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THE SEMILATTICE TENSOR PRODUCT OF DISTRIBUTIVE LATTICES

BY

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ABSTRACT. We define the tensor product $A \otimes B$ for arbitrary semilattices A and B. The construction is analogous to one used in ring theory (see [4], [7], [8]) and different from one studied by A. Waterman [12], D. Mowat [9], and Z. Shmuely [10]. We show that the semilattice $A \otimes B$ is a distributive lattice whenever A and B are distributive lattices, and we investigate the relationship between the Stone space of $A \otimes B$ and the Stone spaces of the factors A and B. We conclude with some results concerning tensor products that are projective in the category of distributive lattices.

1. Preliminaries. For terminology and basic results of lattice theory and universal algebra, consult Birkhoff [3] and Grätzer [5], [6]. The join and meet of elements $a_1, \ldots, a_n$ of a lattice are denoted by $\Sigma_{i=1}^n a_i$ and $\Pi_{i=1}^n a_i$ respectively. All semilattices considered are join-semilattices. The smallest and largest elements of a lattice, if they exist, are denoted by 0 and 1 respectively. We denote by 2 the two element lattice consisting of 0 and 1. The category of distributive lattices is denoted by $\mathcal{D}$.

2. Existence of the semilattice tensor product.

DEFINITION 2.1. Let A, B and C be semilattices. A function $f: A \times B \longrightarrow C$ is a *bihomomorphism* if the functions $g_a: B \longrightarrow C$ defined by $g_a(b) = f(a, b)$ and $h_b: A \longrightarrow C$ defined by $h_b(a) = f(a, b)$ are homomorphisms for all $a \in A$ and $b \in B$.

DEFINITION 2.2. Let A and B be semilattices. A semilattice C is a *tensor product* of A and B if there is a bihomomorphism $f: A \times B \longrightarrow C$ such that C is generated by $f(A \times B)$ and for any semilattice D and any bihomomorphism $g: A \times B \longrightarrow D$ there is a homomorphism $h: C \longrightarrow D$ satisfying $g = hf$.

Note that since $f(A \times B)$ generates C, the homomorphism h is necessarily unique.

THEOREM 2.3. *Let A and B be semilattices. Then a tensor product of A and B exists and is unique up to isomorphism.*

PROOF. Let K be the free semilattice on $A \times B$ and let w be the canonical inclusion map of $A \times B$ into K. Let ρ be the set of all ordered pairs of the

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form $(w(a, b), w(a_1, b_1) + w(a_2, b_2))$, where $a, a_1, a_2 \in A$; $b, b_1, b_2 \in B$; and either $a = a_1 + a_2$ and $b = b_1 = b_2$, or $a = a_1 = a_2$ and $b = b_1 + b_2$. Let σ be the smallest congruence relation in K containing ρ. Let $C = K/\sigma$, u be the canonical homomorphism from K onto C, and $f = uw$.

By the choice of σ it is clear that f is a bihomomorphism. Let D be any semilattice and let $g: A \times B \longrightarrow D$ be any bihomomorphism. There is a unique homomorphism $s: K \longrightarrow D$ such that $g = sw$. Since g is a bihomomorphism, the kernel relation of s, $\ker s$, contains σ, where $\ker s$ is defined to be $\{(x, y) \in K \times K: s(x) = s(y)\}$. Therefore $\ker u = \sigma \subseteq \ker s$, so that $s = hu$ for some unique homomorphism $h: C \longrightarrow D$. Then h is such that $g = sw = huw = hf$. Finally, since $w(A \times B)$ generates K, $uw(A \times B)$ generates K/σ, so that $f(A \times B)$ generates C. This shows that the semilattice C and the bihomomorphism f satisfy the conditions of the definition of a tensor product.

The uniqueness of a tensor product is clear from its definition as a solution of a universal problem.

The tensor product of A and B is denoted by $A \otimes B$ and the image of (a, b) under the canonical bihomomorphism $f: A \times B \longrightarrow A \otimes B$ is written as $a \otimes b$. In this notation the proof of Theorem 2.3 shows that $A \otimes B$ is the semilattice generated by the elements $a \otimes b$ $(a \in A, b \in B)$, subject to the bihomomorphic conditions $(a_1 + a_2) \otimes b = (a_1 \otimes b) + (a_2 \otimes b)$ and $a \otimes (b_1 + b_2) = (a \otimes b_1) + (a \otimes b_2)$ for all $a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$. Every element of $A \otimes B$ can be written in the form $\Sigma_{i=1}^n (a_i \otimes b_i)$ for some $a_i \in A$ and $b_i \in B$, $i = 1, \ldots, n$.

