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THE SEMIGROUP OF VARIETIES OF BROUWERIAN SEMILATTICES

BY

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ABSTRACT. It is shown that the semigroup of varieties of Brouwerian semilattices is free.

Semigroups of varieties have first been studied in the case of groups, where Neumann, Neumann and Neumann–and independently Šmel′kin–have discovered a surprising result: The semigroup of varieties of groups is a free monoid with zero ([13], [14], [18]). Since then various authors have investigated semigroups of varieties of group-like structures such as quasigroups, rings, lattice-ordered groups and Lie-algebras ([2], [5], [8], [15], [19]).

It was Mal′cev who considered the general case, and he succeeded in giving a sufficient condition under which the subvarieties of a given variety form a semigroup [7]. Exploiting this idea Köhler studied the semigroup of varieties of Brouwerian algebras and Blok and Köhler did this for the semigroup of varieties of generalized interior algebras ([6], [1]). Nearly all the cited papers centered around the question whether a result similar to the one for groups could be obtained. This paper continues these efforts in giving a positive answer to the question above for the variety of Brouwerian semilattices.

The paper is divided into two parts. The first one introduces the notion of an extension of a Brouwerian semilattice. Based on ideas originally introduced by Nemitz [10] and most elegantly generalized by Schmidt ([16], [17]) it is proven that every extension of a Brouwerian semilattice by another can be imbedded into some special kind of extension which we call strongly splitting extension. The second part introduces the multiplication of varieties of Brouwerian semilattices, thus giving the "set" of varieties a semigroup structure. Based on results from §1 and using techniques originally developed in [6] it is finally proven that the semigroup of varieties is a free monoid with zero.

0. Preliminaries. A Brouwerian semilattice is an algebra $\langle S, \cdot, *, 1\rangle$ where $\langle S, \cdot, 1\rangle$ is a meet-semilattice with the greatest element 1, and where the

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binary operation $*$ denotes relative pseudocomplementation, i.e. $z \leqslant x * y$ holds for elements x, y, z of S if and only if $zx \leqslant y$. It is well known that the class **BS** of Brouwerian semilattices is a variety, equations determining **BS** have first been given by Monteiro **[9]**. The following rules of computation–which will be frequently used throughout the paper–may be found in **[10]**:

$$
\begin{aligned}
&x \leqslant y \Leftrightarrow x * y = 1,\\
&1 * x = x,\\
&x * y \geqslant y,\\
&x(x * y) = xy,\\
&xy * z = x * (y * z),\\
&x * yz = (x * y)(x * z),\\
&x * (y * z) = (x * y) * (x * z),\\
&x \leqslant y \Rightarrow y * z \leqslant x * z \quad \text{and} \quad z * x \leqslant z * y.
\end{aligned}
$$

It is known also from **[10]** that congruences on Brouwerian semilattices are in 1-1-correspondence with filters. To be more precise: If F is a filter of the Brouwerian semilattice S, then the relation

$$\theta_F = \{\langle x, y\rangle | x, y \in S, (x * y)(y * x) \in F\}$$

is a congruence of S with $[1]_{\theta_F} = F$. Moreover, the mapping $F \mapsto \theta_F$ is an isomorphism between the lattice $\mathfrak{F}(S)$ of all filters of S and the congruence lattice of S. This explains our notation L/F for the quotient algebra and $[x]_F$ for the congruence class of x modulo θ_F. Consequently S is subdirectly irreducible if and only if $S \setminus \{1\}$ has a greatest element.

If f: $S_1 \to S_2$ is a homomorphism between the Brouwerian semilattices S_1 and S_2 then the kernel of f

$$\ker f = \{x \in S_1 | fx = 1\}$$

is a filter of S_1. If f is onto S_2 then the Homomorphism Theorem can be stated as $S_1/\ker f \cong S_2$. For any element a of a Brouwerian semilattice S the mapping f_a: $S \to S$ defined by $f_a x = a * x$ is an endomorphism of S and $\ker f = [a) = \{x \in S | a \leqslant x\}$ is the principal filter generated by a. Also the constant mapping **1**–which is equal to f_0 if S has a smallest element 0–is an endomorphism of S. As usual End S will denote the endomorphism monoid of S.

The variety **BS** is known to be locally finite, i.e. finitely generated Brouwerian semilattices are finite **[12]**. Clearly this also holds for every subvariety of **BS**. If S is a finite subdirectly irreducible Brouwerian semilattice then $V(S)$–the variety generated by S–splits in the lattice of subvarieties of **BS**, i.e. the class **BS**: S of all Brouwerian semilattices which do not contain an isomorphic copy of S as a subalgebra is a variety [12].

For any natural number n the set $\mathbf{n} = \{0, \ldots, n-1\}$ endowed with the natural order is a subdirectly irreducible Brouwerian semilattice. The two papers [11] and [12] emphasize the importance of the splitting varieties **BS**: $\mathbf{n}$. Here we just note that $\mathbf{C}_2 = V(\mathbf{2}) = \mathbf{BS}\colon \mathbf{3}$ is the smallest nontrivial subvariety of **BS**.

