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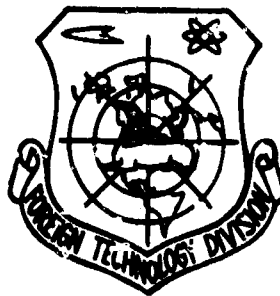
THE SCREW CALCULUS AND ITS APPLICATIONS IN MECHANICS

by

F. M. Dimentberg

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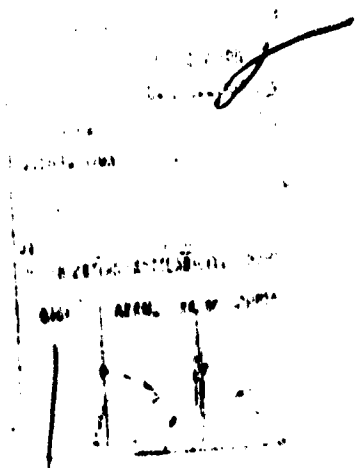
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ABSTRACT

This book is intended for the reader familiar with vector algebra and the basics of the theory of functions of a vector argument. The author has made an attempt to sets forth the basic propositions of screw calculus on the basis of the elementary apparatus of modern vector algebra and to indicates certain of its applications. The book sets forth material from the theory of sliding vectors, the algebra of complex numbers of the form $a + \omega a^0$ with a special multiplier ω that possesses the property $\omega^2 = 0$, the algebra of screws, fundamentals of the differential geometry of the ruled surface, which are necessary for the kinematics of solids, the foundations of screw analysis, and, finally, certain data from the classical theory of screws in its geometrical aspect, with indication of a number of applications in mechanics.

Orig. art. has: 380 equations, 41 figures.

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FOREWORD

The method of screws made its appearance as a method of mechanics during the 'Seventies of the last century. The screw calculus proper was formulated in its definitive form during the 'Nineties, based on the ideas of W. Clifford, A.P. Kotel'nikov and E. Study, and is a generalization of vector calculus. It is based both on the general theory of screws and a special "transfer principle," which establishes correspondence bet'ween the free vectors and the screws in such a way that if they are given a special complex form, all relationships of the vector domain are formally preserved for the screws. As a result, one "screw" equation with no differences in form from a vector equation is equivalent not to three but to six scalar equations, which imparts particular compactness and clarity to all of the expressions.

Despite the long time that has elapsed since the origin of screw calculus, there is still only a select group of persons to whom it is familiar owing to the lack of the necessary literature on the problem.

The author has made an attempt to set forth the basic propositions of screw calculus on the basis of the elementary apparatus of modern vector algebra and to indicate certain of its applications. The book sets forth material from the theory of sliding vectors, the algebra of complex numbers of the form $a + \omega a^0$ with a special multiplier ω that possesses the property $\omega^2 = 0$, the algebra of screws, fundamentals of the differential geometry of the ruled surface, which are necessary for the kinematics of solids, the foundations of screw analysis, and, finally, certain data from the classical theory of screws in its geometrical aspect, with indication of a number of applications in mechanics.

The author's purpose was to popularize (if a bit late) screw calculus among specialists in mechanics; it is hoped that a large group of readers working in various fields of general and applied mechanics will become conversant with it.

In compilation of the book, the work of A.P. Kotel'nikov and D.N. Zeylinger was referred to most frequently, followed by the papers of R. Ball, N. Zanchevskiy, E. Study, R. Mises, S.O. Kislitsyn and other authors. Also included are certain results obtained by the present author, some of which will be published at a later date.

The book is intended for the reader familiar with vector al-

gebra and the basics of the theory of functions of a vector argument.

INTRODUCTION

The theory of screws made its appearance at the beginning of the last century following the appearance of the papers of Poinsot, Chasles and Möbius, in which the theory of force couples and infinitesimally small rotations was studied and the analogy between the force and a small rotation and, as a corollary, the analogy of their addition, were established for the first time. The work of these authors established the equivalence of longitudinal displacement of a body to screw displacement, and laid the foundations for study of kinematics and statics; the notion of the screw, which was subsequently developed further in the papers of Plücker, was also formulated.

Plücker studied a ruled space, i.e., a space whose element is a straight line. To describe the line, Plücker introduced special coordinates (Plücker coordinates), which in the general case define a screw; apart from the screw, he also considered other figures of linear geometry (surfaces, congruences, complexes).

As the combination of a vector and a couple whose plane is perpendicular to the vector, the screw is a geometrical figure that describes both arbitrary displacement of a solid body and an arbitrary system of forces acting on the body. In the study of motion, the screw as displacement is in many cases the most natural generalized displacement on which operations are performed directly; at the same time, the force screw is the corresponding generalized force. This gives rise to the method of mechanics in which all displacements and their derivatives as well as the forces are expressed by screws, and which yields results that can be treated in the language of screws.

Beginning in 1870, the theory of screws was studied comprehensively in the papers of R. Ball, who published the monumental work [1] in 18;6.

Examining arbitrary displacements of the body, Ball reduces them to a combination of certain base screws, attaining clarity in the geometrical interpretation and good mechanical palpability in the results. In a story that he wrote [2] to popularize the essentials of the method of screws, Ball very cleverly juxtaposes the method of screws to the cartesian-coordinate method. The story runs as follows: a certain technical commission was given the task of determining the dynamic properties of a solid body (the housing of a machine), which was secured to its base in a rather complicated fashion. For this purpose, it was first necessary to ascer-

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tain the number of degrees of freedom of the body. One of the members of the commission was a Cartesian. After a long and laborious study of the mobility of the body with the aid of his "tested" coordinate trihedron, he eventually arrived at a result that was summed up in six numbers expressed in degrees and minutes of arc and inches and providing a numerical expression for the possible rotational and translational motions along the coordinate axes; however, this result told nothing of the essential nature of the motion and won disapproving remarks from the president of the commission. At the same time, another member of the commission, one Helix, making use of the fact that any motion of a body is equivalent to a screw motion, established, by comparatively simple matching of screws with nuts of the appropriate pitch, several possible variants of "screwing down," i.e., spatial motions of the body, thus giving a clear interpretation of the motion independently of the coordinate system.

In the Russian literature, Ball's theory found its reflection in the work of I.O. Zanchevskiy [3], who related the theory of screws with the theory of the ruled complex.

Several years before the appearance of Ball's classical work, W. Clifford [4] had given a highly interesting description of screws using special complex numbers. It must be remembered that vector calculus was only developing during this period, and had not at that time acquired the simple form in which we know it today. Vector calculus was approached progressively from various directions: on the one hand, with the aid of geometrical conceptions, and, on the other, with the aid of specially invented "hypercomplex" numbers or "quaternions," which consist of a scalar part and a part that contains three more quantities of a different nature. Clifford introduced the multiplier ω, whose square is equal to zero, as well as complex numbers that consist of a real number and the product of a real number by ω. If the components of the quaternion are considered as complex rather than real in the sense just indicated, the quaternions become biquaternions, which have the same relation to the theory of screws as quanternions have to ordinary vector theory. Clifford did not develop the theory of screws in its applications to mechanics; his subsequent research was concerned with application of the operation that he had introduced and the biquaternions to noneuclidian geometry.

The monumental work of A.P. Kotel'nikov [5] in which screw calculus proper was constructed for the first time made its appearance in 1895. This study used the above complex numbers with the multiplier ω, by means of which a vector is transformed into a screw. The principal service rendered by Kotel'nikov consisted in the fact that for the first time he formulated in its complete form the special "transfer principle" on whose basis all operations of screw calculus can be constructed in exact correspondence with operations of vector calculus if all real quantities in the latter are replaced by complex quantities with the multiplier ω. As a result, it becomes possible to substitute not three equations, as in the case of vector calculus, but six scalar equations for one equation, and the solutions of rather complex problems become more compact.

Kotel'nikov gave an even broader geometrical interpretation
to the transfer principle that he had formulated — the principle
establishes correspondence between geometrical figures in spaces
with different numbers of dimensions, and, in particular, between
objects of point and line spaces, and enables us to study the ge-
ometry of one space with the aid of the geometry of another.

The major work of the prominent German geometrician E. Study
[6] on the geometrical theory of screws appeared in its first edi-
tion in 1901 and its second in 1903. In this volume, which runs
to more than 600 pages, about 50 are devoted to exposition of a
method of describing screws and linear spaces with the aid of com-
plex numbers with the multiplier ω (Study calls them dual numbers),
and a transfer principle similar to that mentioned earlier is for-
mulated. In the second edition, in a short historical note obvi-
ously occasioned by the appearance of a number of papers on the
same problem, the author makes an attempt to establish his pri-
ority in the application of complex numbers to screws. He cites
his work on the application of complex numbers in the linear ge-
ometry of euclidian and noneuclidian spaces, but nothing is said
concerning his formulation at some earlier time of the actual
transfer principle. The following references are made in this
brief outline: to a short paper of F. Schilling [7] dating from
1891, in which formulas of spherical trigonometry are first de-
rived for complex angles, and then to the above-mentioned work by
A.P. Kotel'nikov [5], which Study cites from a short abstract in
"Fortschritte der Mathematik," 1896, in connection with which
nothing is said of Kotel'nikov's formulation of the transfer prin-
ciple, and also to an 1896 paper of R. Saussure [8], where complex
numbers are used, although, in his opinion, not quite correctly.
Incidentally, Saussure's paper actually does submit the idea of
applying a transfer principle to one problem of the displacements
of a solid body.

In his later work [9] (published posthumously in 1950), A.P.
Kotel'nikov makes the following remark: "The transfer principle in
all its generality was discovered and formulated independently
and, apparently, simultaneously by Study and myself. It must be
supposed that the transfer principle was already known to Study
when he wrote... his paper "Ueber neue Darstellung der Kräfte" [A
New Representation of Forces].* But he formulated this principle
quite definitely in his paper "Ueber Nicht-Euklidische und Linien-
geometrie" [Noneuclidian and Linear Geometry]."** The first of
these papers dates from 1899 and, as can be seen from its text,
the transfer principle has not yet been formulated. As for the
second paper, which the author of the present volume has not been
able to obtain, it was published in 1900 and, in all probability,
is the work in which E. Study first gave his formulation of the
transfer principle.

Note should be taken of the well-known work of R. Mises,
which appeared in the form of two articles in 1924 [12] and [13],
which sets forth the general part and applications of the so-
called "motor" calculus ("motor" is a combination of the words
"moment" and "vector," i.e., the screw). In this work, the author
first proceeds from geometrical description of the motor using
two straight lines and then introduces six coordinates of the mo-

tor and operations on the motors — scalar and motor multiplication. This is followed by the introduction of motor dyads and affine-transformation matrices. In motor calculus, as in screw calculus, analogies with vector operations are discernible. However, the transfer principle was not reflected in the work of Mises. Mises examined applications to the dynamics of the solid body, elasticity theory, the structural mechanics of rod systems, fluid dynamics, etc.

Soon after A.P. Kotel'nikov (beginning in 1897), D.N. Zeylinger began to develop the notions of screw calculus; in 1934, he published his definitive work [14], which gives the results of extensive investigations in linear geometry obtained by screw calculus and indicates interesting applications to kinematics. Some information on application of complex numbers with the multiplier ω in linear geometry is given in the book by Study's student W. Blaschke [15]; a description of complex vectors will also be found in M. Lagalli's book [16].

Unfortunately, apart from D.N. Zeylinger, a contemporary and adherent of A.P. Kotel'nikov, and certain other geometricians, it can be said that screw calculus remained almost totally unrecognized over a span of forty years. This is explained in large part by the extreme rarity of the published works of A.P. Kotel'nikov, which came out at Kazan' at the end of the last century and have for the most part been lost; the work of Study, as an obscure geometrical treatise, also failed to attract the notice of those who might have used the ideas embodied in it. Another highly probable factor is that at the beginning of this century, many investigators were attempting to adapt various concepts and methods of geometry for the most part to the developing mechanics of continuous media, while screw calculus, which was associated with linear geometry, was not suitable for description of the ordinary continuous medium; the need to use screw calculus for the mechanics of the solid developed much later.

Only in 1937 did papers begin to appear that might be regarded as a continuation of the theory of screw calculus. S.G. Kislitsyn developed "screw affinors" [17], which represent an extension of the operators of affine geometry to the screw space. Complex numbers with the multiplier ω serve as elements of the matrices of the corresponding affine transformation.

Finally, in 1947, studies of applications of screw calculus to the problems of technical mechanics began to appear (the theory of hinge mechanisms, the theory of gear meshing). These include papers by the author of the present book [18], [19], [20], by S.G. Kislitsyn [21], [22], [23], [24], by F.L. Litvin [25] and certain others.

Among recent papers on the application of screw theory to investigation of mechanisms, we might cite that of A. Yang and F. Freidenstein [26].

Independently of screw calculus, the method of screws had been applied to the theory of mechanisms somewhat earlier, in 1940, by Ya.B. Shor and the present author [27], [28].

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Thus, the notions of screw calculus have been accorded a certain amount of recognition in the literature and have already begun to find applications. Nevertheless, the number of investigators working in this area is quite limited, and for the most part the screw calculus remains unknown to an enormous number of persons concerned with the mechanics of the solid body and the continuous medium, and even more so to engineers working in industry. It can nevertheless be assumed that the recent appearance of many papers on application of the screw calculus will contribute substantially to popularization of this calculus. The author hopes that the present book will also play a part in this trend.

Manu-script Page No.	Footnotes
5*	[10] in References list.
5**	[11] in References list.

Chapter 1

THE SLIDING VECTOR. THE MOTOR AND THE SCREW

§1. Moment of a Vector with Respect to a Point. The Sliding Vector. The Sliding-Vector System. The Principal Vector and Principal Moment of the System

We shall assume that the reader is familiar with the definition of the vector, as well as all operations on free vectors as taught in conventional courses in vector algebra.

Let us recall certain information on the moment of a vector with respect to a point and on systems of sliding vectors. The moment r_O^0 of a vector $r = \vec{AB}$, where A is a given origin and B is the end of the vector, with respect to some point O is the vector equal to the dot product of the radius vector $\rho = \vec{OA}$ by the given vector, i.e.,

$$r_0^0 = \rho \times r. \tag{1.1}$$

By definition, the moment is perpendicular to the plane of triangle OAB and points in the direction from which the circuit of the triangle in the direction of the vector appears to be counterclockwise, and that the magnitude of the moment is equal to twice the area of triangle OAB.

It also follows from the definition of the moment that the moment of a vector with respect to any point will not change if the vector is displaced along its line in an arbitrary fashion.

Two vectors which are equal and whose moments with respect to any point of the space are also equal, are said to be equivalent.

Thus, displacing a vector to any position along its line, we obtain equivalent vectors.

In many problems of the mechanics of the solid body, the conditions of the problem remain in force if the vectors representing various quantities are replaced by equivalent vectors. Vectors that are defined accurate to equivalence, i.e., vectors that can be displaced along the lines of their action, are known as sliding vectors. As an example of a sliding vector, we might cite the vector representing the angular velocity of a solid body. Its position in space is characterized by the position of the body's axis

of revolution; at the same time, it may be placed anywhere we please on this axis.

This book will consider sliding vectors and systems of sliding vectors.

The moment of a vector with respect to a point O' is expressed in terms of the moment with respect to point O as follows:

$$r_{O'}^0 = \rho' \times r = (\overrightarrow{O'O} + \rho) \times r = r_O^0 + \overrightarrow{O'O} \times r. \qquad (1.2)$$

It follows from Formula (1.2) that for two equal vectors to be equivalent, it is sufficient that their moments with respect to a given point in the space be equal.

Let there be given an arbitrary system of sliding vectors $r_1, r_2, \ldots, r_n$. Let us take an arbitrary point O of the space and relate two vectors to it: the principal vector of the system, which is the geometrical sum of all vectors of the system

$$r = \sum_{k=1}^{n} r_k, \qquad (1.3)$$

and the principal moment of the system with respect to O, which is equal to the geometrical sum of the moments of all sliding vectors of the system with respect to the point

$$r_O^0 = \sum_{k=1}^{n} r_{Ok}^0 = \sum_{k=1}^{n} \rho_k \times r_k, \qquad (1.4)$$

where $\rho_1, \rho_2, \ldots, \rho_n$ are the radius vectors of the initial points of the vectors from O.

The relation between the principal moment of a system of sliding vectors with respect to a new point O' and the principal moment of this same system with respect to point O is as follows:

$$r_{O'}^0 = \sum_{k=1}^{n} (\rho + \rho_k) \times r_k = \Sigma \rho_k \times r_k + \rho \times \Sigma r_k =$$
$$= r_O^0 + \rho \times r, \qquad (1.5)$$

where ρ is the vector connecting point O' to point O.

For a system of sliding vectors, the scalar product of the principal vector by the principal moment taken with respect to an arbitrary point O of the space is independent of selection of this point. Actually, on scalar multiplication of (1.3) and (1.5) we obtain for any two points O and O'

$$r \cdot r_{O'}^0 = r \cdot r_O^0 + r \cdot (\rho \times r) = r \cdot r_O^0.$$

The scalar product of the principal vector by the principal moment of the sliding-vector system is known as the invariant of the system and denoted by the letter J.

It follows from the above that on any change in the point O, only that component of the principal moment that is perpendicular

to the principal vector can change, while the component parallel
to the principal vector remains unchanged.

The following cases may present themselves as we examine sys-
tems of sliding vectors:

$$
\begin{aligned}
&1)\ r \neq 0,\ r^\bullet \neq 0,\ J \neq 0; \\
&2)\ r = 0,\ r^\bullet \neq 0; \\
&3)\ r \neq 0,\ J = r \cdot r^\bullet = 0; \\
&4)\ r = 0,\ r^\bullet = 0.
\end{aligned}
\qquad (1.6)
$$

In the first case, the principal vector and principal moment
are arbitrary; in the second case the principal vector is zero; in
the third case the principal moment of the system with respect to
any point is perpendicular to the principal vector; the fourth
case characterizes the null system of vectors.

§2. Equivalent Vector System. The Vector Pair

We shall call two systems of sliding vectors equivalent if
their principal vectors are equal and the principal moments with
respect to any point of the space are also equal.

It follows from Formula (1.5) that if the principal vectors
of two systems are equal and the principal moments with respect
to any single point of the space are also equal, the moments with
respect to any point of the space will also be equal in these sys-
tems.

Let us examine an elementary system — a pair of vectors. The
system of two sliding vectors $r_1 = \overrightarrow{AB}$ and $r_2 = \overrightarrow{CD}$ forms a pair if
the figure $ABCD$ is a parallelogram. The distance between lines AB
and CD is the arm of the pair, while the area of $ABCD$ is the mo-
ment of the pair. The moment of the pair is represented by a vec-
tor perpendicular to the plane of $ABCD$ and pointing in the direc-
tion from which the point describing the perimeter of $ABCD$ appears
to be moving counterclockwise. The pair represents the second of
the cases of the system that were enumerated earlier (1.6).

A pair whose arm is zero is known as a null pair. It corres-
ponds to the fourth of the cases of (1.6).

Obviously, the principal vector of a pair is zero. Hence the
principal moment of the pair, on the basis of (1.5), will be the
same for all points of the space. This principal moment is equal
to the moment of the pair.

It follows from equality of the principal vector to zero and
equality of the moments of the pair for any point of the space
that all pairs whose moments are equal are equivalent. Equivalence
is not violated if the pair is transferred and changed in any way
that preserves the direction and magnitude of its moment, i.e.,
if it is transferred with its plane left parallel to itself, and
if the absolute value of its vector and the arm are changed while
preserving the same product.

The combination of two pairs is equivalent to zero if their moments have the same absolute value, are parallel, and point in opposite directions.

It follows from the fact that the same value of moment corresponds to equivalent pairs that we may consider the moment of any pair instead of that pair. Assigning the moment of a pair defines any pair equivalent to the given pair, and therefore replaces assignment of the pair with an accuracy equal to that of equivalence.

§3. Reduction of a System of Sliding Vectors to an Elementary System

There exist elementary geometrical operations by means of which one system of sliding vectors can be replaced by another system equivalent to it, in particular by an elementary system consisting of the least number of vectors. These operations are as follows:.

a) transfer of the vector along its line;

b) adding or dropping two equal and opposed vectors;

c) replacement of several vectors passing through the same point by their geometrical sum, which passes through the same point;

d) replacement of one vector by its components, obtained by the parallelogram law and passing, together with it, through one point.

The above operations do not change the principal vector and principal moment of the system; as a result of applying them, therefore, we obtain a system equivalent to the given system.

Let us examine the transfer of a sliding vector onto a line parallel to its own line. Let r be a sliding vector on line a. On the parallel line a' we construct a null pair consisting of two vectors r' and r'' with a common origin at point O, with the former equal to the assigned vector r. In other words, we add to the given system two equal and opposed vectors, thus performing elementary operation "b" of the above list. The new system, which is equivalent to the sliding vector r, will consist of vectors r, r' and r'' and will represent the combination: r', pair (r, r'').

Thus, the vector r on line a is equivalent to the combination of the equal vector r' on line a', which runs parallel to line a, and the pair (r, r''), whose moment is equal to the moment of vector r with respect to point O. Since the given pair or the pair equivalent to it is defined by its moment, the combination of vector r' on line a' and the pair (r, r'') is replaced by the combination of vector r' on line a' and the moment r^0 of vector r with respect to point O on line a''.

It follows from this that a sliding vector is equivalent to an elementary system composed of a vector originating from the

point with respect to which the moment is taken and the moment. For this vector-equivalent system, it is always the case that $r \cdot r^0 = 0$.

The operation of equivalent substitution of a sliding vector by the above elementary system at a point is known as reduction of the sliding vector to this point.

Let us consider reduction of a system of sliding vectors in the general case (the first case among those listed). Let there be given a system of sliding vectors $r_1, r_2, \ldots, r_n$. Let us select a certain point of space O and reduce each of the vectors of the system to this point. We shall obtain a system of vectors $r_1, r_2, \ldots, r_n$ with a common origin at point O and equal to the given sliding vectors, and a system of moments $r_1^0, r_2^0, \ldots, r_n^0$, equal to the moments of the given sliding vectors with respect to O; the moments assign the corresponding pairs of the reduction.

Adding vectors and determining the sum

$$r = r_1 + r_2 + \ldots + r_n,$$

and adding the moments and determining the sum

$$r^0 = r_1^0 + r_2^0 + \ldots + r_n^0.$$

we arrive at the result that the system of assigned sliding vectors is equivalent to a vector equal to r, which, in accordance with operation "c" passes through point O, and a pair with moment r^0, since the latter determines this pair or its equivalent.

The vector r is the principal vector and the moment r^0 the principal moment of the system with respect to point O.

In the general case, the vectors r and r^0 form an arbitrary angle. Generally speaking, therefore, for a system of vectors $r \cdot r^0 \neq 0$.

If the point of reduction is changed, the moment will change in accordance with Formula (1.5), but the component of the moment in the direction of the principal vector will remain unchanged; only the component perpendicular to the principal vector will change. There exist points of reduction for which the system principal moment is colinear with the system principal vector.

Let the principal vector be r, and let the moment be r_O^0 and not colinear with r for a certain reduction point O. We pass a straight line through point O perpendicular to r and r_O^0 and find a point C on this line for which the radius vector

$$\vec{CO} = \frac{r_O^0 \times r}{r^2}.$$

Taking C as the new point of reduction, we find the corresponding moment from Formula (1.5):

$$r_c^o = r_o^o + \left(\frac{r_o^o \times r}{r^2}\right) \times r = r \frac{r_o^o \cdot r}{r^2}. \qquad (1.7)$$

from which it is seen that the moment r_C^o is colinear with the principal vector r. In addition to point C, there exists an innumerable set of points that possess the same property. Indeed, for any point C' lying with point C on a straight line parallel to r, we shall have

$$r_{C'}^o = r_C^o + \lambda r \times r = r_C^o.$$

A straight line for any point of which the principal moment is colinear to the principal vector is known as the central axis of a system of sliding vectors.

On the basis of application of Formula (1.5), we may arrive at the conclusion that for any point not lying on the central axis, the principal moment will not be colinear to the principal vector. The central axis of the system is the only straight line that satisfies the condition posed above.

The distribution of the principal moments in the space is shown in Fig. 1 as a function of the position of the reduction points.

In a particular case of the system, it may be found that the principal moment is perpendicular to the principal vector for any point of the space. Then $J = r \cdot r^o = 0$ and we have the third of the cases listed above (1.6). On reducing the system to the central axis on the basis of (1.7), we find that $r^o = 0$ for points of the central axis. The system will be equivalent to one sliding vector, and the central axis will be the straight line on which this vector lies.
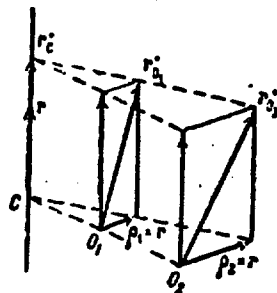
Fig. 1

For example, a system of sliding vectors passing through one point and a system of sliding vectors lying in the same plane reduce to this case, provided that $r \neq 0$.

When a system of sliding vectors is reduced to one equivalent sliding vector, the latter is known as the resultant vector or simply the resultant of the system in question.

Let us consider another method of reducing a system of sliding vectors $r_1, r_2, \ldots, r_n$. We take an arbitrary plane q that is parallel to none of the assigned vectors and consider the points of intersection $A_1, A_2, \ldots, A_n$ of this plane with the lines on which the vectors lie. We then take an arbitrary line a that is not parallel to plane q and is parallel to none of the given vectors. At each of the points A_k, we substitute the sliding vector r_k by its two components according to the parallelogram law (elementary operation "d"), one of which, s_k, lies in plane q, while the other, t_k, is parallel to line a. Instead of the given system of sliding vec-

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tors, we shall have two systems of sliding vectors $s_1, s_2, \ldots, s_n$ and $t_1, t_2, \ldots, t_n$. The first of these is a two-dimensional system equivalent to one resultant s in plane q (provided that it is not equivalent to a pair), while the second is a system of parallel vectors, also equivalent to one resultant t (provided that, like the first, it is not equivalent to a pair). These two resultants present a system equivalent to the given system. In the general case, they lie on crossed lines. Thus, an arbitrary system of sliding vectors is equivalent to a system consisting of two sliding vectors lying on lines that, generally speaking, do not intersect, or, in other words, to a vector cross. Any system can be reduced to a vector cross by an innumerable number of methods.

§4. The Motor and the Screw

The geometric figure-equivalent of a vector system, represented for any point of the space by the principal vector and principal moment of the system with respect to this point, is known as a motor (combination of the words "moment" and "vector"). For simplicity, we shall henceforth use the term motor for the combination of a vector and a moment (r, r^0), referred to some single point, assuming that the origins of r and r^0 are at this point.

If the system of sliding vectors is reduced to a point on the central axis, the principal moment will be colinear with the principal vector.

A motor (r, r^0) whose moment r^0 is colinear to the vector is known as a screw.

The line on which r lies is called the axis of the screw. In other words, a screw is a system consisting of a sliding vector r and a moment r^0 colinear with it.

It follows from all of the above that in the general case, a system of sliding vectors is equivalent to a screw. The axis of the screw is a central axis of the system; the vector of the screw is the principal vector; the moment of the screw is the principal moment of the system with respect to an arbitrary point on the central axis.

Since the vectors r and r^0 are colinear, $r^0 = pr$, where p is a scalar multiplier. This multiplier is called the parameter of the screw. The quantity p will be positive if r and r^0 point in the same direction and negative if they point in opposite directions.

Any sliding vector is, at the same time, a screw with zero parameter, and the straight line on which it lies is the axis of this screw; any moment is a screw with an infinite parameter whose axis may be any straight line parallel to it. Henceforth we shall use the term "screw" for screws with arbitrary parameters, including the zero-parameter screw, i.e., the sliding vector.

A zero-parameter screw whose vector is unity will be called a unit screw (same as unit sliding vector).

Screws will be denoted by upper-case boldface letters.

A screw R fully defines a motor $(\overline{r}, \overline{r})$ for any point in space; this motor, in turn, uniquely defines the screw.

Replacing a screw by the equivalent motor at point O is known as reduction of the screw to point O; the point O to which the motor is referred will be called the reduction point.

The moment r_O^0 is the moment of the screw with respect to point O.

§5. Kinematic Screw and Force Screw

Since the theory of screws has direct applications in mechanics, it will be convenient to make reference here to the kinematic and force interpretations of the screw.

The most general case of displacement of a solid body in space reduces to a screw displacement characterized by the axis, the absolute value of the principal vector and the parameter. A kinematic screw is a screw that characterizes the displacement of a body. The axis of this screw coincides with the axis of the screw displacement, the modulus of the principal vector expresses the magnitude of the body's angle of rotation, and the parameter gives the ratio of the translational displacement (slip) parallel to the axis to the angle of rotation.

If the screw displacement is infinitesimal, its referral to a time increment results in an instantaneous or velocity screw, in which the vector is the angular velocity of the body and the moment its translational velocity. In this case, the velocity of an arbitrary point of the body is represented by the moment of the screw with respect to this point.

The most general system of forces acting on a body can be reduced to a force screw by the rules of reduction of a vector system to a screw, if the vectors represent forces. The moment of a system of forces with respect to any point of the space is the moment of the equivalent force screw with respect to this point, or, what is the same thing, the moment obtained by reduction of the force screw to this point.

§6. Relative Moment of Two Screws

The sum of the summands: a) the projections of the vector of a first screw onto the axis of the moment of a second screw with respect to some point, multiplied by the moment of the second, and b) the projections of the vector of the second screw onto the axis of the moment of the first with respect to the same point, multiplied by the moment of the first, is known as the relative moment of two screws.

If a force screw R acts upon a body performing an elementary displacement characterized by a kinematic screw U, then the work performed by the force screw on the displacement screw will be equal to the relative moment of the force and kinematic screws R and U.

This familiar premise can easily be proven if we reduce both screws to the same point and then examine the sum of the works of the force-screw principal vector on translational displacement of the point and of the force-screw principal moment on angular displacement of the body.

Chapter 2

THE MULTIPLIER ω AND INTRODUCTION OF COMPLEX VECTORS. COMPLEX NUMBERS OF THE FORM $a + \omega a°$. ALGEBRA AND ANALYSIS IN THE DOMAIN OF THESE COMPLEX NUMBERS

§1. The Multiplier ω. The Complex Vector

As we have already stated, the direct definition of a screw by its axis, vector and parameter is replaced by definition of a motor referred to a point of reduction and representing the combination of a vector and a moment. By this substitution we gain an advantage in that operation directly on the screw is replaced by operation on vectors and reduces to a problem of ordinary vector algebra.

Clifford introduced a highly original and important operation by means of which a motor $(r, r°)$ is expressed formally in the form of the complex vector

$$r + \omega r°,$$

where ω is a multiplier whose square is equal to zero.

If we operate with a complex vector of this kind as with a formal sum, then ω will play the part of a number possessing the property $\omega^2 = 0$.

Introduction of the complex vector with this multiplier ω has interesting consequences. Firstly, the results of operations on the motors are found to be independent of the reduction point for which the motor was obtained and, secondly, the "vector" part of the result of an operation on any motor is found equal to the result of the corresponding operation on the vector of the motor.

Since we shall employ the conception of the motor as a complex vector in the exposition to follow, it will be necessary at this point to consider the general properties of complex numbers of the form $a + \omega a°$, where $\omega^2 = 0$.

§2. Operations on Complex Numbers of the Form $a + \omega a°$. Algebra and Analysis

We shall use upper-case letter symbols to denote complex numbers of the form under consideration. Let us examine the complex number

$$A = a + \omega a^{\theta},$$

where $\omega^2 = 0$. The number a is known as the principal part and the number $a^{\theta} = \text{мом} (A)$ is the moment part of the complex number A. If $a^{\theta} = 0$, then the number is said to be real. The ratio $a^{\theta} : a = P(A)$ is known as the parameter of the number A (for $a \neq 0$).

Introducing the parameter $P(a)$, we can present the complex number in the form

$$A = a \left(1 + \omega \frac{a^{\theta}}{a} \right) = a [1 + \omega P(a)]. \qquad (2.1)$$

If $P(a) = 0$, then the number is real.

In defining operations on complex numbers, we shall use, firstly, the invariable principle according to which the equality $A = a + \omega a^{\theta} = 0$ means that the equalities $a = 0$ and $a^{\theta} = 0$ are satisfied simultaneously, and, secondly, we shall consider each complex number formally as a sum and the operation ω as a number possessing the formal property $\omega^2 = 0$.

Addition and subtraction of two of the complex numbers does not differ from addition and subtraction of ordinary complex numbers:

$$A \pm B = (a \pm b) + \omega (a^{\theta} \pm b^{\theta}). \qquad (2.2)$$

For multiplication, we shall use the formula

$$AB = (a + \omega a^{\theta}) (b + \omega b^{\theta}) = ab + \omega (ab^{\theta} + ba^{\theta}). \qquad (2.3)$$

For division (with $b \neq 0$) we obtain

$$\frac{A}{B} = \frac{a}{b} + \omega \frac{a^{\theta}b - b^{\theta}a}{b^2}. \qquad (2.4)$$

The operations of raising to a power and extracting a root will be carried out by the formulas

$$\left. \begin{array}{l} A^n = (a + \omega a^{\theta})^n = a^n + \omega n a^{\theta} a^{n-1}, \\ \sqrt[n]{A} = \sqrt[n]{a + \omega a^{\theta}} = \sqrt[n]{a} + \omega \dfrac{a^{\theta}}{n \sqrt[n]{a^{n-1}}}, \quad (a \neq 0). \end{array} \right\} \qquad (2.5)$$

To define a function of the complex variable $X = x + \omega x^{\theta}$, it will be expedient to represent this function also in the form of a complex variable:

$$F(X) = F(x + \omega x^{\theta}) = f(x, x^{\theta}) + \omega g(x, x^{\theta}), \qquad (2.6)$$

where $f(x, x^{\theta})$ and $g(x, x^{\theta})$ are real functions of the two real variables x and x^{θ}.

Here and below, it will be convenient to consider differentiable functions. For this purpose, it will be necessary to introduce a requirement similar to that introduced in the conventional

theory of functions of the complex variable for analytic functions, namely, that the derivative, i.e., the limit of the ratio of the increment of the function $\Delta F(X)$ to the increment ΔX of the complex variable X as $\Delta X \to 0$, is independent of the ratio $\Delta x^0 : \Delta x$.

Writing the expression for the derivative, we have

$$\frac{dF}{dX} = \frac{df(x, x^0) + \omega\, dg(x, x^0)}{dx + \omega\, dx^0} =$$
$$= \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x^0}\frac{dx^0}{dx}\right) + \omega\left[\frac{\partial g}{\partial x} + \left(\frac{\partial g}{\partial x^0} - \frac{\partial f}{\partial x}\right)\frac{dx^0}{dx} - \right.$$
$$\left. - \frac{\partial f}{\partial x^0}\left(\frac{dx^0}{dx}\right)^2\right]. \qquad (2.7)$$

To satisfy the above condition, it is necessary to set the multipliers before $dx^0 : dx$ equal to zero in Expression (2.7). This will give the relationships*

$$\frac{\partial f}{\partial x^0} = 0, \; \frac{\partial g}{\partial x^0} = \frac{\partial f}{\partial x}. \qquad (2.8)$$

It follows from the first of these that the function f is a function only of the variable x, i.e.,

$$f(x, x^0) = f(x), \qquad (2.9)$$

while the second implies the following expression for the function g:

$$g(x, x^0) = x^0\frac{\partial f}{\partial x} + f^0(x), \qquad (2.10)$$

where $f^0(x)$ is a certain function of x.

Consequently, the general expression for a function of the complex variable

$$X = x + \omega x^0$$

that satisfies the condition formulated will be

$$F(X) = f(x) + \omega\left[x^0\frac{\partial f}{\partial x} + f^0(x)\right]. \qquad (2.11)$$

For X real, i.e., for $x^0 = 0$, the function must have the expression

$$F(X) = f(x) + \omega f^0(x). \qquad (2.12)$$

We shall assume that in the general case, the function of the complex variable $X = x + \omega x^0$ depends both on the complex variable X and on complex parameters A, B, C, ... and define it with the aid of a Taylor series in which ωx^0 takes the part of the increment and all terms containing ω in powers higher than the first are set equal to zero. Thus,

$$F(X, A, B, C, \ldots) = F(x, a, b, c, \ldots) +$$
$$+ \omega \left(x^0 \frac{\partial F}{\partial x} + a^0 \frac{\partial F}{\partial a} + b^0 \frac{\partial F}{\partial b} + c^0 \frac{\partial F}{\partial c} + \cdots \right). \qquad (2.13)$$

Comparing (2.11) and (2.13), we find

$$\left. \begin{array}{l} f(x) = F(x, a, b, c, \ldots), \\ f^0(x) = a^0 \frac{\partial f}{\partial x} + b^0 \frac{\partial f}{\partial x} + c^0 \frac{\partial f}{\partial x} + \ldots \end{array} \right\} \qquad (2.14)$$

Thus, the principal part of the function is equal to a function of the principal parts of the quantities on which it depends.

From Formulas (2.11) and (2.13), we recognize an important fact — namely, that the function of the complex variable $x + \omega x^0$ is fully defined by a function of the principal part x.

It follows from this that if the principal parts f and φ of two functions F and Φ are identically equal, the functions themselves are also equal. Indeed, the equality $f^0 = \varphi^0$ follows on the basis of (2.14) from the equality $f = \varphi$, and we may conclude on the basis of (2.11) that

$$F = \Phi.$$

An important theorem follows from the above.

Theorem 1. *All identities pertaining to differentiable functions are preserved in the domain of complex quantities of the form $a + \omega a^0$.*

For the function e^x we obtain the expression

$$e^X = e^{x + \omega x^0} = e^x + \omega x^0 e^x = e^x (1 + \omega x^0).$$

On the other hand,

$$e^X = e^{x + \omega x^0} = e^x e^{\omega x^0},$$

from which it follows that

$$e^{\omega x^0} = 1 + \omega x^0, \qquad (2.15)$$

or, in general, for any number p

$$e^{\omega p} = 1 + \omega p. \qquad (2.15')$$

Comparison with Formula (2.1) indicates that any complex number A has the form

$$A = a + \omega a^0 = a \left(1 + \omega \frac{a^0}{a} \right) = a(1 + \omega p) = a e^{\omega p}, \qquad p = \frac{a^0}{a}.$$

It follows from this formula that

$$P(ABCD \ldots) = P(A) + P(B) + P(C) + P(D) + \ldots,$$

i.e., that the parameter of the product is equal to the sum of the parameters of the cofactors, while the parameter of a fraction is equal to the difference between the parameters of the numerator and the denominator.

Since the parameter of the number ωa is ∞, the theorems derived are not applicable to complex numbers that do not have a principal part. As a consequence, the modulus of the number's principal part may be taken as the modulus of the number, and hence complex numbers with the modulus zero are singularities.

For functions of the complex argument X we obtain

$$
\left.
\begin{aligned}
&\sin X = \sin x + \omega x^0 \cos x, \quad P(\sin X) = x^0 \operatorname{ctg} x, \\
&\cos X = \cos x - \omega x^0 \sin x, \quad P(\cos X) = -x^0 \operatorname{tg} x, \\
&\operatorname{tg} X = \operatorname{tg} x + \frac{\omega x^0}{\cos^2 x}, \quad P(\operatorname{tg} X) = \frac{x^0}{\sin x \cos x}, \\
&\ln X = \ln x + \omega \frac{x^0}{x}, \quad P(\ln X) = \frac{x^0}{x \ln x}, \\
&\sin AX = \sin ax + \omega(x^0 a + a^0 x) \cos ax, \\
&e^{A+B} = e^A e^B, \quad e^{iX} = \cos X + i \sin X, \\
&\sin^2 X + \cos^2 X = 1
\end{aligned}
\right\}
\tag{2.16}
$$

and so forth.

On the basis of (2.7) and Relationships (2.8), (2.9) and (2.10), we obtain an expression for the derivative of the function $F(X)$. We have

$$
\begin{aligned}
\frac{dF(X)}{dX} &= \frac{\partial f}{\partial x} + \omega \frac{\partial}{\partial x}\left(x^0 \frac{\partial f}{\partial x} + f^0(x)\right) = \\
&= \frac{\partial f}{\partial x} + \omega\left(x^0 \frac{\partial^2 f}{\partial x^2} + \frac{\partial f^0}{\partial x}\right).
\end{aligned}
\tag{2.17}
$$

It is seen from this formula that differentiation with respect to the complex variable X reduces to differentiation with respect to the real variable x.

If a certain function $\varphi(x)$, which is the principal part of $\Phi(X)$, is identically equal to $\partial f/\partial x$, then this will imply that the function $\Phi(x)$ is equal to dF/dX. Indeed, differentiating Equality (2.14) with respect to x, we shall have on the basis of the equality $\varphi = \partial f/\partial x$

$$
\frac{\partial f^0}{\partial x} = a^0 \frac{\partial}{\partial a}\left(\frac{\partial f}{\partial x}\right) + b^0 \frac{\partial}{\partial b}\left(\frac{\partial f}{\partial x}\right) + \cdots = a^0 \frac{\partial \varphi}{\partial a} + \\
+ b^0 \frac{\partial \varphi}{\partial b} + \cdots = \varphi^0(x),
$$

from which, substituting in (2.17),

$$
\begin{aligned}
\frac{dF}{dX} &= \frac{\partial f}{\partial x} + \omega\left[x^0 \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial f^0}{\partial x}\right] = \\
&= \varphi + \omega\left[x^0 \frac{\partial \varphi}{\partial x} + \varphi^0(x)\right] = \Phi.
\end{aligned}
$$

Finally, if F is a given function of the complex variable X and the complex parameters A, B, C, ..., then a function G of the same quantities that identically satisfies the equation

$$dG = F\,dX, \qquad (2.18)$$

will be called the integral of FdX and written as follows:

$$G = \int F\,dX = g + \omega\left[x^0\,\frac{\partial g}{\partial x} + a^0\,\frac{\partial g}{\partial a} + b^0\,\frac{\partial g}{\partial b} + \cdots\right]. \qquad (2.19)$$

It follows from (2.19) that

$$g\,(x) = \int f\,(x)\,dx.$$

If $\Phi = \Phi(X, A, B, \ldots)$, with $\varphi = g$, i.e., the principal part of the function Φ is equal to the integral of the principal part of the function F, the function itself is equal to the integral of F.

Indeed, substituting the function φ in (2.19) instead of g, we obtain

$$G = \varphi + \omega\left[x^0\,\frac{\partial \varphi}{\partial x} + a^0\,\frac{\partial \varphi}{\partial a} + b^0\,\frac{\partial \varphi}{\partial b} + \cdots\right] = \Phi.$$

On the basis of the above, we can formulate the following theorem.

Theorem 2. *All theorems of differential and integral calculus are preserved in the domain of complex numbers of the form $a + \omega a^0$.*

For example, for the complex quantities

$$\left.\begin{array}{l}\frac{d(X^M)}{dX} = MX^{M-1}, \quad \frac{d}{dX}\,(e^X) = e^X, \quad d\ln X = \frac{1}{X}, \\[2mm] \frac{d\sin X}{dX} = \cos X, \quad \frac{d\cos X}{dX} = -\sin X, \\[2mm] \int X^A\,dX = \frac{X^{A+1}}{A+1} + C, \quad \int\cos(AX)\,dX = \frac{\sin(AX)}{A} + C \end{array}\right\} \qquad (2.20)$$

and so forth.

We note the following peculiarities in the calculus of the complex numbers: a) the product of complex numbers can be equal to zero not only when one of the cofactors is equal to zero, but also when the principal parts of the two cofactors are equal to zero; thus $\omega a \omega b = 0$; b) division by ωa is impossible for any a.

§3. Algebraic Equations

Let us dwell on certain properties and singularities of algebraic equations with complex coefficients.

Let the coefficients in the $n\underline{th}$-degree equation

$$F\,(X) = AX^n + BX^{n-1} + CX^{n-2} + \ldots + RX + S = 0 \qquad (2.21)$$

be complex numbers:

$$A = a + \omega a^\circ, B = b + \omega b^\circ, C = c + \omega c^\circ, \ldots,$$
$$R = r + \omega r^\circ, S = s + \omega s^\circ. \tag{2.22}$$

Generally speaking, the roots of such an equation are complex numbers of the same form.

If we replace the coefficients in Eq. (2.21) by their complex expressions (2.22) and X by the complex number $x + \omega x^\circ$, we obtain two equations on separation of the principal and moment parts:

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + rx + s = 0, \tag{2.23}$$
$$[nax^{n-1} + (n-1)bx^{n-2} + \ldots + r]x^\circ +$$
$$+ a^\circ x^n + b^\circ x^{n-1} + c^\circ x^{n-2} + \ldots + r^\circ x + s^\circ = 0. \tag{2.24}$$

Solution of Eq. (2.21) reduces to determination of the principal part α of the root of the real equation (2.23) (it may be real or complex in the form $\alpha' + i\alpha''$), and then, after substituting it in Eq. (2.24), to determination of the moment part of the root:

$$x^\circ = \frac{-(a^\circ x^n + b^\circ x^{n-1} + \ldots + r^\circ x + s^\circ)}{nax^{n-1} + (n-1)bx^{n-2} + \ldots + r} = \alpha^\circ. \tag{2.25}$$

As we see, solution is possible unless we have the singular case in which the discriminant of the real equation (2.23) vanishes. In this case, of course, the equation has a multiple root that causes the derivative of the left member of Eq. (2.23) to vanish, as well as the left member itself. But the derivative of the left member of Eq. (2.23) appears as a multiplier before the x° in Eq. (2.24) and in the denominator of Expression (2.25) for α°, so that in this case determination of the corresponding moment part of the root becomes meaningless.

But it is obvious that when the discriminant of Eq. (2.25) vanishes, a solution nevertheless exists if the numerator of Expression (2.25) vanishes simultaneously with the denominator. This is possible only in the case in which the multiple root of Eq. (2.23) is simultaneously a root of Eq. (2.24). Let us suppose that α is a root of the kth multiplicity of Eq. (2.23) and also a root of the $(k-1)$th multiplicity of Eq. (2.24).

