THE SEISMOLOGICAL IMPLICATIONS OF AEOLOTROPY IN CONTINENTAL STRUCTURE

Robert Stoneley

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Summary

The seismological consequences of a departure from isotropy in the surface layers of the Earth are discussed. It is noted that erroneous values of the focal depth and the thickness of the layers might be found in near earthquakes. For a transversely isotropic continent it is shown that SH waves would travel with a different velocity from that of SV, and that the law of variation of velocity with direction is different. With body waves the sharp distinction into compressional and distortional waves does not hold, and an explosion would generate an S wave as well as a P wave; an apparent difference in the instant of generation of P and S might arise.

It is proved that waves of the Rayleigh type can be propagated over the surface of a transversely isotropic body in which the axis of circular symmetry is normal to the free surface, supposed plane; in such waves the diminution of amplitude with depth is different from that in an isotropic body, so that the amplitudes of the surface waves generated by a source at a given depth will be different from what they would be in an isotropic medium. Love waves can be propagated as in isotropic media.

1. *Introduction.*—In discussion of the propagation of elastic waves through the continent it is usually supposed that the material is isotropic. Further, Professor H. Jeffreys has pointed out * that this is a reasonable assumption to make if the continental igneous rocks have their crystalline constituents orientated in random directions. It is questionable whether seismological recording has reached the stage of precision at which a departure from isotropy might be expected to show up in near earthquake studies, although seismic prospecting might reveal it. Mr Willmore's study of the Heligoland explosion † indicates the wide variation that may be expected in the composition and thickness of the sedimentary and granitic layers; the possibility must not be overlooked that in any one material the velocity of propagation of a disturbance may depend on the direction of travel.

There are some grounds for suspecting that aeolotropy might exist in the continents.‡ One obvious consideration is that crystalline materials deposited in water may settle in a preferential orientation; the same possibility arises if crystals settle from a magma. It is conceivable that the stress system set up by cooling may have engendered some aeolotropy in a manner analogous to the production of double refraction in glass by the application of stress.

For a preliminary investigation it will be sufficient to suppose that the elastic properties of a substance are symmetrical with respect to one fixed direction, which is here taken as the vertical axis $x=y=0$. Such a medium is called

* H. Jeffreys, *The Earth*, 2nd Ed., p. 184, 1929.

† P. L. Willmore, *Nature*, **159**, 707, 1947.

‡ Since this paper was written, I have come across a suggestion by Professor P. Byerly, that double refraction in the crust may be the explanation of a transition that he has observed from SH to SV in the earthquake of 1934 July 6. See *Addendum* to this paper.

"transversely isotropic", and in the notation of Love's *Elasticity** the strain energy function W is given by

$$2W = A(e_{xx}^2 + e_{yy}^2) + Ce_{zz}^2 + 2F(e_{xx} + e_{yy})e_{zz} + 2(A - 2N)e_{xx}e_{yy} + L(e_{yz}^2 + e_{zx}^2) + Ne_{xy}^2. \quad (1)$$

The theory of the propagation of disturbances through the interior of an aeolotropic solid is well known, and it will suffice to mention that corresponding to any assigned wave-normal there are three velocities of wave propagation. For an isotropic solid the characteristic equation has one root corresponding to a compressional wave and a repeated root corresponding to a distortional wave. For an aeolotropic medium this separation into a wave for which the curl of the displacement vanishes and a wave for which the divergence of the displacement vanishes does not in general occur. The nature of the waves at a free surface has a special seismological interest, and it will be more informative if the problem in two dimensions is discussed from the beginning.

2. *The Equations of Motion.*—The equations of small motion in a medium of density ρ are three of the type

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}, \quad (2)$$

in which $X_x = \partial W/\partial e_{xx}$; $Y_z = \partial W/\partial e_{yz}$; etc., and the effect of body forces has been neglected.

The stresses are thus

$$\left.\begin{aligned} X_x &= Ae_{xx} + (A-2N)e_{yy} + Fe_{zz}; & X_y &= Y_x = Ne_{xy}, \\ Y_y &= (A-2N)e_{xx} + Ae_{yy} + Fe_{zz}; & Y_z &= Z_y = Le_{yz}, \\ Z_z &= F(e_{xx} + e_{yy}) + Ce_{zz}; & Z_x &= X_z = Le_{zx}, \end{aligned}\right\} \quad (3)$$

so that the equations of motion are

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}\{Ae_{xx} + (A-2N)e_{yy} + Fe_{zz}\} + \frac{\partial}{\partial y}(Ne_{xy}) + \frac{\partial}{\partial z}(Le_{zx}), \\ \rho\frac{\partial^2 v}{\partial t^2} &= \frac{\partial}{\partial x}(Ne_{xy}) + \frac{\partial}{\partial y}\{(A-2N)e_{xx} + Ae_{yy} + Fe_{zz}\} + \frac{\partial}{\partial z}(Le_{yz}), \\ \rho\frac{\partial^2 w}{\partial t^2} &= \frac{\partial}{\partial x}(Le_{zx}) + \frac{\partial}{\partial y}(Le_{yz}) + \frac{\partial}{\partial z}\{F(e_{xx} + e_{yy}) + Ce_{zz}\}. \end{aligned}\right\} \quad (4)$$

