Ninth International Conference on
Geometry, Integrability and Quantization
June 8–13, 2007, Varna, Bulgaria
Ivaïlo M. Mladenov, Editor
SOFTEX, Sofia 2008, pp 198–209

THE SECTIONAL CURVATURE OF THE TANGENT BUNDLES WITH GENERAL NATURAL LIFTED METRICS

SIMONA L. DRUŢĂ

Faculty of Mathematics, "Al. I. Cuza" University, 700506 Iaşi, Romania

Abstract. We study some properties of the tangent bundles with metrics of general natural lifted type. We consider a Riemannian manifold (M, g) and we find the conditions under which the Riemannian manifold (TM, G), where TM is the tangent bundle of M and G is the general natural lifted metric of g, has constant sectional curvature.

1. Introduction

In the geometry of the tangent bundle TM of a smooth n-dimensional Riemannian manifold (M, g) one uses several Riemannian and pseudo-Riemannian metrics, induced by the Riemannian metric g on M. Among them, we may quote the Sasaki metric, the Cheeger-Gromoll metric and the complete lift of the metric g. The possibility to consider vertical, complete and horizontal lifts on the tangent bundle TM (see [18]) leads to some interesting geometric structures, studied in the last years (see [1–3, 8, 9, 17]), and to interesting relations with some problems in Lagrangian and Hamiltonian mechanics. On the other hand, the natural lifts of g to TM (introduced in [5, 6]) induce some new Riemannian and pseudo-Riemannian geometric structures with many nice geometric properties (see [4, 5]).

Oproiu [11–13] has studied some properties of a natural lift G, of diagonal type, of the Riemannian metric g and a natural almost complex structure J of diagonal type on TM (see also [15, 16]). In [10], the same author has presented a general expression of the natural almost complex structures on TM. In the definition of the natural almost complex structure J of general type there are involved eight parameters (smooth functions of the density energy on TM). However, from the condition for J to define an almost complex structure, four of the above parameters can be expressed as (rational) functions of the other four parameters. A Riemannian metric G which is a natural lift of general type of the metric g depends on other six parameters. In [14] we have found the conditions under which the Kählerian manifold (TM, G, J) has constant holomorphic sectional curvature.

In the present paper we study the sectional curvature of the tangent bundle of a Riemannian manifold (M, g). Namely, we are interested in finding the conditions under which the Riemannian manifold (TM, G), where G is the general natural lifted metric of g, has constant sectional curvature. We obtain that the sectional curvature of (TM, G) is zero and the base manifold must be flat.

2. Preliminary Results

Consider a smooth n-dimensional Riemannian manifold (M, g) and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of a $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from usual local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n, y^1, \dots, y^n)$, where the local coordinates x^i, y^j, $i, j = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e., $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates y^j, $j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it. The M-tensor fields are defined by their components with respect to the induced local charts on TM (hence they are defined locally), but they can be interpreted as some (partial) usual tensor fields on TM. However, the essential quality of an M-tensor field on TM is that the local coordinate change rule of its components with respect to the change of induced local charts is the same as the local coordinate change rule of the components of an usual tensor field on M with respect to the change of local charts on M. More precisely, an M-tensor field of type (p, q) on TM is defined by sets of n^{p+q} components (functions depending on x^i and y^i), with p upper indices and q lower indices, assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the local coordinate change rule of these components (with respect to induced local charts on TM) is that of the local coordinate components of a tensor field of type (p, q) on the base manifold M (with respect to usual local charts on M), when a change of local charts on M (and hence on TM) is performed (see [7] for further details); e.g., the components y^i, $i = 1, \dots, n$, corresponding to the last n local coordinates of a tangent vector y, assigned to the induced local chart $(\tau^{-1}(U), \Phi)$ define an M-tensor field of type $(1, 0)$ on TM. An usual tensor field of type (p, q) on M may be thought of as an M-tensor field of type (p, q) on TM. If the considered tensor field on M is covariant only, the corresponding M-tensor field on

TM may be identified with the induced (pullback by τ) tensor field on TM. Some useful M-tensor fields on TM may be obtained as follows. Let $u : [0,\infty) \longrightarrow \mathbb{R}$ be a smooth function and let $\|y\|^2 = g_{\tau(y)}(y,y)$ be the square of the norm of the tangent vector $y \in \tau^{-1}(U)$. If δ^i_j are the Kronecker symbols (in fact, they are the local coordinate components of the identity tensor field I on M), then the components $u(\|y\|^2)\delta^i_j$ define an M-tensor field of type $(1,1)$ on TM. Similarly, if $g_{ij}(x)$ are the local coordinate components of the metric tensor field g on M in the local chart (U,φ), then the components $u(\|y\|^2)g_{ij}$ define a symmetric M-tensor field of type $(0,2)$ on TM. The components $g_{0i} = y^k g_{ki}$ define an M-tensor field of type $(0,1)$ on TM.