For other notions from Universal Algebra we refer to [3].

1. Extensions of Brouwerian semilattices. Let S, S_1, S_2 be Brouwerian semilattices. Then S is called an *extension of* S_1 *by* S_2 if there exists a short exact sequence

$$\mathbf{1} \to S_1 \xrightarrow{f} S \xrightarrow{g} S_2 \to \mathbf{1},$$

or equivalently, if S has a filter F such that $F \cong S_1$ and $S/F \cong S_2$. S is a *splitting extension* if g is a retraction, i.e. there exists a homomorphism h: $S_2 \to S$ such that $gh = \mathrm{id}_{S_2}$. If, moreover, g is a closure retraction, i.e. $hgx \geqslant x$ for all $x \in S$, then S will be called a *strongly splitting extension of* S_1 *by* S_2. Strongly splitting extensions of Brouwerian semilattices have been deeply investigated by Schmidt ([16], [17]) who called them quasi-decompositions. We recall the following construction and characterization theorems, which are essentially due to him [17, Theorem 10.1]. In fact, Schmidt traces them back to Nemitz [10], who treated the special case of a strongly splitting extension of a Brouwerian semilattice by a Boolean algebra.

We begin with some definitions. Let S_1, S_2 be Brouwerian semilattices. A mapping φ: $S_1 \to \mathrm{End}\, S_2$ will be called *admissible*, if φ is a homomorphism of semigroups with identity, i.e. $\varphi 1 = \mathrm{id}_{S_2}$ and $\varphi(ab) = (\varphi a)(\varphi b)$ hold, and if, moreover, $\varphi(a)x \geqslant x$ for every $a \in S_1$ and every $x \in S_2$. Every admissible mapping φ: $S_1 \to \mathrm{End}\, S_2$ induces an equivalence relation R_φ on $S_1 \times S_2$–we will write simply R if no ambiguity can occur–where R is defined by

$$\langle a, x\rangle R \langle b, y\rangle \Leftrightarrow a = b \quad \text{and} \quad \varphi(a)x = \varphi(b)y.$$

R is even a semilattice congruence, so that the quotient set $S_1 *_\varphi S_2 = S_1 \times S_2/R$ becomes a semilattice. In addition, we have:

THEOREM 1.1. *Let* S_1, S_2 *be Brouwerian semilattices and let* φ: $S_1 \to \mathrm{End}\, S_2$ *be admissible. Then the definition*

$$R\langle a, x\rangle * R\langle b, y\rangle = R\langle a * b, \varphi(a)(x * y)\rangle$$

makes $S_1 *_\varphi S_2$ *a Brouwerian semilattice.* $S_1 *_\varphi S_2$ *is a strongly splitting extension of* S_2 *by* S_1.

Two standard examples should be mentioned. Clearly the direct product of S_1 and S_2 is a strongly splitting extension which is obtained by letting $\varphi a = \mathrm{id}_{S_2}$ for all $a \in S_1$. On the other hand, if we define φ by

$$\varphi a = \begin{cases} \mathrm{id}_{S_2} & \text{if } a = 1, \\ \mathbf{1} & \text{if } a \neq 1, \end{cases}$$

then φ is admissible. The resulting algebra $S_1 \underset{\varphi}{*} S_2$ will be denoted by $S_1 \dagger S_2$–as poset $S_1 \dagger S_2$ is isomorphic to the disjoint union of $S_1 \setminus \{1\}$ and S_2 with all elements of S_2 greater than those of $S_1 \setminus \{1\}$.

THEOREM 1.2. *Let S be a strongly splitting extension of S_2 by S_1. Then there exists an admissible mapping φ: $S_1 \to \mathrm{End}\, S_2$ such that $S \cong S_1 \underset{\varphi}{*} S_2$.*

There is still the problem to give a construction for arbitrary extensions–perhaps similar to the known factor set multiplication in the case of groups. We can prove, however, that any extension can be imbedded into a suitable strongly splitting extension.

To do this let us introduce the following notion: If S is a Brouwerian semilattice let $S^- = \Pi\langle [a)\rangle_{a \in S}/\theta$ where θ is the congruence defined by

$$\langle x_a \rangle_{a \in S}\, \theta\, \langle y_a \rangle_{a \in S} \Leftrightarrow \exists a \in S\; \forall b \in S\, (b \leqslant a \Rightarrow x_b = y_b).$$

It is easy to see that S is isomorphic to a subalgebra of S^-, and if S has a smallest element then $S^- \cong S$. Also for reference in §2 note that $S^- \in V(S)$.

