THE SECOND CONTINUOUS JACOBI TRANSFORM

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ABSTRACT: This paper continues the work started in [1]; a second

continuous Jacobi transform is defined for suitable functions f(x).

Properties of the transform are studied. In particular, the first

continuous Jacobi transform in [1] and the second continuous Jacobi

transform are shown to be inverse to each other. The paper concludes with

an extension of Campbell's sampling theorem [2].

*KEY WORDS AND PHRASES: Continuous Jacobi transform, inverse Jacobi

transform, discrete Jacobi transform, sampling theorem.*

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1.   INTRODUCTION.

In this paper, the second continuous Jacobi transform, $\hat{f}^{(\alpha,\beta)}$, of a

function f(x) is developed along similar lines of Butzer, Stens, and

Wehrens [3]. The results generalize the work in [3] as well as the work of

Debnath [4] on the discrete Jacobi transform. Basic properties of $\hat{f}^{(\alpha,\beta)}$

will be derived including an inversion formula tersely given by

$(\hat{f}^{(\alpha,\beta)}(\cdot))^{\wedge(\alpha,\beta)} = f(\cdot)$. The results are then applied to an extended form of

Campbell's sampling theorem [2].

The paper is divided as follows. Section two includes basic notations

and results obtained in [1] that will be used in the sequel. Section three

is devoted to the study of the second continuous Jacobi transform. In this

section, the first and the second continuous Jacobi transforms are shown to

be inverse to each other. Section four is devoted to a sampling theorem

based on Jacobi transforms and an estimate of a truncation error.

2.   PRELIMINARIES.

    In this section we recall all the necessary background material on

Jacobi functions and the first continuous Jacobi transform as studied in [1].

For the sake of completeness, we repeat some of the basic notions of hyper-

geometric functions.

    For each a, b, c real numbers with $c \neq 0, -1, -2, \ldots$ the hypergeometric

function is given by

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad |z| < 1$$

The above series converges absolutely and uniformly on each compact subset

of $(-1,1)$. Also,

$$\lim_{z \to 1^-} F(a,b;c;z) = F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

where the gamma function is always assumed to be a well-defined function of

its argument.

    The Jacobi function is defined by

$$P_\lambda^{(\alpha,\beta)}(x) = \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)} F(-\lambda,\lambda+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}), \quad x \in (-1,1]$$

where α, $\beta > -1$. The following are some of the properties of $P_\lambda^{(\alpha,\beta)}(x)$

derived in [1] and will be used in the sequel. We refer the reader to [1]

for proofs.

    LEMMA 2.1. [Lemma 2.2, [1], page 148]. For any $x \in (-1,1)$ and

$\lambda \geq -\frac{\alpha+\beta+1}{2}$, we have for

    (i) $-1 < \beta \leq 0$, $|P_\lambda^{(\alpha,\beta)}(x)| \leq \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)} + M(\lambda,\alpha,\beta) \log(\frac{2}{1+x})$

    (ii) $\beta \geq 0$, $(\frac{1+x}{2})^\beta |P_\lambda^{(\alpha,\beta)}(x)| \leq \frac{\Gamma(\lambda+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\lambda+1)} + M'(\lambda,\alpha,\beta) \log(\frac{2}{1+x})$

where $M(\lambda,\alpha,\beta)$ and $M'(\lambda,\alpha,\beta)$ are constants depending on λ, α, and β.

    We will denote the weighted $L^p(-1,1)$ $(p \geq 1)$ space with weight

$w(x) \equiv (1-x)^\alpha (1+x)^\beta$ by $L_w^p(-1,1)$. The norm on $L_w^p(-1,1)$ is given by

$$\| f \|_p = (\frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x) | f(x) |^p dx)^{1/p} .$$

It was shown in [4] that

LEMMA 2.2.   (Lemma 2.4, [1], p.150). $P_\lambda^{(\alpha,\beta)}(x) \in L_w^p(-1,1)$ for all $p \geq 1$ and for all α and β such that $\alpha p+1>0$ and $-1/p<\beta<1/p$.