LEMMA 2.4. *Let A and B be semilattices and let $a, a_i \in A$ and $b, b_i \in B$ for $i = 1, \ldots, n$. Then $a \otimes b \leqslant \Sigma_{i=1}^n (a_i \otimes b_i)$ if and only if for every bihomomorphism $g: A \times B \longrightarrow 2$, $g(a, b) = 1$ implies $g(a_i, b_i) = 1$ for some i.*

PROOF. We recall that if S is a semilattice and $x, y \in S$, then $x \leqslant y$ if and only if for every homomorphism $h: S \longrightarrow 2$, $h(x) = 1$ implies $h(y) = 1$. Also note that there is a one-to-one correspondence between homomorphisms $h: A \otimes B \longrightarrow 2$ and bihomomorphisms $g: A \times B \longrightarrow 2$. Then

$$
\begin{aligned}
a \otimes b \leqslant \sum_1^n (a_i \otimes b_i)
&\iff \text{for every homomorphism } h: A \otimes B \longrightarrow 2,\\
&\qquad h(a \otimes b) = 1 \text{ implies } h\left(\sum_1^n (a_i \otimes b_i)\right) = 1\\
&\iff \text{for every homomorphism } h: A \otimes B \longrightarrow 2,\\
&\qquad h(a \otimes b) = 1 \text{ implies } h(a_i \otimes b_i) = 1 \text{ for some } i\\
&\iff \text{for every bihomomorphism } g: A \times B \longrightarrow 2,\\
&\qquad g(a, b) = 1 \text{ implies } g(a_i, b_i) = 1 \text{ for some } i.
\end{aligned}
$$

Next let A and B be semilattices and let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. If

$a_1 \leqslant a_2$ and $b_1 \leqslant b_2$, then it follows from the order preserving properties of the canonical bihomomorphism that $a_1 \otimes b_1 \leqslant a_2 \otimes b_2$.

We now restrict our attention to distributive lattices. For any positive integer n, let $\mathbf{n}$ be the set $\{1, \ldots, n\}$.

THEOREM 2.5. *Let A and B be distributive lattices and let $a, a_i \in A$ and $b, b_i \in B$ for $i = 1, \ldots, n$. Then $a \otimes b \leqslant \Sigma_{i=1}^{n} (a_i \otimes b_i)$ if and only if there exist nonempty subsets $S_1, \ldots, S_m$ of $\mathbf{n}$ such that $a \leqslant \Sigma_{j=1}^{m} \Pi_{i \in S_j} a_i$ and $b \leqslant \Pi_{j=1}^{m} \Sigma_{i \in S_j} b_i$.*

PROOF. We recall that if L is a distributive lattice and $x, y \in L$, then $x \nleqslant y$ if and only if there is a lattice homomorphism $h : L \longrightarrow 2$ such that $h(x) = 1$ and $h(y) = 0$.

Now assume that $a \otimes b \leqslant \Sigma_1^n (a_i \otimes b_i)$. If we suppose that $a \nleqslant \Sigma_1^n a_i$, then there is a lattice homomorphism $h : A \longrightarrow 2$ such that $h(a) = 1$ and $h(\Sigma_1^n a_i) = 0$. So $h(a_i) = 0$ for all i. Let $g : A \times B \longrightarrow 2$ be defined by $g(x, y) = h(x)$. Then g is a lattice bihomomorphism, $g(a, b) = 1$ and $g(a_i, b_i) = 0$ for all i. Hence $a \otimes b \nleqslant \Sigma_1^n (a_i \otimes b_i)$ by Lemma 2.4 and we have a contradiction. Thus we must have $a \leqslant \Sigma_1^n a_i$. Similarly $b \leqslant \Sigma_1^n b_i$. Let $S_1, \ldots, S_m$ be all the nonempty subsets S of $\mathbf{n}$ such that $b \leqslant \Sigma_{i \in S} b_i$. Then $b \leqslant \Pi_{j=1}^{m} \Sigma_{i \in S_j} b_i$.

We claim that $a \leqslant \Sigma_{j=1}^{m} \Pi_{i \in S_j} a_i$. This is equivalent by the distributivity of A to showing that $a \leqslant \Sigma_{k=1}^{m} a_{i_k}$ for each choice of $i_1 \in S_1, \ldots, i_m \in S_m$. Suppose that for some $i_1 \in S_1, \ldots, i_m \in S_m$ we have $a \nleqslant \Sigma_{k=1}^{m} a_{i_k}$. Let $I = \{i_1, \ldots, i_m\}$ and let $I' = \{x \in \mathbf{n} : x \notin I\}$. Note that since $a \leqslant \Sigma_1^n a_i$, $I \neq \mathbf{n}$, so that $I' \neq \varnothing$.

Now if $b \nleqslant \Sigma_{i \in I'} b_i$ then there is a lattice homomorphism $h : B \longrightarrow 2$ such that $h(b) = 1$ and $h(\Sigma_{i \in I'} b_i) = 0$. Hence $h(b_i) = 0$ for all $i \in I'$. Also since $a \nleqslant \Sigma_{k=1}^{m} a_{i_k}$ there is a lattice homomorphism $f : A \longrightarrow 2$ such that $f(a) = 1$ and $f(\Sigma_{k=1}^{m} a_{i_k}) = 0$. So $f(a_{i_k}) = 0$ for $k = 1, \ldots, m$. Let $g : A \times B \longrightarrow 2$ be defined by $g(x, y) = f(x)h(y)$. Then g is a lattice bihomomorphism, $g(a, b) = 1$, and $g(a_i, b_i) = 0$ for all $i \in \mathbf{n}$. Hence $a \otimes b \nleqslant \Sigma_1^n (a_i \otimes b_i)$ by Lemma 2.4 and we have a contradiction. Thus $b \leqslant \Sigma_{i \in I'} b_i$.