THEOREM 1.3. *Let S, S_1, S_2 be Brouwerian semilattices such that S is an extension of S_2 by S_1. Then S can be imbedded into a strongly splitting extension of S_2^- by S_1.*

PROOF. Without loss of generality we may assume that S_2 is a filter of S and S_1 is the quotient S/S_2. First observe that for any $x \in S$ the mapping $\alpha(x)$: $S_2^- \to S_2^-$ defined by

$$\alpha(x)\big[\langle z_a \rangle_{a \in S_2}\big]_\theta = \big[\langle (a * x) * z_a \rangle_{a \in S_2}\big]_\theta$$

is an endomorphism of S_2^-. In fact, we only have to show that $\alpha(x)$ is well defined, but this is immediate from the definition of θ. Moreover, let $x, y \in S$ such that $x \in [y]_{S_2}$. Then $(x * y)(y * x) \in S_2$ and an easy computation shows that for any $b \in S_2$ with $b \leqslant (x * y)(y * x)$ we have $b * x = b * y$. This proves that $x \in [y]_{S_2}$ implies $\alpha(x) = \alpha(y)$. Hence we can define φ: $S_1 \to \mathrm{End}\, S_2^-$ by $\varphi[x]_{S_2} = \alpha(x)$. Also it is easy to see that $\varphi[1]_{S_2} = \mathrm{id}_{S_2^-}$, as well as $\varphi[x]_{S_2}[\langle z_a \rangle_{a \in S_2}]_\theta \geqslant [\langle z_a \rangle_{a \in S_2}]_\theta$. Moreover, for $x, y \in S$ we have

$$\begin{aligned}
\varphi[x]_{S_2}\varphi[y]_{S_2}\big[\langle z_a \rangle_{a \in S_2}\big]_\theta &= \varphi[x]_{S_2}\big[\langle (a * y) * z_a \rangle_{a \in S_2}\big]_\theta \\
&= \big[\langle (a * x) * ((a * y) * z_a) \rangle_{a \in S_2}\big]_\theta = \big[\langle (a * x)(a * y) * z_a \rangle_{a \in S_2}\big]_\theta \\
&= \big[\langle (a * xy) * z_a \rangle_{a \in S_2}\big]_\theta = \varphi[xy]_{S_2}\big[\langle z_a \rangle_{a \in S_2}\big]_\theta.
\end{aligned}$$

Thus φ is an admissible mapping.

Now define a mapping $h: S \to S_1 \underset{\varphi}{*} S_2^-$ by

$$hx = R\left\langle [x]_{S_2}, \left[\langle (a * x) * x\rangle_{a \in S_2}\right]_\theta \right\rangle -$$

note that $(a * x) * x \geqslant a$. We show that h is a 1-1-homomorphism. So let $x, y \in S$. Then

$$h(xy) = R\left\langle [xy]_{S_2}, \left[\langle (a * xy) * xy\rangle_{a \in S_2}\right]_\theta \right\rangle$$

and

$$hx \cdot hy = R\left\langle [x]_{S_2}[y]_{S_2}, \left[\langle ((a * x) * x)((a * y) * y)\rangle_{a \in S_2}\right]_\theta \right\rangle.$$

In order to prove $h(xy) = hx \cdot hy$ it thus suffices to show that for any $a \in S_2$

$$(a * xy) * ((a * xy) * xy) = (a * xy) * ((a * x) * x)((a * y) * y).$$

This is easily done:

$$\begin{aligned}
&(a * xy) * ((a * x) * x)((a * y) * y)\\
&\quad = \big((a * x)(a * y) * ((a * x) * x)\big) \cdot \big((a * x)(a * y) * ((a * y) * y)\big)\\
&\quad = \big((a * y) * ((a * x) * x)\big) \cdot \big((a * x) * ((a * y) * y)\big)\\
&\quad = (a * x)(a * y) * xy = (a * xy) * xy\\
&\quad = (a * xy) * ((a * xy) * xy).
\end{aligned}$$

Similarly

$$h(x * y) = R\left\langle [x * y]_{S_2}, \left[\langle (a * (x * y)) * (x * y)\rangle_{a \in S_2}\right]_\theta \right\rangle$$

and

$$hx * hy = R\left\langle [x]_{S_2} * [y]_{S_2}, \varphi[x]_{S_2}\left[\langle ((a * x) * x) * ((a * y) * y)\rangle_{a \in S_2}\right]_\theta \right\rangle$$

and again it will be sufficient to show that for any $a \in S_2$

$$\begin{aligned}
&(a * (x * y)) * ((a * (x * y)) * (x * y))\\
&\qquad = (a * (x * y)) * \big((a * x) * (((a * x) * x) * ((a * y) * y))\big).
\end{aligned}$$

For simplicity let us abbreviate the left-hand side with b and the right-hand side with c. Then

$$\begin{aligned}
c &= (a * xy) * \big(((a * x) * x) * ((a * y) * y)\big)\\
&= (a * y) * \big(x * ((a * y) * y)\big)\\
&= x * ((a * y) * y) = x(a * y) * y,
\end{aligned}$$

and

$$b = (a * (x * y)) * (x * y)$$
$$= (x * (a * y))x * y = x(a * y) * y.$$

Thus $b = c$, and so h is a homomorphism.

