BULL. AUSTRAL. MATH. SOC. 20M10
VOL. 33 (1986), 261-272.

THE SEMIDIRECT PRODUCT OF AN INVERSE SEMIGROUP AND A GROUP

G.B. PRESTON

It is shown that a semidirect product of an inverse semigroup and a group, in that order, contains an inverse subsemigroup that is a retract and that, together with the retraction mapping, forms a free inverse morphic image of the semidirect product. The congruence determined by the retraction mapping is shown to be determined by the semigroup of idempotents of the semidirect product.

Introduction.

In [3] the author included a theorem of R. G. Wilkinson that states that the semidirect product $T\ _{\theta}\times S$ of two groups T and S is a group if and only if the antimorphism $\theta \mapsto End\ T$ is such that $S\theta \subseteq Aut\ T$, the automorphism group of T. When θ does not satisfy this condition it was shown (Theorem 4) that $T\ _{\theta}\times S$ is a left group. In analogy with Wilkinson's result for groups, and extending a result of W. R. Nico for inverse monoids [2], it was also shown in [3], Theorem 6, that when T

Received 26 June 1985. This work was carried out in December 1984 at the Computing Laboratory of the University of Kent at Canterbury. The author thanks the director of the Laboratory, Professor E.B. Spratt, for his hospitality and the stimulating work conditions provided.

and S are inverse semigroups then $T \,_\theta\!\times S$ is inverse if and only if $\theta: S \to End\ T$ again satisfies $S\theta \subseteq Aut\ T$. Left open in [1] is the problem of what kind of semigroup $T \,_\theta\!\times S$ is when this condition is not satisfied. There seems to be no simple overall description of such semidirect products. In this paper we look at one special case only, namely when S is a group. The results we obtain are not too dissimilar from those for the case when just T is a group, but they are sufficiently different to require an independent treatment.

1. Preliminaries

If T and S are semigroups and $\theta: S \to End\ T$ that is θ is an antimorphism of S into the endomorphism semigroup of T, then we denote by $T \,_\theta\!\times S$ the set $T \times S$ equipped with the product

$$(t,s)(t_1,s_1) = (tt_1^s, ss_1) ,$$

where t_1^s denotes, for t_1 in T and s in S, the image $t_1(s\theta)$ of t_1 under the endomorphism $s\theta$ of T. This product is called the semidirect product of T and S (with structure map θ).

In Theorem 6 of [3] it was shown that $T \,_\theta\!\times S$ is an inverse semigroup if and only if (i) S and T are inverse semigroups and (ii) $S\theta \subseteq Aut\ T$, the automorphism group of T. Necessary and sufficient conditions for $T \,_\theta\!\times S$ to be regular, again extending a result of Nico [2], were found in [3], Theorem 5. Applied to the situation in which S and T are inverse Theorem 5 of [3] simplifies to

PROPOSITION 1. *Let S and T be inverse semigroups and $\theta: S \to End\ T$. Then $T \,_\theta\!\times S$ is regular if and only if for all idempotents e of S and for all t in T we have $t \in Tt^e$.*

This condition for regularity of $T \,_\theta\!\times S$ is a strong one. When S is a group then S has a unique idempotent, 1 say, and so we have

PROPOSITION 2. *Let S be a group and T an inverse semigroup and $\theta: S \to End\ T$. Then $T \,_\theta\!\times S$ is regular if and only if for all t*

in T *we have* $t \in Tt^1$.

Alternatively, when T is a group, Proposition 1 immediately gives

PROPOSITION 3. *Let* S *be inverse and* T *a group and* $\theta : S \to End\ T$. *Then* $T\ {}_\theta\!\times S$ *is regular.*

Propositions 2 and 3 show there is a basic difference between $T\ {}_\theta\!\times S$ when T is a group and S inverse and when S is a group and T inverse. In this paper we restrict ourselves to the latter situation.

2. An inverse retract.

From now on the second factor of the semidirect products we consider will be a group and, to emphasize this, we denote it by G. T is an inverse semigroup and θ is an antimorphism $G \to End\ T$. The identity of G will be denoted by 1 and sometimes we shall denote 1θ by β. The inverse semigroup $T\beta$ will be denoted by H. We denote by $f|_X$ the restriction of a mapping f to X.