It follows that $I' = S_j$ for some $j = 1, \ldots, m$. But then we have both $i_j \in I$ and $i_j \in I'$. This is a contradiction. Therefore $a \leqslant \Sigma_{k=1}^{m} a_{i_k}$ for every choice of $i_1 \in S_1, \ldots, i_m \in S_m$, so that $a \leqslant \Sigma_{j=1}^{m} \Pi_{i \in S_j} a_i$, and the first half of the proof is complete.

Conversely, assume there exist nonempty subsets $S_1, \ldots, S_m$ of $\mathbf{n}$ such that $a \leqslant \Sigma_{j=1}^{m} \Pi_{i \in S_j} a_i$ and $b \leqslant \Pi_{j=1}^{m} \Sigma_{i \in S_j} b_i$.

Since B is distributive,

$$b \leqslant \sum_{i_k \in S_k; 1 \leqslant k \leqslant m} \prod_{k=1}^{m} b_{i_k}.$$

Hence

$$a \otimes b \leqslant \left(\sum_{j=1}^{m} \prod_{i \in S_j} a_i\right) \otimes \left(\sum_{i_k \in S_k; 1 \leqslant k \leqslant m} \prod_{k=1}^{m} b_{i_k}\right)$$
$$= \sum_{j=1}^{m} \sum_{i_k \in S_k; 1 \leqslant k \leqslant m} \left(\prod_{i \in S_j} a_i \otimes \prod_{k=1}^{m} b_{i_k}\right).$$

Let j be such that $1 \leqslant j \leqslant m$ and let $i_1 \in S_1, \ldots, i_m \in S_m$. Then

$$\prod_{i \in S_j} a_i \otimes \prod_{k=1}^{m} b_{i_k} \leqslant \prod_{i \in S_j} a_i \otimes b_{i_j} \leqslant a_{i_j} \otimes b_{i_j} \leqslant \sum_{1}^{n} (a_i \otimes b_i).$$

Summing over all terms, we obtain

$$\sum_{j=1}^{m} \sum_{i_k \in S_k; 1 \leqslant k \leqslant m} \left(\prod_{i \in S_j} a_i \otimes \prod_{k=1}^{m} b_{i_k}\right) \leqslant \sum_{1}^{n} (a_i \otimes b_i).$$

Hence $a \otimes b \leqslant \Sigma_1^n (a_i \otimes b_i)$.

We remark that the proof of Theorem 2.5 shows that if $a \otimes b \leqslant \Sigma_1^n (a_i \otimes b_i)$ then $a \leqslant \Sigma_1^n a_i$ and $b \leqslant \Sigma_1^n b_i$.

THEOREM 2.6. *Let A and B be distributive lattices. Then $A \otimes B$ is a distributive lattice. If $\Sigma_1^m (a_i \otimes b_i)$ and $\Sigma_1^n (c_j \otimes d_j)$ are arbitrary elements of $A \otimes B$, then their greatest lower bound is $\Sigma_{(i,j) \in \mathbf{m} \times \mathbf{n}} (a_i c_j \otimes b_i d_j)$.*

PROOF. It is easy to verify that $\Sigma_{i,j} (a_i c_j \otimes b_i d_j)$ is a lower bound for $\Sigma_1^m (a_i \otimes b_i)$ and $\Sigma_1^n (c_j \otimes d_j)$.

To prove that $\Sigma_{i,j} (a_i c_j \otimes b_i d_j)$ is the greatest lower bound, it is enough to show that for any $a \in A$, $b \in B$, $a \otimes b \leqslant \Sigma_1^m (a_i \otimes b_i)$ and $a \otimes b \leqslant \Sigma_1^n (c_j \otimes d_j)$ imply that $a \otimes b \leqslant \Sigma_{i,j} (a_i c_j \otimes b_i d_j)$. So assume that $a \otimes b \leqslant \Sigma_1^m (a_i \otimes b_i)$ and $a \otimes b \leqslant \Sigma_1^n (c_j \otimes d_j)$. Then by Theorem 2.5, there exist nonempty subsets $S_1, \ldots, S_r$ of $\mathbf{m}$ such that $a \leqslant \Sigma_{k=1}^r \Pi_{i \in S_k} a_i$ and $b \leqslant \Pi_{k=1}^r \Sigma_{i \in S_k} b_i$. There exist nonempty subsets $T_1, \ldots, T_s$ of $\mathbf{n}$ such that $a \leqslant \Sigma_{l=1}^s \Pi_{j \in T_l} c_j$ and $b \leqslant \Pi_{l=1}^s \Sigma_{j \in T_l} d_j$. Since A and B are distributive, we have

$$a \leqslant \left(\sum_{k=1}^{r} \prod_{i \in S_k} a_i\right) \left(\sum_{l=1}^{s} \prod_{j \in T_l} c_j\right)$$
$$= \sum_{k=1}^{r} \sum_{l=1}^{s} \left[\left(\prod_{i \in S_k} a_i\right) \left(\prod_{j \in T_l} c_j\right)\right] = \sum_{k=1}^{r} \sum_{l=1}^{s} \prod_{(i,j) \in S_k \times T_l} a_i c_j$$

and

$$b \leqslant \left(\prod_{k=1}^{r} \sum_{i \in S_k} b_i\right) \left(\prod_{l=1}^{s} \sum_{j \in T_l} d_j\right)$$
$$= \prod_{k=1}^{r} \prod_{l=1}^{s} \left[\left(\sum_{i \in S_k} b_i\right) \left(\sum_{j \in T_l} d_j\right)\right] = \prod_{k=1}^{r} \prod_{l=1}^{s} \sum_{(i,j) \in S_k \times T_l} b_i d_j.$$

Hence if we consider the family of sets $S_k \times T_l$ $(k = 1, \ldots, r; l = 1, \ldots, s)$ as a collection of nonempty subsets of $\mathbf{m} \times \mathbf{n}$, it follows from Theorem 2.5 that $a \otimes b \leq \Sigma_{i,j}(a_i c_j \otimes b_i d_j)$.