Finally let

$$hx = 1 = R\langle [1]_{S_2}, [\langle 1\rangle_{a\in S_2}]_\theta \rangle.$$

Then $x \in S_2$ and $\langle (a * x) * x\rangle_{a\in S_2}\theta\langle 1\rangle_{a\in S_2}$, i.e. there exists some $a \in S_2$ such that $(b * x) * x = 1$ whenever $b \in S_2$, $b \leqslant a$. In particular $ax \in S_2$–since $x \in S_2$–and $ax \leqslant a$, hence $1 = (ax * x) * x = 1 * x = x$. Thus h is a 1-1-homomorphism, and the theorem is proven.

Next, we will discuss the functorial aspect of the construction of strongly splitting extensions; we will construct homomorphisms between strongly splitting extensions from appropriate pairs of homomorphisms between the constituents.

THEOREM 1.4. *Let S_1, S_2, S_3, S_4 be Brouwerian semilattices, and let φ: $S_1 \to \mathrm{End}\, S_2$, ξ: $S_3 \to \mathrm{End}\, S_4$ be admissible. Suppose that f: $S_1 \to S_3$, g: $S_2 \to S_4$ are homomorphisms such that for every $a \in S_1$ the diagram*

$$(*)\qquad \begin{array}{ccc} S_2 & \xrightarrow{\varphi(a)} & S_2 \\ g\downarrow & & \downarrow g \\ S_4 & \xrightarrow[\xi(fa)]{} & S_4 \end{array}$$

is commutative. Then the mapping h: $S_1 *_\varphi S_2 \to S_3 *_\xi S_4$ defined by $hR\langle a, x\rangle = R\langle fa, gx\rangle$ is a homomorphism.

PROOF. First we have to show that h is well defined. So let $\langle a, x\rangle R\langle b, y\rangle$, i.e. $a = b$ and $\varphi(a)x = \varphi(b)y$. But then $(*)$ implies $fa = fb$ and $\xi(fa)gx = g\varphi(a)x = g\varphi(b)y = \xi(fb)gy$, hence $\langle fa, gx\rangle R\langle fb, gy\rangle$. Obviously h is a semilattice homomorphism. Moreover, for any $a, b \in S_1$ and $x, y \in S_2$ we have

$$h(R\langle a, x\rangle * R\langle b, y\rangle) = hR\langle a * b, \varphi(a)(x * y)\rangle$$
$$= R\langle fa * fb, \xi(fa)(gx * gy)\rangle = R\langle fa, gx\rangle * R\langle fb, gy\rangle$$
$$= R\langle f(a * b), g\varphi(a)(x * y)\rangle = R\langle fa * fb, \xi(fa)g(x * y)\rangle$$
$$= hR\langle a, x\rangle * hR\langle b, y\rangle.$$

This completes the proof.

This theorem has an immediate corollary:

COROLLARY 1.5. *Let S_1, S_2 be Brouwerian semilattices, let φ: $S_1 \to \mathrm{End}\, S_2$ be admissible. If S is a subalgebra of S_1, then $S *_\varphi S_2$ is a subalgebra of $S_1 *_\varphi S_2$.*

The following lemma is a partial converse of Theorem 1.4. Roughly spoken, it says that every homomorphic image of a strongly splitting extension is again a strongly splitting extension. For better reference in §2 we will state it in terms of filters rather than in terms of homomorphisms.

Lemma 1.6. *Let S_1, S_2 be Brouwerian semilattices, let $\varphi\colon S_1 \to \operatorname{End} S_2$ be admissible. Suppose that F is a filter of $S_1 *_\varphi S_2$. Then $F = F_1 *_\varphi F_2$ for some filters F_1 of S_1, F_2 of S_2. Moreover, there is an admissible mapping ξ: $S_1/F_1 \to \operatorname{End} S_2/F_2$ such that*

$$S_1 *_\varphi S_2/F_1 *_\varphi F_2 \cong S_1/F_1 *_\xi S_2/F_2.$$

Proof. We define

$$F_1 = \{a | a \in S_1, \exists x \in S_2\, R\langle a, x\rangle \in F\},$$
$$F_2 = \{x | x \in S_2, \exists a \in S_1\, R\langle a, x\rangle \in F\}.$$

It is easily checked that both of F_1 and F_2 are filters, and that $F = F_1 *_\varphi F_2$. Now let $a \in [b]_{F_1}$, then $R\langle (a * b)(b * a), 1\rangle \in F$. Moreover, for any $x \in S_2$ we have

$$\begin{aligned}\varphi((a * b)(b * a))&((\varphi(a)x * \varphi(b)x)(\varphi(b)x * \varphi(a)x))\\ &= (\varphi(ab)x * \varphi(ab)x)(\varphi(ab)x * \varphi(ab)x) = 1.\end{aligned}$$

This shows that

$$\begin{aligned}R\langle&(a * b)(b * a), 1\rangle\\ &= R\langle (a * b)(b * a), (\varphi(a)x * \varphi(b)x)(\varphi(b)x * \varphi(a)x)\rangle \in F.\end{aligned}$$