We will confine ourselves to the case in which A, C, F, L, N are constants, and the motion takes place in two dimensions (x, z). Then we write $v = 0$; $\partial/\partial y = 0$, so that $e_{yy} = 0$; $e_{xy} = 0$; $e_{yz} = 0$, and the second of equations (4) is satisfied identically. The remaining equations are

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial x}\left\{A\frac{\partial u}{\partial x} + F\frac{\partial w}{\partial z}\right\} + \frac{\partial}{\partial z}\left\{L\frac{\partial u}{\partial z} + L\frac{\partial w}{\partial x}\right\}; \\ \rho\frac{\partial^2 w}{\partial t^2} &= \frac{\partial}{\partial x}\left\{L\frac{\partial u}{\partial z} + L\frac{\partial w}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{C\frac{\partial w}{\partial z} + F\frac{\partial u}{\partial x}\right\}. \end{aligned}\right\} \quad (5)$$

For an isotropic body with Lamé parameters λ, μ we have $A = C = \lambda + 2\mu$; $F = \lambda$; $L = N = \mu$, and it is readily verified that the foregoing equations reduce to the familiar form.

* A. E. H. Love, *Elasticity*, 3rd Ed., p. 158, 1920.

Now set

$$u=\frac{\partial\phi}{\partial x}+\frac{\partial\chi}{\partial z}; \qquad w=\frac{\partial\phi}{\partial z}-\frac{\partial\chi}{\partial x}, \tag{6}$$

so that $\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=$ the dilatation $\Delta=\nabla^2\phi$;

$$\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=\nabla^2\chi,$$

and suppose that for a wave of period $2\pi/\sigma$ the displacements (u, w) contain a time factor $\exp(i\sigma t)$. Then we have

$$\left.\begin{aligned} -\rho\sigma^2(\phi_x+\chi_z)&=A(\phi_{xxx}+\chi_{zxx})+F(\phi_{zzx}-\chi_{zxx})+L(2\phi_{zzx}+\chi_{zzz}-\chi_{zxx});\\ -\rho\sigma^2(\phi_z-\chi_x)&=L(\chi_{zzx}-\chi_{xxx}+2\phi_{zxx})+C(\phi_{zzz}-\chi_{zzx})+F(\phi_{zxx}+\chi_{zzx}),\end{aligned}\right\} \tag{7}$$

where subscripts x, z denote the corresponding partial derivatives.

By putting $\phi=0$ in these equations we obtain two equations in χ that can be satisfied simultaneously only if $A=F+2L=C$. This result illustrates the remark made earlier that in general there is not a purely compressional and a purely distortional wave, as with an isotropic solid.

3. *Plane Waves in an Infinite Solid.*—For waves propagated in a direction given by the direction cosines $(l, 0, n)$ we will suppose ϕ, χ to vary as $\exp\{i\kappa(lx+nz\pm ct)\}$, where $c^2=\sigma^2/\kappa^2$. Then equations (7) give

$$\left.\begin{aligned} \rho c^2(l\phi+n\chi)&=A(l^3\phi+l^2n\chi)+F(ln^2\phi-l^2n\chi)+L(2ln^2\phi+n^3\chi-l^2n\chi);\\ \rho c^2(n\phi-l\chi)&=L(2l^2n\phi+ln^2\chi-l^3\chi)+C(n^3\phi-ln^2\chi)+F(l^2n\phi+ln^2\chi),\end{aligned}\right\} \tag{8}$$

in which $l^2+n^2=1$.

The condition of consistency of these equations gives the wave-velocity equation

$$\begin{aligned} &n^2\{(A-J)l^2+Ln^2-\rho c^2\}\{(J+L)l^2+Cn^2-\rho c^2\}\\ &\qquad+l^2\{Al^2+(J+L)n^2-\rho c^2\}\{(C-J)n^2+Ll^2-\rho c^2\}=0, \end{aligned} \tag{9}$$

where J has been written for $F+L$ in order to simplify some of the later algebra.

Two special cases of (9) may be noted at once:

(i) for a wave propagated in the x-direction, $l=1$, $n=0$, so that $c^2=A/\rho$ and $c^2=L/\rho$ are the solutions;

(ii) for a wave propagated in the z-direction, $l=0$, $n=1$, giving the solutions $c^2=C/\rho$ and $c^2=L/\rho$.