We write short forms of (2.23) and (2.24) with the assumption made above:

$$f(x) = (x - \alpha)^k g(x) = 0, \tag{2.26}$$
$$x^\circ f'(x) + f^\circ(x) = [k(x - \alpha)^{k-1}g(x) +$$
$$+ (x - \alpha)^k g'(x)]x^\circ + (x - \alpha)^{k-1} h(x) = 0, \tag{2.27}$$

where $g(x)$ and $h(x)$ are polynomials that do not contain the multiplier $(x - \alpha)$.

Differentiating the left members of Eqs. (2.26) and (2.27) $k - 1$ times, we obtain

$$f^{(k-1)}(x) = k!(x - \alpha)g(x) + \ldots + (x - \alpha)^k g^{(k-1)}(x). \tag{2.28}$$

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$$x^0 f^{(k)}(x) + f^{(k-1)}(x) = [k! \, g(x) + \ldots + (x-a)^k g^{(k)}(x)] \, x^0 + \\ + (k-1)! \, h(x) + \ldots + (x-a)^{k-1} h^{(k-1)}(x). \qquad (2.29)$$

The left members of Eqs. (2.26) and (2.27) vanish for $x = a$ and any x^0.

The derivatives of all orders up to and including the $(k-1)$th of the left member of Eq. (2.26) vanish for $x = a$.

The derivatives of all orders up to and including the $(k-2)$th of the left member of Eq. (2.27) vanish for $x = a$ and any value of a^0. However, the derivatives of the $(k-1)$th order of (2.29) vanish for

$$x = a, \quad x^0 = a^0 = -\frac{h(a)}{kg(a)},$$

where $g(a)$ and $h(a)$ are the values of the polynomials $g(x)$ and $h(x)$ for $x = a$.

We present Eq. (2.27) in the form

$$(x-a)^{k-1} \{[kx^0 g(x) + h(x)] + x^0 (x-a) g'(x)\} = 0. \qquad (2.30)$$

The expression in the square brackets,

$$H(x) = h(x) + kx^0 g(x) \qquad (2.31)$$

vanishes for $x = a$ and $x^0 = a^0 = -h(a)/kg(a)$

$$H(a) = h(a) - k\frac{h(a)}{kg(a)} g(a) = 0,$$

hence for $x^0 = a^0$, Expression (2.31) must be equal to the product of $(x-a)$ by a certain polynomial $g^0(x)$, i.e.,

$$H(x) = h(x) + ka^0 g(x) = (x-a) g^0(x),$$

from which

$$h(x) = (x-a) g^0(x) - ka^0 g(x).$$

In this case, Eq. (2.27) or (2.30) can be represented in the form

$$(x-a)^{k-1} [kg(x) x^0 + (x-a) g^0(x) - ka^0 g(x) + \\ + x^0 (x-a) g'(x)] = 0. \qquad (2.32)$$

But we may conclude at once from the above that the left member of the basic complex equation (2.21) can be presented as the product of a certain complex polynomial by the kth power of the difference $X - A$, i.e.,

$$F(X) = (X-A)^k G(X) = 0 \qquad (2.33)$$

and that its $(k-1)$th derivative will be

$$F^{(k-1)}(X) = k!(X-A)G(X) + \ldots + (X-A)^k G^{(k-1)}(X) \qquad (2.34)$$

where $A = \alpha + \omega\alpha^0$ is a complex root that causes Eq. (2.33) and all of its derivatives up to and including the $(k-1)$th order to vanish, while the function $G(X)$ takes the form

$$G(X) = g(x) + \omega\, [x^0 g'(x) + g^0(x)]. \qquad (2.35)$$

Actually, expanding the complex expression (2.33), we obtain two real algebraic equations:

$$(x-\alpha)^k g(x) = 0, \qquad (2.36)$$

$$(x-\alpha)^{k-1}[k(x^0-\alpha^0)g(x) + x^0(x-\alpha)g'(x) + $$
$$+ (x-\alpha)g^0(x)] = 0. \qquad (2.37)$$

These equations are the same as (2.26) and (2.32). It is seen at once from Eq. (2.33) that it is satisfied for $x = \alpha$ and for x^0 equal to any number, since on substitution of X by the quantity $A + \omega n = \alpha + \omega(\alpha^0 + n)$, where n is an arbitrary real number, the left member will be equal to zero:

$$F(A + \omega n) = (\omega n)^k G(A + \omega n) = 0,$$

since any power of ω higher than the first is equal to zero.

It follows from the fact of the existence of a multiple root of the complex equation $F(X) = 0$ that the discriminant of this equation is equal to zero.[8] Representing the discriminant as the resultant of the equation and its derivatives, we obtain

$$R(F, F') = 0, R(F, F'') = 0, \ldots, R(F, F^{(k-1)}) = 0,$$

relationships that are expressed accurate to the sign by the determinants

$$\begin{vmatrix} A & B & C & \cdots & \cdots \\ 0 & nA & (n-1)B & \cdots & \cdots \\ 0 & A & B & C & \cdots \\ 0 & 0 & nA & (n-1)B & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad (2.38)$$

and so forth.

If an equation with complex coefficients has a real root, then in the nonsingular case, i.e., when the discriminant of its real part is nonzero, the equations

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + rx + s = 0,$$
$$f^0(x) = a^0 x^n + b^0 x^{n-1} + c^0 x^{n-2} + \ldots + r^0 x + s^0 = 0$$

must be satisfied for x equal to this root. For this it is necessary and sufficient that the resultant of these equations be equal to zero, i.e.,

$$\begin{vmatrix} a & b & c & d & \cdots & \cdots \\ a^0 & b^0 & c^0 & d^0 & \cdots & \cdots \\ 0 & a & b & c & d & \cdots \\ 0 & a^0 & b^0 & c^0 & d^0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \qquad (2.39)$$

As an example, let us consider the simplest quadratic equation

$$F(X) = AX^2 + BX + C = 0, \qquad (2.40)$$

which is decomposed into two real equations:

$$f(x) = ax^2 + bx + c = 0, \qquad (2.41)$$
$$x^0 f'(x) + f^0(x) = (2ax + b)\, x^0 + a^0 x^2 + b^0 x + c^0 = 0. \qquad (2.42)$$

If the discriminant of Eq. (2.41) is zero, i.e., if

$$b^2 = 4ac, \qquad (2.43)$$

then Eq. (2.41) has a double root $x = -b/2a$, which also satisfies the derivative equation

$$2ax + b = 0, \qquad (2.44)$$

and it is necessary for solvability of the starting equation (2.40) that the equation

$$a^0 x^2 + b^0 x + c^0 = 0 \qquad (2.45)$$

have the same root. But since Eq. (2.45) cannot have common roots with Eq. (2.41) other than the double root indicated above, it is sufficient for solvability that the resultant of Eqs. (2.41) and (2.45) or the resultant of Eqs. (2.44) and (2.45) be equal to zero. Taking the latter, we obtain

$$\begin{vmatrix} a^0 & b^0 & c^0 & 0 \\ 0 & 2a & b & 0 \\ 0 & a^0 & b^0 & 0 \\ 0 & 0 & 2a & b \end{vmatrix} = 0.$$

Expanding the determinant and applying (2.43), we obtain

$$2b^0 - 4a^0 c - 4ac^0 = 0. \qquad (2.46)$$

Together with (2.43), (2.46) is equivalent to the complex relation

$$B^2 - 4AC = 0, \qquad (2.47)$$

which is the multiplicity condition of the root of the starting complex quadratic equation. Expanding (2.47), we obtain (2.43) and (2.46).

To summarize everything said above concerning algebraic equations, we may regard the following theorem as having been proven.

Theorem 3. a. *In the general case, an algebraic equation with complex coefficients of the form $a + \omega a^\circ$ has complex roots $\alpha + \omega \alpha^\circ$ of the same form (α and α° are real numbers or complex numbers with the imaginary unity $\sqrt{-1}$).*

b. *The principal part α of a root is a root of a real equation representing the principal part of the given complex equation, while the moment part α°, if the discriminant of the real equation cited is nonzero, is determined uniquely by the moment part of the starting complex equation.*

c. *If the discriminant of the principal part of the starting complex equation is equal to zero, then the principal part of the equation has a multiple root (it is also the principal part of the complex root of the starting equation), but in this case determination of the moment part of the root is, generally speaking, impossible and the solution of the equation loses its meaning. In this case, if the multiple root also causes the moment part of the equation to vanish, the moment part of the root is indeterminate.*

d. *If a root of the principal part of the equation is of multiplicity k and it is at the same time a root of multiplicity k — 1 of the moment part of the equation, then the discriminant of the starting complex equation is zero. In this case, the complex root has multiplicity k. This root also causes to vanish those equations whose left members are successive derivatives, including the (k — 1)th derivative, of the left member of the equation, while the starting equation itself is satisfied by the real part of the root for arbitrary moment part of the unknown.*

e. *For an algebraic equation with complex coefficients to have a real root, it is necessary and sufficient that the resultant of the principal part of the equation and the equation obtained by substituting the moment parts for the principal parts of the coefficients in the latter vanish.*

We note that the properties of algebraic equations with complex coefficients that were considered above have a kinematic interpretation that will be set forth later (see Chapter 4).

Manuscript Page No.	Footnotes
19	These relationships are analogous to the Cauchy-Riemann regularity conditions for functions of the complex variable $z + iz^\circ$, satisfaction of which over the entire range of variation of the function determines analyticity of the function.
25	Which follows from extension of the theorems of the algebra of real numbers to the algebra of complex numbers.

Chapter 3

OPERATIONS ON SCREWS — COMPLEX VECTOR ALGEBRA

§1. General Remarks

After having established the notion of the screw, construction of an algebra in which the screw is the object of various operations requires defining operations directly on the screws.

We shall base all operations on screws on operations on the motors corresponding to these screws. In defining two or more screws, we shall select one common point of reduction in the space and refer the motors of all screws to it. Any algebraic operation on screws (multiplication by a number, addition and multiplication) will be defined as the operation on the motors of these screws, and since each motor, as we have already stated, is formally expressed by a complex vector, the algebra of screws will be reduced to an algebra of complex vectors.

It is found that application of the basic vector operations to the complex vectors (motors) results in quantities that possess the following properties: firstly, they do not depend on the points to which the screws have been reduced and, secondly, the principal part of the quantity obtained as a result of the operations is the quantity obtained by the corresponding operation on the principal parts of the complex vectors. These properties are a consequence of the property of the selected multiplier ω that is expressed by the equality $\omega^2 = 0$.

Expressing the motor by a complex vector, we perform the operation on it formally as on the sum of two vectors. In multiplication, we use the distributive property of the product.

In particular, for the unit screw $E \rightarrow (e, e^0)$, $e \cdot e^0 = 0$, where the sign $\rightarrow$ indicates correspondence of the motor (e, e^0) to the screw E, we shall have

$$E^2 = (e + \omega e^0)^2 = e^2 + 2\omega e \cdot e^0 = 1. \qquad (3.1)$$

§2. Multiplication of a Screw by a Number

We shall define multiplication of a screw by a real number as the construction of a screw whose vector is equal to the vector of the given screw multiplied by this number, and whose moment with respect to any point of the space is equal to the moment of the given screw with respect to the same point, multiplied by the same number.

According to this definition, if E is a unit screw and (e, e^0) is its motor for any point, with $e^2 = 1, e \cdot e^0 = 0$, then the motor (er, e^0r) for the same point will correspond to the screw $R = Er$, where r is a real number.

Applying Formula (1.1) for the moment of the vector with respect to the new point, we arrive at the conclusion that this definition is independent of the point for which the moment was taken, i.e., that the screw Er that satisfies the condition of the definition for some single point will satisfy it for any point of the space.

Using (1.7) to find the point of the central axis, we can easily derive that the moment of the screw Er with respect to the axis of screw E is zero, and, consequently, the axis of the screw Er is at the same time the axis of screw E (zero parameter). It follows from this that multiplication by a real number does not change the axis of the unit screw.

On the basis of (3.1), we have

$$E^2 = 1, \; R = Er, \; R^2 = r^2; \qquad (3.2)$$

if r is a positive number, then the directions of E and Er coincide; if r is a negative number, the directions of E and Er are opposed.

To multiply an arbitrary screw R whose motor is $(r, r^0), \; r \cdot r^0 \neq 0$ by a real number a, we construct the screw Ra, for which the corresponding motor will be (ra, r^0a) by definition. Expressing the motor in terms of the complex vector, we shall have

$$R \rightarrow r + \omega r^0, \; Ra \rightarrow ra + \omega r^0 a, \qquad (3.3)$$

where the sign $\rightarrow$ indicates correspondence of the motor to the given screw. As can be shown, the axis of the screw is preserved in multiplication by a.

Let us give the definition of multiplication of a unit screw E by a complex number $R = r + \omega r^0$.

Expressing the motor (e, e^0) of the screw E in terms of the complex vector

$$E \rightarrow e + \omega e^0, \; e^2 = 1, \; e \cdot e^0 = 0,$$

we define the screw $R = ER$ as the screw corresponding to the motor of screw E multiplied by the complex number R, i.e.,

$$ER \rightarrow (e + \omega e^0)(r + \omega r^0) = er + \omega (e^0 r + er^0). \qquad (3.4)$$

For points on the axis of screw E, the moment e^0 is equal to zero; hence the motor of the screw $R = ER$ for these points

$$er + \omega er^0 = e(r + \omega r^0)$$

will also represent a screw, since the vector and moment are co-

linear.

It follows from this that as a result of multiplying the unit screw E by the complex number $R = r + \omega r^0$ we obtain a screw R whose axis is the axis of screw E and which can be represented by the complex vector

$$R = ER = E(r + \omega r^0) = Ere^{\omega p}, \quad p = \frac{r^0}{r}. \tag{3.5}$$

If r is a positive number, the direction of the vector of screw R coincides with the direction of E; if it is negative, the direction of this vector is opposed.

If a screw is defined by a motor $r + \omega r^0$ for an arbitrary reduction point O, the parameter of the screw will be determined by the formula

$$p = \frac{r \cdot r^0}{r^2}.$$

The complex number $|r|e^{\omega p}$, in which the principal part is equal to the modulus of the screw vector and the parameter is equal to the parameter of the screw, will be known as the complex modulus of the screw $R = Ere^{\omega p}$.

Multiplication of an arbitrary screw $R = Ere^{\omega p}$ by the complex number $A = a + \omega a^0$ will be defined as construction of a screw whose motor is obtained for an arbitrary point by multiplying the motor $(er, e^0 r + er^0)$ of the given screw for the same point by this complex number. Presenting the motor as a complex vector, we obtain

$$\begin{aligned} RA &\rightarrow [er + \omega (e^0 r + er^0)] (a + \omega a^0) = \\ &= era + \omega (e^0 ra + er^0 a + era^0), \end{aligned} \tag{3.6}$$

where $e \cdot e^0 = 0$.

Again in this case, we can easily satisfy ourselves that the definition is independent of the reduction point for which the motor was taken.

For points belonging to the axis of screw R, the moment e^0 is zero; hence for this axis we shall have a motor

$$RA \rightarrow era + \omega e (r^0 a + ra^0),$$

in which the moment is colinear with the vector, i.e., a screw. Consequently, the axis of screw RA is the axis of screw R and, consequently, the axis remains unchanged on multiplication of an arbitrary screw by a complex number.

For the screw RA, we obtain

$$RA = Era\left[1 + \omega \left(\frac{r^0}{r} + \frac{a^0}{a}\right)\right] = Erae^{\omega r^0/r} e^{\omega a^0/a},$$
$$|RA| = |r| |a| e^{\omega (r^0/r + a^0/a)}. \tag{3.7}$$

i.e., the absolute magnitude of the principal vector is multiplied by the absolute magnitude of the principal part of the multiplier, and the parameter of the multiplier is added to the screw parameter.

To summarize briefly:

a) on multiplication of a screw by a real number, the axis of the screw remains unchanged, while the vector and moment are multiplied by this number;

b) when a screw of zero parameter is multiplied by a complex number, the axis of the screw remains unchanged, the vector is multiplied by the principal part of the multiplier, and the parameter becomes equal to the multiplier parameter;

c) when an arbitrary screw is multiplied by a complex number, its axis remains unchanged, the vector is multiplied by the principal part of the multiplier, and the multiplier parameter — a complex number — is added to the screw parameter.

A screw in which the vector is zero and, consequently, the parameter is an infinitely large number will be called singular. The principal part of the modulus of a singular screw is equal to zero.

We shall henceforth denote the complex moduli of screws by the ordinary upper-case italic forms of the upper-case boldface letters used for the screws, and the principal parts of the moduli are the corresponding lower-case letters, namely,

$$|R| = R = re^{\omega p},$$

where p is the parameter and r is a positive number.

§3. Complex Angle Between Two Axes. The Brush

In our terminology, the complex angle A between two axes whose unit screws are E_1 and E_2 will refer to the figure formed by these axes and the straight line segment mn intersecting these axes at right angles, where m is a point on the first axis and n is a point on the second axis (Fig. 2).
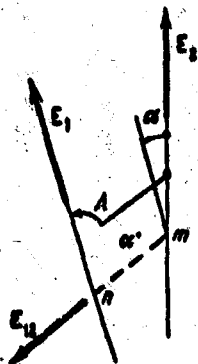
We assign the direction of line mn by the unit screw E_{12} and call it the axis of the complex angle.

To bring E_1 to coincidence with E_2, it is necessary to impart a screw motion to axis E_1, consisting of rotation about the axis E_{12} through the angle α between the directions of E_1 and E_2 and translational motion over a distance α^0 equal to the length of segment mn.

A complex angle is defined by a screw

Fig. 2

$$A = E_{12} A = E_{12}(\alpha + \omega\alpha^0), \qquad (3.8)$$

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and the complex number $\Lambda = \alpha + \omega \alpha^0$ is taken as a measure of the complex angle between axes $\pmb{E}_1$ and $\pmb{E}_2$.

As a convention for the signs of the numbers α and α^0, the former will be considered positive if the rotation appears to be counterclockwise to an observer at whom the unit screw $\pmb{E}_{12}$ is pointed, and the latter as positive if the translational motion takes place in the positive direction of $\pmb{E}_{12}$.

Obviously,

$$\angle (E_1, E_2) = - \angle (E_2, E_1).$$

The set of axes crossing the same axis with unit screw $\pmb{E}$ at right angles is called a brush. The axis of $\pmb{E}$ is the axis of the brush, and axes belonging to the brush are rays of the brush.

The above implies that the following relation obtains between the angles formed by the rays of the brush and defined by unit screws $E_1, E_2, \ldots, E_n$:

$$\angle (E_1, E_2) + \angle (E_2, E_3) + \ldots + \angle (E_{n-1}, E_n) +$$
$$+ \angle (E_n, E_1) = 0. \qquad (3.9)$$

We can express the trigonometric functions of the complex angle on the basis of Formulas (2.16):

$$\left.\begin{aligned}
\cos A &= \cos \alpha - \omega \alpha^0 \sin \alpha, \\
\sin A &= \sin \alpha + \omega \alpha^0 \cos \alpha, \\
\operatorname{tg} A &= \operatorname{tg} \alpha + \omega \frac{\alpha^0}{\cos^2 \alpha} = \operatorname{tg} \alpha + \omega \alpha^0 (1 + \operatorname{tg}^2 \alpha).
\end{aligned}\right\} \qquad (3.10)$$

Note. In defining the quantity α, we have a freedom in that the rotation of the axis $\pmb{E}_1$ to coincidence with the axis $\pmb{E}_2$ can be performed by either of two different paths. If the rotation through the angle $\alpha (< \pi)$ is performed counterclockwise, rotation through the supplementary angle $2\pi - \alpha$ will be performed clockwise, and the corresponding angle of rotation will be $-(2\pi - \alpha) = \alpha - 2\pi$, but the trigonometric functions of the angle (3.9) will be preserved. As a convention, we can take the angle smaller than two right angles as the angle α.

§4. Scalar Multiplication of Screws

We shall use the term scalar product of two screws for the complex number equal to the scalar product of their motors, which are referred to some reduction point.

We shall indicate scalar multiplication of screws by the center dot.

Given the two screws

$$R_1 = E_1 r_1 e^{\omega p_1}, \quad R_2 = E_2 r_2 e^{\omega p_2}$$

with the complex moduli

$$R_1 = r_1 e^{\omega p_1}, \; R_2 = r_2 e^{\omega p_2}, \; r_1 > 0, \; r_2 > 0,$$

and axes 1 and 2 of these screws forming a complex angle

$$A = \alpha + \omega \alpha^\circ.$$

We take an arbitrary point O and refer the motors of the given screws to it. Connecting point O with points m and n of axes 1 and 2, where mn is the shortest line segment between these axes, we obtain the radius vectors of points m and n from O (Fig. 3):

$$\vec{Om} = p_1, \; \vec{On} = p_2, \; \vec{mn} =$$
$$= p_2 - p_1 = E_{12}\alpha^\circ,$$
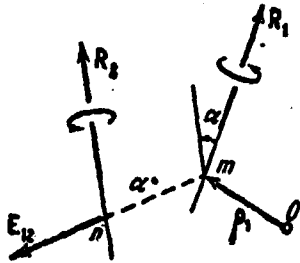
where E_{12} is the unit screw of line mn.

We shall express the motors of screws R_1 and R_2 referred to point O as complex vectors. Thus, we shall have a substitution of the screws by the complex vectors

$$R_1 \to r_1 + \omega \, (p_1 r_1 + p_1 \times r_1),$$
$$R_2 \to r_2 + \omega \, (p_2 r_2 + p_2 \times r_2). \tag{3.11}$$

Performing scalar multiplication, we obtain in accordance with our definition

$$\begin{aligned}
R_1 \cdot R_2 &= r_1 \cdot r_2 + \omega \; [(p_1 + p_2) \, r_1 \cdot r_2 + p_1 r_1 r_2 + r_1 p_2 r_2] = \\
&= r_1 \cdot r_2 + \omega \; [(p_1 + p_2) \, r_1 \cdot r_2 - (p_2 - p_1) \, r_1 r_2] = \\
&= r_1 r_2 \cos\alpha + \omega \; [(p_1 + p_2) \, r_1 r_2 \cos\alpha - r_1 r_2 \alpha^\circ \sin\alpha] = \\
&= r_1 r_2 e^{\omega p_2} e^{\omega p_1} (\cos\alpha - \omega \alpha^\circ \sin\alpha) = R_1 R_2 \cos A. \tag{3.12}
\end{aligned}$$

Hence a theorem.

Theorem 4. *The scalar product of two screws is equal to the product of their complex moduli by the cosine of the complex angle between them.*

The expression for the scalar product of screws in terms of the moduli and angle agrees exactly with the expression for the scalar product of free vectors provided that the real moduli in the latter are replaced by complex moduli and ordinary angles by complex angles.

As we see from the expanded expression (3.12), the principal part of the scalar product of two screws is the ordinary scalar product of the vectors of these screws, while the moment part is the relative moment of the screws

$$r_1 r_2 \; [(p_1 + p_2) \cos\alpha - \alpha^\circ \sin\alpha] \tag{3.13}$$

a quantity that does not depend on the point for which the motors are taken. The multiplier in the square brackets in Expression

(3.13) is known as the "possible coefficient" of the two screws (Ball).

If the screws R_1 and R_2 on which scalar multiplication is to be performed are given by the general expression of the motors, we obtain

$$R_1 \cdot R_2 = (r_1 + \omega r_1^0) \cdot (r_2 + \omega r_2^0) =$$
$$= r_1 \cdot r_2 + \omega (r_1 \cdot r_2^0 + r_2 \cdot r_1^0). \qquad (3.14)$$

The expression $r_1 \cdot r_2^0 + r_2 \cdot r_1^0$ is the relative moment of the two screws and is equal to Expression (3.13).

The scalar product of two screws for which the principal parts of the moduli are not equal to zero vanishes if cos A = 0, and, consequently, if

$$\alpha = \frac{\pi}{2}, \quad \alpha^0 = 0,$$

i.e., if the axes of the screws being multiplied intersect at right angles.

It follows from Formula (3.12) that $R_1 \cdot R_2 = R_2 \cdot R_1$.

If $\rho_1 = \rho_2 = 0$, i.e., if the screws being multiplied are sliding vectors, the scalar product assumes the form

$$R_1 \cdot R_2 = r_1 r_2 \cos \Lambda. \qquad (3.15)$$

Expanding Expression (3.15), we obtain

$$R_1 \cdot R_2 = r_1 r_2 \cos \alpha - \omega r_1 r_2 \alpha^0 \sin \alpha,$$

i.e., the scalar product of two sliding vectors gives the scalar product of these vectors in the principal part and the relative moment of these vectors in the moment part.

If we perform scalar multiplication of the screw R by itself, we obtain

$$R^2 = R^2 \cos 0 = (r e^{\omega \rho})^2 = r^2 e^{2\omega \rho}, \qquad (3.16)$$

i.e., the square of the complex modulus of the screw.

If the screw R is given by the motor $r + \omega r^0$, then the scalar square of the screw has the expression

$$R^2 = (r + \omega r^0)^2 = r^2 + 2\omega r \cdot r^0. \qquad (3.16')$$

For the square of a unit screw E defined by the motor $e + \omega e^0$, $e \cdot e^0 = 0$, we have the formula derived above:

$$E^2 = (e + \omega e^0)^2 = e^2 + 2\omega e \cdot e^0 = 1. \qquad (3.1)$$

The square of the moment

$$(\omega r)^2 = \omega^2 r^2 = 0.$$

If one of the screws on which the scalar multiplication is to be performed has an infinitely large parameter, the product will be

$$\omega r_1^0 \cdot E r_s e^{\omega p_s} = \omega r_1^0 r_s e^{\omega p_s} \cos A = \omega r_1^0 r_s \cos \alpha.$$

The principal part of the product is zero.

§5. Orthogonal Component of Screw Along Straight Line and Projection of Screw onto Axis

Let R be the given screw and let a be a straight line in the space whose unit screw is $\boldsymbol{E}$. We reduce the screw to a certain point A lying on this line; let (r, r^0) be the corresponding motor. Let us project the vector r and the moment r^0 orthogonally onto line a. The component of the vector r will be denoted by r_a and the moment component r^0 by r_a^0.

The screw (r_a, r_a^0) with its central axis on line a

$$R_a = E(r_a + \omega r_a^0) \tag{3.17}$$

will be called the orthogonal component of screw R on line a. It is obvious that neither r_a nor r_a^0 depends on the selection of the reduction point A on line a.

The complex number $r_a + \omega r_a^0$ by which the unit screw $\boldsymbol{E}$ is to be multiplied in order to obtain screw R_a will be called the orthogonal projection (or simply the projection) of screw R onto the axis defined by the unit screw $\boldsymbol{E}$. For the same directions of the screw vector R_a and the vector $\boldsymbol{E}$, the number r_a is positive; if they have opposite directions it is negative.

Let R be a screw and $\boldsymbol{E}$ a unit screw. We form their scalar product

$$R \cdot E = R \cos A = r e^{\omega p} (\cos \alpha - \alpha^0 \sin \alpha) =$$
$$= r \cos \alpha + \omega r (p \cos \alpha - \alpha^0 \sin \alpha), \tag{3.18}$$

where $A = \alpha + \omega \alpha^0$ is the complex angle between the axes R and $\boldsymbol{E}$. The complex expression (3.18) has the following geometrical sense: its principal part is the projection, onto the axis $\boldsymbol{E}$, of the screw vector, while the moment part is the projection of the screw moment with respect to a point lying on the axis onto the same axis. This expression, therefore, is the projection of screw R onto axis $\boldsymbol{E}$ by the definition just given.

Hence the projection of a screw onto the axis is a complex quantity equal to the product of the screw complex modulus by the cosine of the complex angle formed by the axis of the screw with the given axis.

For the case in which R is a screw of zero parameter (i.e.,

a sliding vector), $p = 0$ and Formula (3.18) is simplified, assuming the form

$$R \cdot E = r \cos \alpha - \omega r x^0 \sin \alpha, \qquad (3.19)$$

i.e., the principal part of the complex projection is equal to the projection of the sliding vector onto the axis, while the moment part is equal to the moment of the vector with respect to the axis. Multiplication by the cosine of the complex angle automatically yields both the projection and the moment.

§6. Screw Multiplication of Screws

In our terminology, the screw product of two screws will be the screw whose motor is equal, for an arbitrary point of the space, to the vector product of the motors of the given screws for the same point.

We shall use the cross to indicate screw multiplication.

In order to determine the screw product of two screws R_1 and R_2, it is necessary to perform vector multiplication of the motors of Expressions (3.11). Since we are dealing with motors referred to point O, the motor obtained as the result of multiplication will also be referred to point O. We transfer point O to the point n through which the sliding vector E_{12} (Fig. 3) passes, i.e., we refer the final result to point m.

For point O, we shall have

$$R_1 \times R_2 = [r_1 + \omega \, (p_1 r_1 + \rho_1 \times r_1)] \times$$
$$\times \, [r_2 + \omega \, (p_2 r_2 + \rho_2 \times r_2)] = r_1 \times r_2 + \omega \, [r_1 \times$$
$$\times \, (p_2 r_2 + \rho_2 \times r_2) + (p_1 r_1 + \rho_1 \times r_1) \times r_2]. \qquad (3.20)$$

For the final results to be referred to point m, we add the moment of the vector $r_1 \times r_2$, which must be imagined to pass through O, with respect to point m. We obtain

$$R_1 \times R_2 = r_1 \times r_2 + \omega \, [r_1 \times (p_2 r_2 + \rho_2 \times r_2) +$$
$$+ \, (p_1 r_1 + \rho_1 \times r_1) \times r_2] - \omega \rho_1 \times (r_1 \times r_2) =$$
$$= r_1 \times r_2 + \omega \, \{(p_1 + p_2) \, r_1 \times r_2 + (\rho_2 - \rho_1) \, (r_1 \cdot r_2) +$$
$$+ \, r_2 \, [(\rho_1 - \rho_2) \cdot r_1]\} = r_1 \times r_2 + \omega \, [(p_1 + p_2) \, r_1 \times r_2 +$$
$$+ \, (\rho_2 - \rho_1) \, (r_1 \cdot r_2)]. \qquad (3.21)$$

Exactly the same result would be obtained if we had made the transfer to point n or to any point on line mn.

In the expression obtained, the vector $r_1 \times r_2$ appears in the principal part and the linear combination of the vectors $r_1 \times r_2$ and $\rho_2 - \rho_1$, i.e., a moment parallel to E_{12}, appears in the moment part. It follows from this that the line mn, the axis of unit screw E_{12}, is the axis of the screw product $R_1 \times R_2$. As a result, we may write

$$R_1 \times R_2 = E_{12} r_1 r_2 \, \{\sin \alpha + \omega \, [(p_1 + p_2) \sin \alpha +$$
$$+ \, \alpha^0 \cos \alpha]\} = E_{12} r_1 r_2 e^{\omega p_1} e^{\omega p_2} (\sin \alpha + \omega x^0 \cos \alpha) =$$
$$= E_{12} R_1 R_2 \sin A. \qquad (3.22)$$

Thus the theorem has been proven.

Theorem 5. *The screw product of two screws is the screw whose axis intersects the axes of the screws being multiplied at right angles and whose vector has the direction of the vector product of the vectors of these screws, while its complex modulus [is equal] to the product of the complex moduli of these screws by the sine of the complex angle formed by their axes.*

It follows from Formula (3.22) that

$$R_1 \times R_2 = - R_2 \times R_1.$$

If the vectors of screws R_1 and R_2 are not equal to zero, then, according to Formula (3.22), the screw product of the two screws can vanish only if the axes of these screws coincide.

If one of the screws has an infinitely large parameter, the screw product will be a screw of infinite parameter, since its modulus

$$| R_1 \times R_2 | = \omega r_1^0 r_2 e^{\omega p_2} \sin A = \omega r_1^0 r_2 \sin \alpha \qquad (3.23)$$

has no principal part. Such a screw is a pair, and any line in the space that is perpendicular to the axes of the screws being multiplied may serve as its axis.

§7. Addition of Screws

A screw R is called the sum of the given screws $R_1, R_2, \ldots, R_n$

$$R = R_1 + R_2 + \ldots + R_n, \qquad (3.24)$$

if its vector is equal to the sum of the vectors of these screws and the moment with respect to any point in space is equal to the sum of the moments of the added screws with respect to this same point, i.e.,

$$\left. \begin{array}{l} r = r_1 + r_2 + \ldots + r_n, \\ r^0 = r_1^0 + r_2^0 + \ldots + r_n^0. \end{array} \right\} \qquad (3.25)$$

We can satisfy ourselves that the geometrical figure determined is indeed a screw on the basis of the fact that the scalar product $r \cdot r^0$ does not depend on the point of reduction. Actually, for any reduction point the scalar product

$$r \cdot r^0 = r_1 \cdot r_1^0 + r_2 \cdot r_2^0 + \ldots + r_n \cdot r_n^0 +$$
$$+ \sum_{i,k=1}^{n} (r_i \cdot r_k^0 + r_k \cdot r_i^0), \quad i \neq k \qquad (3.26)$$

consists of the sum of the invariants of the screws and the sum of the relative moments of all possible pairs of screws, and, consequently, it does not depend on the reduction point.

On reduction of the screws to a new point $O' (\overrightarrow{O'O} = \rho)$, we shall have on the basis of Formula (1.5)

- 37 -

$$r^{01} = r^0 + p \times r = \sum_{i=1}^{n} r_i^0 + p \times \sum_{i=1}^{n} r_i =$$

$$= \sum_{i=1}^{n} (r_i^0 + p \times r_i) = \sum_{i=1}^{n} r_i. \qquad (3.27)$$

from which we may conclude that if the condition of the definition as regards the principal moment is satisfied for any one point of the space, it will be satisfied for any point of the space.

Let us consider the scalar product of the sum of several screws $R = R_1 + R_2 + \ldots + R_n$ by a screw S. Substituting motors referred to a certain point for the screws, we obtain

$$R \cdot S = (R_1 + R_2 + \ldots + R_n) \cdot S = [(r_1 + \omega r_1^0) +$$
$$+ (r_2 + \omega r_2^0) + \ldots + (r_n + \omega r_n^0)] \cdot (S + \omega S^0) =$$
$$= (r_1 + \omega r_1^0) \cdot (S + \omega S^0) + \ldots$$
$$\ldots + (r_n + \omega r_n^0) \cdot (S + \omega S^0) = R_1 \cdot S + R_2 \cdot S + \ldots$$
$$\ldots + R_n \cdot S. \qquad (3.28)$$

From this follows the distributive property of the scalar product: the scalar product of the sum of several screws by a certain screw is equal to the sum of the scalar products of the added screws by this screw. In particular, the projection of the sum of several screws onto the axis is equal to the sum of the projections of the added screws onto this axis.

We can satisfy ourselves in a similar manner of the existence of the distributed property of the screw product.

Finding the screw R, the sum over given screw terms $R_1, R_2, \ldots$ $\ldots, R_n$, reduces to determination of the central axis of this screw, the modulus of its vector and its parameter.
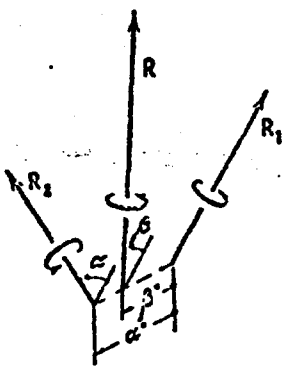
Fig. 4

Applying the distributive property of the scalar product of screws, let us derive a formula for addition of screws by the use of which we can construct a screw equal to the sum of two given screws. This formula is an analogue of the familiar triangle formula for the sum of vectors. Given: screws R_1 and R_2 and required: determine the sum screw R

$$R = R_1 + R_2. \qquad (3.29)$$

Performing scalar multiplication of screw R by E_{12}, the unit screw of the axis of the angle formed by screws R_1 and R_2,

$$R \cdot E_{12} = (R_1 + R_2) \cdot E_{12} = R_1 \cdot E_{12} + R_2 \cdot E_{12} = 0, \qquad (3.30)$$

we find that the axis of screw R intersects at right angles with the axis of the angle formed by R_1 with R_2 (Fig. 4). We denote the complex angles (R_1, R_2) and (R_1, R) by $A = \alpha + \omega \alpha^0$ and $B = \beta + \omega \beta^0$, respectively.

We project all three screws onto the axis of screw R_1 and then onto the axis of screw R_2, performing scalar multiplication of these screws by the unit vectors of the axes of screws R_1 and R_2 by Formula (3.18). On the basis of the distributive property of the scalar product we obtain

$$R \cos B = R_1 + R_2 \cos A,$$
$$R \cos (A - B) = R_1 \cos A + R_2, \quad R = |R|, \quad R_1 = |R_1|, \quad (3.31)$$
$$R_2 = |R_2|,$$

from which

$$R \sin B = R_2 \sin A,$$
$$R \sin (A - B) = R_1 \sin B,$$

or, combining the two relationships obtained, we find

$$\frac{R}{\sin A} = \frac{R_1}{\sin (A - B)} = \frac{R_2}{\sin B}. \quad (3.32)$$

Thus, we may regard the following theorem as having been proven:

Theorem 6. *If R is the sum of two screws R_1 and R_2, Relationship (3.32), which is analogous to the relation between the sides and angles in a triangle, but with the real quantities replaced by complex quantities, applies between the complex moduli of these screws and the complex angles formed by their axes.*

On the basis of (3.29) we have

$$R^2 = (R_1 + R_2)^2 = R_1^2 + R_2^2 + 2R_1 \cdot R_2 =$$
$$= R_1^2 + R_2^2 + 2R_1 R_2 \cos A. \quad (3.33)$$

On the basis of

$$R = r e^{\omega p}, \quad R_1 = r_1 e^{\omega p_1}, \quad R_2 = r_2 e^{\omega p_2}$$

we obtain the relation

$$r^2 e^{2\omega p} = r_1^2 e^{2\omega p_1} + r_2^2 e^{2\omega p_2} + 2 r_1 r_2 e^{\omega (p_1 + p_2)} (\cos \alpha - \omega x \sin \alpha), \quad (3.34)$$

from which, on separation of the principal and moment parts, we find the magnitude of the principal vector and the parameter of the sum screw R:

$$r^2 = r_1^2 + r_2^2 + 2 r_1 r_2 \cos \alpha, \quad (3.35)$$

$$p = \frac{p_1 r_1^2 + p_2 r_2^2 + r_1 r_2 [(p_1 + p_2) \cos \alpha - \alpha^0 \sin \alpha]}{r_1^2 + r_2^2 + 2 r_1 r_2 \cos \alpha}. \quad (3.36)$$

The complex angle B is determined from (3.32):

$$\sin B = \frac{R_2 \sin A}{R} = \frac{r_2 e^{\omega p_2} (\sin \alpha + \omega x^0 \cos \alpha)}{r e^{\omega p}} = \sin \beta + \omega \beta^0 \cos \beta.$$

Separating the principal and moment parts, we get

$$\sin\beta = \frac{r_2\sin\alpha}{r} = \frac{r_2\sin\alpha}{\sqrt{r_1^2 + r_2^2 + 2r_1 r_2\cos\alpha}}, \qquad (3.37)$$

moreover,

$$\alpha\beta^0 - \sin\alpha\beta^0 = \sin(B-\beta) = \sin B\cos\beta - \cos B\sin\beta$$
$$= \frac{R_2\sin A}{R}\cdot\frac{r_1 + r_2\cos\alpha}{r} - \frac{r_1 + R_2\cos A}{R}\cdot\frac{r_2\sin\alpha}{r}.$$

After transformations,

$$\beta^0 = r_2\frac{r_1[(p_2-p_1)\sin\alpha + r^0\cos\alpha] + \alpha r_2}{r_1^2 + r_2^2 + 2r_1 r_2\cos\alpha}. \qquad (3.38)$$

In exactly the same way

$$\sin(\alpha-\beta) = \frac{r_1\sin\alpha}{r} = \frac{r_1\sin\alpha}{\sqrt{r_1^2 + r_2^2 + 2r_1 r_2\cos\alpha}}, \qquad (3.39)$$

$$\alpha^0 - \beta^0 = r_1\frac{r_2[(r_1-p_2)\sin\alpha + r^0\cos\alpha] + \alpha r_1}{r_1^2 + r_2^2 + 2r_1 r_2\cos\alpha}. \qquad (3.40)$$

We have obtained a simple result: the equation of closure of the vector triangle and the equation of moments are embodied in a single screw equation (3.29), which simultaneously expresses the parallelogram law and the lever law.

As is easily seen, this result proceeds directly from the formula for scalar multiplication of screws and from the distributive property of scalar multiplication, interpreted as equality of the projection of the sum of the screws onto the axis to the sum of the projections of the terms onto the same axis.

Fig. 5

The relationships given above can be regarded as formulas for the "expanded" triangle. This figure is obtained by parallel translation of the sides of a plane triangle in the direction perpendicular to its plane (Fig. 5).

Denoting the complex angles of the triangle (i.e., the angles together with the segments onto which the sides have been transferred) by the corresponding upper-case letters and assigning complex values equal to the complex moduli of the corresponding screws to the sides of the triangle, we find that when the quantities appearing in it are given the complex treatment, the familiar trigonometric relationship (3.32) expresses equality of one of the screws to the sum of the other two. With the use of complex numbers, therefore, the geometry of the simple triangle becomes the geometry of the "expanded" triangle.

We obtain the formula for the difference of screws $R_1 - R_2$ by substituting $-R_2$ for R_2 in all of the formulas pertaining to addition of screws.

§8. Orthogonal Projections of a Screw onto Two Mutually Perpendicular Axes

Let us imagine a screw R with modulus $R = re^{\omega p}$, that intersects a certain axis z at right angles, and two axes x and y such that xyz forms a coordinate system with its origin at point O. Obviously, the axis of the screw R is parallel to the xy-plane (Fig. 6).
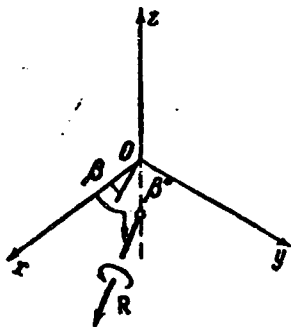
Let the complex angle formed by the axis of screw R with the x-axis be $B = \beta + \omega\beta^0$; then the angle formed by the axis of the screw with the y-axis will be $B - \pi/2$.

Let us determine the magnitudes of the projections X and Y of screw R onto the axes x and y.

Fig. 6

According to Formula (3.18), we have

$$R_x = R \cos B = re^{\omega p}(\cos\beta - \omega\beta^0 \sin\beta) =$$
$$= r\cos\beta\,[1 + \omega\,(p - \beta^0\,\mathrm{tg}\,\beta)],$$
$$R_y = R \cos\left(B - \frac{\pi}{2}\right) = R\sin B = re^{\omega p}(\sin\beta + \omega\beta^0\cos\beta) =$$
$$= r\sin\beta\,[1 + \omega\,(p + \beta^* \quad)].$$

Regarding the orthogonal components of screw R along the x- and y-axes as the screws iR_x and jR_y, where i and j are the unit vectors of the x- and y-axes, and knowing the complex moduli R_x and R_y of these screws and the parameters

$$p - \beta^0\,\mathrm{tg}\,\beta,\ p + \beta^0\,\mathrm{ctg}\,\beta,$$

let us add these screws, i.e., find a screw equal to the sum

$$R' = iR_x + jR_y.$$

The angle formed by the axes of these screws is

$$A = \alpha + \omega x^0 = \frac{\pi}{2},$$

and, consequently, $\alpha = \pi/2,\ \alpha^0 = 0,\ \cos\alpha = 0,\ \sin\alpha = 1$. On the basis of the addition formulas (3.33), (3.35), (3.37) and (3.38), we find the length of the vector r', the parameter p and the complex angle $B' = \beta' + \omega\beta^{0'}$ of the resultant screw with the x-axis. We have

$$R'^2 = R_x^2 + R_y^2 + 2R_xR_y\cos\frac{\pi}{2} = R_x^2 + R_y^2 = R^2,$$
$$r'^2 = r^2,$$

$$p' := \frac{1}{r^2}\left[(p - \beta^0 \, \mathrm{tg}\,\beta)\, r^2 \cos^2\beta + (p + \beta^0 \, \mathrm{ctg}\,\beta)\, r^2 \sin^2\beta + \right.$$
$$\left. + r^2 \cos\beta \sin\beta \left\{[2p + \beta^0 (\mathrm{ctg}\,\beta - \mathrm{tg}\,\beta)]\cos\alpha - x^0 \sin\alpha\right\}\right] =$$
$$= p\cos^2\beta + p\sin^2\beta = p,$$

$$\sin\beta' = \frac{r\sin\beta}{r} = \sin\beta,$$

$$\beta^{0'} = \frac{1}{r^2}\left[r\sin\beta\, r\cos\beta\, \beta^0 (\mathrm{tg}\,\beta + \mathrm{ctg}\,\beta)\right] = \beta^0.$$

It follows from the resulting formulas that the unknown screw R' is identical with the original screw R. Hence follows a theorem.

Theorem 7. *A screw whose axis intersects the z-axis of a rectangular coordinate system at right angles is equal to the sum of its orthogonal components on the x- and y-axes.*

§9. Linear Combination of Two Screws. The Brush. The Cylindroid

The linear combination of two screws is the generalization of their sum.

Let R_1 and R_2 be two arbitrary screws and let A and B be complex numbers. Consider the linear combination

$$R = AR_1 + BR_2. \tag{3.41}$$

in which A and B are varied at will.

Let us take a unit screw E_{12} whose axis intersects the axes of screws R_1 and R_2 at right angles. Expressing the scalar product of screw R by E_{12}, we obtain

$$R \cdot E_{12} = AR_1 \cdot E_{12} + BR_2 \cdot E_{12} = 0, \tag{3.42}$$

from which it follows that the linear combination of screws R_1 and R_2 for any values of A and B intersects the unit screw E_{12} at right angles, i.e., as A and B vary, the axis of screw R describes a brush having the axis E_{12}.