Denote by $\dot{\nabla}$ the Levi-Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{1}$$

of the tangent bundle to TM into the **vertical distribution** $VTM = \operatorname{Ker}\tau_*$ and the **horizontal distribution** HTM defined by $\dot{\nabla}$. The set of vector fields $\left(\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^n}\right)$ on $\tau^{-1}(U)$ defines a local frame field for VTM and for HTM we have the local frame field $\left(\frac{\delta}{\delta x^1},\ldots,\frac{\delta}{\delta x^n}\right)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}, \qquad \Gamma^h_{0i} = y^k\Gamma^h_{ki}$$

and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $\left(\frac{\partial}{\partial y^1},\ldots,\frac{\partial}{\partial y^n},\frac{\delta}{\delta x^1},\ldots,\frac{\delta}{\delta x^n}\right)$ defines a local frame on TM, adapted to the direct sum decomposition (1). Remark that

$$\frac{\partial}{\partial y^i} = \left(\frac{\partial}{\partial x^i}\right)^V, \qquad \frac{\delta}{\delta x^i} = \left(\frac{\partial}{\partial x^i}\right)^H$$

where X^V and X^H denote the vertical and horizontal lift of the vector field X on M respectively. We can use the vertical and horizontal lifts in order to obtain invariant expressions for some results in this paper. However, we should prefer to work in local coordinates since the formulas are obtained easier and, in a certain sense, they are more natural.

We can easily obtain the following

Lemma 1. *If $n > 1$ and u, v are smooth functions on TM such that*

$$ug_{ij} + vg_{0i}g_{0j} = 0$$

on the domain of any induced local chart on TM, then $u = 0$, $v = 0$.

Remark. In a similar way we obtain from the condition

$$u\delta^i_j + vg_{0j}y^i = 0$$

the relation $u = v = 0$.
Consider the energy density of the tangent vector y with respect to the Riemannian metric g

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^i y^k, \qquad y \in \tau^{-1}(U).$$

Obviously, we have $t \in [0, \infty)$ for all $y \in TM$.

3. The Sectional Curvature of the Tangent Bundle with General Natural Lifted Metric

Let G be the general natural lifted metric on TM, defined by

$$\begin{aligned}
G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) &= c_1 g_{ij} + d_1 g_{0i} g_{0j} = G^{(1)}_{ij} \\
G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) &= c_2 g_{ij} + d_2 g_{0i} g_{0j} = G^{(2)}_{ij} \\
G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) &= G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = c_3 g_{ij} + d_3 g_{0i} g_{0j} = G^{(3)}_{ij}
\end{aligned} \tag{2}$$

where c_1, c_2, c_3, d_1, d_2, d_3 are six smooth functions of the density energy on TM.
The Levi-Civita connection ∇ of the Riemannian manifold (TM, G) is obtained from the formula

$$\begin{aligned}
2G(\nabla_X Y, Z) = X(G(X, Z)) + Y(G(X, Z)) - Z(G(X, Y)) + G([X, Y], Z) \\
- G([X, Z], Y) - G([Y, Z], X)
\end{aligned}$$

for all $X, Y, Z \in \chi(M)$ and is characterized by the conditions

$$\nabla G = 0, \qquad T = 0$$

where T is the torsion tensor of ∇.
In the case of the tangent bundle TM we can obtain the explicit expression of ∇.
The symmetric $2n \times 2n$ matrix

$$\begin{pmatrix} G^{(1)}_{ij} & G^{(3)}_{ij} \\ G^{(3)}_{ij} & G^{(2)}_{ij} \end{pmatrix}$$

associated to the metric G in the base $\left(\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right)$ has the inverse

$$\begin{pmatrix} H^{ij}_{(1)} & H^{ij}_{(3)} \\ H^{ij}_{(3)} & H^{ij}_{(2)} \end{pmatrix}$$

where the entries are the blocks

$$\begin{aligned} H^{kl}_{(1)} &= p_1 g^{kl} + q_1 y^k y^l \\ H^{kl}_{(2)} &= p_2 g^{kl} + q_2 y^k y^l \\ H^{kl}_{(3)} &= p_3 g^{kl} + q_3 y^k y^l. \end{aligned} \tag{3}$$

Here g^{kl} are the components of the inverse of the matrix (g_{ij}) and p_1, q_1, p_2, q_2, p_3, $q_3 : [0, \infty) \to \mathbb{R}$, some real smooth functions. Their expressions are obtained by solving the system

$$\begin{aligned} G^{(1)}_{ih} H^{hk}_{(1)} + G^{(3)}_{ih} H^{hk}_{(3)} &= \delta^k_i \\ G^{(1)}_{ih} H^{hk}_{(3)} + G^{(3)}_{ih} H^{hk}_{(2)} &= 0 \\ G^{(3)}_{ih} H^{hk}_{(1)} + G^{(2)}_{ih} H^{hk}_{(3)} &= 0 \\ G^{(3)}_{ih} H^{hk}_{(3)} + G^{(2)}_{ih} H^{hk}_{(2)} &= \delta^k_i \end{aligned}$$

in which we substitute the relations (2) and (3). By using Lemma 1, we get p_1, p_2, p_3 as functions of c_1, c_2, c_3

$$p_1 = \frac{c_2}{c_1 c_2 - c_3^2}, \qquad p_2 = \frac{c_1}{c_1 c_2 - c_3^2}, \qquad p_3 = -\frac{c_3}{c_1 c_2 - c_3^2} \tag{4}$$