Equation (9) gives on reduction *

$$(\rho c^2)^2-\rho c^2(L+Al^2+Cn^2)+L(Cn^4+Al^4)+l^2n^2(L^2+AC-J^2)=0, \tag{10}$$

the roots of which are given by

$$\begin{aligned} 2\rho c^2&=Al^2+Cn^2+L\pm[(Al^2+Cn^2+L)^2-4\{ALl^4+CLn^4+l^2n^2(L^2+AC-J^2)\}]^{\frac12}\\ &=Al^2+Cn^2+L\pm[\{(A-L)l^2-(C-L)n^2\}^2+4J^2l^2n^2]^{\frac12}, \end{aligned} \tag{11}$$

from which it follows that both values of c^2 are real and positive. For an isotropic solid this equation reduces, as it should, to $2\rho c^2=2\lambda+4\mu$ or 2μ, so that the upper sign in (11) may be taken to correspond, for very nearly isotropic media, to the compressional wave, and the lower sign to the distortional wave.

* This equation can be found at once, without introducing ϕ and χ; it is sufficient to suppose that u and w are proportional to $\exp i\kappa(lx+nz\pm ct)$, and to substitute directly in equation (4).

This dependence of velocity upon direction would be of seismological importance. For instance, it would affect the angle of emergence, and in discussions of near earthquakes it would alter estimates of the thicknesses of the surface layers made on the usual assumption of isotropy.

Consider for simplicity a uniform isotropic material 2 with a uniform transversely isotropic surface layer 1 of constant thickness h, and let the wave velocities in 1 and 2 be c_1 and c_2 respectively. Suppose that a wave is travelling horizontally in 2 just below the interface and is refracted through medium 1 to a station on the surface, so as to make an angle θ with the vertical in medium 1. Then θ is determined by making the time of the ray stationary, that is, $c_1^{-1}\sec\theta - c_2^{-1}\tan\theta$ must be stationary. The equation for θ is thus

$$\sin\theta = \frac{c_1}{c_2} + \frac{\cos\theta}{c_1^2}\frac{dc_1}{d\theta},$$

in which c_2 is constant and c_1 is given by (11), with $l=\sin\theta$; $n=\cos\theta$. For assigned values of the elastic constants this equation can be solved numerically, and it is here exhibited in a form suitable for successive approximation when $dc_1/d\theta$ is small, i.e. when the medium is nearly isotropic. The time of passage is then easily found. If the focus is just below the interface and distant X horizontally from the station the time of transit will be

$$\frac{X}{c_2} + \frac{h\sec\theta}{c_1} - \frac{h\tan\theta}{c_2},$$

which reduces to

$$\frac{X}{c_2} + \frac{h}{c_2}\left\{\cot\theta + \frac{c_2}{c_1^2}\cdot\frac{dc_1}{d\theta}\operatorname{cosec}\theta\right\}.$$

Thus the departure from isotropy will lead to a different value of h from what would be found on the assumption of an isotropic layer. Further, an attempt to find an independent value of c_1 by recording the time of transit of an explosion wave in the layer will give $c_1^2 = A/\rho$, as already stated, and not the value that is needed for the preceding formula.

The distance-time curve for waves propagated in a uniformly layered isotropic continent is sensibly straight, except for stations very near the epicentre, for epicentral distances up to about 10°, and its curvature gives the depth of focus.* If, however, the velocity depends on direction this method will give an incorrect value of the focal depth.

The necessary modification of Knott's theory of the reflection and refraction of elastic waves at an interface could be made, as just indicated, by applications of the theorem of stationary time: the amplitudes could be calculated in the usual way by using the appropriate boundary conditions. The formulae would presumably be very complicated, and as the problem is one of diffraction rather than simply reflection and refraction in practice, it does not seem to be of seismological importance to pursue the question further at present.

From (8) we have

$$-\phi = \frac{n}{l}\chi\cdot\frac{(A-J)l^2 + Ln^2 - \rho c^2}{Al^2 + (J+L)n^2 - \rho c^2},$$

in which we substitute for c^2 from (11). Since $(\partial u/\partial x) + (\partial w/\partial z) = \nabla^2\phi = -\kappa^2\phi$, and since the expression just found for ϕ does not vanish in general, it follows that the root with the lower sign (which for nearly isotropic media gives a wave

* P. Byerly, *Bull. Seis. Soc. Amer.*, **29**, 451, 1939.

much like the distortional wave in isotropic media) does not correspond to a wave in which the dilatation vanishes. This point may be of some seismological significance. In an aeolotropic medium the wave ordinarily identified as P will not be wholly compressional. If a disturbance starts purely by a local compression, as in seismic prospecting, both P and S waves will be generated, the proportions being such that initially the stress over the explosion cavity is purely normal. Thus, from the explosion of a buried change there may arise a "direct S" in addition to the direct P.