Formula (3.41) is the complex analogue of the conventional formula for the linear combination of two vectors:

$$r = ar_1 + br_2. \tag{3.43}$$

which describes a plane or a flat bundle of vectors on variation of the real numbers a and b if the vectors r_1 and r_2 are constructed from a common origin.

On complex treatment of Formula (3.43), i.e., on substitution of Formula (3.41) for it, the geometrical locus of the lines on which the screws lie will be a brush, which is thus the complex analogue of the plane or flat bundle of vectors.

Let us now consider that particular case of the linear combination of screws (3.41) in which it is formed with the aid of real multipliers a and b, i.e.,

$$R = aR_1 + bR_2. \tag{3.44}$$

where R_1 and R_2 are the given screws.

Screw R intersects the axis of the unit screw B_{12} at right angles, the latter intersecting the axes of screws R_1 and R_2 at right angles. Assigning different values to the numbers a and b, we cause the axis of the screw to describe a certain geometric locus. As can be seen from Formula (3.44), neither the direction nor the position of the axis changes on proportional variation of the numbers a and b, so that the change in one parameter — the ratio a/b — will be essential. One direction and one point of intersection of the axis of R with the axis of B_{12} will correspond to each value of this parameter. From this it follows that for all possible variations of the numbers a and b, the geometric locus described by the axis of screw R will be a ruled surface all of whose generators intersect the axis of the shortest distance between the axes of screws R_1 and R_2 at right angles. This surface is known as a cylindroid. Let us determine certain of its properties.

Let the complex moduli of screws R_1 and R_2, which we shall call the basic screws, be $e^{\omega p_1}$ and $e^{\omega p_2}$. Let R' and R'' be any two screws defined by Formula (3.41), i.e.,

$$R' = a'R_1 + b'R_2,$$
$$R'' = a''R_1 + b''R_2.$$

We write the scalar product of screws R' and R'':

$$R' \cdot R'' = a'a''e^{2\omega p_1} + b'b''e^{2\omega p_2} + (a'b'' + a''b')e^{\omega(p_1+p_2)}\cos\Theta,$$

where Θ is the complex angle between the axes of the basic screws R_1 and R_2.

Equating the scalar product to zero and dividing by $a'a''e^{\omega(p_1+p_2)}$, we have

$$e^{\omega(p_1-p_2)} + \lambda\mu e^{\omega(p_2-p_1)} + (\lambda+\mu)\cos\Theta = 0, \tag{3.45}$$

where $\lambda = b'/a'$, $\mu = b''/a''$.

Separating the principal and moment parts of (3.45), we obtain two equations, which yield

$$\lambda + \mu = \frac{2(p_1 - p_2)}{\theta^0 \sin\theta - (p_1 - p_2)\cos\theta},$$
$$\lambda\mu = -\frac{\theta^0 \sin\theta + (p_1 - p_2)\cos\theta}{\theta^0 \sin\theta - (p_1 - p_2)\cos\theta}. \tag{3.46}$$

Since the quantity $(\lambda + \mu)^2 - 4\lambda\mu$ is, as can be verified, essentially positive, λ and μ are always real, and therefore (3.45) can always be satisfied. Thus among the screws appearing in the linear combination (3.44) and lying on a cylindroid, there are always two screws R_1 and R_2 whose axes intersect at right angles. Two such screws are known as the principal screws of the cylindroid, and their parameters p' and p'' as the principal parameters;

the point of intersection is the center of the surface.

Principal parameters can be determined as follows. Since the numbers λ and μ are known, we can take arbitrary values of a' and a'', for example, unit values, and obtain from them the values of $b' = \lambda$ and $b'' = \mu$. We shall have

$$R' = R_1 + \lambda R_2, \; R'' = R_1 + \mu R_3.$$

By the rule of addition of screws on the basis of Formula (3.36), we find

$$p' = \frac{p_1 + \lambda^2 p_2 + \lambda\,[(p_1 + p_3)\cos\theta - 0^0\sin\theta]}{1 + \lambda^2 + 2\lambda\cos\theta},$$
$$p'' = \frac{p_1 + \mu^2 p_2 + \mu\,[(p_1 + p_2)\cos\theta - 0^0\sin\theta]}{1 + \mu^2 + 2\mu\cos\theta}. \qquad (3.47)$$

Thus, the principal parameters are known. Taking the axes of the principal screws R' and R'' as the axes x and y, the axis of the surface as the z-axis, and the center as the coordinate origin, we derive the equation of the cylindroid, i.e., we express the position of its generator as a function of its angle with the x-axis and find an expression for the parameter of the screw whose axis lies on this generator.

We take principal screws $R' = R_1$ and $R'' = R_2$ such that the moduli of their vectors are equal to unity, i.e.,

$$R_1 = e^{\omega p}, \; R_2 = e^{\omega p_2}; \qquad (3.48)$$

then the modulus of the vector of screw R — the linear combination

$$R = aR_1 + bR_2 \qquad (3.49)$$

will be determined by the formula

$$R^2 = r^2 e^{2\omega p} = a^2 e^{2\omega p_1} + b^2 e^{2\omega p_2}, \qquad (3.49')$$

or, on separation of the principal part from the moment part,

$$r^2 = a^2 + b^2, \;\; r^2 p = a^2 p_1 + b^2 p_2, \qquad (3.50)$$

from which

$$p = \frac{a^2}{a^2 + b^2}\, p_1 + \frac{b^2}{a^2 + b^2}\, p_2, \qquad (3.51)$$

which gives an expression for the parameter p of screw R in terms of the angle φ and the principal parameters p_1 and p_2:

$$p = p_1 \cos^2\varphi + p_2 \sin^2\varphi. \qquad (3.52)$$

The screw R intersects the z-axis at right angles and forms a complex angle $\Phi = \varphi + \omega\varphi^0$ with the x-axis. The projections of the screw R onto the x- and y-axes will be, respectively,
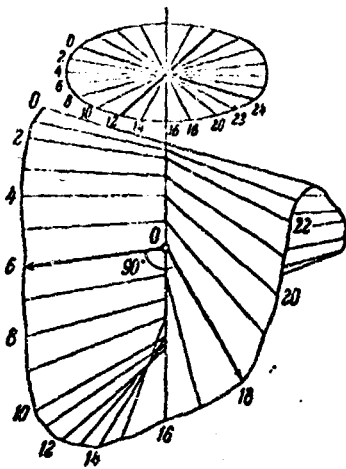
$$\left.\begin{array}{l} R\cos\Phi = a\,e^{\omega p_1}, \\ R\sin\Phi = b\,e^{\omega p_2}, \\ \dfrac{b}{a} = \operatorname{tg}\varphi. \end{array}\right\} \quad (3.53)$$

From (3.53) we obtain

$$\operatorname{tg}\Phi = e^{\omega(p_2-p_1)}\operatorname{tg}\varphi. \quad (3.54)$$

Equation (3.54) is the cylindroid equation referred to the principal screws and the center. From it we obtain the distance of the generators as a function of the angle φ. We have

$$e^{\omega(p_2-p_1)} = e^{\omega 3\tau^{y\prime}\sin 2\varphi},$$

from which

$$\varphi^0 = \frac{p_2-p_1}{2}\sin 2\varphi. \quad (3.55)$$

Fig. 7

On variation of the angle φ, the generator describes a cylindroid surface, which can be represented palpably as follows.

On the vertical axis, we lay off from point O — the center — a segment $OA = (p_2 - p_1)/2$ upward and the same segment OA' downward. The cylindroid generator is a straight line forming a right angle with the vertical axis, which, as φ varies, rotates uniformly about the axis and at the same time slides along the axis, executing a harmonic motion within the limits of the segment AA', with two down-up motions completed in one revolution of the generator. Generators passing through the center O cross at right angles. Part of the surface is shown in Fig. 7.
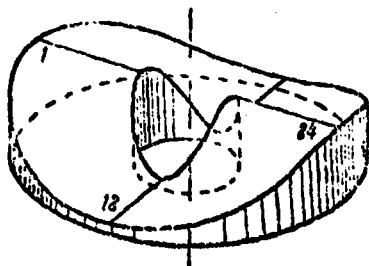
Fig. 8

If we map the surface of a circular cylinder whose axis coincides with the axis of the cylindroid, the cylindroid surface intersects the cylinder surface along a curve whose involute will be a sinusoid with two periods around the circumference of the cylinder (Fig. 8).

§10. Projections of Screw onto Axes of Rectangular Coordinate System. Complex Coordinates of a Straight Line

Let the screw

$$R = ER = Ere^{\omega p}$$

be given and let the complex angles formed by its axis with the x-, y- and z-axes of the rectangular coordinate system be, respectively,

$$A = \alpha + \omega\alpha^0, \, B = \beta + \omega\beta^0, \, \Gamma = \gamma + \omega\gamma^0.$$

The projections of the screw onto these axes will be

$$
\left.
\begin{aligned}
R_x &= r_x + \omega r_x^0 = R \cos A = \\
&= r\,[\cos\alpha + \omega\,(p\cos\alpha - \alpha^0\sin\alpha)], \\
R_y &= r_y + \omega r_y^0 = R \cos B = \\
&= r\,[\cos\beta + \omega\,(p\cos\beta - \beta^0\sin\beta)], \\
R_z &= r_z + \omega r_z^0 = R \cos \Gamma = \\
&= r\,[\cos\gamma + \omega\,(p\cos\gamma - \gamma^0\sin\gamma)].
\end{aligned}
\right\}
\qquad (3.56)
$$

Expressions (3.56) are the complex orthogonal projections or rectangular coordinates of the screw. The principal parts of these expressions

$$r_x = r\cos\alpha, \, r_y = r\cos\beta, \, r_z = r\cos\gamma \qquad (3.57)$$

are the rectangular coordinates of the vector r, while the moment parts of these expressions

$$r_x^0 = r\,(p\cos\alpha - \alpha^0\sin\alpha), \, r_y^0 = r\,(p\cos\beta - \beta^0\sin\beta),$$
$$r_z^0 = r\,(p\cos\gamma - \gamma^0\sin\gamma) \qquad (3.58)$$

are rectangular coordinates of the moment pr of the screw about the coordinate origin or the moments of the screw with respect to the coordinate axes.

Theorem 8. *A screw R is equal to the geometric sum of its orthogonal components on the axes of the rectangular coordinate system.*

This theorem is easily proven if we reduce the screw to the coordinate origin:

$$R \to r + \omega\,(pr + \dot{\rho} \times r),$$

where ρ is the radius vector of an arbitrary point on the screw axis from the coordinate origin. The vector r and the moment $\omega\,(pr + \rho \times r)$ are the sums of their orthogonal components on the coordinate axes; hence the same can be said regarding screw R, which is equivalent to them.

Thus, we may write

$$
\begin{aligned}
R &= iR_x + jR_y + kR_z = i\,(r_x + \omega r_x^0) + \\
&\quad + j\,(r_y + \omega r_y^0) + k\,(r_z + \omega r_z^0).
\end{aligned}
\qquad (3.59)
$$

Taking the scalar square of this equality, we obtain

$$
\begin{aligned}
R^2 &= R_x^2 + R_y^2 + R_z^2 = r_x^2 + r_y^2 + r_z^2 + \\
&\quad + 2\omega\,(r_x r_x^0 + r_y r_y^0 + r_z r_z^0).
\end{aligned}
\qquad (3.60)
$$

Consequently, the square of the complex modulus of the screw decomposes into the square of the vector length and the scalar product of the vector by the moment.

After introduction of the complex coordinates of the screw, we see that any screw equality is equivale t to three complex sc lar equalities. But each complex equality decomposes into two real ones, and hence any screw equality is equivalent to six scalar equalities.

On the basis of Equalities (3.56), we obtain

$$R^2 = R^2 (\cos^2 A + \cos^2 B + \cos^2 \Gamma),$$

from which

$$\cos^2 A + \cos^2 B + \cos^2 \Gamma = 1, \qquad (3.61)$$

i.e., the sum of the squares of the complex direction cosines is unity, as in the real domain. Equality (3.61) breaks down into two:

$$\left. \begin{array}{l} \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \\ \alpha^\circ \cos \alpha \sin \alpha + \beta^\circ \cos \beta \sin \beta + \gamma^\circ \cos \gamma \sin \gamma = 0. \end{array} \right\} \qquad (3.62)$$

If $R = 1$, then R_x, R_y, R_z are complex coordinates of the unit screw R and are equal in magnitude to the corresponding direction cosines:

$$\begin{array}{l} R_x = X = x + \omega x^\circ = \cos A, \\ R_y = Y = y + \omega y^\circ = \cos B, \\ R_z = Z = z + \omega z^\circ = \cos \Gamma. \end{array}$$

The equality

$$X^2 + Y^2 + Z^2 = (x + \omega x^\circ)^2 + (y + \omega y^\circ)^2 + \\ + (z + \omega z^\circ)^2 = 1, \qquad (3.63)$$

which is equivalent to (3.61), decomposes into the two following equalities:

$$\left. \begin{array}{l} x^2 + y^2 + z^2 = 1, \\ xx^\circ + yy^\circ + zz^\circ = 0, \end{array} \right\} \qquad (3.64)$$

which express the equality of the squared vector length to unity and the equality to zero of the scalar product of the vector by the moment with respect to the coordinate origin.

If we have a given axis with unit screw B, whose coordinates will be A, B, C ($A^2 + B^2 + C^2 = 1$), then any axis with the coordinates X, Y, Z of the unit screw ($X^2 + Y^2 + Z^2 = 1$) that satisfies the equation

$$AX + BY + CZ = 0, \qquad (3.65)$$

will intersect the given axis at right angles; the aggregate of all such axes forms a brush.

Consequently, Eq. (3.65) is the complex equation of a brush (analogous to the equation of a plane in the case of real quantities).

§11. Expression of the Scalar and Screw Products of Screws in Terms of the Complex Rectangular Coordinates of the Screws

For two screws defined by complex rectangular coordinates $R_{1x}, R_{1y}, R_{1z}, R_{2x}, R_{2y}, R_{2z}$, we have

$$R_1 = iR_{1x} + jR_{1y} + kR_{1z},$$
$$R_2 = iR_{2x} + jR_{2y} + kR_{2z}.$$

We immediately find the scalar product

$$R_1 \cdot R_2 = R_{1x}R_{2x} + R_{1y}R_{2y} + R_{1z}R_{2z} =$$
$$= (r_{1x} + \omega r^0_{1x})(r_{2x} + \omega r^0_{2x}) + (r_{1y} + \omega r^0_{1y})(r_{2y} + \omega r^0_{2y}) +$$
$$+ (r_{1z} + \omega r^0_{1z})(r_{2z} + \omega r^0_{2z}) = r_{1x}r_{2x} + r_{1y}r_{2y} + r_{1z}r_{2z} +$$
$$+ \omega (r_{1x}r^0_{2x} + r_{1y}r^0_{2y} + r_{1z}r^0_{2z} + r^0_{1x}r_{2x} + r^0_{1y}r_{2y} + r^0_{1z}r_{2z}). \qquad (3.66)$$

The scalar product of two screws decomposes into the scalar product of the vectors of these screws and into their relative moment, which is equal to the sum of the scalar products of the vector of each by the moment of the other, taken with respect to a definite point, in this case the coordinate origin.

The screw product of two screws is the screw

$$R = i(R_{1y}R_{2z} - R_{2y}R_{1z}) + j(R_{2x}R_{1z} - R_{1x}R_{2z}) +$$
$$+ k(R_{1x}R_{2y} - R_{2x}R_{1y}). \qquad (3.67)$$

Indeed, performing scalar multiplication of screw R by screw R_1 and by screw R_2, we obtain

$$R \cdot R_1 = (R_{1y}R_{2z} - R_{2y}R_{1z})R_{1x} + (R_{2x}R_{1z} - R_{1x}R_{2z})R_{1y} +$$
$$+ (R_{1x}R_{2y} - R_{2x}R_{1y})R_{1z} = 0,$$
$$R \cdot R_2 = (R_{1y}R_{2z} - R_{2y}R_{1z})R_{2x} + (R_{2x}R_{1z} - R_{1x}R_{2z})R_{2y} +$$
$$+ (R_{1x}R_{2y} - R_{2x}R_{1y})R_{2z} = 0,$$

and taking the square of the modulus we shall have

$$R^2 = (R_{1y}R_{2z} - R_{2y}R_{1z})^2 +$$
$$+ (R_{2x}R_{1z} - R_{1x}R_{2z})^2 + (R_{1x}R_{2y} - R_{2x}R_{1y})^2 =$$
$$= (R^2_{1x} + R^2_{1y} + R^2_{1z})(R^2_{2x} + R^2_{2y} + R^2_{2z}) -$$
$$- (R_{1x}R_{2x} + R_{1y}R_{2y} + R_{1z}R_{2z})^2 =$$
$$= R^2_1 R^2_2 - (R_1 R_2 \cos \Lambda)^2 = (R_1 R_2 \sin \Lambda)^2, \qquad (3.68)$$

from which it follows that the axis of screw R intersects the axes of screws R_1 and R_2 at right angles, and that its complex modulus is equal to the product of the complex moduli of the factor screws by the sine of the complex angle Λ between them. Taking the principal part of (3.68), we satisfy ourselves that the vector r is the vector product $r_1 \times r_2$.

Consequently, the screw R is the screw product of screws R_1 and R_2.

§12. Compound Multiplication of Screws. Morley-Petersen Theorem. Formulas of Complex Spherical Trigonometry

Using the complex rectangular coordinates, we can easily derive expressions for more complicated screw products: mixed (scalar-screw), double screw, the scalar product of two screw products and the screw product of two screw products.

On the basis of Formulas (3.66) and (3.67), we can write the expression for the mixed product

$$R_1 R_2 R_3 = R_1 \cdot (R_2 \times R_3)$$

of three screws defined by their complex coordinates:

$$R_1 = iR_{1x} + jR_{1y} + kR_{1z},$$
$$R_2 = iR_{2x} + jR_{2y} + kR_{2z},$$
$$R_3 = iR_{3x} + jR_{3y} + kR_{3z}.$$

We have

$$R_2 \times R_3 = \begin{vmatrix} i & j & k \\ R_{2x} & R_{2y} & R_{2z} \\ R_{3x} & R_{3y} & R_{3z} \end{vmatrix} = i(R_{2y}R_{3z} - R_{3y}R_{2z}) +$$
$$+ j(R_{3x}R_{2z} - R_{2x}R_{3z}) + k(R_{2x}R_{3y} - R_{3x}R_{2y}),$$
$$R_1 \cdot (R_2 \times R_3) = R_{1x}(R_{2y}R_{3z} - R_{3y}R_{2z}) +$$
$$+ R_{1y}(R_{3x}R_{2z} - R_{2x}R_{3z}) + R_{1z}(R_{2x}R_{3y} - R_{3x}R_{2y}) =$$
$$= \begin{vmatrix} R_{1x} & R_{1y} & R_{1z} \\ R_{2x} & R_{2y} & R_{2z} \\ R_{3x} & R_{3y} & R_{3z} \end{vmatrix}. \qquad (3.69)$$

Since in Determinant (3.69), the sign is not changed by any "end-around" transposition of rows, it is accordingly possible to transpose the parentheses and signs of scalar and screw multiplication in the mixed product, i.e.,

$$R_1 R_2 R_3 = R_1 \cdot (R_2 \times R_3) = R_2 \cdot (R_3 \times R_1) = R_3 \cdot (R_1 \times R_2). \qquad (3.70)$$

Using the coordinate expressions for the scalar and screw products, we shall be able to obtain formulas for compound screw products.

Double screw product:

$$R_1 \times (R_2 \times R_3) = R_2(R_1 \cdot R_3) - R_3(R_1 \cdot R_2). \qquad (3.71)$$

Scalar product of two screw products:

$$(R_1 \times R_2) \cdot (R_3 \times R_4) = (R_1 \cdot R_3)(R_2 \cdot R_4) - (R_1 \cdot R_4)(R_2 \cdot R_3). \qquad (3.72)$$

Screw product of two screw products:

$$(R_1 \times R_2) \times (R_3 \times R_4) = R_3(R_1 R_2 R_4) - R_4(R_1 R_2 R_3) = R_2(R_1 R_3 R_4) - R_1(R_2 R_3 R_4). \qquad (3.73)$$

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The following theorem, which is known as the Morley-Petersen theorem [29], [30], is a geometrical interpretation of a property of three double screw products of screws, and consists in the following.

Let E_1, E_2, E_3 be three unit screws whose axes do not belong to the same brush. Let R_1, R_2, R_3 be arbitrary screws whose axes have right-angle intersections with the respective pairs of axes (E_2, E_3). $(E_3, E_1), (E_1, E_2)$. Then three screws S_1, S_2, S_3, whose axes have right-angle intersections with the pairs $(E_1, R_1), (E_2, R_2), (E_3, R_3)$. belong to the same brush, i.e., there exists a line that intersects the axes of the screws S_1, S_2, S_3 at right angles.

For the proof, we recall that screw products may be taken for the screws R_1, R_2, R_3 , i.e.,

$$R_1 = E_2 \times E_3, \; R_2 = E_1 \times E_3, \; R_3 = E_1 \times E_2. \qquad (3.74)$$

and screw products for the screws S_1, S_2 and B_3:

$$S_1 = E_1 \times R_1, \; S_2 = E_2 \times R_2, \; S_3 = E_3 \times R_3.$$

Replacing R_1, R_2 and R_3 by their expressions (3.74), we obtain

$$S_1 = E_1 \times (E_2 \times E_3), \; S_2 = E_2 \times (E_3 \times E_1),$$
$$S_3 = E_3 \times (E_1 \times E_2). \qquad (3.75)$$

Expanding the double screw products by Formulas (3.71) and then adding Equalities (3.75), we find

$$S_1 + S_2 + S_3 = E_2(E_1 \cdot E_3) - E_3(E_1 \cdot E_2) +$$
$$+ E_3(E_1 \cdot E_2) - E_1(E_1 \cdot E_3) + E_1(E_2 \cdot E_3) -$$
$$- E_2(E_1 \cdot E_3) = 0.$$

from which it follows that the screws S_1, S_2 and B_3 are linearly dependent and hence belong to the same brush. The axis of this brush will intersect the axes of screws S_1, S_2 and S_3 at right angles.

Let us now consider application of Formulas (3.72) and (3.73) for the scalar and screw products of two vector products to derivation of a formula of complex spherical trigonometry.

Let us replace all screws in Formula (3.72) by unit screws E_1, E_2, E_3, E_4 and assume that $E_4 = E_1$. Remembering that scalar products of unit screws are equal to the cosines of the corresponding complex angles, we obtain the relation

$$(E_1 \times E_2) \cdot (E_3 \times E_2) = E_1 \cdot E_3 - (E_1 \cdot E_2)(E_2 \cdot E_3)$$

or

$$\sin \Lambda_{12} \sin \Lambda_{23} \cos \theta = \cos \Lambda_{13} - \cos \Lambda_{12} \cos \Lambda_{23}.$$

from which

$$\cos \Lambda_{13} = \cos \Lambda_{12} \cos \Lambda_{23} + \sin \Lambda_{12} \sin \Lambda_{23} \cos \theta. \qquad (3.76)$$

where θ is the angle between the axes of angles A_{12} and A_{23}.

Formula (3.76) is an analogue of a familiar formula of spherical trigonometry. It is obtained as a corollary of the known formula for the scalar product of two vector products, but it could have been obtained without derivation from the ordinary spherical-trigonometry formula by putting all angles complex, i.e., by moving the sides of the angles apart (Fig. 9).
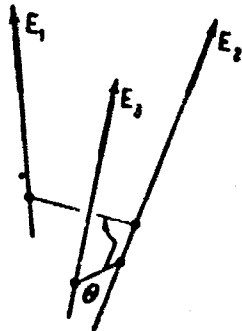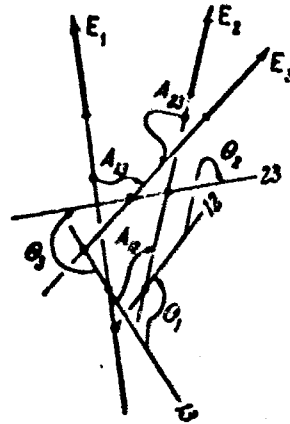
Fig. 9   Fig. 10

Now let us consider the same triplet of unit screws E_1, E_2 and E_3 and write the obvious relationship

$$\sin \Theta_1 = \frac{|(E_1 \times E_2) \times (E_1 \times E_3)|}{|E_1 \times E_2||E_1 \times E_3|} = \frac{E_1 E_2 E_3}{\sin A_{12} \sin A_{13}},$$

$$\sin \Theta_2 = \frac{|(E_1 \times E_2) \times (E_2 \times E_3)|}{|E_1 \times E_2||E_2 \times E_3|} = \frac{E_1 E_2 E_3}{\sin A_{12} \sin A_{23}},$$

$$\sin \Theta_3 = \frac{|(E_1 \times E_3) \times (E_2 \times E_3)|}{|E_1 \times E_3||E_2 \times E_3|} = \frac{E_1 E_2 E_3}{\sin A_{13} \sin A_{23}}.$$

From these formulas we obtain a relationship that is an analogue of those that form the familiar theorem of sines in spherical trigonometry (Fig. 10):

$$\frac{\sin \Theta_1}{\sin A_{23}} = \frac{\sin \Theta_2}{\sin A_{13}} = \frac{\sin \Theta_3}{\sin A_{12}}. \tag{3.77}$$

§13. Transformation of Complex Rectangular Coordinates of a Screw

Having given expressions for the complex rectangular coordinates of a screw, we can easily derive formulas for conversion from one system of rectangular coordinates to another.

Let there be given a system of rectangular coordinates with its origin at point O and with unit vectors of the axes i, j, k (unit screws). Let the coordinates of unit screw E in this system be $\cos A$, $\cos B$, $\cos \Gamma$; these will be the complex direction cosines. Screw E can be expressed as follows:

$$E = i \cos A + j \cos B + k \cos \Gamma$$

The scalar square of screw E will be

$$E^2 = \cos^2 A + \cos^2 B + \cos^2 \Gamma = 1. \qquad (3.79)$$

For two unit screws E_1 and E_2 defined by the coordinates $\cos A_1, \cos B_1, \cos \Gamma_1$ and $\cos A_2, \cos B_2, \cos \Gamma_2$, the scalar product will have the expression

$$E_1 \cdot E_2 = \cos A_1 \cos A_2 + \cos B_1 \cos B_2 + \cos \Gamma_1 \cos \Gamma_2. \qquad (3.80)$$

The condition of intersection of E_1 with E_2 at right angles will be

$$\cos A_1 \cos A_2 + \cos B_1 \cos B_2 + \cos \Gamma_1 \cos \Gamma_2 = 0. \qquad (3.81)$$

Visualize another system of rectangular coordinates with its origin at point O' and with the unit vectors of the axes i', j', k', with O' not coincident with O. Let the coordinates of the unit screw E in the second system be $\cos A'$, $\cos B'$, $\cos \Gamma'$. In the second system, screw E will be expressed as follows:

$$E = i' \cos A' + j' \cos B' + k' \cos \Gamma'. \qquad (3.82)$$

The axes of the second coordinate system form nine complex angles with the axes of the first system; their complex cosines are equal to the scalar products of each pair of unit vectors (unit screws) taken one from each system. Let

$$\left.\begin{array}{lll}
i \cdot i' = \cos A_1, & j \cdot i' = \cos A_2, & k \cdot i' = \cos A_3, \\
i \cdot j' = \cos B_1, & j \cdot j' = \cos B_2, & k \cdot j' = \cos B_3, \\
i \cdot k' = \cos \Gamma_1, & j \cdot k' = \cos \Gamma_2, & k \cdot k' = \cos \Gamma_3,
\end{array}\right\} \qquad (3.83)$$

where $A_1 = \alpha_1 + \omega \alpha_1^0$, $A_2 = \alpha_2 + \omega \alpha_2^0$ and so forth.

The following relations apply between these nine cosines:

$$\left.\begin{array}{l}
\cos^2 A_1 + \cos^2 A_2 + \cos^2 A_3 = 1, \\
\cos^2 A_1 + \cos^2 B_1 + \cos^2 \Gamma_1 = 1. \\
\cos A_1 \cos B_1 + \cos A_2 \cos B_2 + \cos A_3 \cos B_3 = 0, \\
\cos A_1 \cos A_2 + \cos B_1 \cos B_2 + \cos \Gamma_1 \cos \Gamma_2 = 0, \\
\cos^2 B_1 + \cos^2 B_2 + \cos^2 B_3 = 1, \\
\cos^2 A_2 + \cos^2 B_2 + \cos^2 \Gamma_2 = 1, \\
\cos A_1 \cos \Gamma_1 + \cos A_2 \cos \Gamma_2 + \cos A_3 \cos \Gamma_3 = 0, \\
\cos A_1 \cos A_3 + \cos B_1 \cos B_3 + \cos \Gamma_1 \cos \Gamma_3 = 0, \\
\cos^2 \Gamma_1 + \cos^2 \Gamma_2 + \cos^2 \Gamma_3 = 1, \\
\cos^2 A_3 + \cos^2 B_3 + \cos^2 \Gamma_3 = 1, \\
\cos B_1 \cos \Gamma_1 + \cos B_2 \cos \Gamma_2 + \cos B_3 \cos \Gamma_3 = 0, \\
\cos A_2 \cos A_3 + \cos B_2 \cos B_3 + \cos \Gamma_2 \cos \Gamma_3 = 0,
\end{array}\right\} \qquad (3.84)$$

which are equivalent to twenty-four real relationships. These equalities state that all axis vectors are unit vectors and that

the systems of axes are rectangular.

Six real quantities are sufficient to define the position of the second system with respect to the first; hence twelve real relationships must obtain between the nine complex angles (i.e., between the eighteen real quantities). Consequently, of the 24 real relationships of (3.84), twelve will be independent.

Applying Formula (3.78), we can write formulas for the transformation of unit screws:

$$
\left.\begin{aligned}
i' &= i\cos\Lambda_1 + j\cos\Lambda_2 + k\cos\Lambda_3, \\
i &= i'\cos A_1 + j'\cos B_1 + k'\cos\Gamma_1, \\
j' &= i\cos B_1 + j\cos B_2 + k\cos B_3, \\
j &= i'\cos A_2 + j'\cos B_2 + k'\cos\Gamma_2, \\
k' &= i\cos\Gamma_1 + j\cos\Gamma_2 + k\cos\Gamma_3, \\
k &= i'\cos A_3 + j'\cos B_3 + k'\cos\Gamma_3.
\end{aligned}\right\}
\qquad (3.85).
$$

To obtain coordinate transformation formulas for any screw R, we represent the screw in first one system and then the other:

$$
R = iR_x + jR_y + kR_z = i'R_x' + j'R_y' + k'R_z'. \qquad (3.86)
$$

Performing successive scalar multiplications of this equation by i, j, k and by i', j', k', we obtain the formulas

$$
\left.\begin{aligned}
R_x' &= R_x\cos\Lambda_1 + R\ \cos\Lambda_2 + R_z\cos\Lambda_3, \\
R_x &= R_x'\cos\Lambda_1 + R_y'\cos B_1 + R_z'\cos\Gamma_1, \\
R_y' &= R_x\cos B_1 + R_y\cos B_2 + R_z\cos B_3, \\
R\ &= R_x'\cos\Lambda_2 + R_y'\cos B_2 + R_z'\cos\Gamma_2, \\
R_z' &= R_x\cos\Gamma_1 + R_y\cos\Gamma_2 + R_z\cos\Gamma_3, \\
R_z &= R_x'\cos\Lambda_3 + R_y'\cos B_3 + R_z'\cos\Gamma_3.
\end{aligned}\right\}
\qquad (3.87)
$$

The determinant of this transformation is

$$
D = \begin{vmatrix}
\cos\Lambda_1 & \cos\Lambda_2 & \cos\Lambda_3 \\
\cos B_1 & \cos B_2 & \cos B_3 \\
\cos\Gamma_1 & \cos\Gamma_2 & \cos\Gamma_3
\end{vmatrix}.
$$

Expressing its square and applying (3.84), we obtain

$$
D^2 = \begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix} = 1,
$$

from which it follows that $D = \pm 1$. Screw displacement of the coordinate system corresponds to the $+$ sign.

The formulas of transformation (3.87) are written as follows in matrix form:

$$\begin{vmatrix} R_x' \\ R_y' \\ R_z' \end{vmatrix} = \begin{vmatrix} \cos A_1 & \cos A_2 & \cos A_3 \\ \cos B_1 & \cos B_2 & \cos B_3 \\ \cos \Gamma_1 & \cos \Gamma_2 & \cos \Gamma_3 \end{vmatrix} \begin{vmatrix} R_x \\ R_y \\ R_z \end{vmatrix},$$

$$\begin{vmatrix} R_x \\ R_y \\ R_z \end{vmatrix} = \begin{vmatrix} \cos A_1 & \cos B_1 & \cos \Gamma_1 \\ \cos A_2 & \cos B_2 & \cos \Gamma_2 \\ \cos A_3 & \cos B_3 & \cos \Gamma_3 \end{vmatrix} \begin{vmatrix} R_x' \\ R_y' \\ R_z' \end{vmatrix} \qquad (3.88)$$

or, more concisely,

$$R' = A'R, \quad R = AR'. \qquad (3.89)$$

The complex-element matrices A and A' considered here effect an affine orthogonal transformation, one which, unlike that effected by matrices with real elements, is a screw displacement that preserves the complex moduli of the screws and the complex angles between the axes of any two screws.

§14. The Screw Dyad. The Screw Affinor

Let us consider an arbitrary triplet (base) of screws A, B, C with the condition that $(ABC)^2 \neq 0$.[*] Any given screw R can be represented as a linear combination

$$R = R_x A + R_y B + R_z C. \qquad (3.90)$$

The complex numbers R_x, R_y, R_z are scalar products of the screw R by the screws $A^\bullet, B^\bullet, C^\bullet$, which are reciprocal to the given triplet and defined by the formulas

$$A^\bullet = \frac{B \times C}{ABC}, \quad B^\bullet = \frac{C \times A}{ABC}, \quad C^\bullet = \frac{A \times B}{ABC}. \qquad (3.91)$$

We then obtain

$$A = \frac{B^\bullet \times C^\bullet}{A^\bullet B^\bullet C^\bullet}, \quad B = \frac{C^\bullet \times A^\bullet}{A^\bullet B^\bullet C^\bullet}, \quad C = \frac{A^\bullet \times B^\bullet}{A^\bullet B^\bullet C^\bullet}, \qquad (3.92)$$

and

$$(ABC)(A^\bullet B^\bullet C^\bullet) = 1. \qquad (3.93)$$

Thus,

$$R_x = R \cdot A^\bullet, \quad R_y = R \cdot B^\bullet, \quad R_z = R \cdot C^\bullet. \qquad (3.94)$$

A screw R' referred to the base A', B', C' and having the same coordinates as R can be brought into correspondence with a screw R expressed by Formula (3.90):

$$R' = R_x A' + R_y B' + R_z C', \qquad (3.95)$$

so that the following expression may be written for R':

$$R' = R \cdot A^\bullet A' + R \cdot B^\bullet B' + R \cdot C^\bullet C' \qquad (3.96)$$

$$R' = A'A' \cdot R + B'B' \cdot R + C'C' \cdot R. \tag{3.97}$$

Formulas (3.96) and (3.97) represent an affine transformation that brings the three screws A', B', C' into correspondence with the three scress A, B, C. Screw R' in the new system of base screws A', B', C' has the same expression as screw R in the old system A, B, C.

The expressions

$$\Phi = A'A' + B'B' + C'C', \tag{3.98}$$

$$\overline{\Phi} = A'A' + B'B' + C'C' \tag{3.99}$$

are known as conjugate screw dyads; they are sums of dyad products.

Transformations (3.96) and (3.97) are written symbolically as follows:

$$R' = R \cdot \Phi = \overline{\Phi} \cdot R. \tag{3.100}$$

Screw R' can be referred to the original base A, B, C with the aid of the coordinates. Denoting the coordinates of screw R in this base by R'_x, R'_y, R'_z, we obtain

$$R' = R'_x A + R'_y B + R'_z C. \tag{3.101}$$

The quantities R'_x, R'_y, R'_z are defined if we know the expressions for screws A', B', C' in terms of A, B, C. Let

$$
\begin{aligned}
A' &= A_{11}A + A_{12}B + A_{13}C, \\
B' &= A_{21}A + A_{22}B + A_{23}C, \\
C' &= A_{31}A + A_{32}B + A_{33}C,
\end{aligned}
\right\} \tag{3.102}
$$

where the numbers A_{ik} are the coordinates of the "new" base with respect to the "old" one; further, $D^2 \neq 0$, where

$$D = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix},$$

then, substituting (3.102) into (3.95), we obtain

$$
\begin{aligned}
R' = R_x (A_{11}A + A_{12}B + A_{13}C) + \\
+ R_y (A_{21}A + A_{22}B + A_{23}C) + \\
+ R_z (A_{31}A + A_{32}B + A_{33}C).
\end{aligned}
\tag{3.103}
$$

or

$$R' = (A_{11}R_x + A_{21}R_y + A_{31}R_z) A + (A_{12}R_x + A_{22}R_y + \tag{3.104}$$
$$+ A_{32}R_z) B + (A_{13}R_x + A_{23}R_y + A_{33}R_z) C.$$

In accordance with (3.101), we shall have expressions for the coordinates of screw R' in the new system in terms of its coordinates in the old system:

$$R'_x = A_{11}R_x + A_{21}R_y + A_{31}R_z,$$
$$R'_y = A_{12}R_x + A_{22}R_y + A_{32}R_z,$$
$$R'_z = A_{13}R_x + A_{23}R_y + A_{33}R_z.$$

(3.105)

The resulting affine transformation of the screw can also be written as a multiplication:

$$R' = R \cdot A = \bar{A} \cdot R,$$

(3.106)

where A is a transformation matrix with complex elements $A_{ik} = a_{ik} + \omega a_{ik}^0$

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}.$$

(3.107)

Screw R' is a linear screw function of screw R, while the operator A defined by the matrix in (3.107) is known as a screw affinor.

Screw affinors have been investigated and applied by S.G. Kislitsyn.

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54	Here the usual requirement $ABC \neq 0$ is insufficient, since the case in which the principal part of this mixed product is zero is not excluded, and then division by it is impossible.

Chapter 4

THE TRANSFER PRINCIPLE AND ITS APPLICATION TO
THE GEOMETRY AND KINEMATICS OF THE SOLID BODY

§1. Transfer Principle in Complex Vector Algebra

On examining the formulas expressing the results of opera-
tions on screws, we are struck by their identity to the formulas
of ordinary vector algebra. This identity was found to be a con-
sequence of substitution of the vector in the vector-algebra for-
mulas by the motor and its formal expression in the form of a com-
plex vector with a special type of multiplier ω, whose square is
equal to zero, and also of introduction of the complex modulus of
the vector and the complex angle between straight lines in space.

The formulas expressing the sum and the scalar and screw
products of screws in terms of "internal" quantities — moduli and
angles — were found to be quite identical to the corresponding
formulas for the sum and scalar and vector products of vectors on
condition that the modulus of the vector is replaced in the latter
by the complex screw modulus and the ordinary angle between the
lines by the complex angle. The identity of the basic formulas of
vector and screw algebras is illustrated by the table on the next
page.

The complete parallelism of the formulas that is seen in this
table results in parallelism in a multitude of other formulas,
principally the formulas for more complex products of vectors and
screws (scalar-screw, double screw, scalar and screw products,
two screw products, and so forth), as well as in many other for-
mulas of vector and screw algebra.

This parallelism results in a highly important general propo-
sition that constitutes the transfer principle of complex vector
algebra, the algebra of screws. The principle that will be our
subject here is one of many examples of the familiar transfer
principle, which can be characterized as follows. Let there be
formulas that link analytically the elements of some space — vari-
ous geometrical figures (points, lines, etc.), and let us assume
that the corresponding relationships are also preserved if the
elements linked by them are substituted by other elements — total-
ly different geometrical figures — not excluding geometrical fig-
ures with a different number of dimensions. In this case, the same
formulas will express the relationships of two totally different
geometries, and these two geometries become identical to one an-
other. If some theorem is known for one geometry, it is automati-

Vector algebra	Screw algebra
Vector r	Screw R
Modulus of vector $|r| = r$	Complex modulus of screw $|R| = R$
Angle between two straight lines α	Complex angle between two straight lines A
Scalar product of two vectors $r_1 \cdot r_2 = r_1 r_2 \cos \alpha$	Scalar product of two screws $R_1 \cdot R_2 = R_1 R_2 \cos A$
Vector product of two vectors $r_1 \times r_2 = e_{12} r_1 r_2 \sin \alpha$ (e_{12} is the unit vector forming right angles with the vectors r_1 and r_2)	Screw product of two screws $R_1 \times R_2 = E_{12} R_1 R_2 \sin A$ (E_{12} is the unit screw, whose axis intersects the axes of screws R_1 and R_2 at right angles)
Sum of vectors $r = r_1 + r_2$ (r is a vector whose direction and modulus are determined from the closed triangle)	Sum of screws $R = R_1 + R_2$ (R is a screw whose direction and axis position, as well as its complex modulus, are determined from the "expanded" triangle)

cally transferred to the other geometry, and this second geometry can be studied by means of the first, the only adjustment being that the results are interpreted in the second geometry with the aid of other geometrical concepts.[*]

The transfer (or expansion) principle of the complex vector algebra that is the algebra of screws, as formulated by A.P. Kotel'nikov and somewhat later by E. Study, reduces to the following.

Let us consider a certain collection of vectors $r_1, r_2, \dots,$ whose origin is at a certain common reduction point O. Assume that, together with each of the vectors r_i, we also consider a certain moment r_i^0 attributed to it and referred to point O, with the result that an additional set of moments $r_1^0, r_2^0, \dots,$ referred to point O, makes its appearance, so that we have sets of two vectors, i.e., motors $(r_1, r_1^0), (r_2, r_2^0) \dots$ Each motor (r_i, r_i^0), referred to point O, naturally defines a certain screw R_i — its axis, vector and parameter. The set of motors $(r_1, r_1^0), (r_2, r_2^0) \dots,$ referred to the reduction point O, determines a set of screws $R_1, R_2, \dots$. The ends of all vectors and moments with origins at point O form a six-dimensional point space, while the axes of the screws defined by them form a four-dimensional linear space, with a two-dimensional space of screws corresponding to each axis and, consequently, the screw space will be six-dimensional. Thus, with the aid of the reduction point, we establish correspondence between the space of vector doublets or motors (or point pairs) on the one hand and the space of screws on the other. To each motor in the first space there

If the moments r_i^0 are equal to zero, we shall have an ordinary vector (point) space and operations on the vectors will give the relationships of vector algebra for the vectors r_i. If, however, the moments r_i^0 are not equal to zero, then, as was shown in Chapter 3, we may form complex vectors $r_i + \omega r_i^0$ for which the basic vector-algebra formulas may be written similarly, but will at the same time also be formulas for the screws R_i corresponding to these complex vectors. We were able to satisfy ourselves that by virtue of a property of the fortunately introduced multiplier ω, the basic formulas for the screws exactly reproduce the formulas for the principal parts of the screws, i.e., for the vectors. Hence the basic formulas of vector algebra, which are written "lower-case," serve simultaneously as basic formulas for the theory of screws when they are rewritten "upper-case."

The basic formulas are: a) the formula for the scalar square of a vector (or screw) and b) the formula for the angle between two vectors (or between screw axes), expressed with the aid of the scalar product. Here we recognize that the complex modulus of the screw corresponds to the modulus of the vector and the complex angle between the screw axes to the angle between vectors, i.e.,

$$r^2 = r_x^2 + r_y^2 + r_z^2 = r^2, R^2 = R_x^2 + R_y^2 + R_z^2 =$$
$$= R^2 = r^2 e^{2\omega p},$$

$$\cos(r_1, r_2) = \frac{r_{1x}r_{2x} + r_{1y}r_{2y} + r_{1z}r_{2z}}{\sqrt{r_{1x}^2 + r_{1y}^2 + r_{1z}^2}\,\sqrt{r_{2x}^2 + r_{2y}^2 + r_{2z}^2}},$$

$$\cos(R_1, R_2) = \frac{R_{1x}R_{2x} + R_{1y}R_{2y} + R_{1z}R_{2z}}{\sqrt{R_{1x}^2 + R_{1y}^2 + R_{1z}^2}\,\sqrt{R_{2x}^2 + R_{2y}^2 + R_{2z}^2}}.$$

If, however, identity exists between the basic formulas of vector and screw algebras, we may conclude that identity exists between all formulas that can be reduced to a finite number of these basic formulas. This means that at least all of the formulas of vector algebra, written "lower-case," will serve as formulas for the algebra of screws if they are rewritten "upper-case;" here the complex modulus of the screw will correspond to the modulus of the vector in the new formulas, and the complex angle between the axes of two screws will correspond to the angle between two vectors.

The above constitutes the transfer principle for complex vector algebra, the algebra of screws. On the basis of this principle, the correspondence table given above can be extended for many other formulas in such a way that its right-hand column, which pertains to the screws ("upper-case letters") will always correspond to its left column, which pertains to the vector ("lower-case letters"); substitution of upper for lower case signifies substitution of complex quantities for real quantities. The vector-algebra formulas may be regarded as "unexpanded" formulas of screw algebra: writing the former in upper case, we impart com-

plex value to them and then expand them. As a result, we obtain complex formulas for transformation of coordinates, formulas of a more general complex affine transformation, formulas of complex spherical trigonometry, and others.

It is necessary to note here that transfer of the vector-algebra formulas to the algebra of screws loses its significance in cases in which the vector moduli vanish. In these cases, the corresponding screws are degenerate. A special analysis is required for such exceptional cases.

It can be seen on the basis of the above that the transfer principle establishes correspondence between the vector (point) space and the screw space.