and q_1, q_2, q_3 as functions of c_1, c_2, c_3, d_1, d_2, d_3, p_1, p_2, p_3

$$\begin{aligned} q_1 = &- \frac{c_2 d_1 p_1 - c_3 d_3 p_1 - c_3 d_2 p_3 + c_2 d_3 p_3 + 2 d_1 d_2 p_1 t - 2 d_3^2 p_1 t}{c_1 c_2 - c_3^2 + 2 c_2 d_1 t + 2 c_1 d_2 t - 4 c_3 d_3 t + 4 d_1 d_2 t^2 - 4 d_3^2 t^2} \\ q_2 = &- \frac{d_2 p_2 + d_3 p_3}{c_2 + 2 d_2 t} \\ &+ \frac{(c_3 + 2 d_3 t)[(d_3 p_1 + d_2 p_3)(c_1 + 2 d_1 t) - (d_1 p_1 + d_3 p_3)(c_3 + 2 d_3 t)]}{(c_2 + 2 d_2 t)[(c_1 + 2 d_1 t)(c_2 + 2 d_2 t) - (c_3 + 2 d_3 t)^2]} \\ q_3 = &- \frac{(d_3 p_1 + d_2 p_3)(c_1 + 2 d_1 t) - (d_1 p1 + d_3 p_3)(c_3 + 2 d_3 t)}{(c_1 + 2 d_1 t)(c_2 + 2 d_2 t) - (c_3 + 2 d_3 t)^2}. \end{aligned} \tag{5}$$

In [14] we obtained the expression of the Levi-Civita connection of the Riemannian metric G on TM.

Theorem 1. *The Levi-Civita connection ∇ of G has the following expression in the local adapted frame* $\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}, \frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\right)$

$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = Q^h_{ij} \frac{\partial}{\partial y^h} + \widetilde{Q}^h_{ij} \frac{\delta}{\delta x^h}, \qquad \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = (\Gamma^h_{ij} + \widetilde{P}^h_{ji}) \frac{\partial}{\partial y^h} + P^h_{ji} \frac{\delta}{\delta x^h}$$

$$\nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = P^h_{ij} \frac{\delta}{\delta x^h} + \widetilde{P}^h_{ij} \frac{\partial}{\partial y^h}, \qquad \nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = (\Gamma^h_{ij} + \widetilde{S}^h_{ij}) \frac{\delta}{\delta x^h} + S^h_{ij} \frac{\partial}{\partial y^h}$$

where Γ^h_{ij} are the Christoffel symbols of the connection $\dot\nabla$ and the M-tensor fields appearing as coefficients in the above expressions are given as

$$\begin{aligned}
Q^h_{ij} &= \frac{1}{2}(\partial_i G^{(2)}_{jk} + \partial_j G^{(2)}_{ik} - \partial_k G^{(2)}_{ij})H^{kh}_{(2)} + \frac{1}{2}(\partial_i G^{(3)}_{jk} + \partial_j G^{(3)}_{ik})H^{kh}_{(3)}\\
\widetilde{Q}^h_{ij} &= \frac{1}{2}(\partial_i G^{(2)}_{jk} + \partial_j G^{(2)}_{ik} - \partial_k G^{(2)}_{ij})H^{kh}_{(3)} + \frac{1}{2}(\partial_i G^{(3)}_{jk} + \partial_j G^{(3)}_{ik})H^{kh}_{(1)}\\
P^h_{ij} &= \frac{1}{2}(\partial_i G^{(3)}_{jk} - \partial_k G^{(3)}_{ij})H^{kh}_{(3)} + \frac{1}{2}(\partial_i G^{(1)}_{jk} + R^l_{0jk}G^{(2)}_{li})H^{kh}_{(1)}\\
\widetilde{P}^h_{ij} &= \frac{1}{2}(\partial_i G^{(3)}_{jk} - \partial_k G^{(3)}_{ij})H^{kh}_{(2)} + \frac{1}{2}(\partial_i G^{(1)}_{jk} + R^l_{0jk}G^{(2)}_{li})H^{kh}_{(3)}\\
S^h_{ij} &= -\frac{1}{2}(\partial_k G^{(2)}_{ij} + R^l_{0ij}G^{(2)}_{lk})H^{kh}_{(2)} + c_3 R_{i0jk}H^{kh}_{(3)}\\
\widetilde{S}^h_{ij} &= -\frac{1}{2}(\partial_k G^{(1)}_{ij} + R^l_{0ij}G^{(2)}_{lk})H^{kh}_{(3)} + c_3 R_{i0jk}H^{kh}_{(1)}
\end{aligned}$$

where R^h_{kij} are the components of the curvature tensor field of the Levi Civita connection $\dot\nabla$ of the base manifold (M, g).

Taking into account the expressions (2), (3) and by using the formulas (4), (5) we can obtain the detailed expressions of P^h_{ij}, Q^h_{ij}, S^h_{ij}, $\widetilde{P}^h_{ij}$, $\widetilde{Q}^h_{ij}$, $\widetilde{S}^h_{ij}$.

