponential term ($\leqslant 1$). In fact, in this case $\delta C^{(h_v)} = \sum_{h=h_{v'}}^{h_v - 1} C^{(h)}$, i.e., the $C^{(\cdot)}(0, 0)$ subtraction in (8.4) is irrelevant.

Suppose $Q_w \neq 0$, $Q_t \neq 0$; then we use

$$\begin{aligned} |\delta C^{(0)}(\xi, \eta)| &= \left| \frac{1}{2\pi} \int \frac{(e^{ip(\xi - \eta)} - 1)}{(1+p^2)(\gamma^2+p^2)} dp \right| \\ &\leqslant \frac{1}{(2\pi)^2} \int \frac{2\,|p(\xi - \eta)|^{1-\varepsilon}}{(1+p^2)(\gamma^2+p^2)} dp \leqslant \text{const } |\xi - \eta|^{1-\varepsilon} \end{aligned} \tag{A5}$$

so that

$$|\delta C_{wt}^{(h)}| \leqslant \text{const} \sum_{\xi \in w} \sum_{\eta \in t} \sum_{p=0}^{h} (\gamma^p |\xi - \eta|)^{1-\varepsilon} \leqslant \text{const}(\gamma^h d(w, t))^{1-\varepsilon} \tag{A6}$$

and (8.9) is proved.

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