Jacobi polynomials with $\lambda=n \in P$, P is the set of all positive integers, satisfy the following orthogonality relation

$$\frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x) P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) dx = \begin{cases} 0 & n \neq m \\ \delta_n & n=m \end{cases} \qquad (2.1)$$

where $\delta_n = \dfrac{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1)}$.

However, Jacobi functions do not satisfy such an orthogonality relation. Instead, the following result holds:

LEMMA 2.3.   (Lemma 2.5, [1], p.150).   Let λ and ν both be greater than $-\dfrac{\alpha+\beta+1}{2}$, $\lambda \neq \nu$ and $\lambda \neq -(\nu+\alpha+\beta+1)$.   For α, $\beta \in (-\frac{1}{2}, \frac{1}{2})$, we have

$$\frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x) P_\lambda^{(\alpha,\beta)}(x) P_\nu^{(\beta,\alpha)}(-x) dx$$

$$= \frac{\Gamma(\lambda+\alpha+1) \Gamma(\nu+\beta+1)}{\pi(\lambda-\nu)(\lambda+\nu+\alpha+\beta+1)} \{ \frac{\sin\pi\lambda}{\Gamma(\nu+1) \Gamma(\lambda+\alpha+\beta+1)} - \frac{\sin\pi\nu}{\Gamma(\lambda+1) \Gamma(\nu+\alpha+\beta+1)} \} .$$

We also recall the definition of the discrete Jacobi transform as studied by Debnath in [4].

$$\hat{f}^{(\alpha,\beta)}(n) = J(f(x)) = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x) P_n^{(\alpha,\beta)}(x) f(x) dx$$

and under appropriate conditions

$$f(x) = \sum_{n=0}^{\infty} \delta_n^{-1} \hat{f}^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x) . \qquad (2.2)$$

Now, for any f, $g \in L_w^2(-1,1)$ and for appropriate α, β

$$\frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x) f(x) g(x) dx = \sum_{n=0}^{\infty} \delta_n^{-1} \hat{f}^{(\alpha,\beta)}(n) \hat{g}^{(\alpha,\beta)}(n) . \qquad (2.3)$$

From (2.3) and Lemma 2.3 together with the identity $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, we obtain

$$\hat{P}_\lambda^{(\alpha,\beta)}(n) = \begin{cases} (-1)^n \dfrac{\Gamma(\lambda+\alpha+1) \Gamma(n+\beta+1) \sin\pi\lambda}{\pi(\lambda-n)(\lambda+n+\alpha+\beta+1) n! \Gamma(\lambda+\alpha+\beta+1)}, & \lambda \neq n \\ \\ \dfrac{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) n! \Gamma(n+\alpha+\beta+1)} & , \lambda = n \end{cases} . \qquad (2.4)$$

The following estimates on $P_\lambda^{(\alpha,\beta)}(x)$ for large λ will be used.

LEMMA 2.4. (Lemma 2.6, [1], page 152). For λ, $\nu \geq -\frac{\alpha+\beta+1}{2}$, $\alpha > -\frac{1}{2}$ and $-\frac{1}{2} < \beta < \frac{1}{2}$, we have

(i)   for each $[a,b] \subset (-1,1)$, there holds for all $x \in [a,b]$

$$|P_\lambda^{(\alpha,\beta)}(x)| = O(\lambda^{-\frac{1}{2}}) \text{ as } \lambda \to \infty.$$

(ii) $\| P_\lambda^{(\alpha,\beta)}(x) \|_1 \leq \| P_\lambda^{(\alpha,\beta)}(x) \|_2 = O(\lambda^{-\frac{1}{2}})$ as $\lambda \to \infty$.