The curvature tensor field K of the connection ∇ is defined by the well known formula

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \qquad X, Y, Z \in \Gamma(TM).$$

By using the local adapted frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right)$, $i, j = 1, \ldots, n$, we obtained in [14], after a standard straightforward computation

$$\begin{aligned}
K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= XXXX^h_{kij}\frac{\delta}{\delta x^h} + XXXY^h_{kij}\frac{\partial}{\partial y^h}\\
K\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= XXYX^h_{kij}\frac{\delta}{\delta x^h} + XXYY^h_{kij}\frac{\partial}{\partial y^h}\\
K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k} &= YYXX^h_{kij}\frac{\delta}{\delta x^h} + YYXY^h_{kij}\frac{\partial}{\partial y^h}\\
K\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k} &= YYYX^h_{kij}\frac{\delta}{\delta x^h} + YYYY^h_{kij}\frac{\partial}{\partial y^h}\\
K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k} &= YXXX^h_{kij}\frac{\delta}{\delta x^h} + YXXY^h_{kij}\frac{\partial}{\partial y^h}\\
K\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k} &= YXYX^h_{kij}\frac{\delta}{\delta x^h} + YXYY^h_{kij}\frac{\partial}{\partial y^h}
\end{aligned}$$

where the M-tensor fields appearing as coefficients denote the horizontal and vertical components of the curvature tensor of the tangent bundle, and they are given by

$$
\begin{aligned}
XXXX^h_{kij} &= \widetilde{S}^h_{il}\widetilde{S}^l_{jk} + P^h_{li}S^l_{jk} - \widetilde{S}^h_{jl}\widetilde{S}^l_{ik} - P^h_{lj}S^l_{ik} + R^h_{kij} + R^l_{0ij}P^h_{lk}\\
XXXY^h_{kij} &= \widetilde{S}^l_{jk}S^h_{il} + \widetilde{P}^h_{li}S^l_{jk} - \widetilde{S}^l_{ik}S^h_{jl} - \widetilde{P}^h_{lj}S^l_{ik} + \widetilde{P}^h_{lk}R^l_{0ij}\\
&\quad - \frac{1}{2}\dot{\nabla}_i R^r_{0jk}G^{(2)}_{rl}H^{(3)}_{hl} + c_3\dot{\nabla}_i R_{j0kh}\\
XXYX^h_{kij} &= \widetilde{P}^l_{kj}P^h_{li} + P^l_{kj}\widetilde{S}^h_{il} - \widetilde{P}^l_{ki}P^h_{lj} - P^l_{ki}\widetilde{S}^h_{jl} + R^l_{0ij}\widetilde{Q}^h_{lk}\\
XXYY^h_{kij} &= \widetilde{P}^l_{kj}\widetilde{P}^h_{li} + P^l_{kj}S^h_{il} - \widetilde{P}^l_{ki}\widetilde{P}^h_{lj} - P^l_{ki}S^h_{jl} + R^l_{0ij}Q^h_{lk} + R^h_{kij}\\
YYXX^h_{kij} &= \partial_i P^h_{jk} - \partial_j P^h_{ik} + \widetilde{P}^l_{jk}\widetilde{Q}^h_{il} + P^l_{jk}P^h_{il} - \widetilde{P}^l_{ik}\widetilde{Q}^h_{jl} - P^l_{ik}P^h_{jl}\\
YYXY^h_{kij} &= \partial_i \widetilde{P}^h_{jk} - \partial_j \widetilde{P}^h_{ik} + \widetilde{P}^l_{jk}Q^h_{il} + P^l_{jk}\widetilde{P}^h_{il} - \widetilde{P}^l_{ik}Q^h_{jl} - P^l_{ik}\widetilde{P}^h_{jl}\\
YYYX^h_{kij} &= \partial_i \widetilde{Q}^h_{jk} - \partial_j \widetilde{Q}^h_{ik} + Q^l_{jk}\widetilde{Q}^h_{il} + \widetilde{Q}^l_{jk}P^h_{il} - Q^l_{ik}\widetilde{Q}^h_{jl} - \widetilde{Q}^l_{ik}P^h_{jl}\\
YYYY^h_{kij} &= \partial_i Q^h_{jk} - \partial_j Q^h_{ik} + Q^l_{jk}Q^h_{il} + \widetilde{Q}^l_{jk}\widetilde{P}^h_{il} - Q^l_{ik}Q^h_{jl} - \widetilde{Q}^l_{ik}\widetilde{P}^h_{jl}\\
YXXX^h_{kij} &= \partial_i \widetilde{S}^h_{jk} + S^l_{jk}\widetilde{Q}^h_{il} + \widetilde{S}^l_{jk}P^h_{il} - \widetilde{P}^l_{ik}P^h_{lj} - P^l_{ik}\widetilde{S}^h_{jl} - \dot{\nabla}_j R^r_{0ik}G^{(2)}_{rl}H^{(3)}_{hl}\\
YXXY^h_{kij} &= \partial_i S^h_{jk} + S^l_{jk}Q^h_{il} + \widetilde{S}^l_{jk}\widetilde{P}^h_{il} - \widetilde{P}^l_{ik}\widetilde{P}^h_{lj} - P^l_{ik}S^h_{jl} - \dot{\nabla}_j R^r_{0ik}G^{(2)}_{rl}H^{(1)}_{hl}\\
YXYX^h_{kij} &= \partial_i P^h_{kj} + \widetilde{P}^l_{kj}\widetilde{Q}^h_{il} + P^l_{kj}P^h_{il} - Q^l_{ik}P^h_{lj} - \widetilde{Q}^l_{ik}\widetilde{S}^h_{jl}\\
YXYY^h_{kij} &= \partial_i \widetilde{P}^h_{kj} + \widetilde{P}^l_{kj}Q^h_{il} + P^l_{kj}\widetilde{P}^h_{il} - Q^l_{ik}\widetilde{P}^h_{lj} - \widetilde{Q}^l_{ik}S^h_{jl}.
\end{aligned}
$$