By means of the transfer principle, a flat bundle of straight lines is transformed (is expanded) into a brush (see §9 of Chapter 3).

It was also shown in the same Chapter 3 that the basic formulas of screw algebra are invariant with respect to selection of the reduction point, i.e., they do not depend on the motor to which the given screw is reduced. With the treatment of the transfer principle just presented, this property is equivalent to the property of all formulas characterizing internal relationships between screws of remaining unchanged on addition of a term $\rho \times r_i$, where ρ is the same vector for all r_i, to each of the moments r_i^0 of the motors. This transformation is equivalent to parallel translation of the screw space. It could also be shown (but we shall not dwell on this here) that the basic formulas of screw algebra remain unchanged on any motion of the space that preserves the complex moduli of the screws and the angles between their axes, or, in other words, on any orthogonal transformation.

Below, in Chapter 5, we shall indicate the possibility of establishing correspondence between the formulas of vector analysis and those of screw analysis, in which complex scalar functions and screw functions of a screw argument figure.

The transfer principle is of great practical importance in the theory of complex vectors. In solving problems in the kinematics of a solid body with a fixed point, the angular velocities are expressed by vectors passing through a common point, and the algebra of free vectors is applied. If it is necessary to solve a problem of motion of a free solid body, velocity screws are substituted for the angular-velocity vectors in the formulas for the corresponding spherical motion, complex angles between screw axes replace angles between vectors, and the kinematic formulas of the free solid body are obtained by simple recasting of the kinematic formulas of a fixed-point body with substitution of "upper-case" for "lower-case" letters: these formulas are then expanded. For many problems in the kinematics of the arbitrarily moving body, it is possible to formulate the corresponding problem of spherical motion by artificial introduction of a fixed point; solution of this simpler problem automatically leads to the solution of the basic problem with the aid of the transfer principle.

Problems of the motion of a system of solid bodies whose relative motions are subject to geometrical linkage conditions can be solved in the same way. As a result, it becomes possible to solve with comparative ease problems concerning the motion of three-dimensional hinge and other mechanisms.

The situation is quite similar in the statics of the solid body, where many problems of the equilibrium of a free solid body can be solved by solving problems of the equilibrium of a point and subsequent application of the transfer principle. It will be appropriate to note here that the attempt to apply the transfer principle to dynamics no longer produces such simple relationships as can be obtained for kinematics and statics. This is because it is necessary in writing the screw equations of solid-body dynamics to establish correspondence between two spaces twice (first between the space of the angular-velocity vectors and the kinematic screw space, and then between the force-vector space and the force-screw space), and because the complex operator linking the kinematic and force screws cannot be obtained from the corresponding affine operator linking the angular velocity vector with the moment by substitution of complex for real quantities.[*] As a result, many dynamic and static problems must be solved on the basis of general screw theory with the screws expressed by means of six Plücker coordinates.

In this chapter and those that follow, examples of application of this transfer principle to certain problems of geometry and kinematics will be presented.

§2. Finite Displacements of a Solid Body

Let us consider application of the transfer principle to the theory of finite displacements of a solid body.

In the kinematics, we shall consider screw displacements. A displacement is expressed by a screw in which the vector is equal to the rotation angle and the moment is equal to the translational-displacement vector; the screw axis coincides with the displacement axis of the body.

A zero-parameter screw (or sliding vector) corresponds to pure rotation of the body without translational displacement. A screw of infinite parameter corresponds to purely translational displacement of the body.

Let us first dwell on the elementary theory of finite rotations of a body with a fixed point.[**]

If a solid body turns through a finite angle about a certain axis, whose unit vector will be denoted by e, we can relate the initial value of the radius vector of a point of the body, $r = \bar{OA}$, where O is a point on the rotation axis, with its final value after the rotation, $r' = OA'$, where A' is the final position of the point (Fig. 11), on the basis of the following theorem.

Theorem 9. *If we introduce the finite-rotation vector*

$$0 = e \operatorname{tg} \frac{\varphi}{2} = e\theta, \tag{4.1}$$

where e is the unit vector of the axis of rotation and φ is the angle of rotation, then the final value r′ of the radius vector is expressed by the following formula in terms of the initial value r: [*]

$$r' = r + \frac{2\theta}{1+\theta^2} \times (r + \theta \times r). \tag{4.2}$$

Actually, if we consider a section passing through point A perpendicular to the axis of rotation, we shall have in it a vector $s = \vec{OA}$ that becomes the vector $\vec{O'A'} = s'$ after the rotation.

We have the relationships

$$r = (r \cdot e) e + s, \quad r' = (r \cdot e) e + s'. \tag{4.3}$$

For the final position we have

$$s' = s \cos\varphi + e \times s \sin\varphi = s \frac{1-\theta^2}{1+\theta^2} + e \times s \frac{2\theta}{1+\theta^2} =$$
$$= s + e \times s \frac{2\theta}{1+\theta^2} - s \frac{2\theta^2}{1+\theta^2}.$$

On the basis of (4.3), and remembering that $e \times s = e \times r$, we can write

$$s' = s + e \times r \frac{2\theta}{1+\theta^2} + \frac{2\theta^2}{1+\theta^2} [(r \cdot e) e - r] =$$
$$= s + e \times r \frac{2\theta}{1+\theta^2} + \frac{2\theta^2}{1+\theta^2} e \times (e \times r).$$

Adding $(r \cdot e)e$ to the left and right members of the equality and substituting θ for $e\theta$, we obtain Formula (4.2), whose validity was to be demonstrated.

In the particular case in which $\varphi = \pi$, i.e., when the body makes a half-revolution, Formula (4.2) gives

$$r' = 2 (e \cdot r) e - r. \tag{4.4}$$

Any rotation of the body can be accomplished by two half-revolutions on the basis of the following theorem.

Theorem 10. *Rotation of a body through an angle φ about a certain axis is equivalent to two successive half-revolutions of the body about axes that intersect at right angles at the same point on the given axis and form an angle φ/2 with one another.*

Let the unit vector of the axis of rotation be $e_{\star}$ and let the unit vectors of the half-revolution axes be e_1 and e_2. The radius vector r of the point of the body lying in plane $e_1 e_2$ is given, according to (4.4), by the following expression after a half-turn about e:

$$r' = 2\,(e_1 \cdot r)\,e_1 - r,$$

and after the second half-turn by

$$r'' = 2\,(e_2 \cdot r')\,e_2 - r' = 2\,\{[2\,(e_1 \cdot r)\,e_1 - r]\cdot e_2\}\,e_2 - $$
$$- 2\,(e_1 \cdot r)\,e_1 + r.$$

If the angle between r and e_1 is denoted by α, we shall have

$$r'' = r - 2e_1 \cos \alpha - 2e_2 \cos\left(\frac{\varphi}{2} - \alpha\right).$$

Forming the vector product of the initial radius vector and the final radius vector, i.e., after the two half-turns, we obtain

$$r \times r'' = -2r \times e_1 \cos \alpha - 2r \times e_2 \cos\left(\frac{\varphi}{2} - \alpha\right) = $$
$$= \left[-2\sin\alpha\cos\alpha + 2\sin\left(\frac{\varphi}{2} + \alpha\right)\cos\left(\frac{\varphi}{2} - \alpha\right)\right]e = e\sin\varphi.$$

It follows from this that the initial and final radius vectors lie in a plane perpendicular to the vector e and form an angle φ, i.e., as a result of the two half-revolutions, the body has performed rotation through an angle φ about the axis e, Q.E.D.
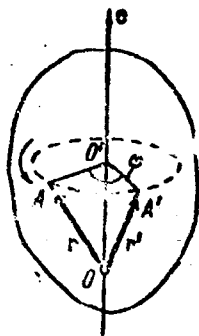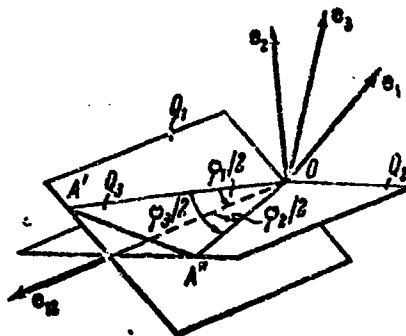
Fig. 11   Fig. 12

Two successive rotations of the body about axes that pass through a common point are equivalent to one rotation about an axis passing through the same point. This rotation is a resultant equivalent to the two rotations, which may be called component rotations.

The following theorem makes it possible, given the unit vectors e_1 and e_2 of the axes and the angles φ_1 and φ_2 of the component rotations, to find the unit vector e_3 of the axis and the angle φ_3 of the resultant rotation.

Suppose that we know the axes e_1 and e_2, which form the angle α, and the corresponding rotation angles φ_1 and φ_2 of the body (Fig. 12). We construct planes Q_1 and Q_2 respectively perpendicular to the vectors e_1 and e_2 at the point O of intersection of the vectors; this will determine an axis with unit vector e_{12}

that coincides with the line of intersection of planes Q_1 and Q_2, with the vector pointing in the direction of the vector product of the vectors e_1 and e_2. In plane Q_1, we draw from point O a ray OA', which on rotation about O through an angle of $\varphi_1/2$ coincides with the axis e_{12}; in plane Q_2, on the other hand, we draw a ray OA'' from point O; the axis e_{12} will coincide with this ray if the former is turned about O through an angle $\varphi_2/2$. We shall denote the unit vectors along the rays OA' and OA'' by e' and e''.

We pass plane Q_3 through rays OA' and OA'' and pass an axis with unit vector e_3 through point O perpendicular to plane Q in the direction of the vector product of the vectors e' and e''. This vector e_3 defines the axis of a rotation equivalent to two rotations about e_1 and e_2, while twice the angle between rays OA' and OA'' determines the magnitude φ_3 of the angle of the unknown rotation of the body.

By way of proof, we note that rotation about e_1 through an angle φ_1 is, according to Theorem 10, equivalent to a half-revolution about e' and a half-revolution about e_{12}; rotation about e_2 through an angle φ_2 is equivalent to a half-revolution about e_{12} and a half-revolution about e'' and, consequently, the complete rotation is equivalent to the above four half-rotations. But the two half-revolutions about e_{12} cancel one another, leaving the half-revolution about e' and that about e'', and these two are equivalent, on the basis of the theorem, to rotation through an angle φ_3 about e_3. It now remains to express the vector of the resultant rotation in terms of the component-rotation vectors.

For this purpose, we write the expression

$$e_3 \frac{\sin(e', e'')}{\cos(e', e'')} = \frac{e' \times e''}{e' \cdot e''} = e_3 \operatorname{tg} \frac{\varphi_3}{2} . \qquad (4.5)$$

We express the vectors e' and e'':

$$e' = \frac{e_1 \times e_3}{\sin \alpha} \cos \frac{\varphi_1}{2} + \frac{e_1 \times e_3}{\sin \alpha} \times e_1 \sin \frac{\varphi_1}{2} , \\ e'' = \frac{e_3 \times e_2}{\sin \alpha} \cos \frac{\varphi_2}{2} + e_2 \times \frac{e_1 \times e_3}{\sin \alpha} \sin \frac{\varphi_2}{2} . \qquad (4.6)$$

Performing vector and then scalar multiplication of Equalities (4.6), we find after rearranging

$$e' \times e'' = e_1 \sin \frac{\varphi_1}{2} \cos \frac{\varphi_2}{2} + e_2 \cos \frac{\varphi_1}{2} \sin \frac{\varphi_2}{2} - \\ - e_1 \times e_2 \sin \frac{\varphi_1}{2} \sin \frac{\varphi_2}{2} , \qquad (4.7)$$

$$e' \cdot e'' = \cos(e', e'') = \cos \frac{\varphi_1}{2} \cos \frac{\varphi_2}{2} - \sin \frac{\varphi_1}{2} \sin \frac{\varphi_2}{2} \cos \alpha. \qquad (4.8)$$

Substituting (4.7) and (4.8) in the numerator and denominator of (4.5), we obtain after canceling $\cos(\varphi_1/2) \cos(\varphi_2/2)$

$$e_3 \operatorname{tg} \frac{\varphi_3}{2} = \frac{e_1 \operatorname{tg} \frac{\varphi_1}{2} + e_2 \operatorname{tg} \frac{\varphi_2}{2} - e_1 \times e_2 \operatorname{tg} \frac{\varphi_1}{2} \operatorname{tg} \frac{\varphi_2}{2}}{1 - e_1 \cdot e_2 \operatorname{tg} \frac{\varphi_1}{2} \operatorname{tg} \frac{\varphi_2}{2}} .$$

Again introducing the finite-rotation vectors

$$\theta_1 = e_1 \, \mathrm{tg} \frac{\varphi_1}{2}, \quad \theta_2 = e_2 \, \mathrm{tg} \frac{\varphi_2}{2}, \quad \theta_3 = e_3 \, \mathrm{tg} \frac{\varphi_3}{2}, \tag{4.9}$$

we obtain the final result — an expression for the vector of the resultant rotation in terms of the vectors of the component rotations:

$$\theta_3 = \frac{\theta_1 + \theta_2 - \theta_1 \times \theta_2}{1 - \theta_1 \cdot \theta_2}. \tag{4.10}$$

This proves the following theorem.

Theorem 11. *If the unit vectors e_1 and e_2 of the axes and the angles φ_1 and φ_2 of successive rotations of a body are given, then the axis of the resultant rotation equivalent to these two rotations is obtained by the following construction: through the point 0 of intersection of e_1 with e_2 we pass two planes perpendicular to these two vectors; in the first plane we draw a ray that forms an angle $-\varphi_1/2$ with the line of intersection of the planes, and in the second plane we draw a ray that forms an angle $\varphi_2/2$ with this line; we pass a plane through these rays, which are defined by the unit vectors e' and e''. The axis with unit vector e_3 perpendicular to this plane at point 0 will be the axis of the resultant rotation and the angle of rotation φ_3 will be twice the angle between e' and e''. The relation between the resultant-rotation vector and the component-rotation vectors is given by Formulas (4.9) and (4.10).*

Formula (4.10) indicates that the resultant rotation depends on the order in which the component rotations were performed. For rotation through small angles, when the products of the angles can be disregarded, we obtain a "linear" addition formula:

$$\theta_3 \approx \theta_1 + \theta_2.$$

On the basis of the transfer principle, Formulas (4.1), (4.2), (4.9) and (4.10) given here may be interpreted as formulas with complex quantities. The finite-rotation angles that appear in them may be put complex, the unit vectors may be made unit screws of axes fixed in space, and the vector moduli may be made complex. In this case, by virtue of the transfer principle, these formulas admit of interpretation in the language of screws, and the theory of finite rotations set forth above becomes a theory of finite screw displacements of a body. Theorems 9, 10 and 11 remain valid in this new interpretation, with the following corrections: firstly, screw displacements are imparted to the body about axes arbitrarily positioned in space and, secondly, the initial and final positions determined are not those of the radius vector of a point, but those of a screw lying on a line belonging to the body.

Thus, we can formulate the following theorems.

Theorem 12. *If we introduce the complex vector of a finite screw displacement*

$$\Theta = E\theta = E\,\mathrm{tg}\frac{\Phi}{2},\qquad\qquad (4.11)$$

where E is the unit screw of the screw-displacement axis and $\Phi = \varphi + \omega\varphi^0$ is the complex rotation angle (displacement screw), then for a screw (or vector) R lying on an arbitrary line of the body, the finite position R' is expressed by the formula

$$R' = R + \frac{2\Theta}{1+\Theta^2} \times (R + \Theta \times R),\qquad\qquad (4.12)$$

which is analogous to Formula (4.2).

A diagram of the displacement under consideration appears in Fig. 13.

Theorem 13. *Screw displacement of a body through a complex angle* $\Phi = \varphi + \omega\varphi^0$ *about an axis whose unit screw is* E *is equivalent to two successive half-revolutions executed about lines with the unit screws* E_1 *and* E_2, *which intersect the axis* E *at right angles and form a complex angle* $\Phi/2$ *with one another (Fig. 14).*
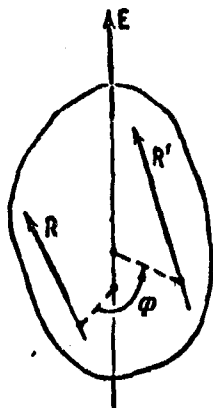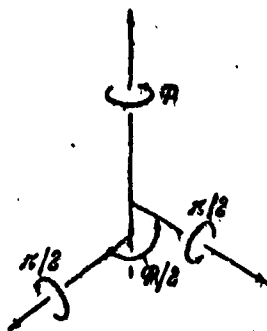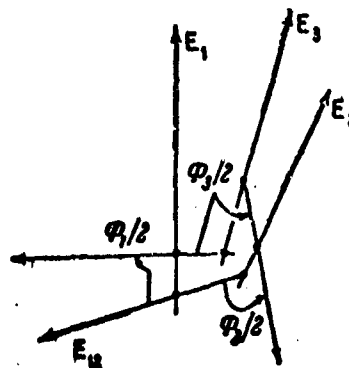
Fig. 13   Fig. 14   Fig. 15

Theorem 14. *Two successive finite screw displacements through complex angles* Φ_1 *and* Φ_2 *about arbitrary axes in a space with unit screws* E_1 *and* E_2 *can be substituted by a single equivalent resultant screw displacement. The axis, whose unit screw will be denoted by* E_3, *and the complex angle* Φ_3 *of the resultant screw displacement are obtained by the following construction (Fig. 15): we draw the axis of the complex angle* $E_1 E_2$ *and then a straight line* a' *that intersects the axis* E_1 *and forms the complex angle* $\Phi_1/2$ *with this angle axis and then a line* a'' *that intersects the axis* E_2 *and forms the complex angle* $\Phi_2/2$ *with this same angle axis; twice the complex angle between* a' *and* a'' *is equal to the complex angle* Φ_3 *of the resultant screw displacement, and the axis of angle* $a'-a''$ *with the unit screw* E_3 *is the axis of this displacement.*

If the complex vectors of the finite screw displacements of the components and the resultant are equal, respectively, to

$$\Theta_1 = E_1\Theta_1 = E_1 \operatorname{tg} \frac{\Phi_1}{2}, \quad \Theta_2 = E_2\Theta_2 = E_2 \operatorname{tg} \frac{\Phi_2}{2},$$

$$\Theta_3 = E_3\Theta_3 = E_3 \operatorname{tg} \frac{\Phi_3}{2}, \tag{4.13}$$

then the relation between these screws is expressed by the formula

$$\Theta_9 = \frac{\Theta_1 + \Theta_2 - \Theta_1 \times \Theta_2}{1 - \Theta_1 \cdot \Theta_2}, \tag{4.14}$$

which is analogous to Formula (4.10).

Theorems 12, 13 and 14 require no proof, since they follow from the analogous formulas pertaining to simple rotations of a body by virtue of the transfer principle.

Analysis of the analogy between the "vector" and "screw" formulas indicates that with the transfer principle, point kinematics become line kinematics and the kinematics of a body with a fixed point become the kinematics of a free body.

§3. Determination of Displacement Screw from Initial and Final Positions of a Solid Body

The problem of displacement of a solid body from one given position to another by means of a single screw displacement is of practical interest for production automation, especially when it is necessary to accomplish a certain technological operation accompanied by an over-all displacement of a workpiece on a machine.* Practical execution of such a displacement requires a converting fixture capable of imparting to the workpiece a single screw displacement that transfers it from one position to the other. Here the initial and final positions are considered to be given, and the problem consists in determining the appropriate displacement screw that effects this translation, i.e., an axis, a rotation angle, and a translational displacement. The initial and final positions of the workpiece can be defined by the initial and final positions of any two straight lines rigidly associated with this workpiece.

First, let us solve a simpler problem, from the solution of which we may then pass to solution of the problem formulated above, using the transfer principle. This simpler problem is as follows: find the finite-rotation vector of a solid body that has a fixed point O if it is known that the two unit vectors $e_1 = \overrightarrow{OA_1}$ and $e_2 = \overrightarrow{OA_2}$, which pass through point O and are inseparably associated with the body, become, after the rotation, the vectors $e'_1 = \overrightarrow{OA'_1}$ and $e'_2 = \overrightarrow{OA'_2}$ (here, naturally, $e_1 \cdot e_2 = e'_1 \cdot e'_2$). This problem is equivalent to the familiar problem of determining the center of a finite rotation of a spherical segment A_1A_2 that becomes a segment $A_1A'_2$ on a sphere of unit radius. For the solution, we first determine the geometric locus of all axes rotation about which can translate vector e_1 into vector e'_1. Obviously, the geometric locus of such axes will be a plane q_1 passing through O and perpendicular to the plane $OA_1A'_1$, with its line of intersection with the latter bisecting the angle between the vectors e_1 and e'_1.

Then we determine the geometric locus of all axes rotation about which can translate vector e_2 into vector e'_2; this will be a plane q_2 passing through O, perpendicular to the plane $OA_2A'_2$, and having a line of intersection with the latter that bisects the angle between vectors e_2 and e'_2. Obviously, the line e of intersection of the planes q_1 and q_2 satisfies the condition that rotation about it translates both e_1 into e'_1 and e_2 into e_2 [sic]. This will be the solution of the problem of determining the finite-rotation axes of a body with a fixed point O.

In concrete terms, the solution on the above scheme is carried out as follows. First we determine the plane q_1 — its unit vector is parallel to the vector $r_1 = e'_1 - e_1$, and then the plane q_2, whose unit vector is parallel to the vector $r_2 = e'_2 - e_2$. The unit vector e, which is perpendicular simultaneously to vectors r_1 and r_2, will obviously be parallel to the line of intersection of planes q_1 and q_2 and will define the point of rotation of a spherical segment A_1A_2 on a sphere of unit radius, with the result that the latter goes over to $A'_1A'_2$. We have

$$r_1 = e'_1 - e_1, \quad r_2 = e'_2 - e_2,$$

$$e = \frac{r_1 \times r_2}{|r_1 \times r_2|} = \frac{(e'_1 - e_1) \times (e'_2 - e_2)}{|(e'_1 - e_1) \times (e'_2 - e_2)|} =$$

$$= \frac{(e'_1 - e_1) \times (e'_2 - e_2)}{|e'_1 - e_1||e'_2 - e_2|\sin(e'_1 - e_1, e'_2 - e_2)},$$

$$|e'_1 - e_1| = 2\sin\tfrac{1}{2}(e'_1, e_1) = \sqrt{2}\sqrt{1 - e'_1 \cdot e_1},$$

$$|e'_2 - e_2| = 2\sin\tfrac{1}{2}(e'_2, e_2) = \sqrt{2}\sqrt{1 - e'_2 \cdot e_2},$$

$$|e'_1 - e_1||e'_2 - e_2| = 2\sqrt{1 - e'_1 \cdot e_1}\sqrt{1 - e'_2 \cdot e_2},$$

$$\cos(e'_1 - e_1, \ e'_2 - e_2) = \frac{(e'_1 - e_1)\cdot(e'_2 - e_2)}{2\sqrt{1 - e'_1 \cdot e_1}\sqrt{1 - e'_2 \cdot e_2}},$$

$$\sin(e'_1 - e_1, \ e'_2 - e_2) =$$

$$= \frac{\sqrt{4(1 - e'_1 \cdot e_1)(1 - e'_2 \cdot e_2) - [(e'_1 - e_1)\cdot(e'_2 - e_2)]^2}}{2\sqrt{1 - e'_1 \cdot e_1}\sqrt{1 - e'_2 \cdot e_2}}.$$

(4.15)

Thus, the unit vector of the axis about which the body must make a finite rotation will be

$$e = \frac{(e'_1 - e_1) \times (e'_2 - e_2)}{\sqrt{4(1 - e'_1 \cdot e_1)(1 - e'_2 \cdot e_2) - [(e'_1 - e_1)\cdot(e'_2 - e_2)]^2}}.$$

(4.16)

It remains to find the finite-rotation angle. For this purpose, we apply Formula (4.2) for the finite rotation, substituting in it the initial and final vectors e_1 and e_2, together with the vector e. We obtain the formula

$$e'_1 = e + \frac{2\theta}{1 + \theta^2}\, e \times (e_1 + e \times e_1 0) =$$

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$$= e + \frac{20}{1+\theta^2} e \times e_1 + \frac{20^2}{1+\theta^2} e \times (e \times e_1) =$$
$$= e + \frac{20}{1+\theta^2} e \times e_1 + \frac{20^2}{1+\theta^2} e (e \cdot e_1) - \frac{20^2}{1+\theta^2} e_1,$$

in which, in this case, all unit vectors are known, and the unknown is the quantity $\theta = tg(\varphi/2)$. Performing scalar multiplication of both sides of the above equality by e_1, we obtain the scalar equation

$$e_1' \cdot e_1 = 1 + \frac{20^2}{1+\theta^2}(e \cdot e_1)^2 - \frac{20^2}{1+\theta^2} = 1 + \frac{20^2}{1+\theta^2}[(e \cdot e_1)^2 - 1],$$

from which

$$\theta^2 = tg^2 \frac{\varphi}{2} = \frac{1 - e_1' \cdot e_1}{1 + e_1' \cdot e_1 - 2(e \cdot e_1)^2}. \qquad (4.17)$$

The problem posed at the outset can be solved very easily after solution of this problem.

Thus, there were given two unit screws E_1 and E_2 lying on two straight lines inseparably connected with the body, screws which, after the body has completed a certain displacement in space, have been translated into the unit screws E'_1 and E'_2, which are known. It is necessary to find the corresponding finite-displacement screw of the body.

We apply the transfer principle, using the procedure employed in solving the previous problem.

First we determine the geometric locus of all axes screw motion with respect to which can translate unit screw E_1 into unit screw E'_1. By virtue of the transfer principle, this will be a plane analogue — a brush Q_1, whose axis will be the axis of a screw $R_1 = E_1' - E_1$. This axis intersects the axis of screw $E_1 \times E_1'$ at right angles and bisects the segment between E_1 and E'_1 on this axis.

We then determine the geometric locus of all axes screw motion with respect to which can transfer unit screw E_2 to unit screw E'_2. Once again, this will be a brush Q_2, whose axis is the axis of a screw $R_2 = E_2' - E_2$. This axis intersects the axis of screw $E_2 \times E_2'$ at right angles and bisects the segment between E_2 and E'_2 on this axis.

The axis E screw motion with respect to which can simultaneously translate E_1 into E'_1 and E_2 into E'_2, i.e., the screw-displacement axis of the body, will belong simultaneously to both of the above brushes and, consequently, this axis must intersect the axes of screws R_1 and R_2 at right angles.

Now it remains to find all of these axes.

By analogy with Formulas (4.15), we have

$$R_1 = E_1' - E_1, \quad R_2 = E_2' - E_2,$$

$$E = \frac{R_1 \times R_2}{|R_1 \times R_2|} = \frac{(E_1' - E_1) \times (E_2' - E_2)}{|E_1' - E_1| \, |E_2' - E_2| \sin(E_1' - E_1, E_2' - E_2)},$$

$$|E_1' - E_1| = \sqrt{2}\sqrt{1 - E_1' \cdot E_1},$$

$$|E_2' - E_2| = \sqrt{2}\sqrt{1 - E_2' \cdot E_2},$$

$$\sin(E_1' - E_1, E_2' - E_2) =$$

$$= \frac{\sqrt{4(1 - E_1' \cdot E_1)(1 - E_2' \cdot E_2) - [(E_1' - E_1) \cdot (E_2' - E_2)]^2}}{2\sqrt{1 - E_1' \cdot E_1}\sqrt{1 - E_2' \cdot E_2}}.$$

(4.18)

The unit screw of the screw-displacement axis of the body is obtained from a formula similar to (4.16):

$$E = \frac{(E_1' - E_1) \times (E_2' - E_2)}{\sqrt{4(1 - E_1' \cdot E_1)(1 - E_2' \cdot E_2) - [(E_1' - E_1) \cdot (E_2' - E_2)]^2}}.$$

(4.19)

Now, taking the initial and final positions of one of the unit screws, namely E_1 and E'_1, and the unit screw E that we have found, we determine the complex angle Φ or, what is the same thing, the modulus of the finite-rotation screw of the body, from a relationship analogous to (4.17),

$$\Theta^2 = \operatorname{tg}^2 \frac{\Phi}{2} = \frac{1 - E_1' \cdot E_1}{1 + E_1' \cdot E_1 - 2(E \cdot E_1)^2},$$

(4.20)

so that the problem has been solved.

§4. Application of the Theory of Finite Screw Displacements to Determination of Relative Displacements of Links in a Three-Dimensional Mechanism

The theory of finite rotations set forth above and its screw analogue, the theory of finite screw displacements, enable us to derive formulas for the relation between the turn angles and slide paths of the links of a three-dimensional mechanism with cylindrical changes.

Let us consider a three-dimensional four-member mechanism with a single rotary hinge 1 and three cylindrical hinges 2, 3 and 4 (Fig. 16). The rotary hinge permits relative rotation of the adjacent links through arbitrary angles, while the cylindrical hinges permit rotation together with slip. The hinge axes occupy arbitrary positions in space. Let us establish the term link for a rigid configuration consisting of two neighboring hinge axes and the shortest-distance line segment between them. Geometrically, therefore, a link is characterized by a complex angle whose principal part is the actual angle between the axes of the hinges at its ends, while the moment part is the length of the link. We shall denote the complex angle of the links 1-2, 2-3, 3-4 and 4-1 by

$$A = \alpha + \omega a^0, \quad B = \beta + \omega b^0, \quad \Gamma = \gamma + \omega r^0, \quad \Delta = \delta + \omega d^0,$$

respectively; link 1-2 will be regarded as the driving link and

link 4-3 as the driven link; we shall assume that link 4-1 is stationary. The angle between links 1-4 and 1-2 will be denoted by $\Phi = \varphi + \omega\varphi^0$, and the angle between links 1-4 and 4-3 by $\Psi = \psi + \omega\psi^0$.
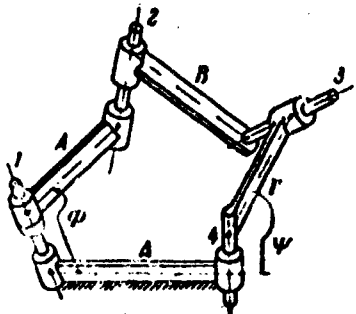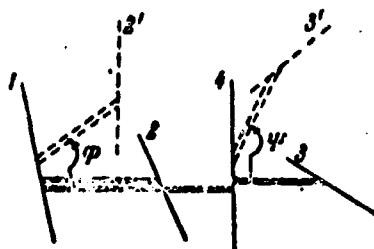
Fig. 16   Fig. 17

We pose the problem of determining the position of the driven link 3-4 as a function of the position of the driving link 1-2, or, in other words, determining the angle Ψ as a function of the angle Φ. The problem is most simply solved as follows. We temporarily remove link 2-3 and "stretch" links 1-2 and 4-3 into line with the fixed link 1-4. Thus the angles Φ and Ψ will have been reduced to zero (Fig. 17). We then give link 1-2 a rotation about axis 1 through a complex angle Φ, and link 4-3 a rotation about axis 4 through the complex angle Ψ. After these rotations, axes 2 and 3 will occupy positions 2' and 3'.

We replace link 2-3, requiring that the configuration of axes 2' and 3' correspond to the configuration of the temporarily removed link 2-3, i.e., that the complex angle between axes 2' and 3' be equal to $B = \beta + \omega\beta^0$, which necessitates that the scalar product of the unit screws of these axes be equal to the cosine of the complex angle in question.

We shall denote the unit screws of the hinge axes in the "stretched" state of the mechanism by E_1, E_2, E_3, E_4.

After rotation about E_1 through an angle Φ, the position of axis E_2 will be expressed as follows on the basis of Formula (4.12):

$$E_2' = E_2 + \frac{2\Theta}{1 + \Theta^2} \times (E_2 + \Theta \times E_2).$$ (4.21)

where $\Theta = E_1 \operatorname{tg}(\Phi/2)$ is the complex finite-rotation vector of link 1-2.

After rotation about E_4 through an angle Ψ, the position of axis E_3, again on the basis of Formula (4.12), will be

$$E_3' = E_3 + \frac{2X}{1 + X^2} \times (E_3 + X \times E_3).$$ (4.22)

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where

$$X = E_4 \operatorname{tg} \frac{\Psi}{2}.$$

Performing scalar multiplication of (4.21) and (4.22) and equating cos B:

$$E'_2 \cdot E'_3 = E_2 \cdot E_3 + \left[\frac{2\Theta}{1+\Theta^2} \times (E_3 + \Theta \times E_2) \right] \cdot E_3 +$$
$$+ E_4 \left[\frac{2X}{1+X^2} \times (E_3 + X \times E_3) \right] +$$
$$+ \left[\frac{2\Theta}{1+\Theta^2} \times (E_3 + \Theta \times E_2) \right] \cdot \left[\frac{2X}{1+X^2} \times (E_3 + X \times E_3) \right] =$$
$$= \cos B.$$

If we expand this product and then remember that the scalar products of the unit screws of the mechanism's axes have the values

$$E'_3 \cdot E'_3 = \cos B, \ E_2 \cdot E_3 = \cos(\Delta - A + \Gamma), \ E_1 \cdot E_3 = \cos A,$$
$$E_1 \cdot E_4 = \cos \Delta, \ E_3 \cdot E_4 = \cos \Gamma,$$

we obtain the following quadratic equation in the unknown complex quantity X:

$$\{[\cos(\Delta - A - \Gamma) - \cos B] + [\cos(\Delta + A - \Gamma) - \cos B] \Theta^2\} X^2 + 4 \sin A \sin \Gamma \Theta X + [\cos(\Delta - A + \Gamma) - \cos B] + [\cos(\Delta + A + \Gamma) - \cos B] \Theta^2 = 0 \qquad (4.23)$$

or, concisely,

$$(M + N\Theta^2) X^2 + 2P\Theta X + (Q + R\Theta^2) = 0. \qquad (4.23')$$

This quadratic equation expresses X as a function of Θ, i.e., properly speaking, the dependence of the turn angle Ψ of the driven link on the turn angle ϕ of the driving link.

We note that the angle $\Phi = \phi + \omega\phi^0$ varies in such a way that the quantity ϕ^0 remains constant (rotary hinge); hence the argument is the real quantity ϕ, while the change in the angle $\Psi = \phi + \omega\phi^0$ represents the change in the angle ϕ and the segment ψ^0 proper.

If we take the principal part of the complex equation (4.23), it will be the same equation, but with real quantities substituted for complex, i.e., with lower-case letters substituted for upper-case letters:

$$\{[\cos(\delta - \alpha - \gamma) - \cos\beta] + [\cos(\delta + \alpha - \gamma) - \cos\beta] \theta^2\} \chi^2 + 4 \sin\alpha \sin\gamma\theta\chi + [\cos(\delta - \alpha + \gamma) - \cos\beta] + [\cos(\delta + \alpha + \gamma) - \cos\beta] \theta^2 = 0 \qquad (4.24)$$

or, in short form,

$$(m + n\theta^2) \chi^2 + 2p\theta\chi + (q + r\theta^2) = 0. \qquad (4.24')$$

where

$$0 = \lg \frac{\varphi}{2}, \quad \chi = \lg \frac{\psi}{2}.$$

Equation (4.24) or (4.24') describes a three-dimensional four-link mechanism whose axes are parallel to the axes of the given mechanism and intersect at one point.

Let us ascertain the conditions under which the discriminant of the complex equation (4.23) vanishes, i.e., the conditions under which the equality

$$P^2\theta^2 - (M + N\theta^2)(Q + R\theta^2) = 0$$

is satisfied. This equality represents an equation in the quantity θ:

$$NR\theta^4 + (MR + QN - P^2)\theta^2 + MQ = 0. \qquad (4.25)$$

We transform the expressions for the coefficients of this equation:

$$NR = 4\sin\frac{\Delta + A - \Gamma + B}{2}\sin\frac{\Delta + A - \Gamma - B}{2} \times$$
$$\times \sin\frac{\Delta + A + \Gamma + B}{2}\sin\frac{\Delta + A + \Gamma - B}{2} =$$
$$= [\cos(B + \Gamma) - \cos(\Delta + A)][\cos(B - \Gamma) -$$
$$- \cos(\Delta + A)] = \varkappa\rho,$$

$$MQ = 4\sin\frac{\Delta - A - \Gamma + B}{2}\sin\frac{\Delta - A - \Gamma - B}{2} \times$$
$$\times \sin\frac{\Delta - A + \Gamma + B}{2}\sin\frac{\Delta - A + \Gamma - B}{2} =$$
$$= [\cos(B + \Gamma) - \cos(\Delta - A)][\cos(B - \Gamma) -$$
$$- \cos(\Delta - A)] = \sigma\tau,$$

$$MR + QN - P^2 =$$
$$= [\cos(\Delta + B) - \cos(A + \Gamma)][\cos(A - B) -$$
$$- \cos(A + \Gamma)] + [\cos(A + B) - \cos(A - \Gamma)] \times$$
$$\times [\cos(\Delta - B) - \cos(A - \Gamma)] - 4\sin^2 A\sin^2 \Gamma =$$
$$= \cos^2 A + \cos 2B + \cos 2\Gamma + \cos 2\Delta - 4\cos A\cos B \times$$

$$\times \cos \Gamma\cos \Delta + [\cos(B + \Gamma) - \cos(\Delta + A)]\times$$
$$\times [\cos(B - \Gamma) - \cos(A - A)] + [\cos(B - \Gamma) -$$
$$- \cos(A + A)][\cos(B + \Gamma) - \cos(\Delta - A)] =$$
$$= \varkappa\tau + \rho\sigma,$$

then Eq. (4.25) assumes the form

$$\varkappa\rho\theta^4 + (\varkappa\tau + \rho\sigma)\theta^2 + \sigma\tau = 0. \qquad (4.25')$$

The roots of this equation will be

$$\theta_1 = \sqrt{-\frac{\tau}{\rho}}, \quad \theta_2 = \sqrt{-\frac{\sigma}{\varkappa}}. \qquad (4.26)$$

Let us ascertain the position of the mechanism's links to which these two values of θ and, consequently, two values of the angle

Φ correspond. We write the expression for the cosine of the angle between the axes of hinges 2 and 4 on the basis of a formula of complex spherical trigonometry

$$\cos(E_2, E_4) = \cos \Lambda \cos \Delta + \sin \Lambda \sin \Delta \cos \Phi =$$
$$= \cos \Lambda \cos \Delta + \sin \Lambda \sin \Delta \frac{1 - \theta^2}{1 + \theta^2}. \qquad (4.27)$$

Now we replace θ^2 by its values θ_1^2 and θ_2^2 (4.26); then we obtain in the one case

$$\cos(E_2, E_4) = \cos \Lambda \cos \Delta + \sin \Lambda \sin \Delta \frac{\pi + \sigma}{\pi - \sigma} =$$
$$= \cos \Lambda \cos \Delta + \sin \Lambda \sin \Delta \frac{2\cos(B + \Gamma) - 2\cos A \cos \Delta}{2 \sin \Lambda \sin \Delta} =$$
$$= \cos(B + \Gamma),$$

and similarly in the other case

$$\cos(E_2, E_4) = \cos(B - \Gamma).$$

The result obtained indicates that for the values of θ and, accordingly, Φ that cause the discriminant of Eq. (4.23) to vanish, the complex angle between axes 2 and 4 is equal to the sum or difference of the angles B and Γ. From this it follows that in this position of the mechanism, axes 2, 3 and 4 are parallel to the same plane, and that links 2-3 and 3-4 have become parallel. This is the "dead" position (Fig. 18, a and b).
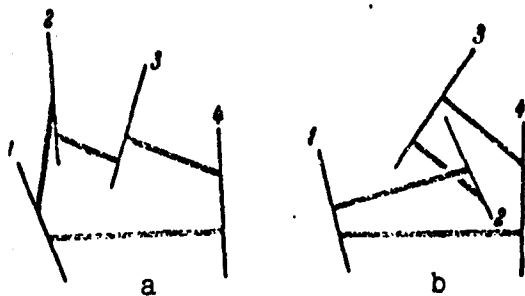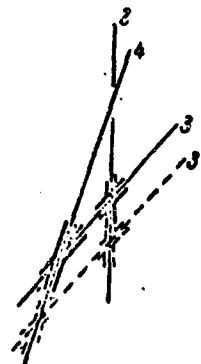
Fig. 18   Fig. 19

On the other hand, since the discriminant of the complex algebraic equation (4.23) is zero, the moment part is unknown, i.e., the moment part X and hence the quantity ψ^0 can be selected arbitrarily on the basis of Theorem 3 (Chapter 2). This purely algebraic property is interpreted in this case as a kinematic fact: when the three cylindrical hinges are positioned parallel to the same plane, two links, together with the middle axis, can slip indefinitely, and, consequently, ψ^0 ceases to be a fixed quantity (Fig. 19). In this case the dead position is a position of indeterminate slip for certain links.

It is also possible to pose the following problem: determine the relationships among the link dimensions (lengths and angles) with which pure rotation will take place in hinge 2 on rotation of the driving link 1-2. Obviously, these relationships will be exceptional, because, generally speaking, rotation with slip must occur in hinge 4; the requirement of pure rotation, however, is a requirement that the slip vanish identically for any value of the driving-link rotation angle.

Let us assume for simplicity that $\Phi^0 = \Psi^0 = 0$, i.e., that links 1-2 and 1-4 on the one hand and links 1-4 and 4-3 on the other come into contact and that the angles Φ and Ψ are real.

Expressing the coefficients and the unknown in Eq. (4.23) in terms of principal and moment parts, we separate these parts:

$$(m + n0^2)\,\chi^2 + 2p0\chi\,(q + r0^2) = 0,$$
$$(m^0 + n^c0^2)\,\chi^2 + 2p^00\chi\,(q^0 + r^00^2) = 0.$$

Since the quantity χ must satisfy both of the above equations, the resultant of these equations must be identically equal to zero, i.e.,

$$\begin{vmatrix} m + n0^2 & 2p0 & q + r0^2 & 0 \\ m^0 + n^c0^2 & 2p^00 & q^0 + r^00^2 & 0 \\ 0 & m + n0^2 & 2p0 & q + r0^2 \\ 0 & m^0 + n^c0^2 & 2p^00 & q^0 + r^00^2 \end{vmatrix} = 0. \qquad (4.28)$$

This will be a condition for a real root of the complex algebraic equation. Expanding the determinant, we obtain a polynomial in θ. Since the determinant must be equal to zero for any θ, all expressions appearing as coefficients of θ and the absolute term must be equated to zero. From this we obtain a number of conditions that will contain only internal parameters of the mechanism, i.e., the link lengths and the angles between the hinge axes. Expanding these conditions, we obtain the necessary relationships for the parameters of the mechanism that satisfies the requirement imposed.

§5. Complex Euler Angles and Euler Kinematic Equations

Complex Euler angles, which are characterized by screw displacements of a body, may be used to determine the position of the body in space. If we take fixed rectangular axes x, y, z (Fig. 20) and axes x', y' and z' that belong to the moving body and may be called the moving axes, the positions of the moving axes relative to the fixed axes can be characterized either by nine complex cosines or three independent Euler angles. The body would have to be rotated through these angles in order to occupy a given position if the axes x', y' and z', which are inseparably associated with the body, coincided with the axes x, y, z in the initial position.

The first such angle is the angle Ψ, which corresponds to screw displacement with respect to the z-axis; after this displacement the axes occupy the positions n, n', z. The second will

Fig. 20

be the angle Θ with respect to the n-axis; after rotation through this angle, the axes will occupy the positions n, n'', z'. The third will be the angle Φ with respect to the axis z'; after this displacement, the axes will occupy the positions x', y', z'.

By virtue of the transfer principle, the relation of the complex Euler angles with the complex rectangular coordinates is formally the same as the relation of the real Euler angles to real rectangular coordinates. We shall therefore represent the conversion from the xyz (fixed) system to the x', y', z' (moving) system in the form of the following table of "complex cosines:"

	X	Y	z
X'	$\cos\Phi\cos\Psi - \sin\Phi\sin\Psi\cos\Theta$	$\cos\Phi\sin\Psi + \sin\Phi\cos\Psi\cos\Theta$	$\sin\Phi\sin\Theta$
Y'	$-\sin\Phi\cos\Psi - \cos\Phi\sin\Psi\cos\Theta$	$-\sin\Phi\sin\Psi + \cos\Phi\cos\Psi\cos\Theta$	$\cos\Phi\sin\Theta$
Z'	$\sin\Theta\sin\Psi$	$\sin\Theta\cos\Psi$	$\cos\Theta$

If we denote the unit vectors of the fixed-system axes by i, j, k, and the unit vectors of the moving-system axes by i', j', k', the cosine of an angle between unit vectors in the former and latter systems is determined at the intersection of the corresponding column and line of the table. For example, $\cos(i, k')$ is equal to $\sin\Theta\sin\Psi$.

The unit-screw coordinates of any straight line belonging to a solid body, given in the fixed system, can be expressed in terms of the coordinates in the moving system by reference to this table.

Without elaborating on this problem, we note that in much the same way as the generally known Rodrigues-Hamilton and Cayley-Klein parameters, complex analogues can be constructed for which conversion to Euler angles and other coordinates is performed in accordance with corresponding formulas with the real quantities replaced by complex quantities.

The projections $\Omega_{x'}, \Omega_{y'}, \Omega_{z'}$ of the velocity screw Ω of an arbitrarily moving body onto axes inseparably associated with the body are connected to the complex Euler angles by the following relationships, which are derived from the familiar Euler kinematic equations for a body having a fixed point by substitution of complex for real quantities:

$$\begin{aligned}
\Omega_{x'} &= \dot{\Psi}\sin\Phi\sin\Theta + \dot{\Theta}\cos\Phi, \\
\Omega_{y'} &= \dot{\Psi}\cos\Phi\sin\Theta - \dot{\Theta}\sin\Phi, \\
\Omega_{z'} &= \dot{\Psi}\cos\Theta + \dot{\Phi}.
\end{aligned} \qquad (4.29)$$

Solving the equation system (4.29) for the Euler angles, we obtain

$$\dot{\Psi} = \frac{1}{\sin\Theta}(\Omega_{x'}\sin\Phi + \Omega_{y'}\cos\Phi),$$
$$\dot{\Theta} = \Omega_{x'}\cos\Phi - \Omega_{y'}\sin\Phi,$$
$$\dot{\Phi} = \Omega_{z'} - (\Omega_{x'}\sin\Phi + \Omega_{y'}\cos\Phi)\operatorname{ctg}\Theta.$$

(4.30)

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No.