As a simple model, suppose that there is a uniform sedimentary layer resting on granite, these two media being distinguished by the suffixes (s) and (g). Then if both P and S type waves (so far as the names are strictly applicable) are sent out from a surface source at the same time the *horizontal* velocities could be found directly from the seismograph records at surface stations. Suppose that one were to apply the customary formula for getting from the travel time curves of P_s, P_g, S_s, S_g the thickness of the layer and the horizontal velocities in the granite. Then if the materials are aeolotropic, not only may the calculated thickness be in error, as pointed out above, but unless special relations hold among the horizontal velocities and the values of P_s, S_s for the particular angles at which the waves (travelling horizontally in granite) are refracted into the sedimentary layer, one would infer that the P and S type waves left their origin at different times. This phenomenon is, of course, well known in near earthquakes; it is not suggested here that it does in fact arise through aeolotropy, but rather that this is one of the hypotheses that must be considered in attempting to correlate the travel-time relations of the various onsets read on records of near earthquakes.

The theory so far sketched relates to waves polarized in the plane of propagation, that is, to waves analogous to P and SV. It remains to ascertain the behaviour of SH waves in such a medium. It is easily verified that the equations of motion (4) are satisfied by supposing that $u=w=0$; $\partial/\partial y=0$; $v=v_0 \exp i\kappa(lx+nz-ct)$, which give on substitution

$$\rho c^2 = l^2 N + n^2 L \tag{11a}$$

for the velocity of waves of SH type, in which the displacement is perpendicular to the plane of propagation. By comparison with equation (10) it is apparent that waves corresponding to SV and SH will be propagated in a given direction with different velocities.

The theory of the reflection and refraction of these SH waves at an interface can be worked out as for waves of P and SV types. It will, in fact, be slightly less complicated. However, in the absence of precise data it would be of little seismological importance to carry through the calculations and subsequent computations.

4. *Surface Waves of Rayleigh Type.*—To investigate the existence of waves analogous to Rayleigh waves, assume

$$\left.\begin{aligned} \phi &= \phi(z) \exp i\kappa(x-ct); \\ \chi &= \chi(z) \exp i\kappa(x-ct), \end{aligned}\right\} \tag{12}$$

and let accents indicate derivatives with respect to z. Then substitution in (7) gives for the determination of ϕ and χ the simultaneous equations

$$\left.\begin{aligned} i\kappa\{(\rho c^2 - A)\kappa^2\phi + (F+2L)\phi''\} + \{L\chi''' + (\rho c^2 - A + F + L)\kappa^2\chi'\} &= 0; \\ \{(\rho c^2 - 2L - F)\kappa^2\phi' + C\phi'''\} + i\kappa\{(L-C+F)\chi'' + (L-\rho c^2)\kappa^2\chi\} &= 0. \end{aligned}\right\} \tag{13}$$

It is readily verified that for isotropic materials the values of ϕ and χ satisfy the equations $\phi'' - \kappa^2(1 - c^2\alpha^{-2})\phi = 0$; $\chi'' - \kappa^2(1 - c^2\beta^{-2})\chi = 0$, as for Rayleigh waves; in these equations $\alpha^2 = (\lambda + 2\mu)/\rho$; $\beta^2 = \mu/\rho$.

In equations (13) put

$$\phi = \phi_0 \exp(\kappa q z); \qquad \chi = \chi_0 \exp(\kappa q z), \tag{14}$$

where ϕ_0, χ_0, q are constants; then

$$\left.\begin{aligned} i\phi_0\{\rho c^2 - A + q^2(F + 2L)\} + q\chi_0\{\rho c^2 - A + F + L + Lq^2\} = 0; \\ iq\phi_0\{\rho c^2 - 2L - F + Cq^2\} - \chi_0\{L - \rho c^2 + q^2(L - C + F)\} = 0. \end{aligned}\right\} \tag{15}$$

The value of c^2 is not yet determined; it depends, of course, on the boundary conditions at the free surface $z = 0$. For an assigned value of c^2 the values of q^2 are given by the condition of consistency of the equations (15), i. e.

$$\{\rho c^2 - A + q^2(F + 2L)\}\{\rho c^2 - L + q^2(C - F - L)\} - q^2\{\rho c^2 - A + F + L + Lq^2\}\{\rho c^2 - 2L - F + Cq^2\} = 0. \tag{16}$$