(iii) for each $[c,d] \subseteq [-\frac{\alpha+\beta+1}{2}, \infty)$, there exists a constant $M > 0$ such that for all λ, $\nu \in [c,d]$

$$\| P_\lambda^{(\alpha,\beta)}(x) - P_\nu^{(\alpha,\beta)}(x) \|_2 \leq M|\lambda - \nu| .$$

The continuous Jacobi transform of the first kind is defined as in [1] by

$$\hat{f}^{(\alpha,\beta)}_{(\lambda)} = \frac{1}{2^{\alpha+\beta+1}} \int_{-1}^{1} w(x) P_\lambda^{(\alpha,\beta)}(x) f(x) dx. \qquad (2.5)$$

for every $f \in L_w^2(-1,1)$ with $\alpha > -\frac{1}{2}$ and $-\frac{1}{2} < \beta < \frac{1}{2}$. We recall that if $\alpha = \beta = 0$, we obtain the results of Butzer, Stens and Wehrens [3] and if $\lambda = n \in P$, we obtain the results of Debnath [4].

Again, it was shown in [1] that for any $f \in L_w^2(-1,1)$ we have

$$|\hat{f}^{(\alpha,\beta)}(\lambda)| = O(\lambda^{-\frac{1}{2}}) \text{ as } \lambda \to \infty. \qquad (2.6)$$

and

$$\hat{f}^{(\alpha,\beta)}(\cdot - \frac{\alpha+\beta+1}{2}) \in C_0(\mathbb{R}^+) \cap L^P(\mathbb{R}^+), \ p > 2. \qquad (2.7)$$

The following Lemma will be essential for our work.

LEMMA 2.5. (Lemma 3.2, [1], page 155). Let $F(x)$ be defined on $\mathbb{R}^+ = [0,\infty)$ such that $\lambda^{\alpha+\beta+\frac{1}{2}} F(\lambda) \in L^1(\mathbb{R}^+)$. Then the function

$$G(x) = \int_0^\infty F(\lambda) P_{\lambda - \frac{\alpha+\beta+1}{2}}^{(\beta,\alpha)}(-x) H(\lambda) \lambda \sin \pi \lambda \, d\lambda$$

belongs to $C(-1,1) \cap L_w^2(-1,1)$ where

$$H(\lambda) = \frac{\Gamma^2(\lambda + \frac{\alpha+\beta+1}{2})}{\Gamma(\lambda + \frac{\alpha-\beta+1}{2}) \Gamma(\lambda + \frac{\beta-\alpha+1}{2})} .$$

From now on we will assume throughout the paper that α and β satisfy

$$\alpha+\beta=0, \quad -\tfrac{1}{2}<\alpha, \quad \beta<\tfrac{1}{2} . \qquad (2.8)$$

If α and β satisfy (2.8), then it was shown in [1] that

$$P_k^{(\alpha,\beta)}(x)=4\int_0^\infty \hat{P}_k^{(\alpha,\beta)}(\lambda-\tfrac{1}{2})P_{\lambda-\tfrac{1}{2}}^{(\beta,\alpha)}(-x)H_0(\lambda)\lambda\sin\pi\lambda d\lambda \qquad (2.9)$$

where

$$H_0(\lambda)=\frac{\Gamma^2(\lambda+\tfrac{1}{2})}{\Gamma(\lambda+\alpha+\tfrac{1}{2})\Gamma(\lambda+\beta+\tfrac{1}{2})}, \quad x\epsilon(-1,1], \quad k\epsilon P.$$

The following is also true.

LEMMA 2.6. (Theorem 3.1, [1], page 158). Let $f\epsilon L_w^2(-1,1)$ be such that $\lambda^{\frac{1}{2}}\hat{f}^{(\alpha,\beta)}(\lambda-\tfrac{1}{2})\epsilon L^1(\mathbb{R}^+)$. Then for almost every $x\epsilon(-1,1)$

$$f(x)=4\int_0^\infty \hat{f}^{(\alpha,\beta)}(\lambda-\tfrac{1}{2})P_{\lambda-\tfrac{1}{2}}^{(\beta,\alpha)}(-x)H_0(\lambda)\lambda\sin\pi\lambda d\lambda. \qquad (2.10)$$

Moreover, if $f(x)$ is continuous then (2.10) holds everywhere on $(-1,1)$.