Footnotes

58 One of the classical examples of the transfer principle
is the familiar principle of duality in projective ge-
ometry on a plane, on the basis of which all considera-
tions remain in force if the points in them are replaced
by lines and lines by points.

61* In the next chapter, we shall present considerations
pertaining to the range of applicability of the trans-
fer principle to solution of problems in mechanics.

61** The presence of a fixed point is not necessary; Formulas
(4.1)-(4.10) apply, strictly speaking, only to displace-
ments due to rotations, irrespective of translational
displacements of the body.

62 This formula was given in A.I. Lur'ye's book [31].

67 Here we set forth a solution the idea for which was sug-
gested by R. Saussure [8].

Chapter 5

ELEMENTS OF THE DIFFERENTIAL GEOMETRY OF THE RULED SURFACE AND CERTAIN RELATIONSHIPS OF THE KINEMATICS OF THE STRAIGHT LINE AND THE SOLID BODY. COMPLEX SCALAR FUNCTIONS AND SCREW FUNCTIONS OF A VECTOR ARGUMENT

§1. The Screw as a Function of a Scalar Argument

Let a screw R be referred to a fixed rectangular coordinate system, and let 'ts complex rectangular coordinates be functions of a certain real scalar parameter t. Then screw R will be a function of t:

$$R = R(t).$$

The screw changes when t changes. When the argument changes from a value t to $t + \Delta t$, the screw acquires a screw increment ΔR, which is added to R:

$$R + \Delta R = R(t + \Delta t).$$

In our nomenclature, the derivative of screw R is the limit of the ratio of the screw increment to the argument increment when the latter approaches zero:

$$\lim_{\Delta t \to 0} \frac{R(t + \Delta t) - R(t)}{\Delta t} = \lim \frac{\Delta R}{\Delta t} = \frac{dR}{dt} = \dot{R}. \tag{5.1}$$

The rules for differentiation of screws are the same as the rules for differentiation of vectors, since a screw can be reduced to a motor and the motor regarded as a complex vector. Thus,

$$\frac{d}{dt}(R_1 + R_2) = \frac{dR_1}{dt} + \frac{dR_2}{dt}, \tag{5.2}$$

$$\frac{d}{dt}(\Lambda R) = \Lambda \frac{dR}{dt}; \quad \Lambda = \text{const.} \tag{5.3}$$

In exactly the same way, we can demonstrate validity of the formulas

$$\frac{d}{dt}(R_1 \cdot R_2) = \frac{dR_1}{dt} \cdot R_2 + R_1 \frac{dR_2}{dt} = \dot{R}_1 \cdot R_2 + R_1 \cdot \dot{R}_2, \tag{5.4}$$

$$\frac{d}{dt}(R_1 \times R_2) = \frac{dR_1}{dt} \times R_2 + R_1 \times \frac{dR_2}{dt} = \dot{R}_1 \times R_2 + R_1 \times \dot{R}_2, \tag{5.5}$$

$$\frac{d}{dt}(R_1 R_2 R_3) = \dot{R}_1 R_2 R_3 + R_1 \dot{R}_2 R_3 + R_1 R_2 \dot{R}_3. \qquad (5.6)$$

The following particular cases may arise in differentiation of a screw:

a) The axis of the screw remains in the same position, and only the complex modulus changes. Then

$$\frac{dR}{dt} = \frac{d}{dt}(ER) = E\frac{dR}{dt} = E\frac{d(re^{\omega p})}{dt} = E(\dot{r} + \omega \dot{p}r)e^{\omega p} =$$
$$= Ere^{\omega p}\left(\frac{\dot{r}}{r} + \omega \dot{p}\right) = Ere^{\omega p}\frac{\dot{r}}{r}e^{\omega \frac{\dot{p}}{\dot{r}}r} = R\Lambda, \qquad (5.7)$$

i.e., the derivative is a screw coaxial with the given screw.

b) The axis of the screw changes position in space and the complex modulus is constant. In this case

$$R^2 = \text{const}, \quad \frac{d}{dt}(R^2) = 2R \cdot \dot{R} = 0, \qquad (5.8)$$

from which it follows that screws R and $\dot{R}$ intersect at right angles.

The function

$$S(t) = \int R(t) \, dt \qquad (5.9)$$

will be called the indefinite integral of the function $R(t)$ if

$$\frac{dS}{dt} = R(t).$$

The function $S(t)$ is determined to within a constant term, which is the screw.

The screw

$$\int_A^B R(t) \, dt = S(B) - S(A) \qquad (5.10)$$

is the definite integral. As in ordinary vector analysis, the generally familiar properties of integrals are also preserved in screw calculus. Thus, the integral of a sum is equal to a sum of integrals, and a constant multiplier can be taken out of the integrand.

§2. The Spherical Curve

Let us recall the basic relationships of the differential geometry of a space curve, limiting the discussion to the particular case in which the curve lies on a sphere of unit radius.

Let a be a point of a curve whose radius vector with respect

to the center O of the sphere will be $\vec{Oa} = r$, then $|r| = r = 1$.

If t is an arbitrary parameter, then the equation

$$r = r(t)$$

is the parametric equation of the curve.

The vector defining the direction of the tangent at point a is the first derivative of r with respect to the parameter

$$\frac{dr}{dt} = \dot{r}(t).$$

As we know, the increment of the radius vector is equal at the limit to the increment of arc length; therefore,

$$\left|\frac{dr}{dt}\right| = \frac{ds}{dt}, \quad \left|\frac{dr}{ds}\right| = 1. \tag{5.11}$$

It follows from this that

$$ds = \sqrt{|dr|^2} = dt\sqrt{|\dot{r}|^2} \tag{5.12}$$

and the arc length

$$s = \int_0^t dt\sqrt{|\dot{r}|^2}. \tag{5.13}$$

where the plus sign must be taken in front of the radical. On the basis of (5.11), we have

$$\frac{dr}{dt} = \tau\frac{ds}{dt}, \quad \frac{dr}{ds} = \tau, \tag{5.14}$$

where τ is the unit vector of the tangent at point a. Thus, introducing the parameter s instead of t, we find that the derivative of the radius vector with respect to this parameter is a unit vector directed along a tangent. Since $|r| = r = 1 = \text{const}$, the direction of the vector τ is perpendicular to that of the vector r.
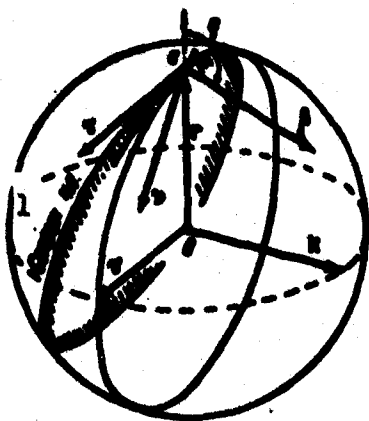
Fig. 21. 1) Osculating plane.

We shall call the plane passing through the center O of the sphere, point a, and the tangent vector the central plane; its intersection with the sphere forms a great circle (Fig. 21); the normal to the curve at point a, which is perpendicular to the central plane, will be called the central normal to the curve. We shall denote the latter's unit vector by k. The triplet of semiaxes on which the unit vectors r, τ and k lie will be called the trihedron of radius vector r. Let us place this trihedron at the center O of the sphere.

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Moving the point a along the curve, we shall vary r, τ and k; the vector τ and its increment define the osculating plane in which the principal normal at point a lies. We shall denote the unit vector of the principal normal by ν, the normal to the curve at point a, which is perpendicular to the tangent and to the principal normal, will be called the binormal; we shall denote its unit vector by β. The triplet of semiaxes on which vectors τ, ν and β lie will be called the natural trihedron of the curve at point a.

In Fig. 21, the osculating plane is intersected on a circle whose plane is indicated by shading; the natural trihedron is placed at point a.

As point a moves along the curve, the variation of the vectors τ, ν, β is determined by the familiar Frenet formulas:

$$\left. \begin{aligned} \frac{d\tau}{ds} &= \frac{\nu}{\rho}, \\ \frac{d\nu}{ds} &= -\frac{\tau}{\rho} + \frac{\beta}{\rho}, \\ \frac{d\beta}{ds} &= -\frac{\nu}{\rho}. \end{aligned} \right\} \tag{5.15}$$

These formulas describe the motion of the natural trihedron along the curve. The kinematic interpretation of these formulas is as follows: the trihedron performs two rotations: one about the binormal, the derivative of whose angle with respect to the arc has an absolute value equal to the curvature $1/\rho_1$ of the curve, where ρ_1 is the radius of the curve, and another about the tangent, the derivative of whose angle with respect to the arc has an absolute value equal to the torsion of the curve, $1/\rho_2$, where ρ_2 is the radius of torsion. On addition, these two motions define the motion of the ends of the trihedron vectors, whose origins are placed at point O.

Let us denote by ds' the element of the arc described by the end of vector τ; we then have

$$\frac{d\tau}{ds} = \nu, \quad \left| \frac{d\tau}{ds} \right| = 1; \tag{5.16}$$

on the basis of Formulas (5.15) and (5.16), we have

$$\frac{ds'}{ds} = \frac{1}{\rho},$$

i.e., the curvature is the derivative of arc s' with respect to s.

It follows from (5.15) that

$$\frac{1}{\rho} = +\sqrt{\left| \frac{d\tau}{ds} \right|}$$

The relative positions of the radius-vector and natural trihedra are determined as follows. We denote the angle between the radius vector r and the unit vector β of the binormal, or, what
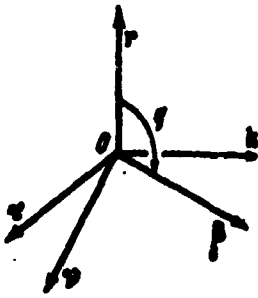
Fig. 22

is the same thing, the angle between the unit vectors k and v of the central normal and principal normal, by q (Fig. 22):

$$q = \angle(r, \beta) = \angle(k, v).$$

We then have

$$\left. \begin{array}{ll} r \cdot \beta = \cos q, & r \cdot v = -\sin q, \\ k \cdot \beta = \sin q, & k \cdot v = \cos q. \end{array} \right\} \quad (5.17)$$

On the basis of Formulas (5.15) and (5.16)

$$r \cdot \frac{dk}{ds} = \frac{r \cdot v}{\rho_1} = -\frac{\sin q}{\rho_1}.$$

On the other hand, differentiating the equality $r \cdot \tau = 0$ with respect to s, we obtain

$$\frac{d}{ds}(r \cdot \tau) = 1 + r \cdot \frac{v}{\rho} = 1 - \frac{\sin q}{\rho} = 0,$$

from which

$$\rho_1 = \sin q, \tag{5.18}$$

i.e., the radius of curvature is equal to the sine of the angle between the radius vector and the binormal. The section shown in Fig. 21 as being cut on the unit sphere by the osculating plane does not coincide with the section cut by the central plane; as these planes move closer together, the angle q will tend to $\pi/2$ and the radius of curvature will tend to unity.

Differentiating the equality

$$r \cdot \beta = \cos q$$

with respect to s, we obtain

$$\frac{dr}{ds} \cdot \beta + r \cdot \frac{d\beta}{ds} = \tau \cdot \beta - \frac{r \cdot v}{\rho} = -\frac{dq}{ds}\sin q,$$

from which, remembering that $\tau \cdot \beta = 0$, and that $r \cdot v = -\sin q$, we find

$$\frac{1}{\rho_2} = -\frac{dq}{ds}, \tag{5.19}$$

i.e., the torsion is equal in magnitude but opposite in sign to the derivative of the angle q with respect to s.

We introduce a fixed system of rectangular cartesian coordinates and examine the projections of the vectors τ, v, β onto the axes of this system:

$$\tau_x, \tau_y, \tau_z; \quad v_x, v_y, v_z; \quad \beta_x, \beta_y, \beta_z.$$

It is obvious that each triplet of these numbers with iden-

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tical indices satisfies the equation system (5.15); thus the above numbers are three systems of integrals of the following differential equations:

$$\frac{dl}{ds} = -\frac{n}{\rho_1}, \quad \frac{dm}{ds} = -\frac{l}{\rho_1} + \frac{p}{\rho_2}, \quad \frac{dp}{ds} = -\frac{m}{\rho_2}, \tag{5.20}$$

where

$$l^2 + m^2 + p^2 = 1. \tag{5.21}$$

If we introduce the variable δ defined by the formula

$$\delta = \frac{l + im}{1 + p}, \quad i = \sqrt{-1}, \tag{5.22}$$

then, by differentiating δ with respect to s and applying the relationships of System (5.20), we can reduce the system to a single equation of the Riccati type:

$$\frac{d\delta}{ds} + i\frac{\delta}{\rho_1} + i\frac{\delta^2 - 1}{2\rho_2} = 0. \tag{5.23}$$

It is assumed that the functions

$$s = s(t), \quad \varphi = \varphi(t), \quad \rho_1 = \sin\varphi, \quad \rho_2 = -\frac{ds}{d\varphi} \tag{5.24}$$

are known. Equations (5.24) are intrinsic equations of the curve, since they do not contain the coordinates.

If δ is found as a result of integration of the equation, we shall be able to find l, m and n from (5.22) by separating the latter into parts containing and not containing i and applying (5.21).

To convert from cartesian coordinates to Euler angles, we follow S.P. Finikov [32] and express the vectors τ, ν, β in terms of the latter:

$$\left.\begin{array}{l}
\tau = i(\cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta) + \\
+ j(\sin\psi\cos\varphi + \cos\psi\sin\varphi\cos\theta) + k\sin\varphi\sin\theta, \\
\nu = -i(\cos\psi\sin\varphi + \sin\psi\cos\varphi\cos\theta) - \\
- j(\sin\psi\sin\varphi - \cos\psi\cos\varphi\cos\theta) + k\cos\varphi\sin\theta, \\
\beta = i\sin\psi\sin\theta - j\cos\psi\sin\theta + k\cos\theta.
\end{array}\right\} \tag{5.25}$$

If x, y and z are the rectangular coordinates of the vector r, then the vector equation

$$\frac{dr}{ds} = \tau$$

yields the following equations in projections:

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$$\frac{dx}{ds} = \cos\psi\cos\varphi - \sin\psi\cos\varphi\cos\theta,$$
$$\frac{dy}{ds} = \cos\varphi\sin\psi + \cos\psi\sin\varphi\cos\theta,$$
$$\frac{dz}{ds} = \sin\varphi\sin\theta.$$

$$(5.26)$$

If we now apply Relationships (5.25), three groups of three equations each can be obtained from the Frenet formulas (5.15) for the rectangular projections of vectors τ, ν, β. However, only three of them will be independent, so that any three of these projections, and, in particular, the projections τ_z, ν_z and β_y, may be used. On substituting them in the Frenet formulas, we obtain

$$\frac{d\theta}{ds} = \frac{1}{\rho_1}\frac{\sin\varphi}{\sin\psi},$$
$$\frac{d\psi}{ds} = \frac{1}{\rho_1}\cos\varphi,$$
$$\frac{d\varphi}{ds} = \frac{1}{\rho_2} - \frac{1}{\rho_1}\frac{\sin\varphi}{\tan\psi}.$$

$$(5.27)$$

The system of equations (5.26) and (5.27) is a Cauchy system for the unknown functions $x, y, z, \psi, \theta, \varphi$. The right-hand members of the system are assumed to be regular functions of s, so that the system admits of a unique regular solution

$$x = x(s), \; y = y(s), \ldots, \varphi = \varphi(s),$$

$$(5.28)$$

which satisfies this system and the initial conditions with $s = s_0$:

$$x = x_0, \; y = y_0, \ldots, \varphi = \varphi_0.$$

$$(5.29)$$

These initial conditions define the initial point of the curve and the initial position of the natural trihedron.

As we know, assigning the two quantities ρ_1 and ρ_2 as functions of arc length determines the curve to within its position in space, while assigning the Cauchy system fully defines the curve, with its "tie-in" to the point and to the given direction.

§3. The Ruled Surface

After this brief exposition on the differential geometry of a curve on a sphere of unit radius, we can go on to the basic concepts and relationships of the differential geometry of ruled surfaces.

A ruled surface is a surface formed by motion of a straight line. This line is known as the generator of the surface.

In analyzing the motion of a point along a spherical curve, we are also dealing with a surface, namely one described by the radius vector of the point from the center of the sphere. In this case, however, the radius vector describes a conical surface; moreover, it is sufficient to follow only the angular displacements of the natural trihedron to characterize the curve. In the

motion of a generator over a ruled surface, the unit screw of the generator executes a three-dimensional motion of general form, and to characterize the motion of the unit screw and some trihedron associated with it, it is necessary to know both the rotational and translational displacements, i.e., generally speaking, screw displacements. Nevertheless, an analogy with the spherical curve is obtained in description of the ruled surface when these screw displacements are expressed with the aid of complex quantities on the basis of the transfer principle.
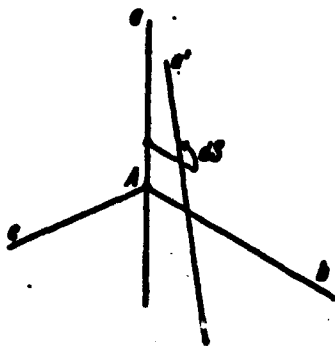
Fig. 23

Let straight line a be the generator of a ruled surface, and let the unit screw lying on a be R (Fig. 23). Let the generator vary together with a certain real parameter t; then $R = R(t)$.

Consider the generator a' corresponding to the parameter value $t + dt$; let its unit screw be R'.

We shall the complex angle (a, a') an element of the complex arc of the surface and introduce the symbol

$$dS = d\theta + \omega d\theta^0 = d\omega,$$ (5.30)

for it; here, $d\theta$ is the real angle between lines a and a', $d\theta^0$ is the shortest distance between these lines and the parameter

$$p = \frac{d\theta^0}{d\theta} = \lim \frac{\Delta\theta^0}{\Delta\theta}$$ (5.31)

is the limit of the ratio of the shortest distance $\Delta\theta^0$ between generators to the angle $\Delta\theta$ between them as the complex angle $\Delta\theta$ between the generators tends to zero. The quantity p is cllled the distribution parameter of the planes tangent to the surface at the points of its generator, or simply the parameter of the generator a.

We note that the principal part $d\theta$ of the complex-arc element of the surface is numerically equal to the length of the elementary arc of the spherical curve that would be described by the end of the unit vector of the surface generator if its origin were placed at the center of the sphere.

It is quickly seen that at the limit, the difference $R' - R = \Delta R$ is a screw dR whose complex modulus is equal to dS. Indeed, since $|R| = R = 1$,

$$\lim \sin (a, a') = \sin (dS) \approx dS = |R \times (R + dR)| =$$
$$= |R \times dR| = dR,$$

so that we have

$$|dR| = dS.$$ (5.32)

We shall denote by b the straight line passing through the axis of complex angle (a, a'), and by A and A' the points of intersection of line b with a and a'; at the limit, line b will be tangent to the surface at point A, which we shall call the center of the generator a. We shall call line b the central tangent and denote its unit screw by K.

Obviously, K may be obtained as the screw product

$$K = \frac{R \times (R + dR)}{\sin(R, R + dR)} = \frac{R \times dR}{dS}; \quad K = 1. \tag{5.33}$$

Thus, the unit screw K is perpendicular to unit screw R. Finally, we construct a vector lying on line c, which is perpendicular to a and b:

$$T = \frac{dR}{dS}. \tag{5.34}$$

According to what was said above, its modulus will be equal to unity; hence T is a unit screw. Since $R =$ const,

$$R \cdot \frac{dR}{dS} = R \cdot T = 0,$$

i.e., the screw dR/dS intersects screw R at right angles; further, performing scalar multiplication of T by K, we obtain

$$T \cdot K = \frac{dR}{dS} \cdot \left(R \times \frac{dR}{dS} \right) = 0.$$

and, consequently, unit screw T intersects R and K at right angles at point A. Line c, the axis of unit screw T, will be called the central normal to the surface.

The geometric locus described by the central normal will be known as the normalia.

The geometric locus of the centers of the generator is known as the line of striction of the surface (or throat line).

The triplet of unit screws R, T and K with a common origin at point A forms a trihedron, which we shall call the generator trihedron. It is easily seen that the unit screw R plays the same role for the surface as the radius vector r does for a spherical curve; the unit screw T of the central normal corresponds to the vector τ of the tangent to the curve, while the unit screw K c the central tangent corresponds to the vector k of the central normal of the curve.

Setting

$$\frac{dR}{dt} = k,$$

we find the following expression for dS:

$$dS = dt \sqrt{|\overline{R}|^2}. \qquad (5.35)$$

In our terms, the complex arc of a surface will be the quantity

$$S = \int_0^t dt \sqrt{|\overline{R}|^2}. \qquad (5.36)$$

where we arbitrarily take the plus sign in front of the radical.

Let (Fig. 24) c be the central normal to the surface, and let is also be the generator of the latter's normalia, i.e., also the generator of a certain surface. Consequently, line c has its own center, which we shall denote by the letter B; the geometric locus of these centers will be the line of striction of the normalia.

We construct at point B the central normal to the c-surface and the central tangent to this same surface. We shall call the former the principal normal to the surface and the latter the binormal of the surface; point B will be called the center of curvature of the a-surface at point A.

We shall denote the unit screws of the principal normal and binormal by N and B, respectively; then we shall have at point B a triplet of unit screws T, N, B; the three half-lines on which they lie will be called the natural trihedron of the surface. This trihedron is perfectly analogous to the same trihedron for a curve.
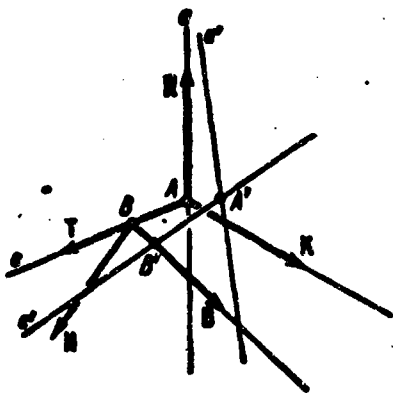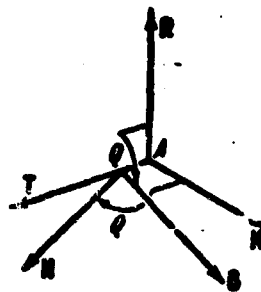
Fig. 24   Fig. 25

We denote by

$$dS' = ds' e^{-i\varphi'} \qquad (5.37)$$

the element of arc described by unit screw T, i.e., the elementary arc of the normalia.

Since the principal normal is at the same time the central normal to the normalia, there exists for N and T a relationship

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analogous to (5.34) between T and R, i.e.,

$$N = \frac{dT}{dS}, \quad N \cdot T = 0. \tag{5.38}$$

Since B is the unit screw of the central tangent to the normalia,

$$B \cdot T = 0, \ B \cdot N = 0.$$

Let us determine the relative positions of the generator and natural trihedra. Let (Fig. 25)

$$Q = \angle (R, B) = \angle (K, N)$$

be the complex angle between the unit screws of the generator and the binormal, or, what is the same thing, between the unit screws of the central tangent and the principal normal. Since the generator and natural trihedra have an axis in common — the central normal — the angle Q fully characterizes the relative inclination of one trihedron to the other.

We find that

$$\left. \begin{array}{l} R \cdot B = \cos Q, \ R \cdot N = -\sin Q, \\ K \cdot B = \sin Q, \ K \cdot N = \cos Q. \end{array} \right\} \tag{5.39}$$

We shall call the complex angle Q the measure of curvature of the surface. On the basis of Formula (5.38), we have

$$\frac{dT}{dS} = \frac{dT}{dS'} \frac{dS'}{dS} = N \frac{dS'}{dS}. \tag{5.40}$$

from which, on the basis of (5.39), we obtain

$$R \cdot \frac{dT}{dS} = R \cdot N \frac{dS'}{dS} = -\frac{dS'}{dS} \sin Q.$$

Further, differentiating the formula $R \cdot T = 0$ with respect to S, we obtain

$$R \cdot \frac{dT}{dS} + T \cdot \frac{dR}{dS} = 0,$$

and since $\frac{dR}{dS} = T$, we have

$$-\frac{dS'}{dS} \sin Q + 1 = 0,$$

whence

$$\frac{dS'}{dS} = \frac{1}{\sin Q}. \tag{5.41}$$

The ratio dS'/dS is the ratio of the rate of change of the unit screw T of the central normal to the rate of change of unit screw R — the generator — and characterizes the curvature of the

surface. We may therefore set

$$\frac{dS}{dS} = \frac{1}{P_1},$$
(5.42)

where P_1 is the radius of curvature of the surface. We have

$$P_1 = \sin Q.$$
(5.43)

Now (5.40) can be rewritten

$$\frac{dT}{dS} = \frac{1}{P_1} N.$$
(5.44)

Differentiating the relation

$$B = T \times N$$

with respect to S, we obtain

$$\frac{dB}{dS} = \frac{dT}{dS} \times N + T \times \frac{dN}{dS} = T \times \frac{dN}{dS}.$$
(5.45)

Since the screw defined by Formula (5.45) simultaneously inter-
sects B and T at right angles, its axis coincides with the axis N,
so that we may write

$$\frac{dB}{dS} = aN.$$
(5.46)

On the other hand, on differentiating one of Equalities (5.39)
with respect to S, we obtain

$$\frac{d}{dS}(R \cdot B) = \frac{dR}{dS} \cdot B + R \cdot \frac{dB}{dS} = -\frac{dQ}{dS} \sin Q$$

or, on the basis of (5.39) and (5.46),

$$R \cdot Na = -\frac{dQ}{dS} \sin Q,$$

from which

$$a = -\frac{dQ}{dS}$$

and, consequently,

$$\frac{dB}{dS} = N \frac{dQ}{dS}.$$
(5.47)

The quantity dQ/dS determines the rapidity of change of the angle
of the binormal to the generator in motion along the surface, and
characterizes the flexure of the surface, a quantity analogous to
the torsion or second curvature of a curve. By analogy with
(5.19), we take

$$\frac{1}{P_3} = -\frac{dQ}{dS}, \quad \frac{dB}{dS} = -\frac{1}{P_3}N. \qquad (5.48)$$

Further, differentiating the equality

$$N = B \times T$$

with respect to S, we obtain

$$\frac{dN}{dS} = B \times \frac{dT}{dS} + \frac{dB}{dS} \times T = B \times \frac{N}{P_1} + T \times \frac{N}{P_3} =$$
$$= -\frac{1}{P_1}T + \frac{1}{P_3}B. \qquad (5.49)$$

Combining Formulas (5.44), (5.48) and (5.49), we obtain a system of complex Frenet formulas for the ruled surface:

$$\left. \begin{array}{l} \frac{dT}{dS} = \frac{N}{P_1}, \\[2mm] \frac{dN}{dS} = -\frac{T}{P_1} + \frac{B}{P_3}, \\[2mm] \frac{dB}{dS} = -\frac{N}{P_3}. \end{array} \right\} \qquad (5.50)$$

The Frenet formulas of the ruled surface characterize the following motion of the natural trihedron: the latter performs a complex rotation (rotation and slip) about the unit screw of the binormal B, whose complex angle has a derivative with respect to the complex arc of the surface whose absolute value is equal to the curvature of the surface, and a complex rotation about the unit screw of the central normal T, whose complex angle has a derivative with respect to the complex arc of the surface whose absolute value is equal to the flexure (second curvature) of the surface.

Formulas analogous to the Frenet formulas can be derived for the motion of the generator trihedron. Thus, first of all, we have Formula (5.34); then, expressing the vector N in terms of the vectors R and K,

$$N = -R \sin Q + K \cos Q,$$

we shall have on the basis of the second of the Frenet formulas and (5.43)

$$\frac{dT}{dS} = \frac{N}{P_1} = \frac{N}{\sin Q} = -R + K \operatorname{ctg} Q. \qquad (5.51)$$

and then, differentiating the equality

$$K = R \times T,$$

we find

$$\frac{dK}{dS} = \frac{dR}{dS} \times T + R \times \frac{dT}{dS} = R \times K \operatorname{ctg} Q = -T \operatorname{ctg} Q. \qquad (5.52)$$

Combining (5.33), (5.51) and (5.52), we obtain the system of relationships

$$
\left.
\begin{aligned}
\frac{dR}{dS} &= \quad T, \\
\frac{dT}{dS} &= -R \;\; + K \, \text{ctg} \, Q, \\
\frac{dK}{dS} &= \quad\;\; -T \, \text{ctg} \, Q.
\end{aligned}
\right\}
\tag{5.53}
$$

This system gives an indication as to the elementary displacement of the generator trihedron. Namely, this displacement consists of two screw displacements — one, dS, with respect to K, and another, $-dS \cot Q = dS^*$, with respect to R. If we add these two displacements and apply (5.41), we obtain

$$
K \, dS - R \, dS \, \text{ctg} \, Q = -N \frac{dS}{\sin Q} = -N \, dS',
\tag{5.54}
$$
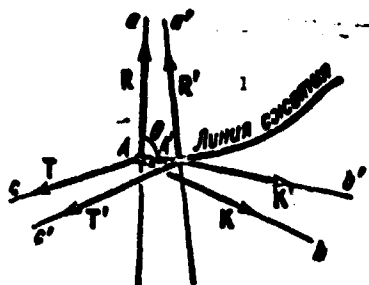
Fig. 26. 1) Line of striction.

from which it is seen that the elementary displacement of the trihedron is a screw displacement dS' with respect to the binormal.

The elementary motion of the generator and central tangent of the surface can be represented as follows. Let R, T, K and R', T', K' be the unit screws of two infinitesimally close trihedra of the surface generator (Fig. 26). The vertices A and A' of the trihedra are points infinitesimally close together on the line of striction. The element $AA' = d\sigma$ is the element of the line of striction and $\theta = \angle (R, \overrightarrow{AA'})$ is the real angle between the central tangent and the tangent to the line of striction. To bring the figure R, T, K and the figure $R'T'K'$ into coincidence, it is necessary to rotate the former about K through a complex angle dS accurate to within second-order infinitesimals, at which point R coincides with R', and then rotate it about R' through a complex angle $-dS \cot Q$.

The (linear) displacement with respect to element AA' along the line of striction is composed of (linear) displacements along K and R that are equal to the moment parts of the screw displacements

$$
dS = ds\omega^0, \quad dS^* = ds^*\omega^0,
$$

so that

$$
\frac{\text{mom}(dS^*)}{\text{mom}(dS)} = \text{ctg}\,\theta = \frac{r\,dr}{\rho\,dS} = \frac{\left(\rho - \frac{1}{\sin q \, \cos q}\right)d\sigma \, \text{ctg}\,q}{\rho \, d\sigma} =
$$
$$
= \text{ctg}\,q - \frac{r}{\rho \sin^2 q}.
\tag{5.55}
$$

Formula (5.55) expresses the relation between the angle formed by the tangent to the line of striction with the central tangent, the generator distribution parameter, and the angle between the generator and binormal. This formula is essential for study of the axoids of a moving solid body.

Introducing the complex rectangular coordinates of the unit screws T, N and B, we can write, on the basis of (5.50), three groups of equations for the nine quantities

$$T_s, T_p, T_m, N_s, N_p, N_s, B_s, B_p, B_s.$$

Of these quantities, each group of three with the same index satisfies the system of differential equations

$$\frac{dL}{dS} = \frac{M}{P_1}, \quad \frac{dM}{dS} = -\frac{L}{P_1} + \frac{P}{P_3}, \quad \frac{dP}{dS} = -\frac{M}{P_3}, \quad (5.56)$$

with

$$L^2 + M^2 + P^2 = 1. \quad (5.57)$$

Introducing, by analogy with (5.22), the new variable

$$\Delta = \frac{L + iM}{1 + P}, \quad i = \sqrt{-1}, \quad (5.58)$$

we reduce System (5.56) to a single complex equation of the Riccati type, which is similar to Eq. (5.23):

$$\frac{d\Delta}{dS} + i\frac{\Delta}{P_1} + i\frac{\Delta^2 - 1}{2P_3} = 0. \quad (5.59)$$

The functions

$$S = S(t), \quad Q = Q(t), \quad P_1 = P_1(t), \quad P_3 = P_3(t) \quad (5.60)$$

must be known; Equalities (5.60) represent the intrinsic equations of the ruled surface, which do not contain coordinates. Assigning the above functions defines the ruled surface accurate to its position in space.

Use may be made of the complex Euler angles Ψ, θ and ϕ, in terms of which the components of vectors T, N and B are expressed, with the aid of formulas similar to (5.25):

$$\left.\begin{array}{l}
T = i(\cos\Psi\cos\phi - \sin\Psi\sin\phi\cos\theta) + \\
\quad + j(\sin\Psi\cos\phi + \cos\Psi\sin\phi\cos\theta) + k\sin\phi\sin\theta, \\
N = -i(\cos\Psi\sin\phi + \sin\Psi\cos\phi\cos\theta) - \\
\quad - j(\sin\Psi\sin\phi - \cos\Psi\cos\phi\cos\theta) + k\cos\phi\sin\theta, \\
B = i\sin\Psi\sin\theta - j\cos\Psi\sin\theta + k\cos\theta.
\end{array}\right\} \quad (5.61)$$

The equation

$$\frac{d\mathbf{R}}{dS} = T,$$

is reduced by taking complex rectangular coordinates of the unit screws R and T, to the system

$$
\left.
\begin{array}{l}
\dfrac{dX}{dS} = \cos\Psi\cos\Phi - \sin\Psi\sin\Phi\cos\Theta, \\[2mm]
\dfrac{dY}{dS} = \sin\Psi\cos\Phi + \cos\Psi\sin\Phi\cos\Theta, \\[2mm]
\dfrac{dZ}{dS} = \sin\Phi\sin\Theta.
\end{array}
\right\}
\qquad (5.62)
$$

Using the Frenet formulas (5.50) and Expressions (5.61), taking three independent ones, namely, the expressions for T_x, N_z and B_y in terms of the complex Euler angles, we obtain

$$
\left.
\begin{array}{l}
\dfrac{d\Psi}{dS} = \dfrac{1}{P_1}\dfrac{\sin\Phi}{\sin\Theta}\,; \\[2mm]
\dfrac{d\Theta}{dS} = \dfrac{1}{P_1}\cos\Phi, \\[2mm]
\dfrac{d\Phi}{dS} = \dfrac{1}{P_3} - \dfrac{1}{P_1}\dfrac{\sin\Phi}{\tan\Theta}\,.
\end{array}
\right\}
\qquad (5.63)
$$

The system of equations (5.62) and (5.63) is a Cauchy system for the unknown complex functions $X, Y, Z, \Psi, \Theta, \Phi$, and is analogous to System (5.26) and (5.27) for the curve. The right-hand members of the system are assumed to be regular functions of S, and the system admits of a unique regular solution

$$
X = X(S), Y = Y(S), \ldots, \Phi = \Phi(S), \qquad (5.64)
$$

which satisfies the system and the initial conditions for $S = S_0$

$$
X = X_0, Y = Y_0, \ldots, \Phi = \Phi_0. \qquad (5.65)
$$

Assigning the Cauchy system fully defines the ruled surface, establishing its initial generator and the corresponding position of the initial natural trihedron.

We note that since the equations given above may be interpreted either geometrically or kinematically, they are equations that also define the position of a solid body in arbitrary motion with consideration of data characterizing its initial position.

As we see from the above, there is full correspondence between the geometry of a curve lying on a sphere of unit radius and a ruled surface. This corollary proceeds from the transfer principle, according to which on transition to the ruled surface a point of the curve must be replaced by a straight line — the generator of this surface — and the unit radius-vector of the curve by a screw lying on the generator, this screw being subject to the condition of equality of its complex modulus to unity (this condition simultaneously expresses unit value of the modulus of the screw vector and zero value of its parameter). Actually, many theorems pertaining to the theory of the ruled surface need not be proven, since they are obtained from theorems pertaining to the spherical curve by the above substitution of objects.

By virtue of the existing correspondence, there is, with minor discrepancies, an analogy in the terms used in connection with the curve and the surface.

The discrepancy reduces to the following: the central normal to the surface corresponds to the tangent to the curve, and the central tangent to the surface to the central normal; the flexure of the surface corresponds to the torsion of the curve. Below we present a table of the corresponding geometric figures for a spherical curve of unit radius and a ruled surface.

Кривая на сфере единичного радиуса 1	Обозначение элемента 2		Линейчатая поверхность 3	Обозначение элемента 2
Точка кривой	a	5	Образующая поверхности	a
Радиус-вектор точки .	r	7	Единичный винт (вектор) образующей	R
Касательная	τ	9	Центральная нормаль . .	T
Центральная нормаль	λ	11	Центральная касательная	K
Главная нормаль . . .	ν	12	Главная нормаль	N
Бинормаль	β	13	Бинормаль	B
Элемент дуги кривой	ds	15	Элемент комплексной дуги поверхности	dS
Элемент дуги касательной	$d\tau$	17	Элемент комплексной дуги центральной нормали	dT
Радиус кривизны . .	ρ_1	18	Радиус кривизны	P_1
Радиус кручения . .	ρ_2	20	Радиус изгиба	P_2

1) Curve on sphere of unit radius; 2) symbol for element; 3) ruled surface; 4) point of curve; 5) generator of surface; 6) radius vector of a point; 7) unit screw (vector) of a generator; 8) tangent; 9) central normal; 10) central normal; 11) central tangent; 12) principal normal; 13) binormal; 14) element of arc of curve; 15) element of complex arc of surface; 16) element of arc of tangent; 17) element of complex arc of central normal; 18) radius of curvature; 19) radius of torsion; 20) radius of flexure.

If we take the principal parts of all formulas pertaining to the ruled surface, they agree with the corresponding formulas of the spherical curve; this spherical curve will be described by the end of a unit vector whose origin is at a fixed point O and which will be parallel to the unit vector of the generator of this surface at corresponding values of t.

§4. Kinematics of a Straight Line and a Solid Body

Let a straight line a with a unit screw R move in space, describing a certain surface, which we shall call the trajectory of line a. Let the various positions of the line and, consequently, of the unit screw R be functions of time t.

The screw

$$V = \frac{dR}{dt} \tag{5.66}$$

will be called the velocity of line a. The screw

$$W = \frac{d^2 R}{dt^2}$$

(5.67)

will be called the acceleration of line a.

Transforming (5.66), we shall have

$$V = \frac{dR}{dt} = \frac{dR}{dS}\frac{dS}{dt} = T\frac{dS}{dt};$$

$$|V| = V = \frac{dS}{dt}, \quad V = TV.$$

(5.68)

From this there follows a theorem.

Theorem 15. *The screw whose complex modulus is equal to the time derivative of the complex element S of the trajectory, while its axis is the central normal to the trajectory, is the velocity of a line. The velocity parameter is equal to the parameter of the line.*

We transform Expression (5.67):

$$\frac{d^2 R}{dt^2} = \frac{d}{dt}\left(\frac{dR}{dt}\right) = \frac{d}{dt}(TV) = T\frac{dV}{dt} + V\frac{dT}{dS}\frac{dS}{dt} =$$

$$= T\frac{dV}{dt} + V^2\frac{dT}{dS}.$$

On the basis of Formula (5.44), we obtain for the derivative of unit screw T

$$W = \frac{d^2 R}{dt^2} = T\frac{dV}{dt} + V^2\frac{N}{P_1}.$$

(5.69)

Consequently, the theorem has been proven.

Theorem 16. *The acceleration of a line is the sum of two screws: the complex modulus of one of them is equal to the time derivative of the modulus of the velocity of the line, while the central normal serves as its axis; the complex modulus of the other is equal to the square of the modulus of the velocity divided by the radius of curvature of the surface, and the principal normal serves as its axis.*

Formula (5.69) is an analogue of the familiar formula for resolution of the acceleration of a point into tangential and normal components.

Let a certain solid body have at time t an instantaneous motion characterized by the screw U, whose unit screw will be E, its modulus U, and its parameter p. Thus, the instantaneous kinematic screw of the body will be

$$U = EU = Eue^{\omega p}.$$

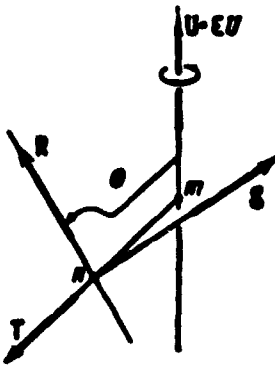Let us determine the velocity of an arbitrary line belonging to the body. Let the unit screw of this line be R.

We shall denote the complex angle between B and R (Fig. 27) by θ; let the axis of the angle θ be a straight line meeting axes B and R at points m and n, respectively, and let its unit screw be T. We pass through point n a straight line perpendicular to mn and to R and denote the unit screw of this line by S.

Now we determine the components of screw U along the axes R and S. By virtue of the perpendicularity of the corresponding unit vectors, the sum of these components will give the screw U (see Chapter 3). We have

$$ U = U' + U'' = RU' + SU'' = RU \cos\theta + SU \sin\theta. $$

The first of the component screws will not change the position of the axis, i.e., the axis of the straight line under consideration, while the second will impart to this line a screw displacement with respect to the axis S characterized by the complex element

$$ dS = ds + \omega\, ds' = U \sin\theta\, dt, $$

from which it follows that the complex modulus of the velocity of a line of the body is

$$ V = \left|\frac{dR}{dt}\right| = \frac{dS}{dt} = U \sin\theta. $$

It follows from the above construction that if R is regarded as the unit screw of the generator of the element of the surface described by the line, then S is the unit screw of the central tangent and T is the unit screw of the central normal. But the axis of the velocity screw V coincides, as we know, with the central normal, i.e., with the axis of the angle between the axes of screws U and R, while the complex modulus of the screw product of these screws will be $U \sin\theta$. Consequently,

$$ V = U \times R = TU \sin\theta = T e^{\omega p} \sin\theta, \qquad (5.70) $$

from which the following theorem proceeds.

Theorem 17. For an instantaneous screw motion of a solid body characterized by screw U, the velocity of any straight line of the body is a screw equal to the screw product of screw U by the unit screw R of this line.

Corollary 1. The central normal to the trajectory of the line meets the axis of instantaneous screw U at right angles. The central normals of the trajectories of all lines of the body form a brush at time t.

Corollary 2. The parameter of the line, i.e., its distribution parameter as the generator of the trajectory, is determined

by the formula

$$\rho' = \frac{d\rho}{d\theta} = \rho + \theta^0 \operatorname{ctg} \theta. \qquad (5.71)$$

This formula is obtained from (5.70) by equating the parameters of the two members.

Let U be an instantaneous screw characterizing the motion of a body A with respect to stationary space and let R be an instantaneous screw characterizing the motion of a certain body B with respect to body A. We visualize two coordinate systems: one fixed, and the other associated with the moving body A. Let us find the relation between the derivative of screw R with respect to the fixed system of coordinates, i.e., the absolute derivative, and the derivative of this screw in the coordinate system attached to moving body A, i.e., the relative derivative (or "apparent" derivative, as it appears to an observer on body A).

This problem is solved by direct application of the transfer principle to the familiar theorem of the absolute and relative derivatives of vector r. According to this theorem,

$$\frac{dr}{dt} = \frac{d'r}{dt} + u \times r,$$

where d/dt is the symbol for the absolute derivative and d'/dt is the symbol of the derivative with respect to the coordinate system whose angular-velocity derivative is u. The vector r may represent various physical quantities. For example, if the above moving coordinate system is associated with a certain solid body a that is in rotation at an angular velocity u, the vector r may represent the angular velocity of another body b with respect to body a (provided that the vectors u and r have a common point). In this case, the theorem gives the relation between the absolute increment of vector r and its increment with respect to the moving body a.

Substituting the screws R and U for the vectors r and u in the above formula and remembering that the conditions of the problem posed correspond exactly to the condition of the above theorem with screws substituted for the vectors (or with screw displacements substituted for the pure rotations), we can write the relationship that we seek:

$$\frac{dR}{dt} = \frac{d'R}{dt} + U \times R. \qquad (5.72)$$

Here dR/dt is the absolute derivative of screw R and $d'R/dt$ is the relative derivative.

In the particular case in which screw R is unchanged in the coordinate system attached to solid body A, the formula assumes the form

$$\frac{dR}{dt} = U \times R. \qquad (5.73)$$

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i.e., the derivative of an instantaneous screw R that retains the same value with respect to a moving body whose instantaneous screw is U is expressed by the screw product $U \times R$. We again arrive at Theorem 17 — the kinematic interpretation of the screw product of two screws.

If screw U is defined by complex rectangular coordinates $U_x, U_y, U_z,$ and screw R by the coordinates $R_x, R_y, R_z,$ then the expressions for the complex coordinates of the rate of change of screw R (or of the straight line of the solid body) will be

$$V_x = U_y R_z - U_z R_y, \quad V_y = U_z R_x - U_x R_z, \quad V_z = U_x R_y - U_y R_x. \qquad (5.74)$$

These formulas represent a generalization of the familiar Euler formulas for the projection of the velocity of a point of a body rotating about a fixed point.

At each point in time during motion of the solid body there exists a straight line with respect to which an instantaneous screw motion of the body is taking place. This line is called the instantaneous screw axis. In continuous motion of a body, the position of the instantaneous screw axis varies and it describes a ruled surface — a fixed axoid — in space. At the same time, the line of the body that coincides at time t with this line, moving together with the body, describes another ruled surface, a moving axoid, in it.