Put $F + L = J$; $\rho c^2 - A = R$; $\rho c^2 - L = S$. Then (16) becomes

$$(1 - q^2)\{LCq^4 + (RC + LS + J^2)q^2 + RS\} = 0. \tag{17}$$

It will be shown later that the factor $(1 - q^2)$ does not lead to any motion of the Rayleigh type. Rejecting this factor, we are left with a quadratic equation in q^2. As a guide to further development it may be noted that for an isotropic body equation (17) reduces to $(q^2 - 1 + c^2\alpha^{-2})(q^2 - 1 + c^2\beta^{-2}) = 0$, where $\alpha^2 = (\lambda + 2\mu)/\rho$; $\beta^2 = \mu/\rho$; this is in agreement with the theory of Rayleigh waves. The values of q^2 that reduce to $1 - c^2\alpha^{-2}$ and $1 - c^2\beta^{-2}$ for an isotropic body will be called q_1^2 and q_2^2 respectively. The explicit values of q^2 are

$$\{-\Gamma \pm (\Gamma^2 - 4LCRS)^{\frac{1}{2}}\}/LC, \tag{20}$$

where Γ is written for $RC + LS + J^2$. By direct substitution and reduction it can be shown that the upper $(+)$ sign corresponds to q_1^2 and the lower $(-)$ sign to q_2^2.

For positive values of q^2 it is evident from (20) that $-\Gamma$ must be positive, which implies that

$$\rho c^2(C + L) < AC - L^2 - (F + L)^2. \tag{21}$$

For an isotropic body this becomes

$$(\lambda + 3\mu)\rho c^2 < (\lambda + 2\mu)^2 + \mu^2 - (\lambda + \mu)^2 = 2\mu(\lambda + 2\mu).$$

Thus, we may expect that the right-hand side of (21) will be positive, and therefore that real values of c and q will exist, for a fairly wide departure from isotropy.

The solutions of (13) are then

$$\left.\begin{aligned} \phi = \phi_1 \exp(\kappa q_1 z) + \phi_2 \exp(\kappa q_2 z); \\ \chi = \chi_1 \exp(\kappa q_1 z) + \chi_2 \exp(\kappa q_2 z), \end{aligned}\right\} \tag{22}$$

in which the positive direction of the axis of z is outwards from the free face, supposed plane, of the solid. Further, from (15),

$$\chi_1 = -i\phi_1\{\rho c^2 - A + q_1^2(F + 2L)\}/q_1\{\rho c^2 - A + F + L + Lq_1^2\} = im_1\phi_1/q_1, \text{ say}, \tag{23}$$

with a corresponding relation $\chi_2 = im_2\phi_2/q_2$. Thus,

$$\chi = (im_1\phi_1/q_1)\exp(\kappa q_1 z) + (im_2\phi_2/q_2)\exp(\kappa q_2 z), \tag{24}$$

and these solutions, which tend to zero with increasing depth, are expressed in terms of the constants ϕ_1, ϕ_2, while $m_1 = m_1(q_1)$ and $m_2 = m_2(q_2)$ are known functions of c.

It remains to introduce the condition that the free surface is free from traction, i.e. at $z=0$, $Z_z=0$, $X_z=0$, or

$$C\frac{\partial w}{\partial z}+F\frac{\partial u}{\partial x}=0; \qquad \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0. \tag{25}$$

Substitution in (25) from (6), (22) and (24) gives

$$\left.\begin{aligned}\phi_1\{C(q_1^2+m_1)-F(1+m_1)\}+\phi_2\{C(q_2^2+m_2)-F(1+m_2)\}=0;\\ \phi_1\{m_1q_1+m_1q_1^{-1}+2q_1\}+\phi_2\{m_2q_2+m_2q_2^{-1}+2q_2\}=0.\end{aligned}\right\} \tag{26}$$

Then the wave-velocity equation is

$$\begin{aligned}\{C(q_1^2+m_1)-F(1+m_1)\}&\{m_2q_2+m_2q_2^{-2}+2q_2\}\\ &=\{C(q_2^2+m_2)-F(1+m_2)\}\{m_1q_1+m_1q_1^{-1}+2q_1\}.\end{aligned} \tag{27}$$

It can be verified that for an isotropic body this equation reduces to the ordinary equation for Rayleigh waves,

$$(2-c^2\beta^{-2})^4=16(1-c^2\alpha^{-2})(1-c^2\beta^{-2}). \tag{28}$$

The values of q^2 are here $1-c^2\alpha^{-2}=q_1^2$; $1-c^2\beta^{-2}=q_2^2$. Also, with this identification of suffixes, $\chi_1=0$ and $\phi_2=0$; $m_1=0$; $m_2=\infty$; it is for this reason that the usual direct derivation of the Rayleigh wave equation is particularly simple.