LEMMA 1. *For* $g \in G$ *set* $g\phi = g\theta|_H$. *Then* $\phi : G \to Aut\ H$.

Proof. We have, for all g in G,

$$H = T\beta = T^{gg^{-1}} = (T^{g^{-1}})^g \subseteq T^g =$$

$$= T^{1g} = (T^g)^1 \subseteq T\beta = H .$$

Hence $T^g = H$, for all g in G. Moreover $\beta^2 = \beta$, so $h\beta = h$, that is

$$(h^g)^{g^{-1}} = (h^{g^{-1}})^g = h ,$$

for all h in H. Hence $g\phi$ is a bijection of H upon H with inverse $(g^{-1})\phi$.

As a corollary, using Theorem 6 of [3], we have

LEMMA 2. $H\ {}_\phi\!\times G$ *is an inverse semigroup. Since product in* $H\ {}_\phi\!\times G$ *coincides with that in* $T\ {}_\theta\!\times G$, $H\ {}_\phi\!\times G$ *is an inverse subsemigroup of* $T\ {}_\theta\!\times G$.

https://doi.org/10.1017/S0004972700003130 Published online by Cambridge University Press

For $(t,g) \in T\ {}_\theta\!\times G$ define α by

$$(t,g)\alpha = (t\beta,g) \; . \tag{1}$$

Then we easily check the following proposition.

PROPOSITION 3. *The map* α *is a surmorphism of* $T\ {}_\theta\!\times G$ *upon* $H\ {}_\theta\!\times G$*, a subsemigroup of* $T\ {}_\theta\!\times G$.

Indeed α, regarded as an endomorphism of $T\ {}_\theta\!\times G$, is idempotent, so $H\ {}_\theta\!\times G$ is a retract of $T\ {}_\theta\!\times G$. Effectively, the retraction α of $T\ {}_\theta\!\times G$ is an extension of the retraction $\beta : T \to H$.

3. The subsemigroup of idempotents

We determine the idempotents of $T\ {}_\theta\!\times G$ and show that they form a subsemigroup which is a strong semilattice of left zero semigroups.

LEMMA 3. *The idempotents of* $T\ {}_\theta\!\times G$ *are the elements* $(t,1)$ *such that* $t = t(t\beta)$.

Proof. From $(t,g)^2 = (tt^g,g^2) = (t,g)$ we have immediately that $g = g^2 = 1$, whence $t = tt^1 = t(t\beta)$. Conversely, when $t = t(t\beta)$, then $(t,1)$ is idempotent.

Observe also

LEMMA 4. *If* $t \in T$ *and* $t = t(t\beta)$*, then* $t\beta$ *is an idempotent.*

Proof. This follows immediately from $\beta^2 = \beta$.

Denote the set of idempotents of H by $E(H)$ and by $\{K_f \mid f \in E(H)\}$ the kernel normal system of β (see [1, §7.4]). Thus $K_f = \{t \in T \mid t\beta = f\}$. Using this notation and Lemma 4, we can reformulate Lemma 3 as follows.

LEMMA 5. *The idempotents of* $T\ {}_\theta\!\times G$ *are the elements* $(t,1)$ *such that* $t \in K_f f$ *for some* f *in* $E(H)$.

Proof. Let $t = uf$, where $u \in K_f$ and $f \in E(H)$. Then $t\beta = (u\beta)(f\beta) = f(f\beta) = f$, since β fixes H. Hence $t(t\beta) = uf^2 = uf = t$. Thus, by Lemma 3, $(t,1)$ is idempotent.

Conversely, if $(t,1)$ is idempotent, setting $t\beta = f$, then from Lemma 4, $f \in E(H)$ and thus, by Lemma 3 again, $t \in K_f f$.

If $A \subseteq T$ and $g \in G$, write $A \times g = \{(a,g) | a \in A\}$.