Hence $\Sigma_{i,j}(a_i c_j \otimes b_i d_j)$ is the greatest lower bound of $\Sigma_1^m(a_i \otimes b_i)$ and $\Sigma_1^n(c_j \otimes d_j)$, and so $A \otimes B$ is a lattice. In view of the expression obtained for the greatest lower bound, it is clear that the lattice $A \otimes B$ is distributive.

We note that Theorem 2.6 implies that $\Pi_1^n(a_i \otimes b_i) = \Pi_1^n a_i \otimes \Pi_1^n b_i$ and in particular $(a_1 \otimes b)(a_2 \otimes b) = a_1 a_2 \otimes b$ and $(a \otimes b_1)(a \otimes b_2) = a \otimes b_1 b_2$.

The word problem for the semilattice tensor product of distributive lattices reduces to the problem of determining when an inequality of the form $a \otimes b \leq \Sigma_1^n(a_i \otimes b_i)$ holds and Theorem 2.5 provides a characterization of such inequalities.

Let A and B be distributive lattices and let A_1 and B_1 be sublattices of A and B respectively. Let $\otimes_1$ be the canonical bihomomorphism from $A_1 \times B_1$ into $A_1 \otimes B_1$ and let $\otimes$ be the restriction to $A_1 \times B_1$ of the canonical bihomomorphism from $A \times B$ into $A \otimes B$. It follows from the definition of the tensor product that there is a canonical homomorphism h from $A_1 \otimes B_1$ into $A \otimes B$ such that for all $a \in A_1$ and $b \in B_1$, $h(a \otimes_1 b) = a \otimes b$. It is easy to see using Theorem 2.5 that h is one-to-one. Thus $A_1 \otimes B_1$ is embedded under the canonical mapping h as a sublattice of $A \otimes B$. Hence we say that $A_1 \otimes B_1$ is canonically isomorphic to a sublattice of $A \otimes B$.

3. **The structure of the semilattice tensor product.** In this section we give several results regarding the structure of the semilattice tensor product. We examine the Stone space of the tensor product and we compare the tensor product to the free product. We also characterize the join-irreducible elements of the tensor product.

We begin with two results concerning congruence relations of the tensor product. The proofs are fairly straightforward, and we leave them as exercises for the reader.

Let A and B be semilattices and let ρ and σ be (semilattice) congruence relations on A and B respectively. The congruence relation on $A \otimes B$ generated by ρ and σ is defined to be the smallest congruence relation on $A \otimes B$ containing all ordered pairs of the form $(a_1 \otimes b_1, a_2 \otimes b_2)$, where $(a_1, a_2) \in \rho$ and $(b_1, b_2) \in \sigma$.

THEOREM 3.1. *Let A and B be semilattices and let ρ and σ be congruence relations on A and B respectively. Let τ be the congruence relation on $A \otimes B$ generated by ρ and σ. Then $A/\rho \otimes B/\sigma$ is isomorphic to $(A \otimes B)/\tau$.*

Now let A and B be distributive lattices and let ρ and σ be lattice congruence relations on A and B respectively. The semilattice congruence relation on

$A \otimes B$ generated by ρ and σ is defined in the same way as before.

THEOREM 3.2. *Let A and B be distributive lattices and let ρ and σ be lattice congruence relations on A and B respectively. Let τ be the semilattice congruence relation on $A \otimes B$ generated by ρ and σ. Then τ is a lattice congruence relation and $A/\rho \otimes B/\sigma$ is isomorphic to $(A \otimes B)/\tau$.*

Let A and B be distributive lattices with a largest element 1. We denote by $A * B$ the free product of A and B in the category $\mathcal{D}_1$ of distributive lattices with 1 and lattice homomorphisms preserving 1. For details concerning the definition and basic properties of free products, see Grätzer [5, pp. 183–186].

THEOREM 3.3. *Let A and B be distributive lattices with a largest element. Then $A \otimes B$ is isomorphic to $A * B$.*

PROOF. Since $A * B$ is unique up to isomorphism it suffices to show that $A \otimes B$ satisfies the conditions of the definition of $A * B$.