It can now be seen that $q^2=1$ is not a relevant root of (17). By itself it does not give an admissible solution, for the corresponding displacements in an isotropic body would be $u=2i\kappa\exp\kappa z$; $w=2\kappa\exp\kappa z$, omitting a constant multiplier. These satisfy both $(\partial u/\partial x)+(\partial w/\partial z)=0$ and $(\partial u/\partial z)-(\partial w/\partial x)=0$, but the boundary conditions at $z=0$, that is, $\lambda\Delta+2\mu\partial w/\partial z=0$ and $(\partial u/\partial z)+(\partial w/\partial x)=0$ cannot be simultaneously satisfied unless the amplitude is zero.

Equally a combination of $q^2=1$ and one of the other roots is not permissible. To see this, put $q_1=1$ in (22), so that $m_1=-1$, while

$$\phi=\phi_1\exp(\kappa z)+\phi_2\exp(\kappa q_2 z);$$
$$\chi=-i\phi_1\exp(\kappa z)+i\phi_2(m_2/q_2)\exp(\kappa q_2 z).$$

Then there is a zero factor on both sides of (27), so that no equation for the wave velocity is obtainable.

The value of m_1 is $-\{R+q_1^2(J+L)\}/\{R+J+Lq_1^2\}$ and correspondingly for m_2. By simplifying $q_1^2+m_1$; $1+m_1$; $q_2^2+m_2$; $1+m_2$ and inserting in (27) it is found that a factor $(q_1^2-1)(q_2^2-1)/(R+J+Lq_1^2)(R+J+Lq_2^2)$ can be removed from both sides of the equation. The Rayleigh wave equation thus becomes

$$\begin{aligned}\{C(R+Lq_1^2)+FJ\}&\{R+q_2^2(L-J)\}q_1\\ &-\{C(R+Lq_2^2)+FJ\}\{R+q_1^2(L-J)\}q_2=0,\end{aligned} \tag{29}$$

which for an isotropic body reduces to the form (28). With regard to the factors that have been removed, $R+J+q_1^2L=0$ would be equivalent for an isotropic medium to $q_1^2=1-\rho c^2/\mu$, which would be possible only if $\lambda=0$. On the other hand, $R+J+q_2^2L=0$ would reduce to $q_2^2=1-\rho c^2/\mu$, which is the case. Thus a factor has been removed which is zero for an isotropic medium, but the equation (29) does in fact reduce for an isotropic medium to the form (28) obtained by Rayleigh.

Evidently the left-hand side of (29) must contain a factor (q_1-q_2). Dividing by this factor we obtain

$$CLFq_1^2q_2^2-(FCR+F^2J+LCR)q_1q_2-CLR(q_1^2+q_2^2)-CR^2-FRJ=0. \tag{30}$$

Substitute $q_1^2 q_2^2 = RS/LC$; $q_1^2 + q_2^2 = -\Gamma/LC$ from (17). Then after some reduction we find

$$R(S+L) = R\rho c^2 = \sqrt{\frac{RS}{LC}}\{RC+F^2\}, \tag{31}$$

or $\rho c^2 = -(S/R)^{\frac{1}{2}}.(CR+F^2)/(CL)^{\frac{1}{2}}$.* (32)

Writing ζ for ρc^2 in this equation, the wave velocity equation for waves of Rayleigh type becomes

$$f(\zeta) \equiv \zeta + \left(\frac{\zeta - L}{\zeta - A}\right)^{\frac{1}{2}} \left\{\frac{C(\zeta - A) + F^2}{(CL)^{\frac{1}{2}}}\right\} = 0. \tag{33}$$

When $\zeta = 0$, $f(\zeta)$ is negative, since $AC > F^2$ and $A > L$ for such media as we are considering. When $\zeta = L$, $f(\zeta)$is positive. Hence there must exist a root of (33) between $\zeta = 0$ and $\zeta = L$. For real roots, also, ζ must either be greater than both A and L or less than both A and L. For all values of ζ greater than A, however, $f(\zeta)$ is positive, so that no root exists in this region.

As a numerical example of an aeolotropic medium not too far removed from what we have contemplated for the continental crust we may quote from Love † the constants found by Voigt for beryl. In dynes/cm.² these are:—

$$c_{11} = A = 2\cdot694 \times 10^{12},$$
$$c_{33} = C = 2\cdot363 \times 10^{12},$$
$$c_{12} = A - 2N = 0\cdot961 \times 10^{12},$$
$$c_{13} = F = 0\cdot661 \times 10^{12},$$
$$c_{44} = L = 0\cdot653 \times 10^{12},$$

giving $J = 1\cdot315 \times 10^{12}$. There is hexagonal symmetry about the z-axis. Putting $\zeta = \eta \times 10^{12}$, the velocity equation for Rayleigh-type waves is

$$\eta = \left(\frac{0\cdot653 - \eta}{2\cdot694 - \eta}\right)^{\frac{1}{2}} (4\cdot773 - \eta).$$

Solved by successive approximation this gives $\eta = 0\cdot60845$. The value of ρ is about 2·7 g./cm.³. The Rayleigh wave velocity c_R is then about 4·75 km./sec.