We mention that we used the character X on a certain position to indicate that the argument on that position was a horizontal vector field and, similarly, we used the character Y for vertical vector fields.

We compute the partial derivatives with respect to the tangential coordinates y^i of of $G^{(\alpha)}_{jk}$ and $H^{jk}_{(\alpha)}$, for $\alpha = 1, 2, 3$

$$
\begin{aligned}
\partial_i G^{(\alpha)}_{jk} &= c'_\alpha g_{0i}g_{jk} + d'_\alpha g_{0i}g_{0j}g_{0k} + d_\alpha g_{ij}g_{0k} + d_\alpha g_{0i}g_{jk}\\
\partial_i H^{jk}_{(\alpha)} &= p'_\alpha g^{jk}g_{0i} + q'_\alpha g_{0i}y^j y^k + q_\alpha \delta^j_i y^k + q_\alpha y^j \delta^k_i\\
\partial_i\partial_j G^{(\alpha)}_{kl} &= c''_\alpha g_{0i}g_{0j}g_{kl} + c'_\alpha g_{ij}g_{kl} + d''_\alpha g_{0j}g_{0k}g_{0l} + d'_\alpha g_{ij}g_{0k}g_{0l} + d'_\alpha g_{0j}g_{ik}g_{0l}\\
&\quad + d'_\alpha g_{0j}g_{0k}g_{il} + d'_\alpha g_{0i}g_{jk}g_{0l} + d'_\alpha g_{0i}g_{0k}g_{jl} + d_\alpha g_{jk}g_{il} + d_\alpha g_{ik}g_{jl}.
\end{aligned}
$$

Next we get the first order partial derivatives with respect to the tangential coordinates y^i of the M-tensor fields P^h_{ij}, Q^h_{ij}, S^h_{ij}, $\widetilde{P}^h_{ij}$, $\widetilde{Q}^h_{ij}$, $\widetilde{S}^h_{ij}$

$$
\begin{aligned}
\partial_i Q^h_{jk} &= \frac{1}{2}\partial_i H^{hl}_{(2)}(\partial_j G^{(2)}_{kl} + \partial_k G^{(2)}_{jl} - \partial_l G^{(2)}_{jk})\\
&\quad + \frac{1}{2}H^{hl}_{(2)}(\partial_i\partial_j G^{(2)}_{kl} + \partial_i\partial_k G^{(2)}_{jl} - \partial_i\partial_l G^{(2)}_{jk})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\partial_i H^{hl}_{(3)}(\partial_j G^{(3)}_{kl} + \partial_k G^{(3)}_{jl}) + \frac{1}{2}H^{hl}_{(3)}(\partial_i\partial_j G^{(3)}_{kl} + \partial_i\partial_k G^{(3)}_{jl})\\
\partial_i \widetilde{Q}^h_{jk} = &\frac{1}{2}\partial_i H^{hl}_{(3)}(\partial_j G^{(2)}_{kl} + \partial_k G^{(2)}_{jl} - \partial_l G^{(2)}_{jk})\\
&+\frac{1}{2}H^{hl}_{(3)}(\partial_i\partial_j G^{(2)}_{kl} + \partial_i\partial_k G^{(2)}_{jl} - \partial_i\partial_l G^{(2)}_{jk})\\
&+\frac{1}{2}\partial_i H^{hl}_{(1)}(\partial_j G^{(3)}_{kl} + \partial_k G^{(3)}_{jl}) + \frac{1}{2}H^{hl}_{(1)}(\partial_i\partial_j G^{(3)}_{kl} + \partial_i\partial_k G^{(3)}_{jl})\\
\partial_i \widetilde{P}^h_{jk} = &\frac{1}{2}\partial_i H^{hl}_{(2)}(\partial_j G^{(3)}_{kl} - \partial_l G^{(3)}_{jk}) + \frac{1}{2}H^{hl}_{(2)}(\partial_i\partial_j G^{(3)}_{kl} - \partial_i\partial_l G^{(3)}_{jk})\\
&+\frac{1}{2}\partial_i H^{hl}_{(3)}(\partial_j G^{(1)}_{kl} + R^r_{0kl}G^{(2)}_{rj})\\
&+\frac{1}{2}H^{hl}_{(3)}(\partial_i\partial_j G^{(1)}_{kl} + R^r_{ikl}G^{(2)}_{rj} + R^r_{0kl}\partial_i G^{(2)}_{rj})\\
\partial_i P^h_{jk} = &\frac{1}{2}\partial_i H^{hl}_{(3)}(\partial_j G^{(3)}_{kl} - \partial_l G^{(3)}_{jk}) + \frac{1}{2}H^{hl}_{(3)}(\partial_i\partial_j G^{(3)}_{kl} - \partial_i\partial_l G^{(3)}_{jk})\\
&+\frac{1}{2}\partial_i H^{hl}_{(1)}(\partial_j G^{(1)}_{kl} + R^r_{0kl}G^{(2)}_{rj})\\
&+\frac{1}{2}H^{hl}_{(1)}(\partial_i\partial_j G^{(1)}_{kl} + R^r_{ikl}G^{(2)}_{rj} + R^r_{0kl}\partial_i G^{(2)}_{rj})\\
\partial_i S^h_{jk} = &-\frac{1}{2}[(\partial_i\partial_r G^{(1)}_{jk} + R^l_{ijk}G^{(2)}_{lr})H^{rh}_{(2)} + (\partial_r G^{(1)}_{jk} + R^l_{0jk}G^{(2)}_{lr})\partial_i H^{rh}_{(2)}]\\
&+c_3' g_{0i}R_{j0kr}H^{rh}_{(3)} + c_3(R_{jikr}H^{rh}_{(3)} + R_{j0kr}\partial_i H^{rh}_{(3)})\\
\partial_i \widetilde{S}^h_{jk} = &-\frac{1}{2}[(\partial_i\partial_r G^{(1)}_{jk} + R^l_{ijk}G^{(2)}_{lr})H^{rh}_{(3)} + (\partial_r G^{(1)}_{jk} + R^l_{0jk}G^{(2)}_{lr})\partial_i H^{rh}_{(3)}]\\
&+c_3' g_{0i}R_{j0kr}H^{rh}_{(1)} + c_3(R_{jikr}H^{rh}_{(1)} + R_{j0kr}\partial_i H^{rh}_{(1)}).
\end{aligned}
$$