As a consequence $(\varphi(a)x * \varphi(b)x)(\varphi(b)x * \varphi(a)x) \in F_2$ or $\varphi(a)x \in [\varphi(b)x]_{F_2}$. This observation makes it clear that we can define an admissible mapping ξ: $S_1/F_1 \to \operatorname{End} S_2/F_2$. So let $a \in S_1$, then we have the following diagram:

$$\begin{array}{ccc} S_2 & \overset{\varphi(a)}{\rightarrow} & S_2 \\ \nu\downarrow & & \downarrow\nu \\ S_2/F_2 & \underset{\gamma_a}{\dashrightarrow} & S_2/F_2 \end{array}$$

Since $\varphi(a)x \geqslant x$ for all $x \in S_2$ it follows that $F_2 = \ker \nu \subseteq \ker(\nu\varphi(a))$. Thus there exists a unique endomorphism γ_a completing the diagram. The argument above shows that γ_a is independent from the choice of $a \in [a]_{F_1}$. Thus the mapping ξ: $S_1/F_1 \to \operatorname{End} S_2/F_2$ given by $\xi[a]_{F_1} = \gamma_a$ is well defined and clearly admissible.

Now the diagram shows that the assumptions of Theorem 1.4 are fulfilled. Thus the mapping h: $S_1 *_\varphi S_2 \to S_1/F_1 *_\xi S_2/F_2$ defined by

$hR\langle a, x\rangle = R\langle [a]_{F_1}, [x]_{F_2}\rangle$ is a homomorphism onto $S_1/F_1 *_\xi S_2/F_2$. Moreover,

$$\begin{aligned}\ker h &= \{R\langle a, x\rangle | [a]_{F_1} = F_1, [x]_{F_2} = F_2\}\\ &= \{R\langle a, x\rangle | a \in F_1, x \in F_2\} = F_1 \underset{\varphi}{*} F_2.\end{aligned}$$

The Homomorphism Theorem proves the rest of the statement.

It should be noted, however, that using the full details of the proof of Theorem 1.2 a shorter proof of this lemma could have been given.

2. Varieties of Brouwerian semilattices. For classes $\mathbf{K}_1$, $\mathbf{K}_2$ of Brouwerian semilattices let $\mathbf{K}_1 \cdot \mathbf{K}_2$ be the class of all extensions of algebras of $\mathbf{K}_1$ by algebras of $\mathbf{K}_2$. If, in particular, $\mathbf{K}_1$ and $\mathbf{K}_2$ are varieties, then $\mathbf{K}_1 \cdot \mathbf{K}_2$ is again a variety, an observation which in full generality is due to Mal′cev [**7**, Theorem 7]. Thus the collection of all subvarieties of **BS** is endowed with a groupoid structure, the resulting groupoid will be denoted by $G(\mathbf{BS})$. In fact, $G(\mathbf{BS})$ is even a semigroup, as also follows from [**7**, Theorem 8]. Clearly multiplication in $G(\mathbf{BS})$ respects inclusion, **BS** is the zero and the trivial variety **T** is the unit of this semigroup. Rephrasing the proofs and the examples given in [**1**] one can see that multiplication from the right distributes over joins and meets taken in the lattice of subvarieties, but left multiplication does not.

A variety $\mathbf{K} \subseteq \mathbf{BS}$ will be called (multiplicatively) *indecomposable* if it cannot be written as a nontrivial product. Clearly $\mathbf{C}_2$ is an indecomposable variety, and as in [**6**] one can prove that for any natural number n we have $\mathbf{C}_2^n = \mathbf{BS}\colon \mathbf{n+2}$. Thus the powers of $\mathbf{C}_2$ generate **BS**, and this also shows that every proper subvariety of **BS** is a finite product of indecomposable varieties. We will even show that this representation is unique, i.e. $G(\mathbf{BS})$ is a free monoid with zero. We will start our investigations with a consequence of Theorem 1.3.

LEMMA 2.1. *Let* $\mathbf{K}_1, \mathbf{K}_2 \in G(\mathbf{BS})$. *Then the product variety* $\mathbf{K}_1 \cdot \mathbf{K}_2$ *is generated by the strongly splitting extensions of algebras of* $\mathbf{K}_1$ *by algebras of* $\mathbf{K}_2$.