In the x-direction the velocities of the two kinds of body waves through the interior are $(A/\rho)^{\frac{1}{2}}$ and $(L/\rho)^{\frac{1}{2}}$, i.e. 10·0 km./sec. and 4·92 km./sec. For an isotropic solid with $\lambda = \mu$, $c_R = 0\cdot9194 . \beta$. For beryl c_R is likewise less than the velocity of the slower of the two body waves, but the ratio of c_R to the velocity of the slower waves is 4·75/4·92 = 0·965.

Knowing the value of c we can now solve equation (17) to find q_1^2 and q_2^2. For beryl the values are 2·06 and 0·0285 respectively. For an isotropic medium, with $\lambda = \mu$, $q_1^2 = 1 - c_R^2 \beta^{-2} = 0\cdot7182$; $q_2^2 = 1 - \frac{1}{3} c_R^2 \beta^{-2} = 0\cdot1547$.

A reference to equations (22) shows that q_1 and q_2 determine the rate at which the amplitude of the vibration falls off with depth; the term in $\exp(\kappa q_2 z)$ will be the more important at depths of several wave-lengths. The example just worked out shows that the rate of diminution of amplitude with depth may be markedly different in aeolotropic media from what occurs in isotropic media. For a deep-focus earthquake, in which, following a reciprocal theorem in dynamics, the amplitudes of the Rayleigh waves should be relatively small, any considerable

* This must reduce to (28) for an isotropic solid. But the algebra is not so easy as might be expected unless one remembers that a factor $(q_1 - q_2)$ has been removed. Note that R and S will be negative for an isotropic solid.

† A. E. H. Love, *Elasticity*, 3rd Ed., p. 161, 1920.

departure from isotropy may give rise to a decidedly larger or smaller amplitude than the usual theory would indicate.

The surface displacements are, apart from the harmonic factors,

$$\left.\begin{aligned} u &= \phi_x + \chi_z = i\kappa\{\phi_1(1+m_1)+\phi_2(1+m_2)\}; \\ w &= \phi_z - \chi_x = \kappa\{\phi_1(q_1+m/q_1)+\phi_2(q_2+m_2/q_2)\}. \end{aligned}\right\} \qquad (34)$$

For a very nearly isotropic body m_1 is small and m_2 is large, and approximately $u = i\kappa\phi_2 m_2$; $w = \kappa\phi_2 m_2/q_2$; thus, the surface particles describe ellipses in a "retrograde" direction. The ratio of the vertical axis to the horizontal axis is $1/q_2$; this agrees, for an isotropic medium, with the usual theory of Rayleigh waves. For beryl the data give $m_1 = -3{\cdot}42$; $m_2 = -2{\cdot}70$. This is greatly different from the values $m_1 = 0$; $m_2 = \infty$ for an isotropic solid. Thus, if there is any considerable amount of aeolotropy in the surface layers of the continents the question of the ratio of the axes of the ellipses described by the surface particle, already heavily complicated by the layering of the crust, will be further complicated by the departure from isotropy.

For a homogeneous medium the waves are non-dispersive, that is, there is no dependence of velocity upon period.

5. *Surface Waves of Love Type.*—It is easy to show that in a transversely isotropic medium waves of the Love type may be propagated just as in an isotropic medium. Suppose that there is a uniform surface layer, of thickness T, underlain by a second uniform medium of infinite depth, and that there is no slipping at the interface. Taking the axes as before, the equations of motion can be satisfied by putting $u = w = 0$; $\partial/\partial y = 0$; $v = V(z)\,.\exp i\kappa(x-ct)$. Then $e_{xx} = e_{yy} = e_{zz} = e_{zx} = 0$; $e_{xy} = i\kappa v$; $e_{yz} = (dV/dz)\,.\exp i\kappa(x-ct)$. Then V satisfies the equation

$$d^2V/dz^2 = \kappa^2 V(N - \rho c^2), \qquad (35)$$

and the theory is the same as that of Love waves in an isotropic medium of rigidity N. If suffixes 1 and 2 refer to the layer and to the subjacent medium respectively, the wave-velocity equation will be

$$\tan\left\{\kappa T\left(\frac{\rho_1 c^2}{N_1} - 1\right)^{\frac{1}{2}}\right\} = \left\{\frac{N_2}{N_1}\cdot\frac{N_2 - \rho_2 c^2}{\rho_1 c^2 - N_1}\right\}^{\frac{1}{2}}, \qquad (36)$$

in which the velocity depends on the wave-length.