3. SECOND CONTINUOUS JACOBI TRANSFORM

In this section we define the second continuous Jacobi transform $\hat{f}^{(\alpha,\beta)}$ of a function f with α and β satisfying (2.8). We also show that $\hat{f}^{(\alpha,\beta)}$ and $f^{\wedge(\alpha,\beta)}$ are inverse of each other.

For each f defined on $\mathbb{R}^+$, we associate the integral

$$I(x)=4\int_0^\infty f(\lambda)P_{\lambda-\tfrac{1}{2}}^{(\beta,\alpha)}(-x)\frac{\Gamma(\lambda+\tfrac{1}{2})}{\Gamma(\lambda+\beta+\tfrac{1}{2})}\lambda\sin\pi\lambda d\lambda .$$

<u>Proposition 3.1</u>. For each f defined on R^+ with f and $\lambda^{-\beta+\tfrac{1}{2}}f\epsilon L^1(\mathbb{R}^+)$, we have $I(x)\epsilon C(-1,1)\cap L_w^2(-1,1)$. (Although the proof of Proposition 3.1 goes along the same lines as Lemma 2.5, we present the proof with the proper modification for the sake of self-containment.

PROOF of Proposition 3.1. We first show that $I(x)$ is well defined. Observe that, for large λ, $\frac{\Gamma(\lambda+\tfrac{1}{2})}{\Gamma(\lambda+\beta+\tfrac{1}{2})}$ behaves like $\lambda^{-\beta}$ (see [1]). Thus for any fixed $T>0$, we have by Lemma 2.4 (ii) and the hypothesis that

$$|I(x)|\leq |4\int_0^T f(\lambda)P_{\lambda-\tfrac{1}{2}}^{(\beta,\alpha)}(-x)\frac{\Gamma(\lambda+\tfrac{1}{2})}{\Gamma(\lambda+\beta+\tfrac{1}{2})}\lambda\sin\pi\lambda d\lambda|$$

$$+|4\int_T^\infty f(\lambda)P_{\lambda-\tfrac{1}{2}}^{(\beta,\alpha)}(-x)\frac{\Gamma(\lambda+\tfrac{1}{2})}{\Gamma(\lambda+\beta+\tfrac{1}{2})}\lambda\sin\pi\lambda d\lambda|$$

$$\leq C_1+C_2\int_T^\infty \lambda^{-\beta+\tfrac{1}{2}}|f(\lambda)|d\lambda<\infty$$

where C_1 and C_2 are some positive constants. Hence I is well-defined.

We show that $I(x) \epsilon C(-1,1)$. For any $x \epsilon (-1,1)$, there exists a $\delta_1 > 0$ such that $x \pm \delta_1 \epsilon (-1,1)$. By lemma 2.4 (i) and for all $|y| < \delta_1$, we have

$$|I(x+y)-I(x)| \le |4 \int_0^\infty f(\lambda)(P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x-y) - P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x)) \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\beta+\frac{1}{2})} \lambda \sin\pi\lambda d\lambda|$$

$$\le 8C \int_0^\infty \lambda^{-\beta+\frac{1}{2}} |f(\lambda)| d\lambda < \infty$$

by hypothesis and C is some positive constant. Thus for λ_0 sufficiently large.

$$|4 \int_{\lambda_0}^\infty f(\lambda)(P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x-y) - P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x) \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\beta+\frac{1}{2})} \lambda \sin\pi\lambda d\lambda|$$

can be made sufficiently small. Thus for $\epsilon > 0$ given, there exists a λ_0 sufficiently large such that the above difference is less than $\epsilon/2$. Fix λ_0. By the continuity of $P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x)$, we have for $\hat{\epsilon} > 0$ that there exists a $\delta_2 > 0$ such that

$$|P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x-y) - P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x)| < \hat{\epsilon} \quad \text{whenever} \quad |y| < \delta_2.$$

Choose $\epsilon = 2\hat{\epsilon} \int_0^{\lambda_0} \lambda^{-\beta+\frac{1}{2}} |f(\lambda)| d\lambda$ and $\delta = \min(\delta_1, \delta_2)$. With this choice. $|I(x+y)-I(x)| < \epsilon$ whenever $|y| < \delta$. Therefore $I(x) \epsilon C(-1,1)$.