It was not convenient to think c_1, c_2, c_3, d_1, d_2, d_3 and p_1, p_2, p_3, q_1, q_2, q_3 as functions of t since **RICCI** did not make some useful factorizations after the command **TensorSimplify**. We decided to consider these functions as well as their derivatives of first, second and third order, as constants, the tangent vector y as a first order tensor, the components $G^{(1)}_{ij}$, $G^{(2)}_{ij}$, $G^{(3)}_{ij}$, $H^{ij}_{(1)}$, $H^{ij}_{(2)}$, $H^{ij}_{(3)}$ as second order tensors and so on, on the Riemannian manifold M, the associated indices being h, i, j, k, l, r, s.

The tensor field corresponding to the curvature tensor field of a Riemannian manifold (TM, G) having constant sectional curvature k, is given by the formula

$$K_0(X,Y)Z = k[G(Y,Z)X - G(X,Z)Y].$$

After a straightforward computation we obtain

$$K_0\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k}=XXXX0^h_{kij}\frac{\delta}{\delta x^h}+XXXY0^h_{kij}\frac{\partial}{\partial y^h}$$
$$K_0\left(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k}=XXYX0^h_{kij}\frac{\delta}{\delta x^h}+XXYY0^h_{kij}\frac{\partial}{\partial y^h}$$
$$K_0\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\delta}{\delta x^k}=YYXX0^h_{kij}\frac{\delta}{\delta x^h}+YYXY0^h_{kij}\frac{\partial}{\partial y^h}$$
$$K_0\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial y^k}=YYYX0^h_{kij}\frac{\delta}{\delta x^h}+YYYY0^h_{kij}\frac{\partial}{\partial y^h}$$
$$K_0\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\delta}{\delta x^k}=YXXX0^h_{kij}\frac{\delta}{\delta x^h}+YXXY0^h_{kij}\frac{\partial}{\partial y^h}$$
$$K_0\left(\frac{\partial}{\partial y^i},\frac{\delta}{\delta x^j}\right)\frac{\partial}{\partial y^k}=YXYX0^h_{kij}\frac{\delta}{\delta x^h}+YXYY0^h_{kij}\frac{\partial}{\partial y^h}$$

where the M-tensor fields appearing as coefficients are the horizontal and vertical components of the tensor K_0 and they are given by

$$XXXX0^h_{kij}=k[G^{(1)}_{jk}\delta^h_i-G^{(1)}_{ik}\delta^h_j],\quad XXXY0^h_{kij}=0$$
$$XXYX0^h_{kij}=k[G^{(3)}_{jk}\delta^h_i-G^{(3)}_{ik}\delta^h_j],\quad XXYY0^h_{kij}=0$$
$$YYXX0^h_{kij}=0,\quad YYXY0^h_{kij}=k[G^{(3)}_{jk}\delta^h_i-G^{(3)}_{ik}\delta^h_j]$$
$$YYYX0^h_{kij}=0,\quad YYYY0^h_{kij}=k[G^{(2)}_{jk}\delta^h_i-G^{(2)}_{ik}\delta^h_j]$$
$$YXXX0^h_{kij}=-kG^{(3)}_{ik}\delta^h_j,\quad YXXY0^h_{kij}=kG^{(1)}_{jk}\delta^h_i$$
$$YXYX0^h_{kij}=-kG^{(2)}_{ik}\delta^h_j,\quad YXYY0^h_{kij}=kG^{(3)}_{jk}\delta^h_i.$$

In order to get the conditions under which (TM,G) is a Riemannian manifold of constant sectional curvature, we study the vanishing of the components of the difference $K-K_0$. In this study it is useful the following generic result similar to the Lemma 1.