It is of some interest to note that for a nearly isotropic body it is the modulus L that resembles the rigidity in Rayleigh wave motion, whereas in Love wave motion the corresponding modulus is N. In fact, equation (35) is satisfied by V = constant, provided that $c^2 = N/\rho$. The numerical example quoted has N/L equal to about $\frac{4}{3}$.

6. *Other Geophysical Implications.*—The notion that the material of the continents might not be isotropic is by no means new. Hecker's observations with horizontal pendulums at Potsdam suggested that the Earth at that place offered more resistance to deforming forces acting east or west than to forces acting north or south. It was shown by Love * that as far as order of magnitude was concerned a sufficient explanation was to be found in the attraction of the tidal wave in the North Atlantic, together with the accompanying excess pressure on the sea bottom. However, some recent work by Tomaschek on the lunar deflexion of gravity in Eastern Europe shows changes from station to station that seem to

* A. E. H. Love, *Geodynamics*, p. 88, 1911.

rule out the possibility of ascribing the whole disturbance to oceanic tides; if these results are confirmed it may be necessary to revive the hypothesis of aeolotropy in continental structure.*

In spite of the large number of careful studies that have been made of well-observed near earthquakes, seismologists are left with an embarrassingly large number of onsets for which there is no convincing explanation. It may be that allowance for lack of isotropy will help to coordinate the records made at different stations, but very great caution would be required, since, as is well known to investigators, it is generally possible to pick out on a record an onset just where one expects to find it. However, the discussion in Section 3 shows that at the present stage of development it is important to ascertain, where possible, the plane of polarization of the S-waves; this requirement, in turn, calls attention to the need of equipping stations with at least three seismographs in order to measure all three components of the ground displacement.

Until definite evidence is available that the continental layers are aeolotropic it seems best to continue to assume that the material is isotropic, but to bear in mind the order of magnitude of the corrections that a departure from isotropy might entail. Perhaps the most direct way of obtaining such evidence would be by seismic exploration, where the conditions are to a large extent under control.

This investigation was carried out during the tenure of a Leverhulme Research Fellowship, and I wish to take this opportunity of expressing my gratitude to the Leverhulme Trustees.

Addendum

Since this paper was written I have searched through a number of "special studies" of earthquakes on the chance of finding whether the writers had noted any anomalous behaviour of P and S, and particularly S, that might bear on the considerations advanced above. I have come across an important paper by Professor Perry Byerly †, in which he notes that in that particular study the S wave generally begins as SH and within about 14 sec. has changed to SV. He mentions that the phenomenon had been noted before ‡, and did not appear to be a function of epicentral distance. He remarks: "Perhaps the surface layers offer the phenomenon of double refraction". I ought to add that as I had received from Professor Byerly an offprint of his paper at the time of publication it may well be that his remark was at the back of my mind when I began this investigation. If Byerly's interpretation of the phenomenon is correct, the fact that the interval between SH and SV does not depend on epicentral distance indicates that any aeolotropy is in the surface layers rather than in the mantle of the Earth. The two stations that are particularly cited are at distances of 27° and 40°.

Much the same observation has been made by Miss Lehmann §, who remarks that no vertical component of the first S-swing has been observed. The readings

* Professor H. Jeffreys, who has kindly read through the MS. of this paper, has expressed to me the opinion that the presence of aeolotropy would be most likely to show up in a systematic difference in the velocities of SV and SH.

† The Earthquake of 1934 July 6: Amplitudes and First Motion, *Bull. Seis. Soc. Amer.*, **28**, 12, 1938.

‡ F. Neumann, *Bull. Seis. Soc. Amer.*, **20**, 15–20, 1930. P. Byerly, *Bull. Seis. Soc. Amer.*, **24**, 98, 1934.

§ I. Lehmann, *Publ. Bur. Cent. Séism. Int.*, Ser. A, No. 12, p. 109, 1935.

of the second phase are as a rule 10 sec., or slightly more, later than the first phase, and she observes no dependence on distance of the interval between the phases. She ascribes the absence of a vertical component in the first S phase to the fact that the angle between the direction of those S oscillations and the horizontal should be small. Whether this is the correct explanation or whether, indeed, the observations made by Professor Byerly and Miss Lehmann refer to different phenomena is a question demanding further investigation. At all events they amply justify an examination of the seismological effects of aeolotropy.

Pembroke College,
Cambridge :
1948 *November* 1.

ERRATUM

G.S., **5**, No. 7, H. Jeffrey's paper:

P. 219, line 12 of Summary, *for* 6371·388 and 6371·099 *read* 6378·388 and 6378·099.