LEMMA 6. *The set of idempotents $K_f f \times 1$, where $f \in E(H)$, forms a left zero subsemigroup of $T\ {}_\theta\!\times G$.*

Proof. Let $t_1, t_2 \in K_f$. Then

$$(t_1 f, 1)(t_2 f, 1) = (t_1 f (t_2 f)^1, 1)$$
$$= (t_1 f, 1),$$

since
$$(t_2 f)^1 = (t_2\beta)(f\beta) = f^2 = f.$$

Let $f, g \in E(H)$ and let $f \geq g$. Then we define the mapping $\phi_{f,g}$ by $(t,1)\phi_{f,g} = (tg,1)$ for $t \in K_f f$.

LEMMA 7. *For $f \geq g$, $\phi_{f,g}$ is a morphism of $K_f f \times 1$ into $K_g g \times 1$. Moreover*

(i) if $f = g$, then $\phi_{f,g}$ is the identity on $K_f f \times 1$;

(ii) if $f \geq g \geq h$, then $\phi_{f,g}\,\phi_{g,h} = \phi_{f,h}$.

Proof. Let $t \in K_f f$ and $f \geq g$. Then $(tg)\beta = (t\beta)(g\beta) = fg = g$, since $f \geq g$. Hence $tg = (tg)g \in K_g g$. Thus $\phi_{f,g} : K_f f \times 1 \to K_g g \times 1$. Let $t_1, t_2 \in K_f f$. Then, using Lemma 6,

$$((t_1,1)(t_2,1))\phi_{f,g} = (t_1,1)(\phi_{f,g}) = (t_1 g, 1)$$

and

https://doi.org/10.1017/S0004972700003130 Published online by Cambridge University Press

$$\begin{aligned}(t_1,1)\phi_{f,g}(t_2,1)\phi_{f,g} &= (t_1g,1)(t_2g,1)\\ &= (t_1g(t_2g)\beta,1)\\ &= (t_1gfg,1)\\ &= (t_1g,1)\ .\end{aligned}$$

Thus $\phi_{f,g}$ is morphic.

Suppose $f = g$. Then for t in $K_f f$, $(t,1)\phi_{f,f} = (tf,1) = (t,f)$, so that (i) holds.

To see (ii), consider $f \geq g \geq h$, and let $t \in K_f f$. Then

$$((t,1)\phi_{f,g})\phi_{g,h} = (tg,1)\phi_{g,h} = (tgh,1)$$

and

$$(t,1)\phi_{f,h} = (th,1) = (tgh,1)\ ,$$

since $g \geq h$ implies $gh = h$.

LEMMA 8. *Let* $f,g \in E(H)$, $t \in K_f f$ *and* $u \in K_g g$. *Then*

$$(t,1)(u,1) = (t,1)\phi_{f,fg}(u,1)\phi_{g,fg}\ .$$

Proof. We easily calculate that

$$(t,1)(u,1) = (t(u\beta),1) = (tg,1) = (tfg,1),$$

since $t = tf$; and

$$\begin{aligned}(t,1)\phi_{f,fg}(u,1)\phi_{g,fg} &= (tfg,1)(ufg,1)\\ &= (tfg(ufg)\beta,1)\\ &= (tfggfg,1)\\ &= (tfg,1)\ .\end{aligned}$$

Denote by F the set of all idempotents of $T\ {}_\theta\!\times G$, so that

$$F = \cup\{K_f f \times 1 \mid f \in E(H)\}\ .$$

Set
$$F_f = K_f f \times 1,\ f \in E(H)\ .$$

Then Lemmas 3 to 8 combine to give the following theorem.

THEOREM 1. *The idempotents of* $T\ {}_\theta\!\times G$ *form a subsemigroup* F *which is a strong semilattice* $\{F_f | f \in E(H)\}$ *of left zero subsemigroups* $F_f = K_f f \times 1$, $f \in E(H)$, *of* $T\ {}_\theta\!\times G$, *with structure morphisms* $\phi_{f,g}$ $f \geq g$, $f,g \in E(H)$.

4. F determines $\alpha \circ \alpha^{-1}$.

We first show that the F_f are determined by F . The next proposition includes this result.