Lemma 2. *If $\alpha_1,\dots,\alpha_{10}$ are smooth functions on TM such that*

$$\alpha_1\delta^h_i g_{jk}+\alpha_2\delta^h_j g_{ik}+\alpha_3\delta^h_k g_{ij}+\alpha_4\delta^h_k g_{0i}g_{0j}+\alpha_5\delta^h_j g_{0i}g_{0k}+\alpha_6\delta^h_i g_{0j}g_{0k}$$
$$+\alpha_7 g_{jk}g_{0i}y^h+\alpha_8 g_{ik}g_{0j}y^h+\alpha_9 g_{ij}g_{0k}y^h+\alpha_{10}g_{0i}g_{0j}g_{0k}y^h=0$$

then $\alpha_1=\dots=\alpha_{10}=0$.

After a detailed analysis of several terms in the vanishing problem of the components of the above difference we can formulate the following proposition.

Proposition 1. *Let (M, g) be a Riemannian manifold. If the tangent bundle TM with the general natural lifted metric G has constant sectional curvature, then the base manifold is flat.*

Proof: For $y = 0$ the difference $XXYY^h_{kij} - XXYY0^h_{kij}$ reduces to R^h_{kij}.
If the sectional curvature of the tangent bundle is constant, this difference vanishes, so the curvature of the base manifold must vanish too. □

4. Tangent Bundles with Constant Sectional Curvature

Theorem 2. *Let (M, g) be a Riemannian manifold. The tangent bundle TM with the natural lifted metric G has constant sectional curvature if and only if the base manifold is flat and the metric G has the associated matrix of the form*

$$\begin{pmatrix} cg_{ij} & \beta g_{ij} + \beta' g_{0i}g_{0j} \\ \beta g_{ij} + \beta' g_{0i}g_{0j} & \alpha g_{ij} + \frac{\alpha'\beta^2+2\alpha'\beta\beta' t-2\alpha\beta'^2 t}{\beta^2} g_{0i}g_{0j} \end{pmatrix}$$

where α and β are two real smooth function depending on the energy density and c is an arbitrary constant. Moreover, in this case, TM is flat, i.e. $k = 0$.

Proof: In Proposition 1 we have proved that the base manifold of the tangent bundle with constant sectional curvature must be flat. By using the `RICCI` package of the program `Mathematica`, we impose the vanishing condition for the curvature tensor of the base manifold in all the differences between the components of the curvature tensors K and K_0 of TM. After a long computation we find some differences in which the third terms are of one of the forms: $\frac{c_3 d_1}{2(c_3^2 - c_1 c_2))} g_{ij}\delta^h_k$ in the case of the differences $YXXX^h_{kij} - YXXX0^h_{kij}$ and $YXYY^h_{kij} - YXYY0^h_{kij}$, $\frac{c_1 d_1}{2(c_1 c_2 - c_3^2)} g_{ij}\delta^h_k$ for the difference $YXXY^h_{kij} - YXXY0^h_{kij}$ and $\frac{c_2 d_1}{2(c_1 c_2 - c_3^2)} g_{ij}\delta^h_k$ for $YXYX^h_{kij} - YXYX^h_{kij}$.
As all the coefficients which appear in these differences must vanish, we obtain $d_1 = 0$, because c_1 and c_3, or c_2 and c_3 cannot vanish at the same time, the metric g being non-degenerated.
If we impose $d_1 = 0$ in $XXXY^h_{kij} - XXXY0^h_{kij}$ we obtain that this difference contains the factors $c_1 c_1'(c_1' c_3 - c_1 c_3' + c_1 d_3)$. Thus, for the annulation of this difference, we have the cases $c_1' = \frac{c_1 c_3' - c_1 d_3}{c_3}$ or $c_1 = \mathrm{const}$ ($c_1 = 0$ being a particular case).
The first case, $c_1' = \frac{c_1 c_3' - c_1 d_3}{c_3}$ is not a favorable one, because the difference $YYYY^h_{kij} - YYYY0^h_{kij}$ contains two summands which cannot vanish

$$\frac{1}{2t} g_{jk}\delta^h_i - \frac{1}{2t} g_{ik}\delta^h_j.$$

In the case $c_1 = \mathrm{const}$ we obtain

$$XXXX^h_{kij} - XXXX0^h_{kij} = -c_1 k(g_{jk}\delta^h_i - g_{ik}\delta^h_j)$$

from which $c_1 = 0$ or $k = 0$. If $c_1 = 0$

$$XYX^h_{kij} - XXYX0^h_{kij} = -k(c_3 g_{jk}\delta^h_i - c_3 g_{ik}\delta^h_j - d_3\delta^h_j g_{0i}g_{0k} + d_3\delta^h_i g_{0j}g_{0k}).$$