It is clear that $A \otimes B$ is in $\mathcal{D}_1$ since $A \otimes B$ has a largest element $1 \otimes 1$. Let $f_1 : A \longrightarrow A \otimes B$ be defined by $f_1(a) = a \otimes 1$. Then

$$f_1(a_1 + a_2) = (a_1 + a_2) \otimes 1 = (a_1 \otimes 1) + (a_2 \otimes 1) = f_1(a_1) + f_1(a_2).$$

Similarly $f_1(a_1a_2) = f_1(a_1)f_1(a_2)$. Also $f_1(1) = 1 \otimes 1$. If $f_1(a_1) = f_1(a_2)$ then $a_1 \otimes 1 = a_2 \otimes 1$, so $a_1 = a_2$. Thus f_1 is a one-to-one $\mathcal{D}_1$ homomorphism. Similarly if we define $f_2 : B \longrightarrow A \otimes B$ by $f_2(b) = 1 \otimes b$ then f_2 is a one-to-one $\mathcal{D}_1$ homomorphism.

Now $f_1(A) \cup f_2(B) = \{a \otimes 1 : a \in A\} \cup \{1 \otimes b : b \in B\}$. Hence for all $a \in A$ and $b \in B$ the sublattice of $A \otimes B$ generated by $f_1(A) \cup f_2(B)$ contains $(a \otimes 1)(1 \otimes b) = a \otimes b$. Thus the sublattice generated by $f_1(A) \cup f_2(B)$ is $A \otimes B$.

Let C be in $\mathcal{D}_1$ and let $g_1 : A \longrightarrow C$, $g_2 : B \longrightarrow C$ be $\mathcal{D}_1$ homomorphisms. Let $g : A \times B \longrightarrow C$ be defined by $g(a, b) = g_1(a)g_2(b)$. Then g is a semilattice bihomomorphism. Hence there is a semilattice homomorphism $h : A \otimes B \longrightarrow C$ such that $h(a \otimes b) = g(a, b) = g_1(a)g_2(b)$. Then if $\Sigma_i(a_i \otimes b_i)$ and $\Sigma_j(c_j \otimes d_j)$ are arbitrary elements of $A \otimes B$, we have

$$\begin{aligned} h\Big(\sum_i (a_i \otimes b_i) \sum_j (c_j \otimes d_j)\Big) &= h\Big(\sum_{i,j} (a_ic_j \otimes b_id_j)\Big) \\ &= \sum_{i,j} h(a_ic_j \otimes b_id_j) = \sum_{i,j} g_1(a_ic_j)g_2(b_id_j) \\ &= \sum_{i,j} g_1(a_i)g_1(c_j)g_2(b_i)g_2(d_j) = \Big[\sum_i g_1(a_i)g_2(b_i)\Big]\Big[\sum_j g_1(c_j)g_2(d_j)\Big] \\ &= \Big[\sum_i h(a_i \otimes b_i)\Big]\Big[\sum_j h(c_j \otimes d_j)\Big] = h\Big(\sum_i (a_i \otimes b_i)\Big) h\Big(\sum_j (c_j \otimes d_j)\Big). \end{aligned}$$

Also $h(1 \otimes 1) = g_1(1)g_2(1) = 1$. Thus h is a $\mathcal{D}_1$ homomorphism. For all $a \in A$ we have $hf_1(a) = h(a \otimes 1) = g_1(a)g_2(1) = g_1(a)$. Thus $hf_1 = g_1$. Similarly $hf_2 = g_2$. Hence $A \otimes B$ satisfies the conditions of the definition of $A * B$ and so $A \otimes B$ is isomorphic to $A * B$.

Let $F_{\mathcal{D}_1}(\alpha)$ denote the free $\mathcal{D}_1$ lattice with α free generators.

COROLLARY 3.4. *$F_{\mathcal{D}_1}(\alpha) \otimes F_{\mathcal{D}_1}(\beta)$ is isomorphic to $F_{\mathcal{D}_1}(\alpha + \beta)$.*

PROOF. Since $F_{\mathcal{D}_1}(\alpha) * F_{\mathcal{D}_1}(\beta)$ is isomorphic to $F_{\mathcal{D}_1}(\alpha + \beta)$, this result follows immediately from Theorem 3.3.

Note that $F_{\mathcal{D}_1}(\alpha)$ is simply the free distributive lattice with α free generators to which a new largest element is adjoined.

We now investigate the relationship between the Stone space of a tensor product and the Stone spaces of its factors. It is convenient to assume that the distributive lattices under consideration have a smallest element. The Stone space $S(L)$ of a distributive lattice L is the set of all prime filters of L, topologized by taking as a basis for the open sets the family $\{x^* : x \in L\}$, where $x^* = \{F \in S(L) : x \in F\}$. Prime filters of L are necessarily nonempty, proper subsets of L. For details, see Stone [11]. Let $S'(L) = S(L) \cup \{L\}$ and topologize $S'(L)$ by using as a basis for the open sets the family $\{x^* : x \in L\}$, where now $x^* = \{F \in S'(L) : x \in F\}$. Thus $S'(L)$ is obtained from $S(L)$ by adding a new element L, and L belongs to every nonempty open set.