We finally show that $I(x) \epsilon L_w^2(-1,1)$.

$$\|I(x)\|_2^2 = \frac{1}{2} \int_{-1}^1 w(x) |I(x)|^2 dx$$

and by Hardy-Littlewood inequality (see [1], page 148) and Lemma 2.4 (ii), we have

$$\|I(x)\|_2 \le \int_0^\infty \lambda^{-\beta+\frac{1}{2}} |f(\lambda)| d\lambda (\frac{1}{2} \int_{-1}^1 (1-x)^\alpha (1+x)^\beta |P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x)|^2 dx)^{\frac{1}{2}}$$

$$\le \int_0^\infty \lambda^{-\beta+\frac{1}{2}} |f(\lambda)| d\lambda < \infty$$

by hypothesis. Hence $I(x) \epsilon C(-1,1) \cap L_w^2(-1,1)$. This completes the proof of Proposition 3.1.

We shall call $I(x)$ the second continuous Jacobi transform and we will denote this by $\hat{f}^{(\alpha,\beta)}$. Thus, we define

$$\hat{f}^{(\alpha,\beta)}(x) = 4 \int_0^\infty f(\lambda) P_{\lambda-\frac{1}{2}}^{(\beta,\alpha)}(-x) \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\beta+\frac{1}{2})} \lambda \sin\pi\lambda d\lambda.$$

From Proposition 3.1 we deduce that $\hat{f}^{(\alpha,\beta)} \epsilon C(-1,1) \cap L_w^2(-1,1)$ with $-\frac{1}{2} < \alpha,\ \beta < \frac{1}{2}$ and $\alpha+\beta=0$. In the next theorem, we will prove an inversion formula for the transform $\hat{f}^{(\alpha,\beta)}$ under appropriate choice of f. The inversion formula proof is analogous to the proof given in [3]. We will employ the Fourier cosine transform of a function f which is given by

$$F_c(f)(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\lambda)\cos\lambda t \, d\lambda.$$

THEOREM 3.1. Assume that $f \epsilon L^1(\mathbb{R}^+)$ is such that $\lambda^{-\beta+\frac{1}{2}}f(\lambda) \epsilon L^1(\mathbb{R}^+)$ and that

$$F_c(f)(t) \equiv 0 \quad \text{for all} \quad \pi \leq t < \infty.$$

Then we have for almost all $\lambda \epsilon \mathbb{R}^+$

$$\frac{1}{2} \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\alpha+\frac{1}{2})} \int_{-1}^1 w(x)\hat{f}^{(\alpha,\beta)}(x)P_{\lambda-\frac{1}{2}}^{(\alpha,\beta)}(x)dx = f(\lambda). \tag{3.1}$$

Moreover, if $f \epsilon C(\mathbb{R}^+)$, then (3.1) holds for all $\lambda \epsilon \mathbb{R}^+$.

PROOF. By employing Fubini's theorem together with Lemma 2.3 (with $\alpha+\beta=0$), we obtain

$$\frac{1}{2} \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\alpha+\frac{1}{2})} \int_{-1}^1 w(x)\{4\int_0^\infty f(t)P_{t-\frac{1}{2}}^{(\beta,\alpha)}(-x) \frac{\Gamma(t+\frac{1}{2})}{\Gamma(t+\beta+\frac{1}{2})} t\sin\pi t \, dt\}P_{\lambda-\frac{1}{2}}^{(\alpha,\beta)}(x)dx$$

$$= \frac{2\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\alpha+\frac{1}{2})} \int_0^\infty tf(t)\sin\pi t \frac{\Gamma(t+\frac{1}{2})}{\Gamma(t+\beta+\frac{1}{2})}\{\int_{-1}^1 w(x)P_{\lambda-\frac{1}{2}}^{(\alpha,\beta)}(x)P_{t-\frac{1}{2}}^{(\beta,\alpha)}(-x)dx\}dt$$