As we considered $c_1 = 0$, we cannot have $c_3 = 0$ because the metric g must be non-degenerated, so the parenthesis cannot vanish and it remains $k = 0$. Now we can conclude that *the tangent bundle with general natural lifted metric cannot have nonzero sectional curvature.*

We continue the study of the general case $c_1 = \mathrm{const}$, since the case $c_1 = 0$ is a particular case only. Because the sectional curvature of the tangent bundle, k, is null, we obtain that the difference $XXYY^h_{kij} - XXYY0^h_{kij}$ vanishes if and only if $d_3 = c_3'$. This condition makes vanish all the differences that we study, except $YYYX^h_{kij} - YYYX0^h_{kij}$. From the annulation of this last difference, we obtain

$$d_2 = c_2' + 2t\frac{c_2' c_3 c_3' - c_2 c_3'^2}{c_3^2}\,.$$

If we denote c_1 by c, c_2 by α and c_3 by β, we obtain that the matrix associated to the metric G has the form given in Theorem 2.

Therefore, Theorem 2 gives the unique form of the matrix associated to the metric G. □

Acknowledgements

Partially supported by the Grant AT No.191/2006, CNCSIS, Ministerul Educaţiei şi Cercetării, România.

References

[1] Abbassi M. and Sarih M., *On Riemannian g-Natural Metrics of the Form $a.g^s + b.g^h + c.g^v$ on the Tangent Bundle of a Riemannian Manifold (M, g)*, Mediterranean J. Math. **2** (2005) 19–43.

[2] Abbassi M. and Sarih M., *On Some Hereditary Properties of Riemannian g-Natural Metrics on Tangent Bundles of Riemannian Manifolds*, Diff. Geom. Appl. **22** (2005) 19–47.

[3] Bejan C. and Oproiu V., *Tangent Bundles of Quasi-Constant Holomorphic Sectional Curvatures*, Balkan J. Geom. Applic. **11** (2006) 11–22.

[4] Kolář I., Michor P. and Slovak J., *Natural Operations in Differential Geometry*, Springer, Berlin, 1993.

[5] Kowalski O. and Sekizawa M., *Natural Transformations of Riemannian Metrics on Manifolds to Metrics on Tangent Bundles – a Classification*, Bull. Tokyo Gakugei Univ. **40** (1988) 1–29.

[6] Krupka D. and Janyška J., *Lectures on Differential Invariants*, Folia Fac. Sci. Nat. Univ. Purkinianae Brunensis, 1990.

[7] Mok K., Patterson E. and Wong Y., *Structure of Symmetric Tensors of Type* $(0,2)$ *and Tensors of Type* $(1,1)$ *on the Tangent Bundle*, Trans. Amer. Math. Soc. **234** (1977) 253–278.

[8] Munteanu M., *Cheeger Gromoll Type Metrics on the Tangent Bundle*, In: Proceedings of Fifth International Symposium BioMathsPhys, Iasi, June 16–17, 2006, 9pp.

[9] Munteanu M., *Old and New Structures on the Tangent Bundle*, In: Proceedings of the Eighth International Conference on Geometry, Integrability and Quantization, June 9-14, 2006, Varna, Bulgaria, I. M. Mladenov and M. de Leon (Eds), SOFTEX, Sofia 2007, pp 264–278.

[10] Oproiu V., *A Generalization of Natural Almost Hermitian Structures on the Tangent Bundles*, Math. J. Toyama Univ. **22** (1999) 1–14.

[11] Oproiu V., *A Kaehler Einstein Structure on the Tangent Bundle of a Space Form*, Int. J. Math. Math. Sci. **25** (2001) 183–195.

[12] Oproiu V., *A Locally Symmetric Kaehler Einstein Structure on the Tangent Bundle of a Space Form*, Contributions to Algebra and Geometry **40** (1999) 363–372.

[13] Oproiu V., *Some New Geometric Structures on the Tangent Bundles*, Publ. Math. Debrecen **55** (1999) 261–281.

[14] Oproiu V. and Druta S., *General Natural Kähler Structures of Constant Holomorphic Sectional Curvature on Tangent Bundles*, An. St. Univ. "Al.I.Cuza" Iasi (S. N.) Matematica **53** (2007) 149–166.

[15] Oproiu V. and Papaghiuc N., *A Kaehler Structure on the Nonzero Tangent Bundle of a Space Form*, Diff. Geom. Appl. **11** (1999) 1–14.

[16] Oproiu V. and Papaghiuc N., *Some Classes of Almost Anti-Hermitian Structures on the Tangent Bundle*, Mediterranean J. Math. **1** (2004) 269–282.

[17] Tahara M., Vanhecke L. and Watanabe Y., *New Structures on Tangent Bundles*, Note di Matematica (Lecce) **18** (1998) 131–141.

[18] Yano K. and Ishihara S., *Tangent and Cotangent Bundles*, M. Dekker, New York, 1973.