Furthermore, we can give a characterization of the product variety which will prove to be very useful in the following. The idea also goes back to Mal′cev [**7**]. So let **K** be a subvariety of **BS** and suppose that S is a Brouwerian semilattice. Then we put

$$D_{\mathbf{K}}(S) = \bigcap \{F | F \in \mathfrak{F}(S), S/F \in \mathbf{K}\}.$$

Then $D_{\mathbf{K}}(S)$ is a filter of S, and, as a subdirect product of the family $\{S/F | F \in \mathfrak{F}(S), S/F \in \mathbf{K}\}$, $S/D_{\mathbf{K}}(S)$ belongs to **K**. In fact, $S/D_{\mathbf{K}}(S)$ is the maximal homomorphic image of S in **K**. Moreover, we have:

LEMMA 2.2. *Let* $\mathbf{K}_1$, $\mathbf{K}_2$ *be subvarieties of* **BS**, *and suppose that* S *is a*

Brouwerian semilattice. Then $S \in \mathbf{K}_1 \cdot \mathbf{K}_2$ *if and only if* $D_{\mathbf{K}_2}(S) \in \mathbf{K}_1$.

PROOF. Let $S \in \mathbf{K}_1 \cdot \mathbf{K}_2$, then there exists $F \in \mathfrak{F}(S)$ such that $F \in \mathbf{K}_1$ and $S/F \in \mathbf{K}_2$. But this implies $D_{\mathbf{K}_2}(S) \subseteq F$ and hence $D_{\mathbf{K}_2}(S) \in \mathbf{K}_2$. The converse part follows immediately from the fact that $S/D_{\mathbf{K}_2}(S) \in \mathbf{K}_2$.

The following lemma is basic for proving that every proper subvariety of **BS** is right cancellable.

LEMMA 2.3. *Let* S_1, S_2 *be Brouwerian semilattices; suppose that* $\varphi\colon S_1 \to \operatorname{End} S_2$ *is admissible. Let* $\mathbf{K}$ *be a subvariety of* **BS**. *Then the following hold*:

(1) *There exists a filter* F *of* S_2 *such that* $D_{\mathbf{K}}(S_2) \subseteq F$ *and*

$$D_{\mathbf{K}}\left(S_1 \underset{\varphi}{*} S_2\right) = D_{\mathbf{K}}(S_1) \underset{\varphi}{*} F.$$

(2) *If* $S_1 \notin \mathbf{K}$ *and if there is some* $a \in D_{\mathbf{K}}(S_1)$ *such that* $\varphi(a) = \mathbf{1}$, *then*

$$D_{\mathbf{K}}\left(S_1 \underset{\varphi}{*} S_2\right) = D_{\mathbf{K}}(S_1) \underset{\varphi}{*} S_2.$$

PROOF. By Lemma 1.6 we have $D_{\mathbf{K}}(S_1 \underset{\varphi}{*} S_2) = F_1 \underset{\varphi}{*} F_2$ for some filters F_1 of S_1, F_2 of S_2. Since

$$S_1 \underset{\varphi}{*} S_2/D_{\mathbf{K}}\left(S_1 \underset{\varphi}{*} S_2\right) \cong S_1/F_1 \underset{\xi}{*} S_2/F_2$$

for some admissible mapping ξ we must have $S_1/F_1 \in \mathbf{K}$ and $S_2/F_2 \in \mathbf{K}$. This implies $D_{\mathbf{K}}(S_1) \subseteq F_1$ and $D_{\mathbf{K}}(S_2) \subseteq F_2$. On the other hand $S_1 \underset{\varphi}{*} S_2/D_{\mathbf{K}}(S_1) \underset{\varphi}{*} S_2 \cong S_1/D_{\mathbf{K}}(S_1) \in \mathbf{K}$ again by Lemma 1.6. Thus $D_{\mathbf{K}}(S_1 \underset{\varphi}{*} S_2) = F_1 \underset{\varphi}{*} F_2 \subseteq D_{\mathbf{K}}(S_1) \underset{\varphi}{*} S_2$, which implies $F_1 \subseteq D_{\mathbf{K}}(S_1)$. So (1) is proven.

To prove (2) observe that $\varphi(a) = \mathbf{1}$ for some $a \in D_{\mathbf{K}}(S_1)$ implies $\langle a, x\rangle R\langle a, 1\rangle$ for all $x \in S_2$. Hence $R\langle a, x\rangle \in D_{\mathbf{K}}(S_1) \underset{\varphi}{*} F_2$ for all $x \in S_2$ and thus $S_2 \subseteq F_2$.

As an immediate consequence of this lemma let us note that the equality $D_{\mathbf{K}}(S_1 \times S_2) = D_{\mathbf{K}}(S_1) \times D_{\mathbf{K}}(S_2)$ holds. Also if $S_1 \notin \mathbf{K}$ we have that $D_{\mathbf{K}}(S_1 \dagger S_2) \cong D_{\mathbf{K}}(S_1) \dagger S_2$. But more importantly:

THEOREM 2.4. *Let* $\mathbf{K}, \mathbf{K}_1, \mathbf{K}_2 \in G(\mathbf{BS})$, *and suppose that* $\mathbf{K} \neq \mathbf{BS}$. *Then* $\mathbf{K}_1 \cdot \mathbf{K} \subseteq \mathbf{K}_2 \cdot \mathbf{K}$ *implies* $\mathbf{K}_1 \subseteq \mathbf{K}_2$ *and* $\mathbf{K}_1 \cdot \mathbf{K} = \mathbf{K}_2 \cdot \mathbf{K}$ *implies* $\mathbf{K}_1 = \mathbf{K}_2$.