$$= 4 \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\alpha+\frac{1}{2})} \int_0^\infty tf(t)\sin\pi t \frac{\Gamma(t+\frac{1}{2})}{\Gamma(t+\beta+\frac{1}{2})} \frac{\Gamma(\lambda+\alpha+\frac{1}{2})\Gamma(t+\beta+\frac{1}{2})}{\pi(\lambda^2-t^2)\Gamma(t+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}$$

$$\cdot \{\sin\pi(t-\frac{1}{2})-\sin\pi(\lambda-\frac{1}{2})\}dt$$

$$= 4\int_0^\infty tf(t)\sin\pi t \{\frac{\cos\pi t-\cos\pi\lambda}{\pi(\lambda^2-t^2)}\} \, dt \equiv Q(\lambda).$$

Set $q_t(\lambda) = \dfrac{\cos\pi t-\cos\pi\lambda}{\pi(\lambda^2-t^2)}$.

We claim that $Q(\lambda)\epsilon L^1(\mathbb{R}^+)$. Indeed, an application of Fubini's theorem together with the assumption that $f \epsilon L^1(\mathbb{R}^+)$ implies that

$$\int_0^\infty |Q(\lambda)| \, d\lambda < \infty.$$

Now, by another application of Fubini's theorem, we obtain

$$F_c[Q](s) = \sqrt{\frac{2}{\pi}} \int_0^\infty Q(\lambda) \cos\lambda s\, d\lambda$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty (4 \int_0^\infty tf(t) \sin\pi t q_t(\lambda) dt) \cos\lambda s\, d\lambda$$

$$= 4\sqrt{\frac{2}{\pi}} \int_0^\infty tf(t) \sin\pi t (\int_0^\infty q_t(\lambda) \cos\lambda s\, d\lambda) dt$$

$$= 4 \int_0^\infty tf(t) \sin\pi t F_c(q_t)(s) dt.$$

We employ the result of [3] and proceed analogously and determine that

$$F_c(q_t)(s) = \sqrt{\frac{2}{\pi}} \frac{1}{2t} \sin(\pi - s)t, \quad 0 \le s \le \pi,\ t > 0.$$

Thus

$$F_c(Q)(s) = 2\sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin\pi t \sin(\pi - s) t\, dt$$

$$F_c(Q)(s) = F_c(f)(s) - F_c(f)(2\pi - s) = F_c(f)(s), \quad 0 \le s \le \pi.$$

Moreover, $F_c(Q)(s) = 0$ for $\pi \le s < \infty$. The assumption $F_c(f)(s) = 0$ for $\pi \le s < \infty$ together with the uniqueness of the Fourier-cosine transform implies that $Q(\lambda) = f(\lambda)$ for almost every $\lambda \epsilon \mathbb{R}^+$. Thus

$$f(\lambda) = \tfrac{1}{2} \frac{\Gamma(\lambda + \alpha + \tfrac{1}{2})}{\Gamma(\lambda + \tfrac{1}{2})} \int_{-1}^1 w(x) \hat{} f^{(\alpha,\beta)}(x) P_{\lambda - \tfrac{1}{2}}^{(\alpha,\beta)}(x) dx.$$

The continuity of f will imply that (3.1) holds for all $\lambda \epsilon \mathbb{R}^+$.

From Theorem 3.1 and Lemma 2.6 we deduce that

$$(\hat{} f^{(\alpha,\beta)}(\cdot))^{\wedge(\alpha,\beta)}(\lambda) = 2 \frac{\Gamma(\lambda + \alpha + \tfrac{1}{2})}{\Gamma(\lambda + \tfrac{1}{2})} f(\lambda). \tag{3.2}$$

and

$$\wedge(\frac{\Gamma(\lambda + \tfrac{1}{2})}{\Gamma(\lambda + \alpha + \tfrac{1}{2})} f^{\wedge(\alpha,\beta)}(\cdot))^{(\alpha,\beta)}(x) = f(x). \tag{3.3}$$

Equations (3.2) and (3.3) reduce to the formulas obtained in [3] whenever $\alpha = \beta = 0$.