PROPOSITION 4. *The mapping* $\gamma : (t,1) \to t\beta$, $(t,1) \in F$ *is a surmorphism of* F *upon the semilattice* $E(H)$. *Moreover, if* $\delta : F \to L$ *is a morphism of* F *into a semilattice* L *then there is a (unique) morphism* $\varepsilon : E(H) \to L$, *say, such that* $\delta = \gamma\varepsilon$.

Proof. Let $(t,1),(u,1) \in F_f$, so that $t\beta = u\beta = f$. Then $(t,1)(u,1) = (t,1)$ and $(u,1)(t,1) = (u,1)$, by Lemma 6. Since L is a semilattice,

$$(t,1)\delta(u,1)\delta = (u,1)\delta(t,1)\delta \ ,$$

that is

$$(t,1)\delta = (u,1)\delta \ .$$

Hence, if we define ε by $(t\beta)\varepsilon = (t,1)\delta$ for $(t,1) \in F$, then ε is well-defined, $\gamma\varepsilon = \delta$, and ε is the sole mapping satisfying this equation. Also ε is clearly a morphism.

COROLLARY. F *determines its subsemigroups* $F_f, f \in E(H)$.

Proof. By the proposition $\{F_f | f \in E(H)\}$ is the set of congruence classes induced by γ and $F\gamma$ is the free semilattice on F .

The next theorem shows how F determines the congruence $\alpha \circ \alpha^{-1}$.

THEOREM 2. *Let* (t,g) *and* $(u,h) \in T\ {}_\theta\!\times G$. *Then* $((t,g),(u,h)) \in \alpha \circ \alpha^{-1}$ *if and only if there exists* $f \in E(H)$ *such that* $(t,g)(t\beta,g)^{-1}$, $(u,h)(u\beta,h)^{-1}$ *and* $(t,g)(u,\beta h)^{-1}$ *belong to* F_f.

Proof. Suppose $((t,g),(u,h)) \in \alpha \circ \alpha^{-1}$. Then $(t\beta,g) = (u\beta,h)$,

https://doi.org/10.1017/S0004972700003130 Published online by Cambridge University Press

so that $g = h$ and $t\beta = u\beta$, $= v$, say.

Now $(t\beta, g)$ belongs to the inverse semigroup $H {}_\phi\!\times G$ and has inverse $((t\beta)^{-1})^{g^{-1}}, g^{-1}) = ((t^{-1}\beta)^{g^{-1}}, g^{-1})$. Similarly, $((u^{-1}\beta)^{g^{-1}}, g^{-1})$ is the inverse of $(u\beta, h) = (u\beta, g)$. Hence

$$(t,g)(t\beta,g)^{-1} = (t(t^{-1}\beta)^{gg^{-1}}, gg^{-1})$$
$$= (t(t^{-1}\beta), 1) ,$$

since $\beta^2 = \beta$; and similarly,

$$(u,h)(u\beta,h)^{-1} = (u(u^{-1}\beta),1) ,$$
$$(t,g)(u\beta,h)^{-1} = (t(u^{-1}\beta),1) .$$

Consider now $(t(t^{-1}\beta),1)$. This is idempotent if $t(t^{-1}\beta)(t(t^{-1}\beta))\beta = t(t^{-1}\beta)$, by Lemma 3. But

$$t(t^{-1}\beta)(t(t^{-1}\beta))\beta = t(t^{-1}\beta)(t\beta)(t^{-1}\beta) = t(t^{-1}tt^{-1})\beta$$
$$= t(t^{-1}\beta) .$$

Similarly, $(u(u^{-1}\beta),1)$ and (hence, since $u\beta = t\beta$), $(t(u^{-1}\beta),1)$ are idempotent.

Since, immediately, we have (with γ as in Proposition 4) $(t(t^{-1}\beta),1)\gamma = (u(u^{-1}\beta),1)\gamma = (t(u^{-1}\beta),1)\gamma = f$, say, where $f \in E(H)$, therefore $(t(t^{-1}\beta),1)$, $(u(u^{-1}\beta),1)$ and $(t(u^{-1}\beta),1)$ all belong to F_f .