THEOREM 3.5. *Let A and B be distributive lattices with a smallest element. Then the spaces $S'(A \otimes B)$ and $S'(A) \times S'(B)$ are homeomorphic.*

PROOF. If $F \in S'(A \otimes B)$, let $F_1 = \{a \in A : a \otimes b \in F$ for some $b \in B\}$ and $F_2 = \{b \in B : a \otimes b \in F$ for some $a \in A\}$. It is readily verified that $F_1 \in S'(A)$ and $F_2 \in S'(B)$. Thus the map $\phi : S'(A \otimes B) \longrightarrow S'(A) \times S'(B)$ given by $\phi(F) = (F_1, F_2)$ is well defined. Note that we have $a \otimes b \in F$ if and only if $a \in F_1$ and $b \in F_2$. Now if $F, G \in S'(A \otimes B)$ and $\phi(F) = \phi(G)$, then $(F_1, F_2) = (G_1, G_2)$, so that $a \otimes b \in F$ if and only if $a \otimes b \in G$, and this implies that $F = G$. Thus ϕ is one-to-one.

Next let $(F_1, F_2) \in S'(A) \times S'(B)$ be given and put $F = \{\Sigma_1^n(a_i \otimes b_i) \in A \otimes B : \Sigma_1^n(a_i \otimes b_i) \geqslant a \otimes b$ for some $a \in F_1$ and $b \in F_2\}$. It is easy to see that F is a filter. To show that $F \in S'(A \otimes B)$, we may suppose without loss of generality that $\Sigma_1^n(a_i \otimes b_i) \in F$, so that $\Sigma_1^n(a_i \otimes b_i) \geqslant a \otimes b$ for some $a \in F_1$, $b \in F_2$. Then by Theorem 2.5 there exist nonempty subsets $S_1, \ldots, S_m$ of n such that $a \leqslant \Sigma_{j=1}^m \Pi_{i \in S_j} a_i$ and $b \leqslant \Pi_{j=1}^m \Sigma_{i \in S_j} b_i$. Hence $\Sigma_{j=1}^m \Pi_{i \in S_j} a_i \in F_1$ and $\Pi_{j=1}^m \Sigma_{i \in S_j} b_i \in F_2$, so that $\Pi_{i \in S_{j_0}} a_i \in F_1$ for some $j_0 \in$ m and $\Sigma_{i \in S_{j_0}} b_i \in F_2$. Hence $b_{i_0} \in F_2$ for some $i_0 \in S_{j_0}$ and $a_{i_0} \in F_1$. Thus $a_{i_0} \otimes b_{i_0} \in F$. Hence F is a prime filter or $A \otimes B$. It is clear that $\phi(F) = (F_1, F_2)$ and so ϕ is onto.

Now

$$\begin{aligned}\phi((a \otimes b)^*) &= \{\phi(F) : F \in (a \otimes b)^*\} = \{\phi(F) : a \otimes b \in F\}\\ &= \{(F_1, F_2) : a \in F_1 \text{ and } b \in F_2\}\\ &= \{(F_1, F_2) : F_1 \in a^* \text{ and } F_2 \in b^*\} = a^* \times b^*.\end{aligned}$$

Thus the image of a basic open set in $S'(A \otimes B)$ is open. Since ϕ is one-to-one and onto, $\phi^{-1}(a^* \times b^*) = (a \otimes b)^*$. Hence the inverse image of a basic open set is open. Thus ϕ is a homeomorphism.

We now characterize the join-irreducible elements in the tensor product.

THEOREM 3.6. *Let A and B be distributive lattices. The join-irreducible elements of $A \otimes B$ are the generators $a \otimes b$ such that a is join-irreducible in A and b is join-irreducible in B.*

PROOF. First, it is clear that a join-irreducible element of $A \otimes B$ must be a generator.

Let $a \otimes b$ be join-irreducible in $A \otimes B$ and suppose $a \leqslant \Sigma_1^n a_i$. Then $a \otimes b \leqslant (\Sigma_1^n a_i) \otimes b = \Sigma_1^n (a_i \otimes b)$. Hence $a \otimes b \leqslant a_i \otimes b$ for some i and so $a \leqslant a_i$. Thus a is join-irreducible. Similarly b is join-irreducible.

Now let a and b be join-irreducible and assume $a \otimes b \leqslant \Sigma_1^n (a_i \otimes b_i)$. Then by Theorem 2.5 there exist nonempty subsets $S_1, \ldots, S_m$ of $\mathbf{n}$ such that $a \leqslant \Sigma_{j=1}^m \Pi_{i \in S_j} a_i$ and $b \leqslant \Pi_{j=1}^m \Sigma_{i \in S_j} b_i$. Since a is join-irreducible, $a \leqslant \Pi_{i \in S_{j_0}} a_i$ for some $j_0 \in \mathbf{m}$; also $b \leqslant \Sigma_{i \in S_{j_0}} b_i$. Since S_{j_0} is nonempty and b is join-irreducible, $b \leqslant b_{i_0}$ for some $i_0 \in S_{j_0}$; and $a \leqslant a_{i_0}$. Hence $a \otimes b \leqslant a_{i_0} \otimes b_{i_0}$ and so $a \otimes b$ is join-irreducible in $A \otimes B$.

It is well known that a finite distributive lattice is isomorphic with the set of all hereditary subsets of its set of nonzero join-irreducible elements, partially ordered by set inclusion [6, p. 72]. Hence the characterization of the join-irreducible elements of $A \otimes B$ given by Theorem 3.6 provides us with a simple way of obtaining a picture of $A \otimes B$ when A and B are finite. We simply form the poset of the nonzero join-irreducible elements of $A \otimes B$, using Theorem 3.6; $A \otimes B$ is then the lattice of all the hereditary subsets of this poset.