4. A SAMPLING THEOREM

In this section we give a proof of a sampling Theorem of Campbell [2] by employing results on the continuous Jacobi transform of the first and second kinds.   Moreover, we will obtain an error estimate for a function f that is band-limited in the sense of the Fourier-cosine transform.

THEOREM 4.1. If $F \varepsilon C(\mathbb{R}^+)$ is given by

$$F(\lambda) = \frac{1}{2} \int_{-1}^{1} w(x) f(x) P_{\mu\lambda-\frac{1}{2}}^{(\alpha,\beta)}(x)\, dx \qquad (4.1)$$

for some $\mu > 0$ and $f \varepsilon L_w^2(-1,1)$, then one obtains for all $\lambda \varepsilon \mathbb{R}^+$ that

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{(2n+1)\Gamma(n+1)\Gamma(\mu\lambda+\alpha+\frac{1}{2})F(\frac{n+\frac{1}{2}}{\mu})}{\pi(\lambda^2\mu^2-(n+\frac{1}{2})^2)\Gamma(n+\alpha+1)\Gamma(\lambda\mu+\frac{1}{2})} \sin\pi(\lambda\mu-(n+\frac{1}{2})) \qquad (4.2)$$

PROOF. From (4.1), (2.3) and (2.4), we obtain (with $\alpha+\beta=0$)

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{n!(2n+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \hat{f}^{(\alpha,\beta)}(n) \hat{P}_{\mu\lambda-\frac{1}{2}}^{(\alpha,\beta)}(n)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n n!(2n+1)\Gamma(n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \frac{\Gamma(\lambda\mu+\alpha+\frac{1}{2})\Gamma(n+\beta+1)\sin\pi(\lambda\mu-\frac{1}{2})}{\pi(\lambda^2\mu^2-(n+\frac{1}{2})^2)n!\Gamma(\lambda\mu+\frac{1}{2})} \hat{f}^{(\alpha,\beta)}(n)$$

$$(\mu\lambda-\frac{1}{2}\neq n)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)\Gamma(n+1)\Gamma(\lambda\mu+\alpha+\frac{1}{2})}{\pi(\lambda^2\mu^2-(n+\frac{1}{2})^2)\Gamma(n+\alpha+1)\Gamma(\lambda\mu+\frac{1}{2})} \sin\pi(\lambda\mu-\frac{1}{2}) F(\frac{n+\frac{1}{2}}{\mu})$$

or

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{(2n+1)\Gamma(n+1)\Gamma(\lambda\mu+\alpha+\frac{1}{2})\sin\pi(\lambda\mu-(n+\frac{1}{2}))}{\pi(\lambda^2\mu^2-(n+\frac{1}{2})^2)\Gamma(n+\alpha+1)\Gamma(\lambda\mu+\frac{1}{2})} F(\frac{n+\frac{1}{2}}{\mu}).$$

REMARK 4.1. Theorem 3.1 when applied to the above situation yields the series representation (4.2) for $F \varepsilon L^1(\mathbb{R}^+)$, $\lambda^{-\beta+\frac{1}{2}} F \varepsilon L^1(\mathbb{R}^+)$ and $F_C(F)(s) = 0$ for $s > \pi\mu$ for some $\mu > 0$.

Let $S_{n,\mu}F$ denote the n-th partial sum of the series in (4.2). We give below an error estimate for approximating F. In particular, we show that

THEOREM 4.2. Let F satisfy hypotheses of Theorem 4.1 for some fixed $\mu > 0$. Then there exists a constant $C > 0$ such that

$$|F(\lambda) - (S_{n,\mu}F)(\lambda)| \leq C(\lambda+1)^{\frac{1}{2}} E_n(f)$$

where

$$E_n(f) = \inf_{P_n \varepsilon P_n} \| f - P_n \|_2 ,$$

P_n is the set of all algebraic polynomials of degree n.