During motion of the body, as we know, the moving and fixed axoids a_2 and a_1 are in contact with one another at each point in time along a common generator a_{12} — the instantaneous screw axis. At time t, two infinitesimally close generators a_2 and a'_2 of the moving axoid coincide with two infinitesimally close generators a_1 and a'_1 of the stationary axoid. If

$$dS_1 = \angle (a_1, a'_1), \quad dS_2 = \angle (a_2, a'_2),$$

then, as we know,

$$dS_1 = dS_2 = dS = ds e^{\omega p}.$$

i.e., surface elements of the two axoids are equal; moreover, the generators a_1 and a_2 have a common center A. It is known that later, in the time interval dt that follows t, the generator a'_2 of the moving axoid slides along the generator a'_1 of the stationary axoid until their centers A'_2 and A'_1 coincide, and rotates about a' until the corresponding generator trihedra coincide.

Let (Fig. 28) b_{12} be the common central tangent of the axoids at time t, let b'_1, b'_2 be the central tangents to them at points A'_1 and A'_2, and let the arc

$$dW = dw e^{\omega p}$$

be the elementary screw displacement of the moving axoid at time $t + dt$ with respect to the common generator of a'_1, a'_2. If dS_2 and
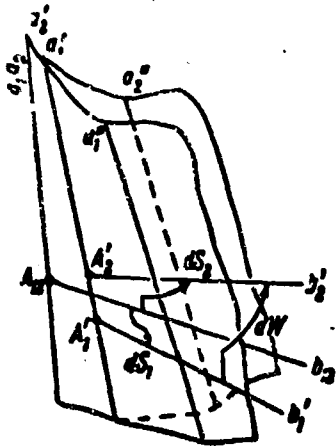
Fig. 28

dS_1 pertain to the moving and stationary axoids, we shall have, considering elementary displacements of the generator trihedra,

$$dW = dS_1 \operatorname{ctg} Q_1 \pm$$
$$\pm dS_2 \operatorname{ctg} Q_2 = dw^{\ldots} =$$
$$= ds^{\ldots} (\operatorname{ctg} Q_1 \pm \operatorname{ctg} Q_2).$$

where Q_1 and Q_2 are the angles between the generators and binormals of the axoids.

Taking the parameters of both parts, we obtain

$$\varkappa = p + P(\operatorname{ctg} Q_1 \pm \operatorname{ctg} Q_2) =$$
$$= p - \frac{\dfrac{q_1}{\sin^2 q_1} \pm \dfrac{q_2}{\sin^2 q_2}}{\operatorname{ctg} q_1 \pm \operatorname{ctg} q_2}.$$

Substituting the values of q_1^0 and q_2^0 expressed in terms of $q_{1, 2}$, $\theta_{1, 2}$ in the above, we obtain on the basis of the formula derived earlier (5.55)

$$\varkappa = p \frac{\operatorname{ctg} \theta_1 \pm \operatorname{ctg} \theta_2}{\operatorname{ctg} q_1 \pm \operatorname{ctg} q_2}, \tag{5.75}$$

at which point the following theorem can be formulated.

Theorem 18. *In arbitrary motion of a body, its moving axoid will roll along the stationary axoid in such a way as to maintain continuous coincidence of pairwise-equal successive elements of the complex arcs of the surfaces of the two axoids. At each point in time, the common generator of the axoids will serve as the axis of the instantaneous screw, and the parameter π will depend on: a) the parameter p of the common generator a_{12}, b) the angles q_1 and q_2 between the generators and binormals of the axoids, and c) the angles θ_1 and θ_2 between the generators and the tangents to the lines of striction of the axoids.*

§5. Phase Portrait of the Motion of a System with Two Degrees of Freedom by Means of a Ruled Surface

When the motion of a system with one degree of freedom is represented on the phase plane, two quantities are used — the coordinates of a point on the plane, which represent, respectively, the generalized coordinate q of the system and its generalized velocity $\dot{q}$. A four-dimensional phase space is necessary for the analogous representation of a system with two degrees of freedom.

A space of straight lines may be used as the four-dimensional phase space, since each line in the space is defined by four quantities.

It is easiest to proceed as follows.* In the space (Fig.
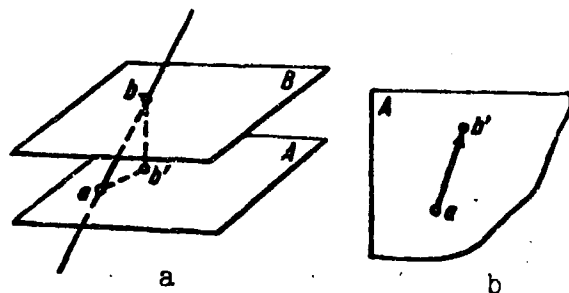
Fig. 29

29a), visualize two planes A and B at a distance H from one another. Intersecting an arbitrary line of the space, these planes cut out of it a segment ab enclosed between A and B, which segment we shall treat as a vector passing from point a to point b. We take plane A for the xy-plane and direct the z-axis from A to B. The Plücker coordinates of the vector $\overrightarrow{ab}$, which is a sliding vector, will be the projections and moments with respect to the axes

$$x, y, H, H\eta, -H\xi, \xi y - \eta x, \qquad (5.76)$$

where ξ and η are the coordinates of point a in plane A.

The above quantities are linked by an identity that expresses the perpendicularity of the vector and the moment,

$$xH\eta - yH\xi + H(\xi y - \eta x) = 0. \qquad (5.77)$$

Since the quantity H is known and serves as a scale, the following four numbers will serve as the coordinates of vector $\overrightarrow{ab}$:

$$x, y, x^0 = H\eta, \quad y^0 = -H\xi. \qquad (5.78)$$

We take these numbers as the phase coordinates of the system with two degrees of freedom, i.e.,

$$\left. \begin{array}{ll} x = q_1, & y = q_2, \\ x^0 = H\eta = \dot{q}_1, & y^0 = -H\xi = \dot{q}_2. \end{array} \right\} \qquad (5.79)$$

In this case, with motion of a system with two degrees of freedom, each system state, which is characterized by the two generalized coordinates q_1, q_2 and the two generalized velocities $\dot{q}_1, \dot{q}_2$, will correspond to a vector $\overrightarrow{ab}$, which can be represented by the component ab' in plane A (Fig. 29b). If the system moves with defined initial data, the state of the system will vary in such a way that the vector $\overrightarrow{ab}$ describes a certain ruled surface — the analogue of the phase curve of the system with one degree of freedom. With other initial data, other ruled surfaces are possible; the aggregate of them will represent a family of phase surfaces.

The coordinates of vector $\overrightarrow{ab}$ can be presented in complex form:

$$\left.\begin{array}{l} X = x + \omega x^0 = x + \omega H \eta, \\ Y = y + \omega y^0 = y - \omega H \xi, \end{array}\right\} \qquad (5.80)$$

and these coordinates will serve as the representation of the system's complex coordinates

$$\left.\begin{array}{l} q_1^\bullet = q_1 + \omega \dot{q}_1, \\ q_2^\bullet = q_2 + \omega \dot{q}_2. \end{array}\right\} \qquad (5.81)$$

It is curious to note that the complex coordinates of a holonomic mechanical system satisfy the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T^\bullet}{\partial \dot{q}_i^\bullet}\right) - \frac{\partial T^\bullet}{\partial q_i^\bullet} = Q_i^\bullet, \qquad (5.82)$$

where

$$T^\bullet = T + \omega \dot{T}, \qquad Q^\bullet = Q + \omega \dot{Q}. \qquad (5.83)$$

We may satisfy ourselves of this by assigning an arbitrary function of several time-dependent complex variables in the complex form

$$\begin{aligned} f(x_1^\bullet, x_2^\bullet, \ldots, x_n^\bullet) &= \\ &= f(x_1, x_2, \ldots, x_n) + \omega\left(\dot{x}_1 \frac{\partial f}{\partial x_1} + \dot{x}_2 \frac{\partial f}{\partial x_2} + \ldots + \dot{x}_n \frac{\partial f}{\partial x_n}\right) - \\ &= f + \omega \frac{df}{dt} = f + \omega \dot{f}. \end{aligned} \qquad (5.84)$$

Further, any complex quantity of the form $x + \omega \dot{x}$ may be regarded as a function of the complex parameter ("complex time")

$$x(t) + \omega \dot{x}(t) = x(t + \omega), \qquad (5.85)$$

as proceeds from the general expression for the function of a complex variable.

If $q_1^\bullet, q_2^\bullet$ are expressed in terms of the parameter t, then the equations

$$q_1^\bullet = q_1^\bullet(t), \qquad q_2^\bullet = q_2^\bullet(t). \qquad (5.86)$$

represent the parametric equations of the phase surface; on the other hand, if we eliminate the real parameter, the equation

$$f(q_1^\bullet, q_2^\bullet) = f(q_1 + \omega \dot{q}_1, q_2 + \omega \dot{q}_2) \qquad (5.87)$$

will be that of the phase surface in the complex coordinates of the motion.

On the mapping plane A, the phase surface will be represented

in the form of a projection – a varying region of segments ab' (Fig. 30).

If at some point in time t the equalities

$$q_1 = \dot{q}_1 = q_2 = \dot{q}_2 = 0, \qquad (5.88)$$

are satisfied, the state of the system will correspond to a "singular" point; then the region representing the phase surface will contract into a point – the coordinate origin. In this case, it would be possible to construct a geometric theory of singular points and trace the behavior of a family of phase surfaces in the neighborhood of a singular point, but we shall not dwell on this.
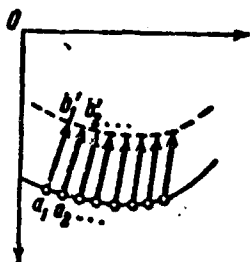
Fig. 30

By way of example, let us consider a motion whose phase surface is the surface formed by uniform rotation of a certain straight line that intersects the z-axis and forms a constant rather small angle δ with the xy-plane, and simultaneously by slip of this line along the z-axis in a harmonic manner such that two periods of oscillation along the z-axis take place during one complete revolution. This will be a surface of the cylindroid type. The region representing the corresponding phase surface will be obtained by cutting this surface with a plane A and a plane B parallel to the former and situated at a height H, and then projecting it onto plane A.

We have the coordinates of the generating line of the surface:

$$\left. \begin{array}{l} X = x + \omega x^0 = H \operatorname{ctg} \delta \cos \Phi = H \operatorname{ctg} \delta (\cos \varphi \quad \omega \varphi^0 \sin \varphi), \\ Y = y + \omega y^0 = H \operatorname{ctg} \delta \sin \Phi = H \operatorname{ctg} \delta (\sin \varphi + \omega \varphi^0 \cos \varphi), \end{array} \right\} \qquad (5.89)$$

where $\Phi = \varphi + \omega \varphi^0$ is the complex angle between the horizontal projection of the generator and the z-axis.
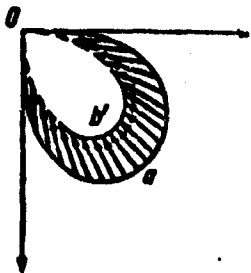
For the cylindroid (see Chapter 3),

$$\varphi^0 = K \sin 2\varphi,$$

where K is a constant. Consequently,

$$\left. \begin{array}{l} X = x + \omega r^0 = \\ = H \operatorname{ctg} \delta (\cos \varphi - \omega K \quad 2\varphi \sin \varphi), \\ Y = y + \omega y^0 = \\ = H \operatorname{ctg} \delta (\sin \varphi + \omega K \sin 2\varphi \cos \varphi). \end{array} \right\} \qquad (5.90)$$

Fig. 31

For the motion represented by the surface taken, we shall have

$$\frac{dq_1}{dq_1} = -\operatorname{ctg} \varphi, \quad \frac{q_1}{q_1} = \operatorname{tg} \varphi,$$

from which

$$\frac{dq_2}{dq_1} = -\frac{q_1}{q_2}, \quad q_2\,dq_2 + q_1\,dq_1 = 0,$$

consequently,

$$q_1^2 + q_2^2 = R^2 = \text{const.}$$

Moreover,

$$\frac{dq_1}{dt} = -HK\,\text{ctg}\,\delta\,\sin 2\varphi\,\sin\varphi.$$

On the other hand,

$$q_1 = R\cos\varphi, \quad \frac{dq_1}{dt} = -R\dot{\varphi}\sin\varphi,$$

so that

$$-HK\,\text{ctg}\,\delta\,\sin 2\varphi\,\sin\dot{\varphi} = -R\dot{\varphi}\sin\varphi,$$

and, consequently,

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{HK}{R}\text{ctg}\,\delta\,\sin 2\varphi, \quad \ln|\text{tg}\,\varphi| = \frac{2HK}{R}\text{ctg}\,t + \ln C,$$
$$\varphi = \text{arctg}\left[C\exp\left(\frac{2HK}{R}\text{ctg}\,\delta\right)t\right].$$

This gives the solution

$$q_1 = R\cos\varphi = \frac{R}{\sqrt{1 + C^2\exp\frac{4HK\,\text{ctg}\,\delta}{R}t}},$$

$$q_2 = \frac{CR\exp\frac{2HK\,\text{ctg}\,\delta}{R}t}{\sqrt{1 + C^2\exp\frac{4HK\,\text{ctg}\,\delta}{R}t}}. \tag{5.91}$$

The phase surface lies in the range $0 < \varphi < \frac{\pi}{2}$ (Fig. 31).

§6. Complex Scalar Functions and Screw Functions of a Screw Argument

It will be shown in the exposition to follow how the familiar concepts of the scalar function and vector function of a vector argument can be extended to functions of a screw argument. It is assumed that the reader is familiar with the basic definitions and formulas of the theory of scalar and vector fields.

Here, as in earlier chapters, we shall begin by assigning screws by means of motors reduced to a single common reduction point O, which is selected once and for all.

The space of the motors (r_i, r_i^0), with all r_i and r_i^0 having a

common origin at point O, is at the same time a space of point pairs, the point being the ends of the vectors r_i and the moments r_i [sic], as well as a space of complex vectors $r_i + \omega r_i^o$. Since a fully defined screw R_i can be brought into correspondence with each motor (r_i, r_i^o), a space of screws R_i each element of which is defined by its own axis, vector and parameter, stands in one-to-one correspondence with this space. Let a certain number, a complex scalar, be assigned in accordance with a certain law to each screw R_i. We shall call the function defining this assignment the complex scalar function $F(R)$ of screw R.

We introduce the following definition: the complex scalar function of screw R is the same as the function of the corresponding motor (r, r^o) at the reduction point O, which [motor] is equivalent to this screw. Expressing the motor by the complex vector, we shall have

$$F(R) = F(r + \omega r^o),$$ (5.92)

and, consequently, the screw function is reduced to a function of a complex vector.

In order to establish certain properties of the function to be determined, let us express the argument in terms of the complex coordinates of the vectors in a rectangular coordinate system with origin at point O, and then apply the formulas for functions of a complex scalar argument, which were given in Chapter 2. In so doing, we shall introduce here the condition formulated earlier for differentiation of a function of a complex scalar argument, namely, independence of the derivative of the direction of differentiation or, in other words, the condition of "analyticity."

The complex coordinates of the screw and, accordingly, of the motor reduced to point O, will be

$$R_x = r_x + \omega r_x^o, \quad R_y = r_y + \omega r_y^o, \quad R_z = r_z + \omega r_z,$$ (5.93)

where $r_x, r_y, r_z, r_x^o, r_y^o, r_z^o$ are the six real Plücker coordinates of the screw. Development of the function gives

$$F(R_x, R_y, R_z) = F(r_x + \omega r_x^o, r_y + \omega r_y^o, r_z + \omega r_z^o) =$$
$$= F(r_x, r_y, r_z) + \omega \left(r_x^o \frac{\partial F}{\partial r_x} + r_y^o \frac{\partial F}{\partial r_y} + r_z^o \frac{\partial F}{\partial r_z} \right).$$

For simplicity, we shall assume at first that the function F becomes real when the coordinates are real, so that $F(r_x, r_y, r_z)$ is a real quantity.

Returning to vector notation, we find

$$F(R) = F(r) + \omega r^o \cdot \nabla F(r) = F(r) + \omega r^o \cdot \text{grad} F(r).$$ (5.94)

In Formula (5.94), the symbol ∇ denotes the familiar hamiltonian operator.

Now let another screw be assigned to each screw R_i. The function defining this assignment will be called the screw function $F(R)$ of screw R. As has already been stated, each screw R_i uniquely defines a motor or complex vector at point O, so that the screw function defined here is simultaneously the motor function, reduced to point O, of the motor corresponding to the screw argument reduced to point O.

Thus, we shall have

$$F(R) = F(r + \omega r^0),\qquad(5.95)$$

and the function is again reduced to a vector function.

Using the coordinate expressions for the argument R and the function F, we obtain

$$F(iR_x + jR_y + kR_z) = iF_x(R_x, R_y, R_z) + jF_y(R_x, R_y, R_z) +$$
$$+ kF_z(R_x, R_y, R_z) = iF_x(r_x, r_y, r_z) + jF_y(r_x, r_y, r_z) +$$
$$+ kF_z(r_x, r_y, r_z) + \omega\Big[i\Big(r_x^0\frac{\partial F_x}{\partial r_x} + r_y^0\frac{\partial F_x}{\partial r_y} + r_z^0\frac{\partial F_x}{\partial r_z}\Big) +$$
$$+ j\Big(r_x^0\frac{\partial F_y}{\partial r_x} + r_y^0\frac{\partial F_y}{\partial r_y} + r_z^0\frac{\partial F_y}{\partial r_z}\Big) +$$
$$+ k\Big(r_x^0\frac{\partial F_z}{\partial r_x} + r_y^0\frac{\partial F_z}{\partial r_y} + r_z^0\frac{\partial F_z}{\partial r_z}\Big)\Big].\qquad(5.96)$$

Let us assume, as in the preceding case, that the function $F(R)$ reverts into a vector with its origin at point O when R reverts to r, i.e., into a vector with origin at point O. Hence $iF_x(r_x, r_y, r_z)$, $jF_y(r_x, r_y, r_z)$ and $kF_z(r_x, r_y, r_z)$ are vectors whose origins lie at point O.

The expressions in square brackets can be presented in the form

$$\Big[(ir_x^0 + jr_y^0 + kr_z^0)\cdot\Big(i\frac{\partial}{\partial r_x} + j\frac{\partial}{\partial r_y} + k\frac{\partial}{\partial r_z}\Big)\Big](iF_x + jF_y + kF_z).\qquad(5.97)$$

Going over to the vector form, we obtain

$$F(R) = F(r) + \omega(r^0\cdot\nabla)F(r).\qquad(5.98)$$

The expression $(r^0\cdot\nabla)F(r)$ is the derivative of the vector $F(r)$ in the direction of vector r^0 multiplied by the vector r^0.

Analyzing Expressions (5.94) and (5.98) for the complex scalar function and for the screw function, we note the following peculiarities of these expressions: firstly, the principal part of the function is equal to a function of the principal part of the screw (i.e., its vector) and, secondly, the screw function is fully defined by a function of its principal part.

It follows from this that if for two complex scalar functions $F(R)$ and $\Phi(R)$ and two screw functions $\boldsymbol{F}(R)$ and $\boldsymbol{\Phi}(R)$ we know the identities

$$F(r) \equiv \Phi(r), \quad \boldsymbol{F}(r) \equiv \boldsymbol{\Phi}(r),$$

they imply the identities

$$F(R) \equiv \Phi(R), \quad \boldsymbol{F}(R) \equiv \boldsymbol{\Phi}(R).$$

Consequently, if the functions F and $\boldsymbol{F}$ are assigned by analytical expressions of the coordinates of vector r, then all of the identities that obtain in the domain of these functions remain in force if the real coordinates of vector r are replaced by complex coordinates, i.e., if the vector r is replaced by screw R.

Let us consider the operator ∇^*, which is analogous to the operator ∇ and has the expression

$$\nabla^* = i\frac{\partial}{\partial R_x} + j\frac{\partial}{\partial R_y} + k\frac{\partial}{\partial R_z}. \tag{5.99}$$

Making the substitution

$$R_x = r_x + \omega r_x^0, \quad R_y = r_y + \omega r_y^0, \quad R_z = r_z + \omega r_z^0 \tag{5.100}$$

in (5.99), we obtain

$$\nabla^* = i\left(\frac{\partial}{\partial r_x} - \omega \frac{\partial}{\partial r_x}\frac{\partial r_x^0}{\partial r_x}\right) + j\left(\frac{\partial}{\partial r_y} - \omega \frac{\partial}{\partial r_y}\frac{\partial r_y^0}{\partial r_y}\right) + $$
$$+ k\left(\frac{\partial}{\partial r_z} - \omega \frac{\partial}{\partial r_z}\frac{\partial r_z^0}{\partial r_z}\right).$$

Assuming that r_x^0, r_y^0, r_z^0 are not dependent on r_x, r_y, r_z, we find that $\nabla^* = \nabla$, i.e., the "complex" hamiltonian operator is the same as the real one.

Applying this operator to the complex scalar function of a screw argument, we obtain

$$\mathbf{grad}\, F(R) = \nabla F(R) = \nabla F(r) + \omega\nabla [r^0 \cdot \nabla F(r)]. \tag{5.101}$$

For the screw function of a screw argument we shall have

$$\mathbf{div}\, \boldsymbol{F}(R) = \nabla \cdot \boldsymbol{F}(R) = \nabla \cdot \boldsymbol{F}(r) + \omega\nabla \cdot [(r^0 \cdot \nabla)\boldsymbol{F}(r)], \tag{5.102}$$

$$\mathbf{rot}\, \boldsymbol{F}(R) = \nabla \times \boldsymbol{F}(R) = \nabla \times \boldsymbol{F}(r) + \omega\nabla \times [(r^0 \cdot \nabla)\boldsymbol{F}(r)]. \tag{5.103}$$

It is seen from Expressions (5.101), (5.102) and (5.103) that differentiation of the screw functions reduces to application of the operator ∇ to a real function - the principal part of the function under consideration.

If it is known concerning the two functions $\boldsymbol{F}(R)$ and $\boldsymbol{\Phi}(R)$ that

$$\nabla \boldsymbol{F}(r) \equiv \boldsymbol{\Phi}(r),$$

then we may conclude the following identity on the basis of (5.101):

$$\nabla F(R) \equiv \Phi(R).$$

Similar conclusions can be reached for the functions $F(R)$ and $\Phi(R)$ or $\Psi(R)$ if it is known that

$$\nabla \cdot F(r) \equiv \Phi(r), \quad \nabla \times F(r) \equiv \Psi(r),$$

then it follows on the basis of (5.102) and (5.103) that

$$\nabla \cdot F(R) \equiv \Phi(R), \quad \nabla \times F(R) \equiv \Psi(R).$$

If the inverse problems are posed — those of determining the scalar from its gradient and the screw from its divergence and curl, we arrive in a similar manner at the conclusion that the solution of these problems for the principal part fully defines the solution.

The above enables us to formulate the following theorem.

Theorem 24. *All formulas and all theorems of vector analysis remain in force in the domain of screws.*

The same singular cases that are encountered in the algebra of screws exist in screw analysis: these are the cases in which the principal part of the screw vanishes. Special investigation is required for such cases.

Let us turn to a complex scalar screw function and assume that it depends on several screws $R_1, R_2, \ldots, R_n$. Omitting the almost obvious derivation from the coordinate expression for the screws, we write the final expression for the function

$$F(R_1, R_2, \ldots, R_n) = F(r_1 + \omega r_1', r_2 + \omega r_2', \ldots, r_n +$$
$$+ \omega r_n') = F(r_1, r_2, \ldots, r_n) + \omega (r_1' \cdot \nabla_1 F + r_2' \cdot \nabla_2 F + \ldots$$
$$\ldots + r_n' \cdot \nabla_n F). \qquad (5.104)$$

where the subscripts to the symbol ∇ signify that differentiation is conducted only with respect to the vector to which the subscript corresponds and that the remaining vectors are assumed constant during this process.

For a screw function of several screws $R_1, R_2, \ldots, R_n$, we obtain the following expression:

$$F(R_1, R_2, \ldots, R_n) = F(r_1 + \omega r_1', r_2 + \omega r_2', \ldots, r_n +$$
$$+ \omega r_n') = F(r_1, r_2, \ldots, r_n) + \omega [(r_1' \cdot \nabla_1) F +$$
$$+ (r_2' \cdot \nabla_2) F + \ldots + (r_n' \cdot \nabla_n) F]. \qquad (5.105)$$

Expressions (5.104) and (5.105) indicate that a function of several screws is fully defined by a function of the vectors of these screws.

Expressions (5.104) and (5.105) indicate that if we consider the variation of the functions F and F as only one of the variable

screws, for example, R_n, varies, while the remaining $n - 1$ screws are set constant, then, generally speaking, F will have a complex value, while, generally speaking, F will be a screw even if R_n reverts into a vector whose origin is point O. We can easily satisfy ourselves that the above properties of the functions F and F remain in force even in this case, i.e., we can dispense with the limiting assumption adopted at the outset according to which F was real and F was a vector with origin at point O when R becomes a vector with origin at point O.

Formulas for the scalar and screw products of two screws can be derived directly from Expressions (5.104) and (5.105), together with other relationships of screw algebra, provided that these relationships are regarded as functionals between screws.

It follows from all of the above that a screw analysis that reproduces ordinary vector analysis exactly can be constructed by substitution of screws for vectors. In this, the correspondence between geometrical objects that was established earlier is obviously preserved: the complex modulus of the screw will correspond to the modulus of the vector and the complex angle between the axes of the screws will correspond to the angle between vectors.

After having ascertained the necessary conditions for analytical notation in the expressions for functions of a screw variable, we can turn to the transfer principle, which we discussed in Chapter 4, and advance general considerations at this point concerning the conditions of applicability of this principle to solution of problems in the mechanics of the solid.

It is clear from the formulation of the transfer principle that it consists in: a) use of one-to-one correspondence between the space of motors (complex vectors) reduced to a certain point and the space of screws and b) transition from the space of vectors with common origin to a space of motors referred to this origin. One-to-one correspondence between the two spaces is a geometrical fact that remains in force through any affine orthogonal transformations, i.e., for any motions that preserve the length of the vector and the angle between two arbitrary vectors, and, consequently, this correspondence obtains for any motions of a solid body. As concerned transition from vectors to motors, on the other hand, it is accomplished with the aid of complex quantities and operations on them, and it is necessary that one or another equation linking the mechanical quantities represented by the vectors become the equation between the quantities represented by the screws on substitution of complex quantities for the real ones. But this is possible only on satisfaction of the condition that the corresponding functional expressions have the respective forms (5.94), (5.98), (5.104) and (5.105), i.e., that they satisfy the "analyticity" condition.

From this we may conclude that the "analyticity" condition of the corresponding equations is, at the same time, a condition of applicability of the transfer principle to the mechanics of the solid body.

Manu-script Page No.	Footnote
99	Here we use a slightly modified method of representing a point in four-dimensional space that was proposed by Ye.S. Fedorov [33].

Chapter 6

SCREW GROUPS. APPLICATIONS TO KINEMATICS AND STATICS

§1. Linear Dependence and Linear Independence of Screws. The Screw Group

Here we shall examine combinations of screws with real multipliers.

If n screws are given $(n \leqslant 6)$

$$R_1, R_2, \ldots, R_n$$

and it is impossible to select n real numbers

$$a_1, a_2, \ldots, a_n,$$

which, without all of them being zero simultaneously, would satisfy the equality

$$a_1 R_1 + a_2 R_2 + \ldots + a_n R_n = 0, \tag{6.1}$$

then the screws in question are said to be linearly independent; otherwise they are said to be linearly dependent.

If the real rectangular (Plücker) coordinates of screws $R_1, R_2, \ldots, R_n$ are

$$
\begin{array}{cccccc}
x_1, & y_1, & z_1, & x_1^0, & y_1^0, & z_1^0, \\
x_2, & y_2, & z_2, & x_2^0, & y_2^0, & z_2^0, \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
x_n, & y_n, & z_n, & x_n^0, & y_n^0, & z_n^0,
\end{array}
$$

then, multiplying like coordinates by $a_1, a_2, \ldots, a_n$, respectively, we obtain instead of Equality (6.1) six homogeneous linear equations between n variables.

If these six equations can be satisfied by even one system of values of the numbers a_k, then Condition (6.1) will be satisfied, and the screws will be linearly dependent; when, on the other hand, the equations are incompatible, the screws will be independent. For $n > 6$, a system of six equations can, generally speaking, be satisfied, and hence seven or more screws are always dependent.

Let there be $n (n \leqslant 6)$ linearly independent screws. We can con-

struct the linear screw combination

$$R = a_1 R_1 + a_2 R_2 + \ldots + a_n R_n. \qquad (6.2)$$

Assigning all possible values to the real numbers $a_1, a_2, \ldots, a_n$, we obtain a nondenumerable set of screws known as an n-member group. Screws $R_1, R_2, \ldots, R_n$ are called the basic screws of the group and the numbers $a_1, a_2, \ldots, a_n$ the coordinates of screw R of the group. Obviously, the basic screws $R_1, R_2, \ldots, R_n$ belong to the group.

Let us prove certain theorems pertaining to groups of screws.

Theorem 19. *If the screws $R_1, R_2, \ldots, R_n$ $(n < 6)$ are linearly independent, then for m screws $(m < n)$*

$$S_k = a_{k1} R_1 + a_{k2} R_2 + \ldots + a_{kn} R_n \quad (k = 1, 2, \ldots, m) \qquad (6.3)$$

to be independent, it is necessary and sufficient that at least one of the $m\underline{th}$-order determinants of the matrix

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdot & \cdot & \ldots & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{vmatrix} \qquad (6.4)$$

be nonzero.

Actually, if all determinants of matrix (6.4) are zero, there exist m quantities $b_1, b_2, \ldots, b_m$ such that

$$a_{1s} b_1 + a_{2s} b_2 + \ldots + a_{ms} b_m = 0, \quad s = 1, 2, \ldots, n,$$

so that, multiplying Equality (6.3) by b_k and summing over k, we obtain

$$b_1 S_1 + b_2 S_2 + \ldots + b_m S_m = 0, \qquad (6.5)$$

from which it follows that the screws S_k are dependent. Conversely, if the screws S_k are dependent, a relation of the form of (6.5) exists between them and can be expressed in terms of the screws R, namely,

$$c_1 R_1 + c_2 R_2 + \ldots + c_n R_n = 0, \qquad (6.6)$$

where $c_s = a_{1s} b_1 + a_{2s} b_2 + \ldots + a_{ms} b_m$ $(s = 1, 2, \ldots, n)$. But all of the quantities c_s must be equal to zero, since screws $R_1, R_2, \ldots, R_n$ are independent by hypothesis. On the other hand, it follows from the equalities $c_s = 0$ that all determinants of Matrix (6.4) are zero.

Theorem 20. *Any n independent screws belonging to a group may be taken as the basic screws of the group.*

Let $R_1, R_2, \ldots, R_n$ be the basic screws of an n-member group $(n < 6)$. Each screw can be defined by six real rectangular (Plücker)

coordinates, which are independent quantities. In this case, each screw can be regarded as a vector in a six-dimensional space. A group of n screws represents an n-dimensional vector space. Obviously, any vector of this space can be expressed linearly in terms of n given linearly independent vectors of a subspace, i.e., in terms of the basic screws of the group; consequently, any screw S of the group can be expressed linearly in terms of $R_1, R_2, \ldots, R_n$. Taking n such screws $S_1, S_2, \ldots, S_n$, also linearly independent, we obtain another system of basic screws of the group.

Theorem 21. *If the parameters of the basic screws of a group are increased by the same amount p, then the parameters of all screws of the group will have been increased by this same amount p.*

For the proof, we multiply Equality (6.2) by $e^{\omega p} = 1 + \omega p$; then basic screws with parameters increased by p will appear in the right member, while the left member will have an arbitrary screw of the group, its parameter also increased by p.

§2. Two-Member and Three-Member Groups

Let us consider two-member and three-member groups of screws. A two-member group is determined by the expression

$$R = a_1 R_1 + a_2 R_2. \tag{6.7}$$

As we have already seen, by assigning all possible values to the real numbers a_1 and a_2, we shall obtain various screws whose axes will lie on a ruled surface, a cylindroid (see Chapter 3). As has already been established, there are among the screws of a two-member group two screws whose axes intersect at right angles. These will be the principal screws of the two-member group. The principal parameters correspond to the principal screws. Taking screws R_1 and R_2 as the principal screws in Formula (6.7), we express the scalar square of screw R:

$$R^2 = r^2 e^{2\omega p} = a_1^2 r_1^2 e^{2\omega p_1} + a_2^2 r_2^2 e^{2\omega p_2}, \tag{6.8}$$

from which

$$r^2 = a_1^2 r_1^2 + a_2^2 r_2^2, \quad pr^2 = p_1 a_1^2 r_1^2 + p_2 a_2^2 r_2^2, \tag{6.9}$$

$$p = \frac{p_1 a_1^2 r_1^2 + p_2 a_2^2 r_2^2}{a_1^2 r_1^2 + a_2^2 r_2^2}. \tag{6.10}$$

For any two screws of a two-member group

$$R' = a_1' R_1 + a_2' R_2,$$
$$R'' = a_1'' R_1 + a_2'' R_2,$$

the screw product will be

$$R' \times R'' = (a_1' a_2'' + a_1'' a_2') R_1 \times R_2, \tag{6.11}$$

and hence the numbers a_1', a_2', a_1'', a_2'' are real, this implies that the screw product of any two screws of the group will be the same

screw to within the real multiplier.

If the complex angle between screws R' and R'' is θ, then, taking the complex moduli and parameters of the left and right members of (6.11), we obtain

$$R'R'' \sin \theta = (a_1'a_1'' + a_2'a_2'')R_1R_2,$$
$$p' + p'' + \theta^0 \operatorname{ctg} \theta = p_1 + p_2, \qquad (6.12)$$

i.e., the sum of the parameters of any two screws of a group added to the sine parameter of the angle between them is equal to the sum of the principal parameters.

On addition of two screws, the cylindroid plays the same role as the plane in vector addition. The screw sum, together with the screw terms, lies on a cylindroid, and the angle that it forms with the axis and the parameter are given by the formulas presented in Chapter 3.

A three-member group of screws is defined by the expression

$$R = a_1R_1 + a_2R_2 + a_3R_3. \qquad (6.13)$$

Let us first assume that the axes of the basic screws R_1, R_2, R_3 of a three-member group intersect at right angles. We take the axes of these screws as the axes of a rectangular coordinate system. We denote the corresponding parameters by p_1, p_2, p_3, and the parameter of screw R by p.

We express the scalar square of screw R according to (6.13):

$$r^2 e^{2\omega p} = a_1^2 r_1^2 e^{2\omega p_1} + a_2^2 r_2^2 e^{2\omega p_2} + a_3^2 r_3^2 e^{2\omega p_3}, \qquad (6.14)$$

from which the parameter of screw R is

$$p = \frac{p_1 a_1^2 r_1^2 + p_2 a_2^2 r_2^2 + p_3 a_3^2 r_3^2}{a_1^2 r_1^2 + a_2^2 r_2^2 + a_3^2 r_3^2}. \qquad (6.15)$$

Let the projections of the radius vector ρ of an arbitrary point on the axis of screw R be ξ, η, ζ. Since the projections of the vector r of screw R onto the coordinate axes are a_1r_1, a_2r_2, a_3r_3, the moments of screw R with respect to the axes will be, respectively,

$$pa_1r_1 + \eta a_3r_3 - \zeta a_2r_2,$$
$$pa_2r_2 + \zeta a_1r_1 - \xi a_3r_3,$$
$$pa_3r_3 + \xi a_2r_2 - \eta a_1r_1.$$

On the other hand, these moments are equal to $p_1a_1r_1$, $p_2a_2r_2$ and $p_3a_3r_3$. From this we obtain the homogeneous equation system

$$\left. \begin{array}{l} (p_1 - p)a_1r_1 + \zeta a_2r_2 - \eta a_3r_3 = 0, \\ -\zeta a_1r_1 + (p_2 - p)a_2r_2 + \xi a_3r_3 = 0, \\ \eta a_1r_1 - \xi a_2r_2 + (p_3 - p)a_3r_3 = 0. \end{array} \right\} \qquad (6.16)$$

Assigning the screw parameter p, we determine the geometric locus of the axes of the group screws having this parameter; for this it is necessary to exclude $a_1 r_1$, $a_2 r_2$, $a_3 r_3$ from the equation system (6.16), which yields

$$(p_1 - p)\xi^2 + (p_2 - p)\eta^2 + (p_3 - p)\zeta^2 + $$
$$+ (p_1 - p)(p_2 - p)(p_3 - p) = 0. \qquad (6.17)$$

This geometric locus, if real, is a hyperboloid of one sheet. The surface will be imaginary if p is larger than the largest or smaller than the smallest of the numbers p_1, p_2, p_3. For the axes of screws whose parameter is zero, the geometric locus is described by the equation

$$p_1 \xi^2 + p_2 \eta^2 + p_3 \zeta^2 + p_1 p_2 p_3 = 0 \qquad (6.18)$$

and will be real if the product of the numbers p_1, p_2, p_3 is negative. A family of hyperboloids including, in particular, a hyperboloid with zero parameter, which is described by Eq. (6.18), will correspond to various values of p in Eq. (6.17).

We have taken as the basic screws of our three-member group three screws whose axes intersect at right angles. But it is easily seen that the most general case of assignment of the three basic screws of the group reduces to the same case, or, in other words, a three-member group of screws in which the basic screws are three screws with mutually perpendicular intersecting axes is the most general case of the three-member group. Actually, as has already been shown in this chapter, any screw can be presented as the sum of its components along the axes of a rectangular coordinate system.

The three arbitrarily selected basic screws of the group may be replaced by three triplets of screws whose axes lie on the axes of the rectangular coordinate system; adding three screws on each axis, we obtain three screws whose axes intersect at right angles and which are equivalent to the sum of the three given basic screws of the three-member group. Since a sum of screws is a linear combination with real multipliers, the sum screws are members of the same group as the summand screws; hence screws lying on the axes of a rectangular coordinate system and equivalent to three arbitrary basic screws of a three-member group are screws of the same three-member group, which proves the hypothesis advanced.

§3. The Linear Complex of Straight Lines and the Congruence. Four-, Five- and Six-Member Screw Groups

Before turning to a description of higher-order groups, let us define certain geometric figures of the ruled space.

As has already been indicated in Chapter 3, any straight line is fully defined by the rectangular coordinates X, Y, Z which are linked by the relationship

$$X^2 + Y^2 + Z^2 = 1,$$

so that only two of them, for example X and Y, can be regarded as

independent.

Assigning all possible values to the numbers x, x^0, y, y^0 with the condition

$$-1 < x < +1, \quad -1 < y < +1,$$

we obtain ∞^4 straight lines. It follows from this that the ruled space is four-dimensional.

Let

$$X = X(u, v, w), \quad Y = Y(u, v, w),$$
$$Z = Z(u, v, w), \quad (6.19)$$

where u, v and w are independent real parameters that may assume all possible values and X, Y and Z are complex functions of these parameters. Taking X, Y and Z as the coordinates of line a and assigning to u, v and w all of the values that they can assume, we obtain a set of ∞^3 straight lines which is known as a line complex. The lines belonging to the complex are known as its rays, Relationships (6.19) as the equations of the complex, and the parameters u, v, w as the real coordinates of the rays.

If A is an arbitrary point of the space and $a(u, v, w)$ is a ray of the complex that passes through point A, then the numbers u, v, w must satisfy two conditions:

$$f_1(u, v, w) = 0, \quad f_2(u, v, w) = 0,$$

from which it follows that only one of the parameters u, v, w can be left arbitrary. Therefore, ∞^1 rays of the complex pass through any given point of the space.

The simplest line complex will be the linear complex, in which all rays passing through a given point of the space lie in the same plane. This plane is called the polar plane of point A.

To construct a linear complex, we take a screw U,

$$|U| = U = e^{\omega p},$$

the modulus of whose vector is unity, and project it onto a certain line of the space having a unit screw E, $|E| = 1$.

If $\Phi = \varphi + \omega \varphi^0$ is the complex angle between U and E, the projection will be expressed by

$$U \cdot E = e^{\omega p} \cos \Phi = \cos \varphi + \omega (p \cos \varphi - \varphi^0 \sin \varphi). \quad (6.20)$$

Let us ascertain for which lines this projection will be real. We note that the expression for the moment part of the projection

$$p \cos \varphi - \varphi^0 \sin \varphi \quad (6.21)$$

is the projection of the moment of screw U about an arbitrary

point A of this line onto the line. Thus, Expression (6.21) vanishes for all lines of the space that pass through point A and lie in a plane Q perpendicular to the moment of the screw about point A. Following similar reasoning as regards each point of the space, we obtain a set of straight lines lying in the same plane and satisfying the condition

$$p \cos \varphi - \varphi^0 \sin \varphi = 0, \tag{6.22}$$

for which the projection of the screw onto the given straight line has a real value, i.e., the component of the screw along this line is a vector. It follows from this that a collection of lines of the space the component of a screw along which is a vector is a linear complex of lines defined by the screw. The axis of the screw is called the axis of the complex.

A complex is defined by five quantities — four real coordinates and a parameter.

It follows from Eq. (6.22) that

$$p = \varphi^0 \operatorname{tg} \varphi.$$

i.e., that the distance between rays of a complex is inversely proportional to the tangents of the angles formed by the rays with the axis of the complex. The quantity p is called the parameter of the complex; it characterizes the "steepness" of rays at a certain distance from the axis.

It also follows that if $\varphi^0 = 0$, i.e., if a ray intersects the axis of the complex, then $\varphi = \pi/2$, i.e., the ray forms a right angle with the axis of the complex. In other words, rays intersecting the complex axis form a brush.

Let Q be an arbitrary plane and R and S arbitrary points in this plane. Let Q_1 and Q_2 be the polar planes of points R and S. Planes Q_1 and Q_2 intersect the plane Q along certain straight lines b_1 and b_2; let the intersection point of these lines be T. It can be seen that plane Q is polar with respect to point T. Indeed, lines RT and ST are rays of a complex, so that the projections of screw U onto them will be real. If the screw is brought to point T, the moment will be perpendicular to both RT and ST and, consequently, it will be perpendicular to plane Q. This means that plane Q is polar with respect to point T.

Point T is called the pole of plane Q.

For a linear complex, therefore, there passes through each point of the space a single plane that contains rays passing through this point and, conversely, all rays of a complex that lie in a given plane pass through one point.

Let A, B and C be the rectangular coordinates of the complex axis and let X, Y, Z be the rectangular coordinates of a ray of this complex.

With the condition (6.22), Relation (6.20) asssumes the form

$$e^{i p\varphi} (AX + BY + CZ) = \cos\varphi, \qquad (6.23)$$

where

$$A^2 + B^2 + C^2 = 1, \quad X^2 + Y^2 + Z^2 = 1.$$

This is the equation of the linear complex.

In the particular case when $p = 0$, Eq. (6.23) becomes

$$AX + BY + CZ = \cos\varphi,$$

from which it follows that the complex consists of rays intersecting its axis.

This will be a degenerate complex.

If we take the z-axis as the axis of the complex, then $A = B = 0$, $C = 1$, and Eq. (6.23) of the complex will be simplified to

$$e^{i p\varphi} Z = \cos\varphi. \qquad (6.24)$$

If two screws U_1 and U_2 are given, each of these screws defines a linear complex. Through each point A of the space, we can pass polar planes Q_1 and Q_2 of this point, which correspond to both complexes. Obviously, the line of intersection of these planes will simultaneously be a ray of either complex. The collection of lines that are rays shared by the two linear complexes is called a congruence. It follows from the above that a single straight line belonging to a congruence passes through a given point of the space.

Now let us pass to a brief characterization of four- and five-member screw groups.

For a four-member group whose basic screws are R_1, R_2, R_3, R_4, it is possible to indicate those lines of the space the projections onto which of the group screws will be real. Obviously, these lines will be common to four complexes defined by the four given screws. The conditions for definition of such lines are expressed in the following manner:

$$\left. \begin{array}{l} \text{mom } [e^{i p_k \varphi_k}(A_k X + B_k Y + C_k Z)] = 0, \\ A_k^2 + B_k^2 + C_k^2 = 1, \quad X^2 + Y^2 + Z^2 = 1, \quad k = 1, 2, 3, 4, \end{array} \right\} \qquad (6.25)$$

where the p_k are the parameters of the screws, A_k, B_k, C_k are the complex rectangular coordinates of the screw axes and X, Y, Z are the complex rectangular coordinates of the sought line.

Separating the principal parts from the moment parts, we obtain four equations

$$\begin{array}{l} (a_k^2 + p_k a_k) x + (b_k^2 + p_k b_k) y + (c_k^2 + p_k c_k) z + a_k x^0 + b_k y^0 + \\ \qquad + c_k z^0 = 0, \quad k = 1, 2, 3, 4. \end{array} \qquad (6.26)$$

Dividing the equations by z^0, we obtain from them expressions for four quantities $x/z^0, y/z^0, z/z^0, x^0/z^0$ in terms of y^0/z^0, and then require

satisfaction of the equality

$$x^2 + y^2 + z^2 = 1,$$

which results in a quadratic equation in the quantity y^0/z^0. We may conclude from this that there exist in the entire space no more than two straight lines (real or imaginary) that satisfy the condition formulated.

Let us now take these lines as axes of linear complexes. The conditions equivalent to (6.25), which indicate zero value of the moment of an arbitrary screw R of the group about these lines, will be

$$
\begin{cases}
p\cos\varphi_1 - \varphi_1^0\sin\varphi_1 = 0, \\
p\cos\varphi_2 - \varphi_2^0\sin\varphi_2 = 0,
\end{cases}
\tag{6.27}
$$

where p is the parameter of screw R, $\varphi_1 + \omega\varphi_1^0$, $\varphi_2 + \omega\varphi_2^0$ are the complex angles formed by the lines with the axis of R.

Comparing with (6.22), we can interpret Conditions (6.27) as the conditions for the axis of screw R to be simultaneously a ray of two complexes whose axes coincide with the two lines indicated, with the common parameter of these complexes equal to p. It follows from this that the axes of all screws of a four-member group form a congruence.