THEOREM 3.7. *Let A and B be chains. Then every element of $A \otimes B$ can be expressed in an essentially unique way as an irredundant sum of generators.*

PROOF. Let $\Sigma_1^n(a_i \otimes b_i)$ and $\Sigma_1^m(c_j \otimes d_j)$ be arbitrary elements of $A \otimes B$. Without loss of generality we may assume that $a_1 < \cdots < a_n$, $b_1 > \cdots > b_n$, $c_1 < \cdots < c_m$, $d_1 > \cdots > d_m$ (re-index and use absorption if necessary). Note that by Theorem 3.6, every generator in $A \otimes B$ is join-irreducible. Suppose that $\Sigma_1^n(a_i \otimes b_i) = \Sigma_1^m (c_j \otimes d_j)$. Then $a_1 \otimes b_1 \leqslant \Sigma(c_j \otimes d_j)$, so $a_1 \otimes b_1 \leqslant c_{j_1} \otimes d_{j_1}$ for some $j_1 \in \mathbf{m}$. Since $c_{j_1} \otimes d_{j_1} \leqslant \Sigma(a_i \otimes b_i)$ we have $c_{j_1} \otimes d_{j_1} \leqslant a_{i_1} \otimes$

b_{i_1} for some $i_1 \in \mathbf{n}$. Then $a_1 \otimes b_1 \leqslant c_{j_1} \otimes d_{j_1} \leqslant a_{i_1} \otimes b_{i_1}$, so that $a_1 = a_{i_1}$, $b_1 = b_{i_1}$ and hence $a_1 \otimes b_1 = c_{j_1} \otimes d_{j_1}$. Similarly we obtain $a_2 \otimes b_2 = c_{j_2} \otimes d_{j_2}, \ldots, a_n \otimes b_n = c_{j_n} \otimes d_{j_n}$ for some $j_2, \ldots, j_n \in \mathbf{m}$. Since $\Sigma(a_i \otimes b_i) = \Sigma(c_j \otimes d_j)$ there can be no further terms in the second sum, and $m = n$. Moreover, in view of our initial assumptions, we have $j_1 = 1, \ldots, j_n = n$, so that $a_1 \otimes b_1 = c_1 \otimes d_1, \ldots, a_n \otimes b_n = c_n \otimes d_n$. Thus after redundant terms have been eliminated every element has a unique representation as a sum of generators, except for the order of the terms.

We point out that Theorem 3.7 is false for arbitrary distributive lattices. As an example, let $A = 2$ and $B =$ the diamond (the four element lattice $\{0, a, b, 1\}$ with a and b incomparable). Then in $A \otimes B$ we have $(1 \otimes a) + (0 \otimes b) = (1 \otimes a) + (0 \otimes 1)$. Thus there is an element in $A \otimes B$ that can be expressed in two essentially different ways.

Finally it should be observed that several of the results in §§2 and 3 that were shown to hold for the semilattice tensor product $A \otimes B$ of distributive lattices A and B remain valid when A and B are arbitrary semilattices. In fact, Lemma 2.4 and the remark following it were proved for arbitrary semilattices, and they can be used to prove that the following results hold in the general case: Theorem 3.6; the remark following Theorem 2.5; the remark following Theorem 2.6 (provided that $\Pi_1^n a_i$ and $\Pi_1^n b_i$ exist); and one direction of Theorem 2.5, namely, if $a \otimes b \leqslant \Sigma_{i=1}^n (a_i \otimes b_i)$ then there exist nonempty subsets $S_1, \ldots, S_m$ of $\mathbf{n}$ such that $b \leqslant \Sigma_{i \in S_j} b_i$ for $j = 1, \ldots, m$ and $a \leqslant \Sigma_{k=1}^m a_{i_k}$ for each choice of $i_1 \in S_1, \ldots, i_m \in S_m$.

4. Projective tensor products.

THEOREM 4.1. *Let A and B be distributive lattices. Then A and B are retracts of $A \otimes B$.*

PROOF. Let $g: A \times B \longrightarrow A$ be defined by $g(a, b) = a$. Then g is a semilattice bihomomorphism. Hence there is a semilattice homomorphism $h: A \otimes B \longrightarrow A$ such that $h(a \otimes b) = a$. Then if $\Sigma_i(a_i \otimes b_i)$ and $\Sigma_j(c_j \otimes d_j)$ are arbitrary elements of $A \otimes B$, we have

$$\begin{aligned} h\Big(\sum_i (a_i \otimes b_i) \sum_j (c_j \otimes d_j)\Big) &= h\Big(\sum_{i,j} (a_i c_j \otimes b_i d_j)\Big) = \sum_{i,j} h(a_i c_j \otimes b_i d_j) \\ &= \sum_{i,j} a_i c_j = \Big(\sum_i a_i\Big)\Big(\sum_j c_j\Big) = \sum_i h(a_i \otimes b_i) \sum_j h(c_j \otimes d_j) \\ &= h\Big(\sum_i (a_i \otimes b_i)\Big) h\Big(\sum_j (c_j \otimes d_j)\Big). \end{aligned}$$

Hence h is a $\mathcal{D}$ homomorphism. Let $f: A \longrightarrow A \otimes B$ be defined by $f(a) = a \otimes b_0$, where b_0 is a fixed element of B. Then f is a $\mathcal{D}$ homomorphism. For all $a \in A$ we have $hf(a) = h(a \otimes b_0) = a$ so that hf is the identity map on A. Thus A is a retract of $A \otimes B$. Similarly B is a retract of $A \otimes B$.