PROOF. Denote by $S_n f$ the n-th partial sum of

$$f(x) \simeq \sum_{n=0}^{\infty} \delta_n^{-1} \hat{f}^{(\alpha,\beta)}(n) P_n^{(\alpha,\beta)}(x)$$

where $\delta_n^{-1} = \dfrac{(n!)^2 (2n+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$.

Thus

$$(S_{n,\mu}F)(\lambda) = \sum_{k=0}^{\infty} \frac{(2k+1)\Gamma(k+1)\Gamma(\lambda\mu+\alpha+\tfrac{1}{2})}{\pi(\lambda^2\mu^2-(k+\tfrac{1}{2})^2)\Gamma(k+\alpha+1)\Gamma(\lambda\mu+\tfrac{1}{2})} (S_n f)^{\wedge(\alpha,\beta)} \hat{P}^{(\alpha,\beta)}_{\lambda\mu-\tfrac{1}{2}}(k) .$$

By Cauchy-Schwartz inequality and Lemma 2.4, we have

$$|F(\lambda)-(S_{n,\mu}F)(\lambda)|^2 = |\tfrac{1}{2}\int_{-1}^{1} w(x)(f(x)-(S_n f)(x))P^{(\alpha,\beta)}_{\lambda\mu-\tfrac{1}{2}}(x)dx|^2$$

$$\leq \tfrac{1}{2}\int_{-1}^{1} w(x)|f(x)-(S_n f)(x)|^2 dx \cdot \tfrac{1}{2}\int_{-1}^{1} w(x)|P^{(\alpha,\beta)}_{\lambda\mu-\tfrac{1}{2}}(x)|^2 dx$$

$$\leq \|f-S_n f\|_2^2 |P^{(\alpha,\beta)}_{\lambda\mu-\tfrac{1}{2}}\|_2^2 \leq E_n^2(f)\|P_{\mu\lambda-\tfrac{1}{2}}\|_2^2.$$
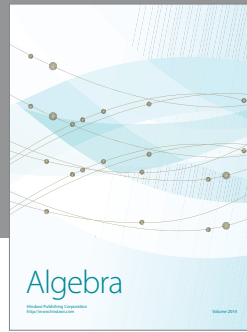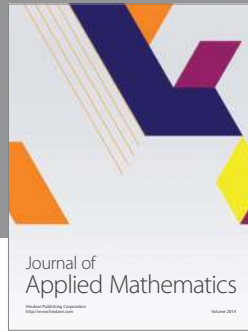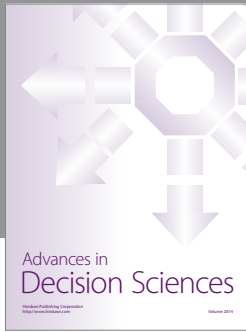
Thus,

$$|F(\lambda)-(S_{n,\mu}F)(\lambda)| \leq CE_n(f)(\lambda+1)^{-\tfrac{1}{2}} \text{ for some constant } C.$$

REMARK 4.2   The estimate obtained in Theorem 4.2 shows that the error becomes smaller for large λ.

References

1.   DEEBA, E.Y. and KOH, E.L. The Continuous Jacobi Transform, Internat. J. Math. & Math. Sci., Vol. 6, No. 1 (1983), 145-160.

2.   CAMPBELL, L.L. A Comparison of the Sampling Theorems of Kramer and Whittaker, J. Soc. Induct. Appl. Math. 12 (1964), 117-130.

3.   BUTZER, P.L., STENS, R.L. and WEHRENS, M. Continuous Legendre Transform, Internat. J. Math. & Math. Sci., Vol. 3, No. 1 (1980), 47-67.

4.   DEBNATH, L. On Jacobi Transform, Bull. Cal. Math. Soc., Vol. 55 (1963) 113-120.

5.   ERDELYI, A. MAGNUS, W., OBERHETTINGER F. and TRICOMI, F.G. Higher Transcendental Functions, Vol. 1, McGraw Hill, New York 1953.