For a five-member group with the basic screws $R_1, R_2, \ldots, R_5$, the condition of real projections of each of these screws onto an arbitrary line of the space or, what is the same thing, zero relative moments, will give five equations of the type (6.26):

$$
\begin{aligned}
(a_k^2 + p a_k) x + (b_k^2 + p_k b_k) y + (c_k^2 + p_k c_k) z + a_k x^0 + b_k y^0 + \\
+ c_k z^0 = 0, \quad k = 1, 2, \ldots, 5.
\end{aligned}
\tag{6.28}
$$

Solving System (6.28), we find the values of the five ratios of the coordinates x, y, z, x^0, y^0, z^0 to one of them, for example, the last, z^0. With the additional condition

$$x^2 + y^2 + z^2 = 1$$

this will define the coordinates of the only line about which the moment of the screws of the five-member group will be zero. Conversely, if the axis of this line is taken as the axis of a certain linear complex, then, proceeding in the same way as in the case of a four-member group, namely, writing the condition for zero relative moment of an arbitrary screw of the group and the line in question in the form of (6.27), we satisfy ourselves that the axes of all screws of a five-member group having the same parameter are rays of the same complex.

Finally, a six-member group of screws is a system from which any screw can be obtained by linear combination with real multipliers.

§4. Reciprocal Screws and Reciprocal Screw Groups

As was already indicated in Chapter 3, the moment part of the scalar product of two screws

$$\text{mom}\ (R_1 \cdot R_2) = r_1 r_2\ [(p_1 + p_2) \cos \alpha - \alpha^\circ \sin \alpha] \tag{6.29}$$

is the relative moment of these screws, which is equal to the sum of the scalar products of the principal vector of the former by the principal moment of the latter and the principal vector of the latter by the principal moment of the former, with the moments of both screws taken with respect to the same pole.

Two screws are said to be reciprocal if their relative moment is equal to zero or, what is the same thing, if their scalar product is equal to a real number.

Theorem 22. *A screw reciprocal to the n independent screws of an n-member group ($n < 6$) is reciprocal to any screw in this group.*

Indeed, let a screw S be reciprocal with screws $R_1, R_2, \ldots$ $\ldots, R_n$ ($n < 6$), i.e.,

$$\text{mom}\ (S \cdot R_1) = \text{mom}\ (S \cdot R_2) = \ldots = \text{mom}\ (S \cdot R_n) = 0.$$

Performing scalar multiplication of screw S by an arbitrary screw of the n-member group formed by the above n screws, we obtain

$$\text{mom}\ (S \cdot R) = \text{mom}[S \cdot (a_1 R_1 + a_2 R_2 + \ldots + a_n R_n)] =$$
$$= a_1 \text{mom}\ (S \cdot R_1) + a_2 \text{mom}\ (S \cdot R_2) + \ldots + a_n \text{mom}(S \cdot R_n).$$

and hence the right-hand member is equal to zero,

$$\text{mom}\ (S \cdot R) = 0,$$

Q.E.D.

If its parameter p is nonzero, screw S cannot itself enter into the n-member group under consideration, because its scalar product by itself is imaginary, and hence cannot be a real number. If, on the other hand, the parameter of screw S is zero, then its axis is a common ray of the complexes corresponding to all screws of the group.

Theorem 23. *The aggregate of screws reciprocal to the screws of an n-member group ($n < 6$) forms a $(6 - n)$-member group.*

Let us prove the theorem for the case $n = 3$. If a screw S with the coordinates $\Xi = \xi + \omega \xi^\circ$, $H = \eta + \omega \eta^\circ$, $Z = \zeta + \omega \zeta^\circ$ is reciprocal to three screws R_1, R_2, R_3 with the coordinates $X_1 = x_1 + \omega \xi_1^\circ$, $Y_1 = y_1 + \omega \eta_1^\circ, \ldots, Z_3 = z_3 + \omega \zeta_3^\circ$, then we have the equations

$$\left.\begin{array}{l} \xi x_1^\circ + \eta y_1^\circ + \zeta z_1^\circ + \xi^\circ x_1 + \eta^\circ y_1 + \zeta^\circ z_1 = 0, \\ \xi x_2^\circ + \eta y_2^\circ + \zeta z_2^\circ + \xi^\circ x_2 + \eta^\circ y_2 + \zeta^\circ z_2 = 0, \\ \xi x_3^\circ + \eta y_3^\circ + \zeta z_3^\circ + \xi^\circ x_3 + \eta^\circ y_3 + \zeta^\circ z_3 = 0, \end{array}\right\} \tag{6.30}$$

and since these three screws R_1, R_2, R_3 are linearly independent, there is at least one third-order nonzero determinant in the matrix of System (6.30). In this case, leaving the three terms of each equation with this determinant in the left member – say, the system in ξ, η, ζ°,– we solve the system for these coordinates, expressing them in terms of $\xi^\circ, \eta^\circ, \zeta$:

$$
\begin{aligned}
\xi &= A_1\xi^\circ + B_1\eta^\circ + C_1\zeta, \\
\eta &= A_2\xi^\circ + B_2\eta^\circ + C_2\zeta, \\
\zeta^\circ &= A_3\xi^\circ + B_3\eta^\circ + C_3\zeta.
\end{aligned}
\right\}
\qquad (6.31)
$$

Here $\xi^\circ, \eta^\circ, \zeta$ may be taken arbitrarily, so that there exist no fewer than three independent systems of values of these numbers, from which three reciprocal screws can be formed; for example, screws with coordinates proportional to the numbers

$$
\begin{aligned}
&A_1, \ A_2, \ 0, \ 1, \ 0, \ A_3, \\
&B_1, \ B_2, \ 0, \ 0, \ 1, \ B_3, \\
&C_1, \ C_2, \ 1, \ 0, \ 0, \ C_3,
\end{aligned}
$$

are possible, and if we form a matrix of these numbers, there exists in it a nonzero third-order determinant

$$
\begin{vmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{vmatrix}.
$$

from which it follows that there are no fewer than three linearly independent screws reciprocal to the three given screws. It is easily seen that there will be no more than three of them, since we cannot take arbitrarily more than three systems of values of the numbers $\xi^\circ, \eta^\circ, \zeta$ in (6.31); any fourth system of values is expressed linearly in terms of these three, and in the matrix formed from the coordinates of the four corresponding screws, all fourth-order determinants will be zero. Thus, the theorem has been proven for $n = 3$. The proof for another value of n is fundamentally no different from that given above.

The relation of the axes of the screws of the reciprocal group to the complex of rays defined by the screws of the group is easily established.

For $n = 1$, the reciprocal group will be five-membered. The condition of reciprocity of the single screw R of a given one-membered group to an arbitrary screw S of the reciprocal group will be

$$
\text{mom} \ (R \cdot S) = rs \left[(p + q) \cos \varphi - \varphi' \sin \varphi \right] = 0, \qquad (6.32)
$$

where p and q are the parameters of screws R and S and $\varphi + \varphi'$ is the complex angle between the axes of the screws. Comparing (6.32) with Formula (6.22), which defines a ray of a linear complex, we find that the axes of the screws S that have the assigned parameter q in the group reciprocal to R are rays of a linear complex whose axis coincides with the axis of screw R, while the parameter is equal to the sum $p + q$.

For $n = 2$, the reciprocal group will be four-membered. Each screw of this four-member group must satisfy two conditions:

$$\text{мом } (R_1 \cdot S) = r_1 s \left[(p_1 + q) \cos \varphi_1 - \varphi_1^0 \sin \varphi_1 \right] = 0, \left. \right\}$$
$$\text{мом } (R_2 \cdot S) = r_2 s \left[(p_2 + q) \cos \varphi_2 - \varphi_2^0 \sin \varphi_2 \right] = 0, \quad (6.33)$$

where R_1 and R_2 are screws of the given two-member group. The first condition states that the axis of screw S is a ray of a complex with the axis of the first screw as its axis, while its parameter is the quantity p_1 increased by q; the second condition states that the axis of screw S is a ray of a complex with the axis of the second screw as its axis and its parameter the quantity p_2 increased by q. Thus, the axes of all the S are rays of two linear complexes and, consequently, belong to a congruence.

For $n = 3$, the given group and the group reciprocal to it will be three-membered. As we have seen, any three-member group can be defined by three basic screws whose axes intersect at right angles at a point. In this case, we shall consider the basic group to be defined by three such screws R_1, R_2, R_3; let the screws S_1, S_2, S_3 on the same axes be the basic screws for the reciprocal group. Any screw of the first group and any screw of the second group will have the expressions

$$R = a_1 R_1 + a_2 R_2 + a_3 R_3, \left. \right\}$$
$$S = b_1 S_1 + b_2 S_2 + b_3 S_3, \quad (6.34)$$

where a_k and b_k are real numbers.

Performing scalar multiplication of Equality (6.34) and equating the moment part to zero, we obtain

$$\text{мом } (R \cdot S) = a_1 b_1 r_1 s_1 (p_1 + q_1) + a_2 b_2 r_2 s_2 (p_2 + q_2) +$$
$$+ a_3 b_3 r_3 s_3 (p_3 + q_3) = 0.$$

Since this last equality must be satisfied for any a_k and b_k, it is necessary that

$$r_1 s_1 (p_1 + q_1) = 0, \quad r_2 s_2 (p_2 + q_2) = 0, \quad r_3 s_3 (p_3 + q_3) = 0,$$

from which (for $r_k \neq 0$ and $s_k \neq 0$)

$$p_1 = -q_1, \quad p_2 = -q_2, \quad p_3 = -q_3. \quad (6.35)$$

The hyperboloid on which the axes of the screws of this system with parameter p lie is defined by Eq. (6.17), while the corresponding hyperboloid of the reciprocal system will, according to (6.35), be defined by the equation

$$(p_1 + p)\xi^2 + (p_2 + p)\eta^2 + (p_3 + p)\zeta^2 +$$
$$+ (p_1 + p)(p_2 + p)(p_3 + p) = 0, \quad (6.36)$$

i.e., it will be the same hyperboloid, but for screws of parameter $-p$.

In solving many problems in mechanics, it is helpful to have a geometrical interpretation of the objects that makes it possible to perform the necessary operations directly on these objects and obtain easily inspected results. A number of problems can be solved effectively by methods similar to the classical methods of graphical statics with the aid of the direct geometrical screw representation to be given below.
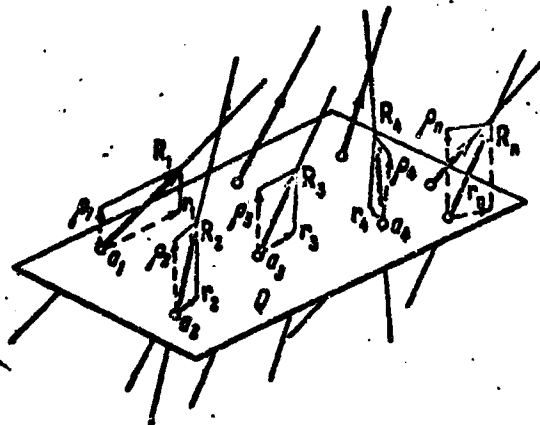

Fig. 32

Let us envision an arbitrary system of sliding vectors $R_1, R_2, \ldots, R_n$. We pass a certain intersecting plane Q (Fig. 32), which will henceforth serve as our plane of representation, and mark the points $a_1, a_2, \ldots, a_n$ of intersection of the lines on which the indicated vectors lie.

Then at each of these points we decompose the corresponding vector into two components: one in plane Q and another perpendicular to plane Q. We shall denote the components in plane Q by $r_1, r_2, \ldots, r_n$, and the components perpendicular to Q by $\rho_1, \rho_2, \ldots, \rho_n$. Thus the given system of sliding vectors has been broken up into two systems: a) a system of [co]planar sliding vectors and b) a system of parallel sliding vectors whose common direction is perpendicular to the plane of the first system. The first system is equivalent to a certain sliding vector r in plane Q, and the second to a vector ρ perpendicular to Q (unless these systems are equivalent to pairs). Consequently, the screw R to which the given system is equivalent is in turn equivalent to a system of two sliding vectors r, ρ (an orthogonal vector cross), as expressed by

$$R \to (r, \rho).$$
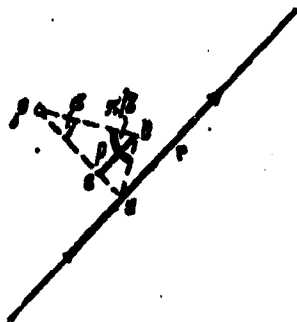
Fig. 33

as well as the trace c of vector ρ with the magnitude ρ assigned to it, we have a representation of the screw that is uniquely defined for the plane Q (Fig. 33). In particular cases in which the line in plane Q is infinitely distant and the trace is also infinitely distant, we shall have equivalent pairs. This representation, in turn, fully defines a screw, i.e., the magnitude of its principal vector, its central axis and its parameter.

Actually, the magnitude of the principal vector of the screw will be $\sqrt{r^2 + \rho^2}$. Furthermore, dropping a perpendicular from point ρ to the axis of r and constructing the right triangle ρab, in which ρa is the hypotenuse and the angle $a\rho b = \alpha =$ arc tan (ρ/r) (Fig. 33), we find the moment of the screw with respect to point c — the base of the perpendicular dropped from vertex b to the hypotenuse:

$$\overrightarrow{c\rho} \times \rho + \overrightarrow{ca} \times r.$$

The ratio of the vertical and horizontal components of the moment is

$$\frac{\left|\overrightarrow{ca} \times r\right|}{\left|\overrightarrow{c\rho} \times \rho\right|} = \frac{r\,(ca)}{\rho\,(c\rho)} = \frac{r}{\rho} \cdot \frac{(\rho/r)(cb)}{(r/\rho)(cb)} = \frac{\rho}{r}\,\frac{cb}{cb} = \frac{\rho}{r}\,,$$

i.e., to the ratio of the vertical and horizontal components of the principal vector, from which it follows that line cb is the projection of the central axis; on the other hand, segment cb, which serves as a proportionality coefficient, is obviously equal to the parameter p of the screw. Finally, the invariant J of the screw, which is equal to the scalar product of its principal vector by the principal moment, will, if point ρ is taken as the moment point, be expressed by

$$J = (r + \rho)\cdot\text{MOM}, \; r = \rho\cdot\text{MOM}, \; r = \pm \rho r\,(\rho a).$$

In view of the equivalence of the cross to the screw, we can use the notation of the corresponding screws for the crosses and speak of operations performed directly on the crosses.

The use of crosses with unit value of the component perpendicular to the image plane, which was proposed by Ya.B. Shor [34], is of great assistance in the constructions. The given screw is reduced to an expression $R = \rho K$, where K is a screw equivalent to the cross

$$\left(\frac{r}{\rho}, \varkappa\right) = (k, \varkappa), \; \varkappa = 1.$$

We shall call the cross $K \to (k, \varkappa)$ at $\varkappa = 1$ the unit cross.

The magnitude of the component of a unit cross in the image plane is $k \cot \alpha$, the modulus of the principal vector is $\sqrt{1 + k^2} = 1/\sin \alpha$, and the invariant is equal to the moment of vector k with respect to the trace $\varkappa$.

... .rmine the relative moment of two screws R_1 and R_2
....re K_1 and K_2. As the moment point we take the
po.n. 4 of intersection of the axes of the components k_1 and k_2.
The relative moment will be equal to the sum of the scalar prod-
uct. of the principal vector of the first cross by the principal
moment of the second relative to point A and of the principal vec-
tor of the second by the principal moment of the first with re-
spect to point A. Expressing the crosses in terms of the unit
crosses, we have

$$\text{MOM} \ (R_1, R_2) = \rho_1\rho_2 \ \text{MOM.} \ (K_1, K_2) =$$
$$= \rho_1\rho_2 \ [k_1 \cdot (\overrightarrow{Ax_2} \times x_2) + k_2 \cdot (\overrightarrow{Ax_1} \times x_1)] =$$
$$= \rho_1\rho_2 \ [(k_1 \times \overrightarrow{Ax_2}) \cdot x_2 + (k_2 \times \overrightarrow{Ax_1}) \cdot x_1] =$$
$$= \rho_1\rho_2 \ (\text{MOM}_{x_2} k_1 + \text{MOM}_{x_1} k_2).$$

(6.37)

The reciprocity condition for two unit crosses is expressed
by the simple relationship

$$\text{MOM}_{x_2} \ k_1 + \text{MOM}_{x_1} \ k_2 = 0.$$

(6.38)

Reciprocity of two unit crosses has the following geometrical
interpretation. Let K_1 and K_2 be two unit crosses with the respec-
tive components k_1, k_2 and the traces κ_1 and κ_2 (Fig. 34a). Obvi-
ously, the relative moment of these
crosses will not change if κ_1 is dis-
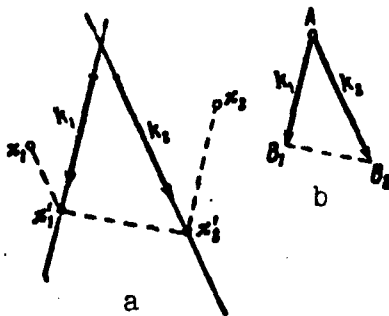placed parallel to k_2 and κ_2 parallel
to k_1.

Fig. 34

Moving these points until κ_1 is
in coincidence with k_1 and κ_2 with k_2,
we obtain instead of points x_1, x_2
points x_1', x_2', and the unit crosses have
become sliding unit vectors. But for
reciprocity of two sliding vectors it
is necessary and sufficient that the
axes of these vectors intersect, and
in this case the line $x_1'x_2'$ must be
parallel to the line B_1B_2 connecting
the ends of vectors k_1 and k_2 drawn
from the common origin A (Fig. 34b). It is this that will be the
necessary and sufficient condition for reciprocity of unit crosses.

The linear combination of two unit crosses

$$K = \xi K_1 + \eta K_2,$$

in which $\xi + \eta = 1$, is obviously also a unit cross. The end of vec-
tor k, reduced to the common origin A with vectors k_1 and k_2, ob-
viously lies on the line connecting the ends of vectors k_1 and k_2,
dividing the corresponding line segment into parts that are in-
versely proportional to the ratio $\xi:\eta$; on the other hand, the
trace κ lies on the line connecting the traces κ_1 and κ_2 and di-
vides the segment $\kappa_1\kappa_2$ in the same proportions. If, in particular,
$\xi = \eta = 1/2$, we obtain a "sum" of unit vectors, and in this case the
end of the resultant vector k and the trace κ lie at the midpoints
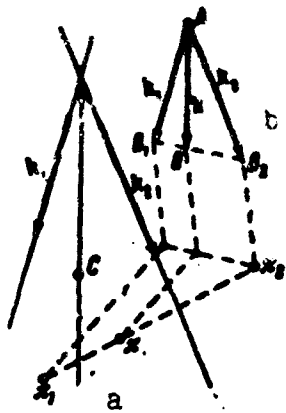of the corresponding segments.

Fig. 35

Problem 1. Construct a unit cross K that is the linear combination of unit crosses K_1 and K_2 whose component k passes through a given point C.

Solution. We draw the axis k (Fig. 35a) through the point of intersection of the axes k_1 and k_2 and point C; then we find the point B of intersection of the component k with the line connecting the ends of B_1 and B_2 of the components k_1 and k_2 reduced to the common origin A (Fig. 36b). Then on line $\kappa_1\kappa_2$ we find the point κ that satisfies the condition $\kappa_1\kappa : \kappa\kappa_2 = B_1B : BB_2$. Point κ is the trace of the unknown unit cross.

Problem 2. Construct the unit cross K that is the linear combination of unit crosses K_1 and K_2 reciprocal to unit cross L.

Solution. Let

$$K = \xi K_1 + \eta K_2,$$ (6.39)

where $\xi + \eta = 1$, and K_1, K_2, L are defined by their components k_1, k_2, l and traces $\kappa_1, \kappa_2, \lambda$ (Fig. 36a). First we construct the unit crosses K' and K'_2, which are reciprocal to unit cross L and such that their components k_1 and k_2 are same as those of unit crosses K_1 and K_2, although the traces κ'_1 and κ'_2 lie on two arbitrary parallel lines passing through traces κ_1 and κ_2. Such unit crosses are easily constructed, as follows: it is necessary first to draw lines u and v through κ_1 and κ_2 parallel to k_1 and k_2, line $\lambda\sigma_1$
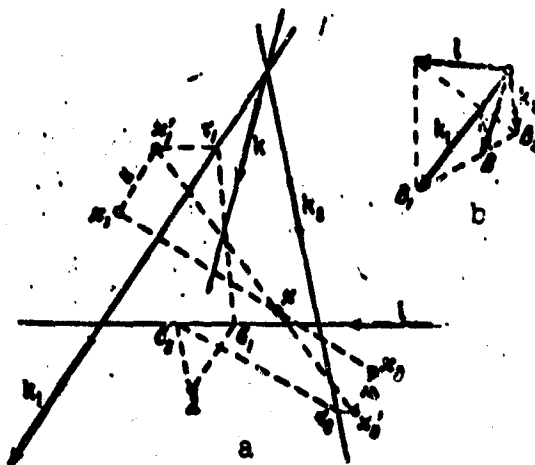

Fig. 36

through point λ parallel to k_1, line $\sigma_1\tau_1$ parallel to the difference between vectors l and k (Fig. 36b), and then a line through τ_1 parallel to l to the intersection with u, which defines a point

- 125 -

κ'_1; on the basis of the geometrical reciprocity principle established above, unit cross K'_1 will be reciprocal to unit cross L. In exactly the same way, it is necessary to pass a line $\lambda\sigma_2$ through point λ parallel to k_2, then a line $\sigma_2\tau_2$ parallel to the difference of vectors l and k_2 (Fig. 36b), and then, through point τ_2, a line parallel to l to the intersection with v, which defines a point κ'_2; unit cross K'_2 will be reciprocal to unit cross l. Thus we shall have two unit crosses K'_1 and K'_2, which will differ from unit crosses K_1 and K_2 only in the position of the traces $\kappa'_1\kappa'_2$. However, the two unit crosses K'_1 and K'_2 are reciprocal to L. Hence the combination

$$\xi K'_1 + \eta K'_2 = K' \qquad (6.40)$$

will also be reciprocal to L. But this combination has the same component k as the unknown combination (6.39), which, by the conditions of the problem, must also be reciprocal to unit cross L.

We connect points κ'_1 and κ'_2 by a straight line and find the point κ of intersection of this line with line $\kappa_1\kappa_2$. Then κ is the trace of the unknown unit cross K, since $\kappa_1\kappa : \kappa\kappa_2 = \kappa_1\kappa : \kappa\kappa_2$ and the point belongs simultaneously to the combinations of (6.39) and (6.40), while the component k corresponding to it is the same for both combinations.

The latter is determined by dividing segment B_1B_2 into parts proportional to $\kappa_1\kappa$ and $\kappa\kappa_2$ (Fig. 36b).

Instead of the above purely graphical construction, Condition (6.38), which results in an equation with one unknown, can be used for solution of the problem.

Problem 3. Construct the unit cross L_{123} reciprocal to the three unit crosses K_1, K_2 and K_3.

Solution. First we construct unit cross L_{12}, which is reciprocal to unit crosses K_1 and K_2. Obviously, this unit cross can be obtained by taking as the axis l_{12} the straight line passing through κ_1 and κ_2, and as the point λ_{12} the point of intersection of k_1 and k_2. Here the magnitude of l_{12} remains undetermined. It can be determined from the condition of reciprocity to K_3 or by the construction described above, or by forming the relative moment of unit cross L_{12} and unit cross K_3 and equating it to zero, at which point the magnitude of l_{12} is determined from an equation with one unknown. The problem is solved.

A second variant of the solution will be unit cross L_{132}, which is constructed in exactly the same way, but with the condition that the unit cross reciprocal to K_1 and K_3 is constructed first, and then subjected to the condition of reciprocity with K_2.

Finally, the third variant will be unit cross L_{231} constructed first as reciprocal to K_2 and K_3 and then subjected to the condition of reciprocity with K_1.

The three unit crosses L_{123}, L_{132} and L_{231} define a three-member

group reciprocal to the three-member group of unit crosses K_1, K_2 and K_3. All of these crosses are shown in Fig. 37.

Problem 4. Construct the unit cross reciprocal to five given unit crosses $K_1, K_2, \ldots, K_5$.

Solution. We first construct the three-member group of unit crosses reciprocal to unit crosses K_1, K_2, K_3, in accordance with Problem 3. Let this be the unit crosses $L_{123}, L_{132}, L_{231}$. We find the unit cross L_{1234}, which is the linear combination of unit crosses L_{123} and L_{132} reciprocal to K_4. This can be done as in Problem 3, which was solved above. In exactly the same way, we find unit cross L_{1324}, which is the linear combination of L_{132} and L_{231} reciprocal to K_4. Thus we shall have two unit crosses L_{1234} and L_{1324} that define a two-member group reciprocal to unit crosses K_1, K_2, K_3, K_4.
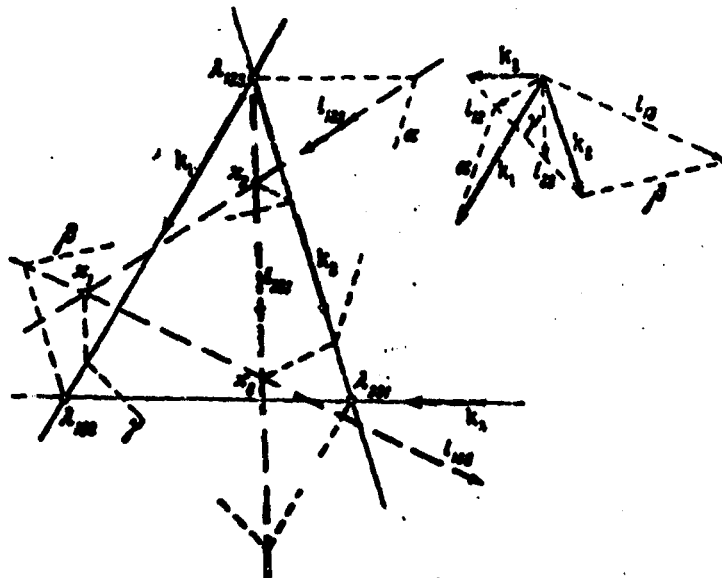

Fig. 37

Let us now form the linear combination of unit crosses L_{1234} and L_{1324} and, subjecting it to the condition of reciprocity with unit cross K_5, still following the solution of Problem 3, find the unique unit cross L_{12345} that is reciprocal to the given unit crosses $K_1, K_2, \ldots, K_5$.

The geometrical constructions indicated here can be used in the problems of three-dimensional statics and kinematics.*

§6. Screw Groups in Kinematics and Statics

The theory of screw groups is closely related to analysis of the properties of motions of a solid body that has one or another number of degrees of freedom (from one to six), and to the properties of force systems acting on a body, including reaction forces if the body is not free.

The most general form of displacement of a solid body is the screw displacement, which is characterized by a screw axis, the modulus of its vector, and the parameter. The elementary turn angle $d\varphi$ serves as the modulus of the vector in an infinitesimally small displacement, the ratio of the translational displacement $d\varphi^0$ to $d\varphi$ as the parameter; defining the screw by its axis and complex modulus with unit principal part and multiplying the complex modulus by $d\varphi$, we obtain a kinematic screw — a screw that expresses an infinitesimally small displacement of the body.

Let the body be able to move along only one screw R with complex modulus $R = e^{\omega p}$; multiplying this screw by $d\varphi$, we obtain the displacement

$$d\Phi = R\,d\varphi, \quad d\Phi = e^{\omega p}\,d\varphi.$$

Knowing the displacement screw, we can determine the displacement of any point of the body as the moment of the screw with respect to this point.

The displacements of all points of the body equidistant from the screw axis are directed along tangents to screw lines constructed on the axis of the screw and having the same pitch. The plane normal to the displacement is the polar plane with respect to the point under consideration; all rays of the complex passing through this point lie in it.

If the body can perform displacements along two screws R_1 and R_2 defined by axes in space and by the complex moduli

$$R_1 = e^{\omega p_1}, \quad R_2 = e^{\omega p_2},$$

then, on imparting to the body two small displacements $d\varphi_1$ and $d\varphi_2$ along these screws, we obtain a resultant motion that will also be a screw motion. The resultant screw is a function of both the axis positions of screws R_1 and R_2 and the elementary displacements $d\varphi_1$ and $d\varphi_2$. Varying the latter, we obtain a set of new screws along which the body can execute displacements. They all lie on a cylindroid that can be constructed from two given screws; among all of these screws, there exist two whose axes intersect at right angles. If, apart from the screws lying on the cylindroid, there are no other screws with respect to which the body could be displaced, the body has two degrees of freedom.

If the body can execute displacements along n screws $R_1, R_2, \ldots, R_n$, then we take any m of these screws $(m < n)$ and impart to the body m screw displacements along them. The resultant displacement will be a screw; let us assume that no matter how we vary the magnitudes of their principal vectors, i.e., the magnitudes of the elementary rotations, the resultant screw always differs from the remaining $n - m$ screws. In this case, the n screws are independent. A body capable of displacement along n indendent screws has freedom of the nth degree.

Consequently, study of the geometrical distribution of all screws along which a body possessing nth-degree freedom can be displaced reduces to study of the distribution of all screws of

an n-member group. In particular, screws along which a body having three degrees of freedom can execute motion are distributed along hyperboloids in such a way that screws with the same parameter lie on each of the hyperboloids; among them there is a zero-parameter hyperboloid, which corresponds to pure rotational motions of the body.

Let us now consider the force interpretation of the screws. A force screw is characterized by the combination of a force vector and a couple whose moment is parallel to the force vector. Thus the principal vector of the screw is a force vector and its principal moment is the moment of a couple. The moment of a screw with respect to a certain point of space is the moment of the motor obtained by reduction of the screw to this point.

It follows from analysis of screw groups that if a body is at equilibrium under the action of n force screws, it is necessary that some one of these screws be in the group formed by the remaining $n - 1$ screws. In particular, we shall have:

a) for two screws, equilibrium is possible only in the case in which the parameters of the screws are equal and their axes lie on the same straight line;

b) for three screws, if their parameters are identical, equilibrium is possible only if their axes lie in the same plane (the cylindroid constructed on two of them must be a plane), and, moreover, the axes must intersect at one point;

c) for four screws with the same parameter, and in particular for four forces, equilibrium is possible if the screw axes lie on the same hyperboloid (this hyperboloid is the hyperboloid of screws with equal parameters for a three-member group formed by three of these screws);

d) for five screws of the same parameter, equilibrium is possible if the axes of these screws lie on rays of one congruence;

e) for six screws of the same parameter, equilibrium is possible if the axes of these screws are rays of the same complex.

Finally, for seven or more screws, we do not obtain any necessary condition, since in the general case any of the seven or more screws is in a group formed by six of these screws if they are linearly independent.

The above equilibrium conditions, which are a corollary of the properties of screw groups, are extremely important for the statics of the solid body, since they imply the most general conclusions for the equilibrium conditions of many structures. For example, they apply directly to structures (trusses, foundations) that are secured to a base by a certain number of members, and provide a basis for judgments as to the invariability (immobility) of the system in the presence of various couplings. By virtue of the analogy between statics and kinematics, these same conditions serve for determination of the mobility of three-dimensional hinged mechanisms and, in particular, enable us to identify cases

if ... "angular" placement of members when motion is possible regard-
less of the presence of redundant couplings in kinematic pairs.

The importance of reciprocal screws can be seen in a particu-
lar example in which we are to find the forces in six rods ar-
ranged arbitrarily in space.

Problem. A solid body is secured to a base by six rods, and
a certain force screw P acts on the body (Fig. 38). It is required
to find the values of the forces $S_1, S_2, ..., S_6$ acting along the re-
taining rods. The problem reduces to resolution of the screw P
along six straight lines of the space.

Let us apply the principle of possible displacements. Con-
sider the system with the 6th rod removed. Then the body obtains
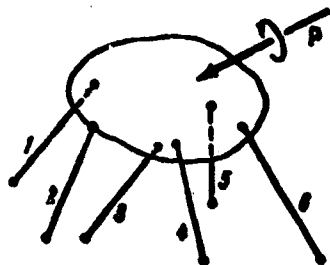one degree of freedom, which is characterized by motion along a

Fig. 38

certain screw T_{12345}. This screw must be
such that the displacement of the points
of the body at which the five remaining
rods are attached will be normal to the
axes of these rods. This means that screw
T_{12345} defines a linear complex whose
rays are these five rods, and that the
displacements of these points take place
in their polar planes. Consequently,
screw T_{12345} is reciprocal to all of the
five screws (in this case of zero param-
eter) whose axes are directed along the
five rods. This screw can be found by
the method indicated above (Problem 4).
To find the force acting along the 6th

rod, it is necessary to decompose force screw P into two compo-
nents: one along screw U, which is reciprocal to screw T_{12345},
and the other along the axis of the 6th rod. This task can be ac-
complished purely graphically, for which it is necessary, repre-
senting the screws by unit crosses, to find the unit cross of U
(in accordance with Problem 2), and then to make the elementary
decomposition of screw P. Then the same method is used to decom-
pose the U component along the axes of the 5th rod and along the
screw reciprocal to the four screws 1, 2, 3, 4, etc. An analytical
solution can also be provided, using reciprocal screws constructed
with the aid of the unit crosses. We form the expression for the
sum of the works done on screw T_{12345} by the external-force screw
P and the force S_6 acting along the missing rod, and, equating it
to zero, we obtain a single equation with the unknown magnitude
of the force in the 6th rod. The forces in the remaining rods are
determined similarly. .

The work of the force screw on the displacement performed
along the kinematic screw is the moment part of the scalar prod-
uct of these screws, or, what is the same thing, the relative mo-
ment of the screws.

The force screw can also be compensated by a smaller number
of screws, i.e., the body may also be at equilibrium with fewer
than six rods. Then, however, it is necessary to satisfy the con-
dition that the force screw acting on the body be in the group

formed by the rod reactions. If we construct the group of screws reciprocal to the screws whose axes are directed along the rods, the effective screw, which satisfies this condition, will be reciprocal to this constructed reciprocal group.

Manu-script Page No.	Footnote
127	In his time, B. Mayor [35] proposed a special method for mapping space vectors and screws onto a plane, but it was more complicated than that presented here. Due to the difficulty of the exposition, B. Mayor's book did not win wide recognition. Subsequently, Mayor's ideas were developed to some extent by R. Mises and V. Prager. In the Soviet literature, the Mayor method has been interpreted and developed elegantly in the interesting book by B.N. Gorbunov and A.A. Umanskiy [36].

Chapter 7

SCREW BINORS AND THE DYNAMICS OF THE SOLID BODY

§1. The Screw Binor

Transformation of a screw with the aid of a dyad and an affinor (see Chapter 3) makes it possible to express the coordinates of the screw in a certain coordinate system in terms of its coordinates with respect to another coordinate system. In the general case, this transformation is determined by nine complex or eighteen real numbers.

If A is an affinor, then multiplication of this affinor by a screw R reduced to a certain motor is expressed as follows:

$$R' = AR = A(r + \omega r^\circ) = Ar + \omega Ar^\circ. \qquad (7.1)$$

A more general transformation of screw R is obtained by its multiplication by the screw binor (A) introduced by S.G. Kislitsyn [17] as a generalization of the screw affinor, namely,

$$R' = (A)R = Ar + A^+r^\circ. \qquad (7.2)$$

This transformation becomes (7.1) in the particular case when

$$A^+ = \omega A. \qquad (7.3)$$

The transformation using the binor is determined by two matrices with nine complex elements each, i.e., by a total of thirty-six real numbers.

Binor transformation of the real components of a screw is accomplished by multiplying the latter by the elements of a square matrix with six rows and six columns. This matrix is found to be identical with the "motor" transformation matrix introduced by R. Mises [12].

When the screw is referred to a rectangular basis with the aid of the coordinates, the affine transformation operation on the vector r and moment r° of the screw to the new system reduces to multiplication of the matrices

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}, \quad \begin{vmatrix} A_{11}^+ & A_{12}^+ & A_{13}^+ \\ A_{21}^+ & A_{22}^+ & A_{23}^+ \\ A_{31}^+ & A_{32}^+ & A_{33}^+ \end{vmatrix}, \qquad (7.4)$$

by these vectors; here

$$A_{ik} = a_{ik} + \omega a_{ik}^0, \quad A_{ik}^+ = a_{ik}^+ + \omega a_{ik}^{+0}.$$

If the product is expanded in the real coordinates, we obtain the transformation

$$
\begin{aligned}
r'_x &= a_{11}r_x + a_{12}r_y + a_{13}r_z + a_{11}^+r_x^0 + a_{12}^+r_y^0 + a_{13}^+r_z^0, \\
r'_y &= a_{21}r_x + a_{22}r_y + a_{23}r_z + a_{21}^+r_x^0 + c_{22}^+r_y^0 + a_{23}^+r_z^0, \\
r'_z &= a_{31}r_x + a_{32}r_y + a_{33}r_z + a_{31}^+r_x^0 + a_{32}^+r_y^0 + a_{33}^+r_z^0, \\
r_x'^0 &= a_{11}^0r_x + a_{12}^0r_y + a_{13}^0r_z + a_{11}^{+0}r_x^0 + a_{12}^{+0}r_y^0 + a_{13}^{+0}r_z^0, \\
r_y'^0 &= a_{21}^0r_x + a_{22}^0r_y + a_{23}^0r_z + a_{21}^{+0}r_x^0 + a_{22}^{+0}r_y^0 + a_{23}^{+0}r_z^0, \\
r_z'^0 &= a_{31}^0r_x + a_{32}^0r_y + a_{33}^0r_z + a_{31}^{+0}r_x^0 + a_{32}^{+0}r_y^0 + a_{33}^{+0}r_z^0.
\end{aligned}
\tag{7.5}
$$

Thus, multiplication of a binor by a screw is equivalent to transformation of its real Plücker coordinates with the aid of the matrix

$$
\begin{vmatrix}
a_{11} & a_{12} & a_{13} & a_{11}^+ & a_{12}^+ & a_{13}^+ \\
a_{21} & a_{22} & a_{23} & a_{21}^+ & a_{22}^+ & a_{23}^+ \\
a_{31} & a_{32} & a_{33} & a_{31}^+ & a_{32}^+ & a_{33}^+ \\
a_{11}^0 & a_{12}^0 & a_{13}^0 & a_{11}^{+0} & a_{12}^{+0} & a_{13}^{+0} \\
a_{21}^0 & a_{22}^0 & a_{23}^0 & a_{21}^{+0} & a_{22}^{+0} & a_{23}^{+0} \\
a_{31}^0 & a_{32}^0 & a_{33}^0 & a_{31}^{+0} & a_{32}^{+0} & a_{33}^{+0}
\end{vmatrix}.
\tag{7.6}
$$

It is quickly recognized that in the case of an affinor, i.e., for $A_{ik}^+ = \omega A_{ik} = \omega a_{ik}$, the transformation matrix (7.6) assumes the following particular form:

$$
\begin{vmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\
a_{11}^0 & a_{12}^0 & a_{13}^0 & a_{11} & a_{12} & a_{13} \\
a_{21}^0 & a_{22}^0 & a_{23}^0 & a_{21} & a_{22} & a_{23} \\
a_{31}^0 & a_{32}^0 & a_{33}^0 & a_{31} & a_{32} & a_{33}
\end{vmatrix}.
\tag{7.7}
$$

If Matrices (7.6) and (7.7) are represented as block matrices, they will have the following respective forms:

$$
\begin{vmatrix} A & A^+ \\ \hline A^0 & A^{+0} \end{vmatrix}; \quad \begin{vmatrix} A & 0 \\ \hline A^0 & A \end{vmatrix}.
\tag{7.8}
$$

Repeated application of the binor multiplies out the matrices:

$$R'' = (B)R' = (B)(A)R = (C)R.$$

$$
\begin{vmatrix} C & C^+ \\ C^0 & C^{+0} \end{vmatrix} R = \begin{vmatrix} B & B^+ \\ B^0 & B^{+0} \end{vmatrix} \begin{vmatrix} A & A^+ \\ A^0 & A^{+0} \end{vmatrix} R = \begin{vmatrix} BA + B^+A^0 & B^0A^+ + B^+A^{+0} \\ B^0A + B^{+0}A^0 & B^0A^+ + B^{+0}A^{+0} \end{vmatrix} R.
\tag{7.9}
$$

A binor is a linear operator that possesses the property

$$(A)\,(R_1 + R_2) = (A)R_1 + (A)R_2,$$
$$(A)\lambda R = \lambda\,(A)R. \qquad\qquad (7.10)$$

where λ is a scalar multiplier.

Let us turn to Transformation (7.2) and express matrices A and A^+ with complex elements in terms of real matrices:

$$A = a + \omega a^0, \quad A^+ = a^+ + \omega a^{+0}.$$

Substituting in (7.2), we obtain

$$R' = (ar + a^+r^0) + \omega\,(a^0r + a^{+0}r^0). \qquad (7.2')$$

Expression (7.2') indicates that on transformation of screw R with the aid of a binor, the principal part of the transformed screw R' is not a result of transformation of the principal part of screw R only, but also depends on the latter's moment part. Consequently, the result of multiplication of a binor by a screw is a function of the screw that is not expressed by Formula (5.98) for the screw function of a screw and, consequently, does not satisfy the analyticity condition of which we spoke at the end of Chapter 5.

§2. Inertia Binor of a Solid Body

A.P. Kotel'nikov [5] introduced the notion of the momentum screw of a system of material points (kinetic screw). A momentum screw is a screw equivalent to a system of sliding vectors whose axes pass through points of the system and which are geometrically equal to the velocities of these points multiplied by the corresponding masses.

If the couplings in the system are such that we can impart a screw displacement to the entire system in any position without changing the relative positions of the points, then we say that a kinematic screw is possible for the system. In particular, if the system is a solid body, this will be a kinematic screw that determines the instantaneous screw motion of the body, and if it is referred to time, it will be a velocity screw.

Let this screw be U with the complex modulus

$$|U| = U = ue^{\omega p} = u + \omega u^0, \quad p = \frac{u^0}{u}, \qquad (7.11)$$

where u is the magnitude of the angular velocity and $u^0 = pu$ is the magnitude of the translational velocity of points of the body lying on the screw axis.

Let us take a certain point O on the central axis of the kinematic screw and denote by ρ the radius vector of an arbitrary point of the body reckoned from point O.

The velocity of a point, as defined by the kinematic screw U, is the moment of the motor obtained by reduction of this screw to the point, with the reduction radius ρ. Hence we obtain the fol-

lowing expression for the velocity of a point:

$$\boldsymbol{v} = \boldsymbol{u}^0 + \boldsymbol{u} \times \boldsymbol{\rho}. \tag{7.12}$$

Assigning an elementary mass dm to the point of the body, we obtain the momentum of the point:

$$\boldsymbol{v}\, dm = \boldsymbol{u}^0\, dm + \boldsymbol{u} \times \boldsymbol{\rho}\, dm. \tag{7.13}$$

The angular momentum of the point with respect to point O is equal to

$$\boldsymbol{\rho} \times \boldsymbol{v}\, dm = \boldsymbol{\rho} \times (\boldsymbol{u}^0 + \boldsymbol{u} \times \boldsymbol{\rho})\, dm. \tag{7.14}$$

Integrating (7.13) and (7.14) over all points, we obtain the vector and angular momentum of the solid body, which form a motor equivalent to the body's momentum screw:

$$K \to \boldsymbol{k} + \omega \boldsymbol{k}^0 = \int \boldsymbol{v}\, dm + \omega \int \boldsymbol{\rho} \times \boldsymbol{v}\, dm. \tag{7.15}$$

Let us write Expressions (7.13) and (7.14) in rectangular coordinates. For projection of the momentum vector onto the axes x, y, z, we shall have the expressions

$$\left.\begin{aligned}
v_x dm &= (u_x^0 + u_y \zeta - u_z \eta)\, dm, \\
v_y dm &= (u_y^0 + u_z \xi - u_x \zeta)\, dm, \\
v_z dm &= (u_z^0 + u_x \eta - u_y \xi)\, dm,
\end{aligned}\right\} \tag{7.16}$$

and for the moments of this vector with respect to the same axes, the expressions

$$\left.\begin{aligned}
(v_z \eta - v_y \zeta)\, dm &= [u_z^0 \eta - u_y^0 \zeta + u_x(\eta^2 + \zeta^2) - u_y \xi \eta - u_z \xi \zeta]\, dm, \\
(v_x \zeta - v_z \xi)\, dm &= [u_x^0 \zeta - u_z^0 \xi - u_x \xi \eta + u_y(\xi^2 + \zeta^2) - u_z \eta \zeta]\, dm, \\
(v_y \xi - v_x \eta)\, dm &= [u_y^0 \xi - u_x^0 \eta - u_x \xi \zeta - u_y \eta \zeta + u_z(\xi^2 + \eta^2)]\, dm,
\end{aligned}\right\} \tag{7.17}$$

where ξ, η, ζ are the projections of radius vector ρ on the axes x, y, z.

Now we integrate Expressions (7.16) and (7.17) over all points, simultaneously adopting the notation

$$\left.\begin{aligned}
\int dm &= m, \quad \int \xi\, dm = S_1, \quad \int \eta\, dm = S_2, \quad \int \zeta\, dm = S_3, \\
\int \eta \zeta\, dm &= D_1, \quad \int \zeta \xi\, dm = D_2, \quad \int \xi \eta\, dm = D_3, \\
\int (\eta^2 + \zeta^2)\, dm &= I_1, \quad \int (\xi^2 + \zeta^2)\, dm = I_2, \\
\int (\xi^2 + \eta^2)\, dm &= I_3
\end{aligned}\right\} \tag{7.18}$$

as abbreviated notations for mass, static moments, products of inertia and moments of inertia with respect to the axes.