It remains to deal with the 'if' part of the theorem. Suppose then that $f \in E(H)$ and that $(t,g)(t\beta,g)^{-1}$, $(u,h)(u\beta,h)^{-1}$ $(t,g)(u\beta,h)^{-1}$ all belong to F_f . Then, with calculations as before, it follows that $g = h$ and

$$(t(t^{-1}\beta),1)\gamma = (u(u^{-1}\beta),1)\gamma = (t(u^{-1}\beta),1)\gamma = f ,$$

that is $$(tt^{-1})\beta = (uu^{-1})\beta - (tu^{-1})\beta = f .$$

Since β is a morphism from the inverse semigroup T to the inverse semigroup H , it follows that $t\beta = u\beta$ ([1.§7.4]). Thus $(t,g)\alpha = (u,h)\alpha$; and this completes the proof of the theorem.

Note that, since $T\ {}_\theta\!\times G$ is not always an inverse semigroup, an element (t,g) does not necessarily have a unique inverse, nor (see Proposition 2) any inverse. So we cannot replace the $(t\beta,g)^{-1}$ and $(u\beta,h)^{-1}$ in the above theorem by $(t,g)^{-1}$ and $(u,h)^{-1}$. The introduction of β has allowed us to give a description of the congruence $\alpha \circ \alpha^{-1}$ that mimics closely the inverse semigroup situation.

5. The kernel normal system of $\alpha \circ \alpha^{-1}$

The kernel normal system of $\beta \circ \beta^{-1}$ is $\{K_f | f \in E(H)\}$. Set $A_f = K_f \times 1$; then $A_f = (f,1)\alpha^{-1}$ and so

$$\mathcal{A} = \{A_f | f \in E(H)\}$$

is the set of inverses of idempotents of $(T\ {}_\theta\!\times G)\alpha$. Furthermore, as we have just seen, $\mathcal{A}$ determines $\alpha \circ \alpha^{-1}$. So, by analogy with inverse semigroup terminology, let us call $\mathcal{A}$ the kernel normal system of $\alpha \circ \alpha^{-1}$.

$A = \cup\mathcal{A}$ is a semilattice of its subsemigroups A_f, whose multiplication induces that of its subsemigroup, the strong semilattice F. The following lemmas give more information on how F sits within A.

First let us define, for $f \in E(H)$, a mapping $\chi_f : A \to A$, $(t,1) \to (tf,1)$, $(t,1) \in A$. Straightforward calculations immediately give the next lemma.

LEMMA 9. *(a) For $f,g \in E(H)$,*

(i) $A_f\chi_f = F_f$;

(ii) $A_g\chi_f = A_{fg}$.

(b) For $f \in E(H)$, χ_f is a morphism of A into A.

LEMMA 10. *F_f is a two sided ideal of A_f. Furthermore, for all x in A_f,*

(i) $xA_f = x\chi_f$;

(ii) χ_f *is the identity mapping on* F_f *;*

(iii) for y *in* A_f, $yx = y\chi_f$ *, whence* $A_f x = F_f$.

Proof. For $(t,1),(u,1)$ in A_f, $(t,1)(u,1) = (tf,1) = (t,1)\chi_f$, from which (i) and (iii) follow. (ii) then follows since if $(t,1) \in A_f$ then $(t,1) \in F_f$ only if $tf = t$, by Lemma 5.

The next lemma determines product in A .

LEMMA 11. *Let* $x,y \in A$ *so that* $x \in A_f$, $y \in A_g$, *for some* f,g *in* $E(H)$. *Then*

$$xy = x\chi_g \in A_{fg} .$$

Hence F *is a right ideal of* A.

Proof. If $x \in A_f$, $y \in A_g$ then there exist $t \in K_f$, $u \in K_g$ such that $x = (t,1)$, $y = (u,1)$. Then

$$\begin{aligned} xy &= (t,1)(u,1) = (t(u\beta),1) = (tg,1) \\ &= x\chi_g . \end{aligned}$$

Moreover, $(tg)\beta = (t\beta)(g\beta) = fg$, that is $tg \in K_{fg}$. Thus $xy \in A_{fg}$.

If also $x \in F$, so that $x \in F_f$ and $tf = t$, then $(tg)fg = (tf)g = tg$ and so, by Lemma 5, $xy = (tg,1) \in F_{fg}$. Thus F is a right ideal of A.