PROOF. First observe that it suffices to show that $\mathbf{K}_1 = \{D_{\mathbf{K}}(S) | S \in \mathbf{K}_1 \cdot \mathbf{K}\}$, since this implies $\mathbf{K}_1 = \{D_{\mathbf{K}}(S) | S \in \mathbf{K}_1 \cdot \mathbf{K}\} \subseteq \{D_{\mathbf{K}}(S) | S \in \mathbf{K}_2 \cdot \mathbf{K}\} = \mathbf{K}_2$. Also—because of Lemma 2.2—only the inclusion $\mathbf{K}_1 \subseteq \{D_{\mathbf{K}}(S) | S \in \mathbf{K}_1 \cdot \mathbf{K}\}$ has to be shown. So let $S \in \mathbf{K}_1$. Since $\mathbf{K} \neq \mathbf{BS}$ there exists a finite algebra $S_1 \in \mathbf{BS} \setminus \mathbf{K}$. We may even assume that S_1 is subdirectly irreducible and $S_1 \cong S_2 \dagger \mathbf{2}$ for some $S_2 \in \mathbf{K}$. Then $S_2 \dagger S \in \mathbf{K}_1 \cdot \mathbf{K}$. By Lemma 2.3 we have $D_{\mathbf{K}}(S_2 \dagger S) \subseteq D_{\mathbf{K}}(S_2) \dagger S = \{1\} \dagger S \cong S$. By Lemma 1.6, $D_{\mathbf{K}}(S_2 \dagger S) = \{1\} \dagger F$ for some filters F of S. Suppose $F \neq S$. Then, again by Lemma 1.6, we have

$S_2 \dagger S / D_{\mathbf{K}}(S_2 \dagger S) \cong S_2 \dagger S/F \in \mathbf{K}$. Since $F \neq S$ implies that $\mathbf{2}$ is isomorphic to a subalgebra of S/F, it follows that $S_2 \dagger \mathbf{2}$ is isomorphic to a subalgebra of $S_2 \dagger S/F$. But this would imply $S_1 \in \mathbf{K}$, contradictory to our choice of S_1. Hence $F = S$ and so $S \cong D_{\mathbf{K}}(S_2 \dagger S)$.

The rest of the paper will be occupied by the proof of the freeness of $G(\mathbf{BS})$.

LEMMA 2.5. *Let* $\mathbf{K}$ *be a subvariety of* $\mathbf{BS}$; *and suppose that* X *is a subclass of* $\mathbf{BS}$ *which is closed under finite direct products. Then*

$$\mathbf{K} \cdot V(X) = V\left\{S_1 *_{\varphi} S_2 \middle| S_1 \in X, S_2 \in \mathbf{K}, \varphi \text{ admissible}\right\}.$$

PROOF. Clearly $\mathbf{K} \cdot V(X)$ is generated by its finitely generated free algebras. Thus it suffices to show that for any natural number n the free algebra $F_{\mathbf{K} \cdot V(X)}(n)$ on n generators of the variety $\mathbf{K} \cdot V(X)$ is isomorphic to a subalgebra of $S_1 *_{\varphi} S_2$ for some algebras $S_1 \in X$, $S_2 \in \mathbf{K}$ and some admissible mapping φ. First note that $F_{\mathbf{K} \cdot V(X)}(n)/D_{V(X)}(F_{\mathbf{K} \cdot V(X)}(n)) \cong F_{V(X)}(n)$ and $D_{V(X)}(F_{\mathbf{K} \cdot V(X)}(n)) \in \mathbf{K}$. Thus Theorem 1.3 implies that $F_{\mathbf{K} \cdot V(X)}(n)$ can be embedded into $F_{V(X)}(n) *_{\varphi} S_2$ for some $S_2 \in \mathbf{K}$ and some admissible mapping φ. In fact since $F_{\mathbf{K} \cdot V(X)}(n)$ is finite we may even take $S_2 = D_{V(X)}(F_{\mathbf{K} \cdot V(X)}(n))$. Now X is closed under finite direct products and $F_{V(X)}(n)$ is finite, thus $F_{V(X)}(n)$ is a subalgebra of some $S_1 \in X$. It remains to show that we can extend φ from $F_{V(X)}(n)$ to S_1. But again since $F_{V(X)}(n)$ is finite this is an immediate consequence of the injectivity of finite Brouwerian semilattices in the category of semilattices [4, Corollary 2.9].

LEMMA 2.6. *Let* $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ *be subvarieties of* $\mathbf{BS}$ *such that* $\mathbf{K}_2 \not\subseteq \mathbf{K}_4$ *and* $\mathbf{K}_1 \cdot \mathbf{K}_2 = \mathbf{K}_3 \cdot \mathbf{K}_4$. *Then there is a nontrivial variety* $\mathbf{K}$ *such that* $\mathbf{K}_1 \cdot \mathbf{K} = \mathbf{K}_3$.