Then we find expressions for the projections of the vector $\boldsymbol{k}$ and moment $\boldsymbol{k}^0$ of the motor corresponding to the body's momentum

- 135 -

screw onto axes x, y, z:

$$
\begin{aligned}
k_x &= & S_3 u_y - S_2 u_z + m u_x^0 & , \\
k_y &= -S_3 u_x & + S_1 u_z & + m u_y^0 & , \\
k_z &= S_2 u_x - S_1 u_y & & + m u_z^0, \\
k_x^0 &= I_1 u_x - D_3 u_y - D_2 u_z & - S_3 u_y^0 + S_2 u_z^0, \\
k_y^0 &= -D_3 u_x + I_2 u_y - D_1 u_z + S_3 u_x^0 & - S_1 u_z^0, \\
k_z^0 &= -D_2 u_x - D_1 u_y + I_3 u_z - S_2 u_x^0 + S_1 u_y^0 & .
\end{aligned}
\right\} \quad (7.19)
$$

Formulas (7.19) indicate that the motor $k + \omega k^0$ is obtained by multiplying by the motor $u + \omega u^0$ the binor (T) determined by the matrix

$$
\begin{vmatrix}
0 & S_3 & -S_2 & m & 0 & 0 \\
-S_3 & 0 & S_1 & 0 & m & 0 \\
S_2 & -S_1 & 0 & 0 & 0 & m \\
I_1 & -D_3 & -D_2 & 0 & -S_3 & S_2 \\
-D_3 & I_2 & -D_1 & S_3 & 0 & -S_1 \\
-D_2 & -D_1 & I_3 & -S_2 & S_1 & 0
\end{vmatrix} ,
$$

and namely

$$
(T) = \left(\left\| \begin{matrix}
\omega I_1 & S_3 - \omega D_3 & -S_2 - \omega D_2 \\
-S_3 - \omega D_3 & \omega I_2 & S_1 - \omega D_1 \\
S_2 - \omega D_2 & -S_1 - \omega D_1 & \omega I_3
\end{matrix} \right\| , \right.
$$
$$
\left. \left\| \begin{matrix}
m & -\omega S_3 & \omega S_2 \\
\omega S_3 & m & -\omega S_1 \\
-\omega S_2 & \omega S_1 & m
\end{matrix} \right\| \right) , \qquad (7.20)
$$

which can be written in abbreviated form as follows:

$$
K = (T)U. \qquad (7.21)
$$

The binor (T) is called the inertia binor of the solid body. Formula (7.21) therefore states the fact that the momentum screw is obtained by multiplying the inertia binor by the kinematic screw.

§3. Equation of Motion of a Solid Body in Screw Form

Differentiating Equalities (7.19) with respect to time, we obtain the time derivatives of the projections and moments of the momentum screw in the left member and the time derivatives of the corresponding products of the inertia binor by the kinematic screw in the right member. The corresponding terms of the right members of the equalities will express the products of the masses and static moments by the projections of the body's center-of-gravity acceleration and the products of the moments of inertia by the angular accelerations. These will be the projections of the acting-forces screw onto the coordinate axes and the moments about these axes. Consequently, passing to the screw equality (7.21), we shall have the relation derived by R. Mises [13],

$$\frac{dK}{dt} = \frac{d}{dt} [(T)U] = R,$$ (7.22)

where R is the acting-forces screw. Equation (7.22) is the screw notation of the law of momentum and the law of angular momentum.

Let the body under consideration have a screw displacement with respect to fixed space determined by screw U; if we wish to express the time derivative of the screw $K = (T)U$ with respect to fixed space in terms of the time derivative with respect to a coordinate system attached to the moving body, we must apply Formula (5.72), obtaining

$$\frac{dK}{dt} = \frac{d'K}{dt} + U \times K = \frac{d}{dt} [(T)U] + U \times [(T)U] = R,$$

where the symbol d'/dt denotes the relative or "apparent" derivative, i.e., the derivative presented to an observer on the moving system. Multiplying out the brackets after the d' sign and noting that in the coordinate system attached to the body the inertia binor is constant, i.e., that $d'(T) = 0$, we obtain

$$(T) \frac{d'U}{dt} + U \times [(T)U] = R.$$ (7.23)

Expansion of Eqs. (7.22) and (7.23) by coordinates results in a system of six dynamic equations.

One remark is in order.

Assume that we wish to obtain from Eq. (7.23) the dynamic equation for a body having a fixed point (vector equation). For this purpose, it would be necessary to assume that the kinematic screw U has become an angular-velocity vector passing through the fixed point. Taking this latter point as the coordinate origin, we must set the translational-displacement coordinates of this point of the body equal to zero, and add the reactions at the fixed point to the projections of the external-force principal vector. Then the dynamic equations decompose into two groups of three equations each. But those three equations that express the relation of the principal part of screw U, i.e., the angular-velocity vector, to the moment of the external forces will be not the principal part of the equations, but rather their moment part. The corresponding vector equation will be not the principal, but the moment part of screw equation (7.23). Thus, the differential equations for the principal part of the kinematic screw are not the principal part of the basic differential equations, but, to the contrary, are their moment part.

This circumstance stems from the fact that if an angular velocity serves as the principal part, i.e., the vector, in a kinematic screw, the principal vector of forces will be the principal part in the force screw; on the other hand, the moment is the generalized force for the angular coordinate. Moreover, multiplication of a binor by a screw places both a vector and a moment in the principal part. Consequently, the binor cannot be obtained from any real operator by substituting complex quantities for real

quantities in it, i.e., a binor is not an operator that possesses the property of "analyticity," and the screw formulas obtained as a result of its application are not a direct generalization of the vector formulas (see §1 of this chapter).

It must be concluded on the basis of the above that it is impossible to obtain a screw equation of the dynamics of an arbitrarily moving body from the dynamics vector equation of a body with a fixed point by application of the transfer principle.

§4. Statics and Small Vibrations of an Elastically Suspended Solid Body

Practical interest attaches to solution of the problem of equilibrium and vibrations of a solid body suspended in space with the aid of a certain number of elastic links or springs; it consists in determination of the forces in the springs when a given force acts upon the body. The positioning of the springs may be arbitrary, but it is an indispensable condition that no displacement of the body be possible without deformation of the springs, i.e., that the entire system be incapable of moving freely as a mechanism.

For more than six springs, the problem of finding the forces is statically indeterminate. If, however, we introduce some additional condition linking the forces in the springs to their elongations (or compressions), the problem may be reduced to a problem of the statics of a solid body.

Let us take the relation expressed by simple proportionality between the force in the spring and the change in its length and assume that the forces in all springs are zero when the body is in its unloaded position. We shall also assume that the displacements of the body are small.

We define the positions of the spring axes in space by their Plücker coordinates — the direction cosines of the unit vector $\boldsymbol{E}_i$ of the axes and the moments of these vectors with respect to the axes of a certain rectangular coordinate system xyz. Let the angles formed by the spring axes with the coordinate axes be $\alpha_i, \beta_i, \gamma_i,$ and let the coordinates of the points of attachment of the springs to the body be $\xi_i, \eta_i, \zeta_i,$ where i is the number assigned to the spring. The unit-vector moments of the spring axes about the coordinate axes will have the expressions

$$l_i = \eta_i \cos\gamma_i - \zeta_i \cos\beta_i, \quad m_i = \zeta_i \cos\alpha_i - \xi_i \cos\gamma_i,$$
$$n_i = \xi_i \cos\beta_i - \eta_i \cos\alpha_i$$

and, consequently, the Plücker coordinates of the spring axes will be

$$\cos\alpha_i, \ \cos\beta_i, \ \cos\gamma_i, \ l_i, \ m_i, \ n_i. \tag{7.24}$$

Let us impart to the body a small screw displacement characterized by an arbitrary screw $\bullet$ defined by the coordinates

$$\varphi_x, \ \varphi_y, \ \varphi_z, \ \delta_x, \ \delta_y, \ \delta_z,$$

where the first three quantities are the projections of the body's angle of rotation onto the coordinate axes and the last three are the projections, onto the same axes, of the displacement of the point of the body that coincides with the coordinate origin.

In order to express the force arising in the ith spring as a result of this displacement, it is necessary to find the displacement of some point inseparably connected with the body and lying on the spring axis (for example, the point of attachment of the spring to the body) and to project this displacement onto the spring axis. We obtain the elongation or compression of the spring, and, multiplying this quantity by the rigidity coefficient c_i of the spring, we find the force S_i of the spring. But a (small) displacement of a point of the body is expressed by the displacement-screw moment with respect to this point, while the projection of the moment onto a straight line passing through the point is the relative moment of the screw and the line. Consequently, for a spring with an axis unit vector E_i, we shall have a force for displacement of the body along screw Φ:

$$S_i = c_i \text{ MOM } (\Phi \cdot E_i) =$$
$$= c_i (\delta_x \cos \alpha_i + \delta_y \cos \beta_i + \delta_z \cos \gamma_i + \varphi_x l_i + \varphi_y m_i + \varphi_z n_i). \qquad (7.25)$$

We express the projections of the forces $S_i = E_i S_i$ onto the coordinate axes and their moments with respect to these axes, multiplying the magnitude of S_i successively by each of the six quantities of (7.24), and then, summing the projections and moments over all springs, we find the six coordinates $-P_x, -P_y, -P_z, -L_x, -L_y, -L_z$ of the force screw that compensates this force screw R which is capable of causing displacement of the body along screw Φ.

We note that in view of the assumed smallness of the displacement, we draw no distinction between the initial, i.e., the unloaded position of the body and its final position, i.e., that which has been reached after the displacement. This assumption is customary in the structural mechanics of rod systems and in the theory of small vibrations; moreover, it corresponds to the first-approximation solution in those cases in which we take into account the nonlinearity associated with the influence of displacement components of the second and higher negative orders.

Performing the multiplications and summations indicated above, we obtain the following system of equilibrium equations for an elastically suspended solid body:

$$
\begin{aligned}
c_{11}\delta_x + c_{12}\delta_y + c_{13}\delta_z + c_{14}\varphi_x + c_{15}\varphi_y + c_{16}\varphi_z - P_x &= 0, \\
c_{21}\delta_x + c_{22}\delta_y + c_{23}\delta_z + c_{24}\varphi_x + c_{25}\varphi_y + c_{26}\varphi_z - P_y &= 0, \\
c_{31}\delta_x + c_{32}\delta_y + c_{33}\delta_z + c_{34}\varphi_x + c_{35}\varphi_y + c_{36}\varphi_z - P_z &= 0, \\
c_{41}\delta_x + c_{42}\delta_y + c_{43}\delta_z + c_{44}\varphi_x + c_{45}\varphi_y + c_{46}\varphi_z - L_x &= 0, \\
c_{51}\delta_x + c_{52}\delta_y + c_{53}\delta_z + c_{54}\varphi_x + c_{55}\varphi_y + c_{56}\varphi_z - L_y &= 0, \\
c_{61}\delta_x + c_{62}\delta_y + c_{63}\delta_z + c_{64}\varphi_x + c_{65}\varphi_y + c_{66}\varphi_z - L_z &= 0,
\end{aligned}
\qquad (7.26)
$$

where

$$c_{11} = \sum c \cos^2 \alpha, \quad c_{22} = \sum c \cos^2 \beta, \quad c_{33} = \sum c \cos^2 \gamma,$$
$$c_{12} = c_{21} = \sum c \cos \alpha \cos \beta, \quad c_{13} = c_{31} = \sum c \cos \alpha \cos \gamma,$$
$$c_{14} = c_{41} = \sum c \cos \alpha l,$$
$$c_{15} = c_{51} = \sum c \cos \alpha m, \quad c_{16} = c_{61} = \sum c \cos \alpha n,$$
$$c_{24} = c_{42} = \sum c \cos \beta l, \quad c_{25} = c_{52} = \sum c \cos \beta m,$$
$$c_{26} = c_{62} = \sum c \cos \beta n,$$
$$c_{34} = c_{43} = \sum c \cos \gamma l, \quad c_{35} = c_{53} = \sum c \cos \gamma m,$$
$$c_{36} = c_{63} = \sum c \cos \gamma n,$$
$$c_{44} = \sum c l^2, \quad c_{55} = \sum c m^2, \quad c_{66} = \sum c n^2,$$
$$c_{45} = c_{54} = \sum c l m, \quad c_{46} = c_{64} = \sum c l n,$$
$$c_{56} = c_{65} = \sum c m n.$$

(7.27)

In Formulas (7.27), the summation is extended over all springs; the index i has been omitted.

Thus, solution of the static problem reduces to determination of the coordinates of an unknown displacement screw Φ from the given coordinates of a force screw R by equation system (7.26).

Equations (7.26) can be expressed in binor form as follows:

$$\begin{vmatrix} c_{14}+\omega c_{44} & c_{15}+\omega c_{54} & c_{16}+\omega c_{64} \\ c_{24}+\omega c_{44} & c_{25}+\omega c_{54} & c_{26}+\omega c_{64} \\ c_{34}+\omega c_{44} & c_{35}+\omega c_{54} & c_{36}+\omega c_{64} \end{vmatrix} \begin{vmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{vmatrix} +$$
$$+ \begin{vmatrix} c_{11}+\omega c_{41} & c_{12}+\omega c_{42} & c_{13}+\omega c_{43} \\ c_{21}+\omega c_{41} & c_{22}+\omega c_{42} & c_{23}+\omega c_{43} \\ c_{31}+\omega c_{41} & c_{32}+\omega c_{42} & c_{33}+\omega c_{43} \end{vmatrix} \begin{vmatrix} \delta_x \\ \delta_y \\ \delta_z \end{vmatrix} - \begin{vmatrix} P_x+\omega L_x \\ P_y+\omega L_y \\ P_z+\omega L_z \end{vmatrix} = 0$$

(7.28)

or, introducing the short notation (C) for the elasticity binor,

$$(C)\Phi = (C, C^+)(\varphi + \omega \delta) = C\varphi + C^+\delta = R. \tag{7.29}$$

It follows from the linearity of the operator (C) that if two states

$$(C)\Phi_1 = R_1, \quad (C)\Phi_2 = R_2,$$

are given, then the state corresponding to linear combination of displacements Φ_1 and Φ_2 will be characterized by the equality

$$(C)(\lambda\Phi_1 + \mu\Phi_2) = \lambda R_1 + \mu R_2,$$

from which it follows that the linear combination of the force screws acting on the body corresponds to the linear combination of the displacement screws. A linear combination of screws with real multipliers is a screw that lies on the same cylindroid as the basic screws; if the multipliers of the linear combination are complex, then the axis of the linear-combination screw describes a brush, i.e., a set of straight lines intersecting at right angles with a certain straight line. Both the axis of the

displacement screw and the axis of the force screw will describe brushes.

The validity of the following proposition is equally verified: if a body is acted upon statically by a force screw R' that causes elastic displacement of the body along a screw Φ', then on static action of a force screw R'' reciprocal to screw Φ', elastic displacement of the body will occur along a screw Φ'', reciprocal to screw R'. For the proof, let us consider states 1 and 2, which correspond to the action of the forces R' and R''; we write the expression for the works done by the external forces of the 1st state on the displacements of the 2nd state and those done by the external forces of the 2nd state on displacements of the 1st state. We have:

a) for the 1st state, the force screw R', and the displacement screw Φ', with $R' = (C)\Phi'$,

b) for the 2nd state, the force screw R'' and the displacement screw Φ'', with $R'' = (C)\Phi''$.

Writing the expression for the work as the moment part of the screw scalar product and expanding it, we obtain

$$
\begin{aligned}
\text{мом } ([(C)\,\Phi']\cdot\Phi'') &= P'_x\delta''_x + P'_y\delta''_y + P'_z\delta''_z + L'_x\varphi''_x + \\
&+ L'_y\varphi''_y + L'_z\varphi''_z = c_{11}\delta'_x\delta''_x + c_{12}\delta'_y\delta''_x + c_{13}\delta'_z\delta''_x + \\
&+ c_{14}\varphi'_x\delta''_x + c_{15}\varphi'_y\delta''_x + c_{16}\varphi'_z\delta''_x + c_{21}\delta'_x\varphi''_x + c_{22}\delta'_y\varphi''_x + \\
&+ c_{23}\delta'_z\varphi''_x + c_{24}\varphi'_x\varphi''_x + c_{25}\varphi'_y\varphi''_x + c_{26}\varphi'_z\varphi''_x + \\
&+ c_{31}\delta'_x\delta''_y + c_{32}\delta'_y\delta''_y + c_{33}\delta'_z\delta''_y + c_{34}\varphi'_x\delta''_y + \\
&+ c_{35}\varphi'_y\delta''_y + c_{36}\varphi'_z\delta''_y + c_{41}\delta'_x\varphi''_y + c_{42}\delta'_y\varphi''_y + \\
&+ c_{43}\delta'_z\varphi''_y + c_{44}\varphi'_x\varphi''_y + c_{45}\varphi'_y\varphi''_y + c_{46}\varphi'_z\varphi''_y + \\
&+ c_{51}\delta'_x\delta''_z + c_{52}\delta'_y\delta''_z + c_{53}\delta'_z\delta''_z + c_{54}\varphi'_x\delta''_z + \\
&+ c_{55}\varphi'_y\delta''_z + c_{56}\varphi'_z\delta''_z + c_{61}\delta'_x\varphi''_z + c_{62}\delta'_y\varphi''_z + \\
&+ c_{63}\delta'_z\varphi''_z + c_{64}\varphi'_x\varphi''_z + c_{65}\varphi'_y\varphi''_z + c_{66}\varphi'_z\varphi''_z.
\end{aligned}
$$

The above expression is symmetrical with respect to the indices $'$ and $''$, from which it follows that

$$
\text{мом } ([(C)\,\Phi']\cdot\Phi'') = \text{мом } ([(C)\,\Phi'']\cdot\Phi'),
$$

or

$$
\text{мом } (R'\cdot\Phi'') = \text{мом } (R''\cdot\Phi'), \qquad (7.30)
$$

which is the well-known reciprocity theorem.

If force screw R'' is reciprocal to the displacement screw Φ', then the right member of (7.30) is equal to zero and, consequently, the left member of (7.30) is also equal to zero, from which it follows that the displacement screw Φ'' is reciprocal to force screw R'.

The above enables us to construct a comparatively simple scheme for determination of screw Φ from the given force screw R. To wit, we impart five displacements to the body along screws $\Phi_1, \Phi_2, \ldots, \Phi_5$, each of which is reciprocal to screw R. These screws

may be: 1) screws ϕ_1 and ϕ_2, whose axes intersect the axis of screw R at right angles and whose parameter is equal to zero (simple rotations); 2) screws ϕ_3 and ϕ_4, whose axes are perpendicular to the axis of screw R and whose parameter is infinity (translational displacements) and, finally, 3) screw ϕ_5, which lies on the same axis with screw R and has a parameter equal in absolute magnitude to the parameter of screw R and opposite to it in sign. This last screw will be reciprocal to R by virtue of the vanishing of Expression (3.12) for a possible coefficient of screws ϕ_5 and R.

To each of the displacement screws listed above, there will correspond a force screw capable, on application to the body, of causing displacement along this screw. Let the force screws corresponding to the five screw displacements listed above be $R_1, R_2, \ldots, R_r$

Now it can be seen on the basis of the above that the screw ϕ reciprocal to screws $R_1, R_2, \ldots, R_4$ will be the sought screw of the displacements due to the action of force screw R.

The entire solution can be carried through using graphoanalytical and even purely graphical operations that employ the geometrical interpretation of screws as orthogonal crosses. In essence, this solution supplants the analytical solution of equation system (7.26).

Let us now examine small vibrations of an elastically suspended solid body due to the action of a force screw $R = R_0 \sin \lambda t$, where K_0 is the amplitude screw.

For this case, it is necessary to use the dynamic equation of the solid body, (7.22), together with the static equation (7.29).

Let us denote the unknown screw by ϕ, and the velocity screw by $\dot{\phi}$. Taking an arbitrary coordinate origin O, we obtain motors $\phi + \omega\delta$ and $\dot{\phi} + \omega\delta$ by reduction to point O. On the basis of Eq. (7.22), we shall have for the fixed coordinate system

$$\frac{d}{dt}[(T)\dot{\phi}] = -R_{ynp} + R_0 \sin \lambda t, \qquad (7.31)$$

where R_{upr} is the screw of the internal elastic forces and is linearly expressible in terms of the displacement screw of the elastically suspended body. On the basis of Eq. (7.29), this screw is expressed as follows:

$$R_{ynp} = (C)\phi. \qquad (7.32)$$

Expanding the expression in brackets under the time-differentiation sign in the left member of Eq. (7.31) and substituting the expression from (7.32) for it, we obtain

$$(T)\frac{d\dot{\phi}}{dt} + \frac{d(T)}{dt}\dot{\phi} + (C)\phi = R_0 \sin \lambda t.$$

Taking the smallness of the vibration amplitudes into account and assuming further that the body's ellipsoid of inertia is not too prolate, it may be assumed for the first-approximation solution of the problem corresponding to its linear formulation that $d(T)/dt$ is a quantity of the second order of smallness and that it can be disregarded. In this case, the differential equation of the body's vibrations will assume the form

$$(T)\ddot{\Phi} + (C)\Phi = R = R_0 \sin \mathcal{U}, \qquad (7.33)$$

which is the differential equation of the "screw" operator and is equivalent to six scalar differential equations.

Here (T) is the binor of inertia and (C) is the binor of elasticity.

For an arbitrarily selected coordinate system xyz, Eq. (7.33) has the following form in binor-matrix notation:

$$\begin{vmatrix} \omega I_1 & S_0 - \omega D_0 & -S_1 - \omega D_1 \\ -S_0 - \omega D_0 & \omega I_2 & S_1 - \omega D_1 \\ S_1 - \omega D_1 & -S_1 - \omega D_1 & \omega I_3 \end{vmatrix} \begin{vmatrix} \ddot{\varphi}_x \\ \ddot{\varphi}_y \\ \ddot{\varphi}_z \end{vmatrix} +$$

$$+ \begin{vmatrix} m & -\omega S_0 & \omega S_2 \\ \omega S_0 & m & -\omega S_1 \\ -\omega S_2 & \omega S_1 & m \end{vmatrix} \begin{vmatrix} \ddot{\delta}_x \\ \ddot{\delta}_y \\ \ddot{\delta}_z \end{vmatrix} +$$

$$+ \begin{vmatrix} c_{11} + \omega c_{11} & c_{12} + \omega c_{12} & c_{13} + \omega c_{13} \\ c_{21} + \omega c_{21} & c_{22} + \omega c_{22} & c_{23} + \omega c_{23} \\ c_{31} + \omega c_{31} & c_{32} + \omega c_{32} & c_{33} + \omega c_{33} \end{vmatrix} \begin{vmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{vmatrix} +$$

$$+ \begin{vmatrix} c_{11} + \omega c_{11} & c_{12} + \omega c_{12} & c_{13} + \omega c_{13} \\ c_{21} + \omega c_{21} & c_{22} + \omega c_{22} & c_{23} + \omega c_{23} \\ c_{31} + \omega c_{31} & c_{32} + \omega c_{32} & c_{33} + \omega c_{33} \end{vmatrix} \begin{vmatrix} \delta_x \\ \delta_y \\ \delta_z \end{vmatrix} -$$

$$- \begin{vmatrix} P_x + \omega L_x \\ P_y + \omega L_y \\ P_z + \omega L_z \end{vmatrix} - \begin{vmatrix} R_x \\ R_y \\ R_z \end{vmatrix}. \qquad (7.34)$$

Given appropriate selection of the coordinate system, we can arrive at a simplified form of either the binor (T) or the binor (C). Let us assume that the body's three principal central axes of inertia have been taken as the coordinate system; then the inertia binor (T) that appears in the first two terms of the left member of Eq. (7.34) will assume the simple form

$$\left(\begin{vmatrix} \omega I_1 & 0 & 0 \\ 0 & \omega I_2 & 0 \\ 0 & 0 & \omega I_3 \end{vmatrix}, \begin{vmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{vmatrix} \right).$$

Assume that we wished to select the coordinate system with a view to simplifying binor (C) in Eq. (7.34). In the general case, if the quantities $\delta_{x,y,z}$ and $\varphi_{x,y,z}$ are taken as the coordinates of the motion and we do not convert to generalized screw coordinates, the matrices cannot be simplified to any appreciable degree, as

can be seen from the analysis given below. This analysis makes it possible to bring out the structure of the solid body's "elastic suspension" and establish the cases in which one or another simplification is possible.

Let us seek translational-displacement directions of the body with which the principal vector of the resultant spring force is parallel to the displacement. Obviously, this will reduce to elimination of nondiagonal elements of the matrix

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix},$$

and we obtain three mutually perpendicular principal directions of suspension rigidity.

In the general case, the resultant of the spring forces in translational displacement of the body will be a screw; if we take the axes of the coordinate system parallel to the three principal directions of suspension rigidity, the coordinates of the axes x', y', z' of the three corresponding screws will be

$$\left. \begin{array}{ll} \eta'_x = \dfrac{\Sigma c \cos \alpha n}{\Sigma c \cos^2 \alpha}, & \zeta'_x = \dfrac{\Sigma c \cos \alpha m}{\Sigma c \cos^2 \alpha}, \\[2mm] \xi'_y = \dfrac{\Sigma c \cos \beta n}{\Sigma c \cos^2 \beta}, & \zeta'_y = \dfrac{\Sigma c \cos \beta l}{\Sigma c \cos^2 \beta}, \\[2mm] \xi'_z = \dfrac{\Sigma c \cos \gamma m}{\Sigma c \cos^2 \gamma}, & \eta'_z = \dfrac{\Sigma c \cos \gamma l}{\Sigma c \cos^2 \gamma}. \end{array} \right\} \tag{7.35}$$

The magnitudes of the vectors and moments of the screws equivalent to the spring resistance forces in translational displacements of the body by one unit along each of the axes x, y, z will have the expressions

$$X'_x = \Sigma c \cos^2 \alpha, \quad Y'_x = 0, \quad Z'_x = 0,$$
$$L'_x = \Sigma c \cos \alpha l, \quad M'_x = 0, \quad N'_x = 0,$$
$$X'_y = 0, \quad Y'_y = \Sigma c \cos^2 \beta, \quad Z'_y = 0, \quad L'_y = 0,$$
$$M'_y = \Sigma c \cos \beta m, \quad N'_y = 0,$$
$$X'_z = 0, \quad Y'_z = 0, \quad Z'_z = \Sigma c \cos^2 \gamma,$$
$$L'_z = 0, \quad M'_z = 0, \quad N'_z = \Sigma c \cos \gamma n.$$

We can easily satisfy ourselves that for pure rotations of the body about axes x^*, y^*, z^* that intersect the pairs of axes $y'z''$, $x'z'$ and $x'y'$ at right angles, the axes of the resultant screws of the spring forces will also be parallel to the axes of rotation.

Let the axes of these screws be x'', y'', z''. The magnitudes of the principal vectors and principal moments of these screws are determined as follows: on rotations about the axes x^*, y^* and z^* through a unit angle, the force in the ith spring will be equal to the relative moments of the unit vector of the spring with the coordinates (7.24) and the unit vectors of the indicated

axes of rotation, which have the Plücker coordinates (η'_z, ζ'_y), (ξ'_z, ζ'_x), (ξ'_y, η'_x), i.e., for the respective axes it will be

$$S_{ix} = c_i (l_i + \cos \beta_i \zeta'_y - \cos \gamma_i \eta'_z),$$
$$S_{iy} = c_i (m_i - \cos \alpha_i \zeta'_x + \cos \gamma_i \xi'_z),$$
$$S_{iz} = c_i (n_i + \cos \alpha_i \eta'_x - \cos \beta_i \xi'_y).$$

Multiplying these quantities one by one by $\cos \alpha_i$, $\cos \beta_i$, $\cos \gamma_i$, l_i, m_i, n_i and summing over all springs, we obtain the projections and moments of the spring resultant-force screws in the xyz coordinate system:

$$
\left.
\begin{aligned}
X'_x &= \sum S_{ix} \cos \alpha_i = \sum c_i \cos \alpha_i l_i, \\
Y'_x &= \sum S_{ix} \cos \beta_i = 0, \quad Z'_x = \sum S_{ix} \cos \gamma_i = 0, \\
T'_x &= \sum S_{ix} l_i = \sum c_i l_i^2 + \zeta'_y \sum c_i \cos \beta_i l_i - \\
 &\qquad\qquad\qquad - \eta'_z \sum c_i \cos \gamma_i l_i, \\
M'_x &= \sum S_{ix} m_i = \sum c_i l_i m_i + \zeta'_y \sum c_i \cos \beta_i m_i - \\
 &\qquad\qquad\qquad - \eta'_z \sum c_i \cos \gamma_i m_i, \\
N'_x &= \sum S_{ix} n_i = \sum c_i l_i n_i + \zeta'_y \sum c_i \cos \beta_i n_i - \\
 &\qquad\qquad\qquad - \eta'_z \sum c_i \cos \gamma_i n_i
\end{aligned}
\right\} \quad (7.36)
$$

and so forth, and then we find the coordinates of the axes of these screws:

$$
\left.
\begin{aligned}
\zeta'_x &= \frac{\sum c l m + \zeta'_y \sum c \cos \beta m - \eta'_z \sum c \cos \gamma n}{\sum c \cos \alpha l}, \\
\eta'_x &= \frac{\sum c l n + \zeta'_y \sum c \cos \beta n - \eta'_z \sum c \cos \gamma n}{\sum c \cos \alpha l}, \\
\zeta'_y &= \frac{\sum c m l - \zeta'_x \sum c \cos \alpha l + \xi'_z \sum c \cos \gamma l}{\sum c \cos \beta m}, \\
\xi'_y &= \frac{\sum c m n - \zeta'_x \sum c \cos \alpha n + \xi'_z \sum c \cos \gamma n}{\sum c \cos \beta m}, \\
\eta'_z &= \frac{\sum c l n + \eta'_x \sum c \cos \alpha l - \xi'_y \sum c \cos \beta l}{\sum c \cos \gamma n}, \\
\xi'_z &= \frac{\sum c m n + \eta'_x \sum c \cos \alpha m - \xi'_y \sum c \cos \beta m}{\sum c \cos \gamma n}.
\end{aligned}
\right\} \quad (7.37)
$$

The magnitudes of the vectors and moments of all restoring forces of the springs in rotations of the body about x^*, y^*, z^* through unit angles will be

$$X'_x = \sum c \cos \alpha l, \quad Y'_x = 0, \quad Z'_x = 0,$$
$$L'_x = \sum c l^2 + \zeta'_y \sum c \cos \beta l - \eta'_z \sum c \cos \gamma l, \quad M'_x = N'_x = 0,$$
$$X'_y = 0, \quad Y'_y = \sum c \cos \beta m, \quad Z'_y = 0, \quad L'_y = 0,$$
$$M'_y = \sum c m^2 - \zeta'_x \sum c \cos \alpha m - \xi'_z \sum c \cos \gamma n, \quad N'_y = 0,$$
$$X'_z = 0, \quad Y'_z = 0, \quad Z_z = \sum c \cos \gamma n, \quad L'_z = M'_z = 0,$$
$$N'_z = \sum c n^2 + \eta'_x \sum c \cos \alpha n - \xi'_y \sum c \cos \beta n.$$

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The axes of these screws are parallel to the axes x, y, z, and their positions are determined by Formulas (7.37).

Thus, unit translational displacements of the body along three mutually perpendicular directions and unit rotations of the body about three given axes parallel to them result in a system of six screws (X', L'), (Y', M'), (Z', N') and (X'', L''), (Y'', M''), (Z'', N'') of the resultant forces in the springs. In the general case, the axes of these screws do not intersect. The positions of the axes are shown schematically in Fig. 39.
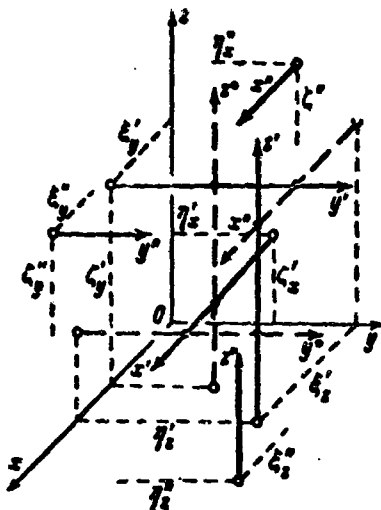
Obviously, this system of six screws characterizes the structure of the body's elastic suspension, i.e., the spring system. In particular cases, the system may be symmetrical with respect to one of the planes — in this case the axes of two of the screws intersect and the parameters of these screws are equal to zero; for a system having two planes of symmetry, two screw axes intersect a third; a quasi-symmetric system in which the axes of the above six screws form two coincident mutually perpendicular sets of three, with the parameters of these screws nonzero, is a possibility. In the latter case, the suspension has a "center" of elasticity, and the axes of the screws are the principal axes of elasticity.

Fig. 39

From examination of the diagram of Fig. 39, we may conclude that in the general case, the equation system does not decompose into independent equations if translational displacements and rotations of the body with respect to the individual axes are taken as the sought coordinates. This is possible only in a particular case — in the presence of a center of elasticity and for coincidence of the principal central axes of inertia of the body with the principal axes of elasticity.

The problem of vibrations of an elastically suspended body consists in determination of the displacement screw Φ from a given force screw R. This problem presents no fundamental difficulty, and we shall not dwell on its analytical solution. Interest does attach to certain properties of the elastically suspended body system under consideration and to geometrical interpretation of its vibrations — the positions of the axes of the displacement screws as functions of the axes of the force screws acting on the system.

In the event that the principal central axes of inertia of the body are taken as the coordinate axes, the system of six differential equations equivalent to the binor equation (7.33), written in vector form after separation of the moment from the principal part, will be

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$$M\ddot{\delta} + C_1\delta + C_2\varphi = P = P_0\sin\lambda t,$$
$$T\ddot{\varphi} + C_3\delta + C_4\varphi = L = L_0\sin\lambda t,$$
(7.38)

where

$$M = \begin{vmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{vmatrix}, \quad T = \begin{vmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{vmatrix}, \quad C_1 = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{21} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix},$$

$$C_2 = \begin{vmatrix} c_{14} & c_{15} & c_{16} \\ c_{24} & c_{25} & c_{26} \\ c_{34} & c_{35} & c_{36} \end{vmatrix}, \quad C_3 = \begin{vmatrix} c_{41} & c_{51} & c_{61} \\ c_{42} & c_{52} & c_{62} \\ c_{43} & c_{53} & c_{63} \end{vmatrix}, \quad C_4 = \begin{vmatrix} c_{44} & c_{45} & c_{46} \\ c_{54} & c_{55} & c_{56} \\ c_{64} & c_{65} & c_{66} \end{vmatrix}.$$

For $P_0 = L_0 = 0$, we have natural vibrations of the body. Since the system has six degrees of freedom, there exists a total of six natural frequencies $\lambda^{(\mu)}$, to each of which there corresponds a "natural screw" $\Phi^{(\mu)} = \Phi_0^{(\mu)}\sin\lambda t$, where $\Phi_0^{(\mu)}$ is an amplitude screw with the coordinates

$$\varphi_x^{(\mu)}, \ \varphi_y^{(\mu)}, \ \varphi_z^{(\mu)}, \ \delta_x^{(\mu)}, \ \delta_y^{(\mu)}, \ \delta_z^{(\mu)},$$
(7.39)

the first three coordinates are the coordinates of its principal vector, and the last three are the coordinates of the principal moment with respect to the coordinate origin. To this natural screw corresponds a force screw equivalent to the system of all elementary inertial forces of a body vibrating along the natural screw. The coordinates of the force screw are expressed as follows:

$$-\lambda^2 m\delta_x^{(\mu)}, \quad -\lambda^2 m\delta_y^{(\mu)}, \quad -\lambda^2 m\delta_z^{(\mu)}, \quad -\lambda^2 I_x\varphi_x^{(\mu)},$$
$$-\lambda^2 I_y\varphi_y^{(\mu)}, \quad -\lambda^2 I_z\varphi_z^{(\mu)},$$
(7.40)

where the first three coordinates are the coordinates of the principal vector of all forces, and the last three are the coordinates of the principal moment of all forces with respect to the coordinate origin.

Since the orthogonality condition

$$I_x\varphi_x^{(\mu)}\varphi_x^{(\nu)} + I_y\varphi_y^{(\mu)}\varphi_y^{(\nu)} + I_z\varphi_z^{(\mu)}\varphi_z^{(\nu)} + m\delta_x^{(\mu)}\delta_x^{(\nu)} +$$
$$+ m\delta_y^{(\mu)}\delta_y^{(\nu)} + m\delta_z^{(\mu)}\delta_z^{(\nu)} = 0$$
(7.41)

applies between the coordinates of the two natural screws corresponding to the natural frequencies $\lambda^{(\mu)}$ and $\lambda^{(\nu)}$, and this condition will be equivalent to the condition of reciprocity of the inertial force screw corresponding to the μth vibration mode to the kinematic screw corresponding to the νth mode if both sides of (7.41) are multiplied by $-\lambda^{(\mu)^2}$, we conclude from this that the screw equivalent to the system of all elementary inertial forces in vibrations of the body along one of the six natural screws is reciprocal to all of the five remaining natural screws.

Relationships similar to those between the external-force and displacement screws for static action of external forces apply for the external-force and displacement screws of a vibrating

body.

Let a body be acted upon by an external force screw $R' = R_0' \sin \lambda t$, and let the displacement screw be $\Phi' = \Phi_0' \sin \lambda t$, where R_0', Φ_0' are amplitude screws. It can be shown that for an external force screw $R'' = R_0'' \sin \lambda t$, reciprocal to Φ', the displacement screw Φ'' will be reciprocal to screw R'.

In fact, we may write the following relationship for the screw coordinates on the basis of the reciprocity theorem:

$$
\begin{aligned}
(P_x' \delta_x'' &+ P_y' \delta_y'' + P_z' \delta_z'' + L_x' \varphi_x'' + L_y' \varphi_y'' + L_z' \varphi_z'') - \\
&- \lambda^2 (m \delta_x' \delta_x'' + m \delta_y' \delta_y'' + m \delta_z' \delta_z'' + I_x \varphi_x' \varphi_x'' + I_y \varphi_y' \varphi_y'' + I_z \varphi_z' \varphi_z'') = \\
&= (P_x'' \delta_x' + P_y'' \delta_y' + P_z'' \delta_z' + L_x'' \varphi_x' + L_y'' \varphi_y' + L_z'' \varphi_z') - \\
&\quad - \lambda^2 (m \delta_x'' \delta_x' + m \delta_y'' \delta_y' + m \delta_z'' \delta_z' + I_x \varphi_x'' \varphi_x' + \\
&\qquad\qquad + I_y \varphi_y'' \varphi_y' + I_z \varphi_z'' \varphi_z').
\end{aligned}
\tag{7.42}
$$

The first terms in the parentheses in the left and right members of Equality (7.42) are the relative moments of the external force screw and the displacement screw of the solid body; the second terms are the relative moments of the inertial-forces screw (time derivative of the kinematic screw) and the displacement screw. These relative moments are expressions for the displacement work of the forces, with the left member of the equality expressing the work done by the first-state forces on displacements of the second state, and the second expressing the work done by the second-state forces on the displacements of the first state.

Since the work of the inertial forces in the right and left members is the same for a given frequency λ, the work of the external force screw of the first state on the displacement screw of the second state is equal to the work of the external force screw of the second state on the displacement screw of the first state, i.e.,

$$\text{mom} (R' \cdot \Phi'') = \text{mom} (R'' \cdot \Phi').$$

But the second work is zero, since by hypothesis screw R'' is reciprocal to screw Φ'', from which it follows that

$$\text{mom} (R' \cdot \Phi'') = 0, \tag{7.43}$$

i.e., that screw Φ'' is reciprocal to screw R'.

The property demonstrated above is perfectly identical with the property of the system under static action of an external-force screw, given equal frequencies of forced vibrations in the first and second states. On the basis of this property, we can find the amplitude screw of the body's displacements for a given amplitude screw of the external forces, using the same scheme as was described in our analysis of the statics of an elastically suspended solid body (pages 141-142).

If screw R represents a linear combination

$$R = \lambda R_1 + \mu R_2 \tag{7.44}$$

with va.....ng real parameters λ and μ, then its axis describes a ruled surface that constantly intersects a certain straight line, the axis of the surface, as already discussed. The displacement screw Φ of the solid body will also describe a surface

$$\Phi = \lambda\Phi_1 + \mu\Phi_2, \qquad (7.45)$$

where Φ_1 and Φ_2 are displacement screws corresponding to separate actions of R_1 and R_2. If the axis of surface (7.45) coincides with the axis of a spring, the force in this spring on variation according to (7.44) will be zero. The following problem can be posed: find a combination of screws of the type (7.44) (for example, this might be a rotating eccentric) in which forces will not arise in a given spring. Indeed, let Φ_1 and Φ_2 be any two displacement screws of the body whose axes intersect the axis of the given spring at right angles. Constructing the two force screws R_1 and R_2 — the resultant forces in the springs — we obtain an axis — the shortest line between the latter screws, which will be intersected by any linear combination of these screws which causes a displacement along the screw — the linear combination of Φ_1 and Φ_2 — that will not give rise to forces in the given spring. We note that the parameters of screws Φ_1 and Φ_2 can be designated arbitrarily, and that this enables us to select the most rational combination of force screws.
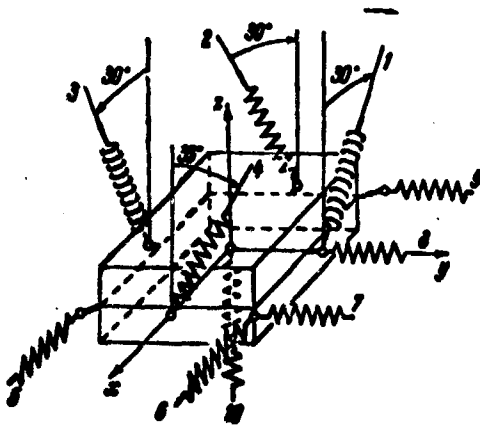

Fig. 40

In conclusion, let us consider an example of three-dimensional system (Fig. 40). In this system, the elastic suspension has three principal axes; the coefficient matrix of Eqs. (7.26) has the following structure:

$$\begin{vmatrix} c_{11} & 0 & 0 & c_{14} & 0 & 0 \\ 0 & c_{22} & 0 & 0 & c_{25} & 0 \\ 0 & 0 & c_{33} & 0 & 0 & c_{36} \\ c_{41} & 0 & 0 & c_{44} & 0 & 0 \\ 0 & c_{52} & 0 & 0 & c_{55} & 0 \\ 0 & 0 & c_{63} & 0 & 0 & c_{66} \end{vmatrix}. \qquad (7.46)$$

The reduced coefficients are the squares of the partial fre-

quencies:

$$\frac{c_{11}}{m} = \lambda_1^2 = 2500, \qquad \frac{c_{22}}{m} = \lambda_2^2 = 3500, \qquad \frac{c_{33}}{m} = \lambda_3^2 = 4000,$$

$$\frac{c_{44}}{m\rho_x^2} = \lambda_4^2 = 4500, \qquad \frac{c_{55}}{m\rho_y^2} = \lambda_5^2 = 5000, \qquad \frac{c_{66}}{m\rho_z^2} = \lambda_6^2 = 6000,$$

$$\frac{c_{14}}{m\rho_x} = \lambda_{14}^2 = 1772, \qquad \frac{c_{25}}{m\rho_y} = \lambda_{25}^2 = 1969, \qquad \frac{c_{36}}{m\rho_z} = \lambda_{36}^2 = 2474.$$

The radii of inertia of the body are

$$\rho_x = 0.5a, \ \rho_y = a, \ \rho_z = \sqrt{0.5a}, \ a = 10 \, \text{cm}.$$

The equation system with Matrix (7.46) breaks up into three independent pairs of equations with two unknowns, from which the natural frequencies are found to be

$$\lambda^{(1)} = 42.28, \ \lambda^{(2)} = 52.10, \ \lambda^{(3)} = 55.92,$$
$$\lambda^{(4)} = 81.93, \ \lambda^{(5)} = 91.01, \ \lambda^{(6)} = 104.27.$$

When the system is acted upon by a harmonic couple whose axis is inclined equally to the axes x, y and z, the axis of the displacement screw is determined from a system of equations with a nonzero right member. As the frequency λ of the forced vibrations
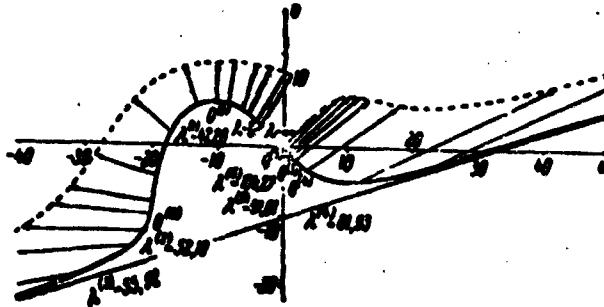

Fig. 41

varies from 0 to ∞, the axis of the screw describes a ruled surface. The coordinates of the points of intersection of this surface with the plane xy are determined from the formulas

$$u = \frac{b_y - p\varphi_y}{\varphi_z}, \quad v = \frac{b_x - p\varphi_x}{\varphi_z}, \qquad (7.47)$$

where p is the parameter of the screw corresponding to the generator in question and is determined by the formula

$$p = \frac{b_x\varphi_x + b_y\varphi_y + b_z\varphi_z}{\varphi_x^2 + \varphi_y^2 + \varphi_z^2}. \qquad (7.48)$$

Constructing vectors proportional to φ_x/φ_z and φ_y/φ_z at each point in the horizontal plane uv, we obtain the horizontal components of vectors lying on generators of the surface (screw axes) whose ends lie in a plane parallel to plane uv.

Figure 41 shows curves in the plane *uv* and in a plane parallel to it that represent the intersection of the ruled surface described by the screw axis on variation of the forced-vibration frequency from 0 to ∞ with the two horizontal planes. The corresponding vectors of the screw axes indicate the upper and lower planes. Thus, on the basis of this drawing we may draw inferences as to the nature of the surface described by the axis of the displacement screw as the frequency varies.

Manu- script Page No.	Transliterated Symbols		
142	упр = upr = uprugiy = elastic		

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