6. $H \ {}_\phi\!\times G$ is a free inverse morphic image of $T \ {}_\theta\!\times G$.

We show that $H \ {}_\phi\!\times G$ is a maximal inverse image of $T \ {}_\theta\!\times G$, or in other words, that $H \ {}_\phi\!\times G$ is a free inverse morphic image of $T \ {}_\theta\!\times G$ (under the morphism α). We have to show that if U is any inverse semigroup and $\tau: T \ {}_\theta\!\times G \twoheadrightarrow U$ is any surmorphism, then there exists a unique morphism κ , say, such that $\alpha\kappa = \tau$.

The next lemma states a useful manipulative result we shall need; indeed it has been used often already.

LEMMA 12. *Let* $(t,g)(t_1,g_1),\ldots,(t_n,g_n)$ *be elements of* $T\ {}_\theta\!\times G$. *Then*

$$(t,g)(t_1,g_1)\ldots(t_n,g_n) = (t,g)(t_1\beta,g_1)\ldots(t_n\beta,g_n)\ .$$

Proof. We have

$$(t,g)(t_1,g_1) = (tt_1{}^g,gg_1) = (t(t_1{}^1)^g,gg_1)$$
$$= (t,g)(t_1{}^1,g_1) = (t,g)(t_1\beta,g_1).$$

The result follows by induction.

LEMMA 13. *Let* U *be an inverse semigroup and let* $\tau\colon T\ {}_\theta\!\times G \twoheadrightarrow U$ *be a surmorphism. Then for all* $(t,g) \in T\ {}_\theta\!\times G$, $(t,g)\tau = (t\beta,g)^\tau$.

Proof. Let $(t,g)\tau = u$ and $(t\beta,g)\tau = v$. Since τ is surjective there exists (t_1,g_1) such that $(t_1,g_1)\tau = u^{-1}$. Then $u = uu^{-1}u = uu^{-1}v$, by Lemma 12. Hence $u \leq v$.

Similarly $v = vv^{-1}v = vv^{-1}u$, by Lemma 12. Hence $v \leq u$. Thus $u = v$, and the proof of the lemma is complete.

COROLLARY. *Let* U *be an inverse semigroup and let* $\tau\colon T\ {}_\theta\!\times G \twoheadrightarrow U$ *be a surmorphism. Let* κ *denote the restriction of* τ *to* $H\ {}_\phi\!\times G$. *Then* $\alpha\kappa = \tau$, *and* κ *is the unique morphism satisfying this equation.*

Hence we have, in summary, the following theorem.

THEOREM 3. *Let* T *be inverse,* G *a group and* $\theta\colon G \nrightarrow \mathrm{End}\ T$. *Let* $1\theta = \beta$ *and* $T\beta = H$. *Set* $\phi = \theta_{|H}$. *Then* $\phi\colon G \nrightarrow \mathrm{Aut}\ H$ *and* $H\ {}_\phi\!\times G$ *is inverse, an inverse subsemigroup of* $T_\theta \times G$.

Moreover $\alpha\colon (t,g)\rightarrow (t\beta,g)$ *belongs to* $\mathrm{End}(T\ {}_\theta\!\times G)$ *and has image* $H\ {}_\phi\!\times G$ *and, since* $\alpha^2 = \alpha$, $H\ {}_\phi\!\times G$ *is a retract of* $T\ {}_\theta\!\times G$.

Furthermore, $H\ {}_\phi\!\times G$ *is a free inverse morphic image of* $T\ {}_\theta\!\times G$, *under the morphism* α.

https://doi.org/10.1017/S0004972700003130 Published online by Cambridge University Press

References

[1] A. H. Clifford and G. B. Preston, "The algebraic theory of semigroups, Vol. 2", *Math.Surveys, No. 7, Amer. Math. Soc.* 1967.

[2] W. R. Nico, "On the regularity of semidirect products", *J. of Algebra* 80 (1983), 29-36.

[3] G. B. Preston, "Semidirect products of semigroups", *Proc. Royal Society of Edinburgh* (to appear).

Department of Mathematics
Monash University
Clayton
Victoria 3168.