PROOF. Define $\mathbf{K} = V\{D_{\mathbf{K}_4}(S) | S \in K_2\}$. Since $\mathbf{K}_2 \not\subseteq \mathbf{K}_4$, $\mathbf{K}$ is nontrivial. Also since by Lemma 2.2, $\mathbf{K}_2 \subseteq \mathbf{K} \cdot \mathbf{K}_4$ we have $\mathbf{K}_3 \cdot \mathbf{K}_4 = \mathbf{K}_1 \cdot \mathbf{K}_2 \subseteq \mathbf{K}_1 \cdot \mathbf{K} \cdot \mathbf{K}_4$ and right cancellation (Theorem 2.4) yields $\mathbf{K}_3 \subseteq \mathbf{K}_1 \cdot \mathbf{K}$.

To prove the converse inclusion first note that by the remark following Lemma 2.3 the class $\{D_{\mathbf{K}_4}(S) | S \in \mathbf{K}_2\}$ is closed under finite direct products. Thus by Lemma 2.5 it suffices to show that

$$\left\{D_{\mathbf{K}_4}(S_2) *_{\varphi} S_1 \middle| S_1 \in \mathbf{K}_1, S_2 \in \mathbf{K}_2, \varphi \text{ admissible}\right\} \subseteq \mathbf{K}_3.$$

So let $S_2 \in \mathbf{K}_2$, $S_1 \in \mathbf{K}_1$, $\varphi\colon D_{\mathbf{K}_4}(S_2) \to \operatorname{End} S_1$ admissible. First we extend φ to φ_1 by defining

$$\varphi_1 x = \begin{cases} \varphi x & \text{if } x \in D_{\mathbf{K}_4}(S_2), \\ \mathbf{1} & \text{if } x \notin D_{\mathbf{K}_4}(S_2). \end{cases}$$

Then φ_1 is admissible too. With $\varphi_2\colon S_2 \to \operatorname{End} S_1$ defined by

$$\varphi_2 x = \begin{cases} \mathrm{id}_{S_1} & \text{if } x = 1, \\ \mathbf{1} & \text{if } x \neq 1, \end{cases}$$

we form $\xi\colon S_2 \times S_2 \to \operatorname{End} S_1$, $\xi\langle x, y\rangle = \varphi_1(x)\varphi_2(y)$, and obviously ξ is again admissible. Then $(S_2 \times S_2) *_\xi S_1 \in \mathbf{K}_1 \cdot \mathbf{K}_2 = \mathbf{K}_3 \cdot \mathbf{K}_4$, and thus $D_{\mathbf{K}_4}((S_2 \times S_2) *_\xi S_1) \in \mathbf{K}_3$. But since $D_{\mathbf{K}_4}(S_2) \neq \{1\}$, there is some $\langle x, y\rangle \in D_{\mathbf{K}_4}(S_2 \times S_2)$ such that $\xi\langle x, y\rangle = \mathbf{1}$. By Lemma 2.3(2) this implies

$$\begin{aligned} D_{\mathbf{K}_4}\big((S_2 \times S_2) *_\xi S_1\big) &= D_{\mathbf{K}_4}(S_2 \times S_2) *_\xi S_1 \\ &= \big(D_{\mathbf{K}_4}(S_2) \times D_{\mathbf{K}_4}(S_2)\big) *_\xi S_1. \end{aligned}$$

Finally $D_{\mathbf{K}_4}(S_2) *_\varphi S_1$ is isomorphic to a subalgebra of $(D_{\mathbf{K}_4}(S_2) \times D_{\mathbf{K}_4}(S_2)) *_\xi S_1$ by Corollary 1.5 and thus $D_{\mathbf{K}_4}(S_2) *_\varphi S_1 \in \mathbf{K}_3$.

Now we can state the announced theorem on the semigroup of subvarieties of **BS**:

THEOREM 2.7. *$G(\mathbf{BS})$ is a free monoid with zero on $2^{\aleph_0}$ generators.*

PROOF. Let $\mathbf{K}_1 \cdot \mathbf{K}_2 = \mathbf{K}_3 \cdot \mathbf{K}_4$ for some proper subvarieties $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ of **BS** where $\mathbf{K}_1$ and $\mathbf{K}_3$ are indecomposable. Then Lemma 2.6 shows that neither $\mathbf{K}_2 \subsetneqq \mathbf{K}_4$ nor $\mathbf{K}_4 \subsetneqq \mathbf{K}_2$ can hold. Thus $\mathbf{K}_2 = \mathbf{K}_4$ and by Theorem 2.4 also $\mathbf{K}_1 = \mathbf{K}_3$.

An obvious induction based on this shows that the representation of a proper subvariety as a finite product of indecomposable varieties is in fact unique. This means that $G(\mathbf{BS})$ is free. Wroński has shown that $G(\mathbf{BS})$ has $2^{\aleph_0}$ elements [**20**].

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