COROLLARY 4.2. *Let A and B be distributive lattices. If $A \otimes B$ is $\mathcal{D}$ projective, then A and B are $\mathcal{D}$ projective.*

PROOF. Since a retract of a $\mathcal{D}$ projective distributive lattice is $\mathcal{D}$ projective, this follows directly from Theorem 4.1.

Corollary 4.2 has the following partial converse; the example given at the end shows that it cannot be extended.

THEOREM 4.3. *Let A and B be finite distributive lattices. If A and B are $\mathcal{D}$ projective then $A \otimes B$ is $\mathcal{D}$ projective.*

PROOF. If A and B are finite, then $A \otimes B$ has a finite number of generators and so is itself finite. Now by [1, Theorem 7.1] a finite distributive lattice is $\mathcal{D}$ projective if and only if the product of any two join-irreducible elements is join-irreducible. Suppose A and B are $\mathcal{D}$ projective and let $a_1 \otimes b_1$ and $a_2 \otimes b_2$ be join-irreducible elements of $A \otimes B$. It follows by Theorem 3.6 that a_1, a_2 and b_1, b_2 are join-irreducible in A and B respectively. Since A and B are $\mathcal{D}$ projective, a_1a_2 and b_1b_2 are join-irreducible. Hence $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$ is join-irreducible. Thus $A \otimes B$ is $\mathcal{D}$ projective.

We give an example to show that the hypothesis of finiteness in Theorem 4.3 is essential. Let A and B be the set of nonnegative integers with the usual ordering, and if x is a nonnegative integer, let x^+ denote $x + 1$. Then A and B are $\mathcal{D}$ projective since a chain is $\mathcal{D}$ projective if and only if it is countable [1, Theorem 8.2]. Now a necessary condition that $A \otimes B$ be $\mathcal{D}$ projective is that every element of $A \otimes B$ be expressible as a finite product of meet-irreducible elements [2, Theorem 3]. But we assert that there are no meet-irreducible elements in $A \otimes B$. For let $\Sigma_1^n(a_i \otimes b_i)$ be an arbitrary element of $A \otimes B$, where we suppose without loss of generality that $a_1 < \cdots < a_n$ and $b_1 > \cdots > b_n$. Then

$$\sum_1^n (a_i \otimes b_i)$$

$$= \left[(a_1 \otimes b_1^+) + \sum_2^n (a_i \otimes b_i)\right]\left[\sum_1^{n-1} (a_i \otimes b_i) + (a_n^+ \otimes b_n)\right].$$

For clearly, $\Sigma_1^n(a_i \otimes b_i)$ is less than or equal to the right-hand side. On the other hand,

$$\left[(a_1 \otimes b_1^+) + \sum_2^n (a_i \otimes b_i)\right]\left[\sum_1^{n-1} (a_i \otimes b_i) + (a_n^+ \otimes b_n)\right]$$

$$= (a_1 \otimes b_1^+) \sum_1^{n-1} (a_i \otimes b_i) + (a_1 \otimes b_1^+)(a_n^+ \otimes b_n)$$

$$+ \sum_2^n (a_i \otimes b_i) \sum_1^{n-1} (a_i \otimes b_i) + (a_n^+ \otimes b_n) \sum_2^n (a_i \otimes b_i)$$

$$\leqslant \sum_1^{n-1} (a_i \otimes b_i) + (a_1 a_n^+ \otimes b_n b_1^+) + \sum_2^n (a_i \otimes b_i) + \sum_2^n (a_i \otimes b_i)$$

$$\leqslant \sum_1^n (a_i \otimes b_i) + (a_1 \otimes b_n) \leqslant \sum_1^n (a_i \otimes b_i) + (a_1 \otimes b_1) \leqslant \sum_1^n (a_i \otimes b_i).$$

Now if $\Sigma_1^n (a_i \otimes b_i) = (a_1 \otimes b_1^+) + \Sigma_2^n (a_i \otimes b_i)$, then $a_1 \otimes b_1^+ \leqslant \Sigma_1^n (a_i \otimes b_i)$, and hence by Theorem 2.5, $b_1^+ \leqslant \Sigma_1^n b_i = b_1$, which is a contradiction. Therefore

$$\sum_1^n (a_i \otimes b_i) < (a_1 \otimes b_1^+) + \sum_2^n (a_i \otimes b_i).$$

Similarly

$$\sum_1^n (a_i \otimes b_i) < \sum_1^{n-1} (a_i \otimes b_i) + (a_n^+ \otimes b_n).$$

Thus $\Sigma_1^n (a_i \otimes b_i)$ is the product of two larger elements, and is therefore not meet-irreducible. So $A \otimes B$ has no meet-irreducible elements; hence it is not $\mathcal{D}$ projective.

5. **Connections with other tensor products.** A tensor product of lattices that is different from the one defined in our paper has been studied by D. Mowat [9], Z. Shmuely [10], and A. Waterman [12]. For complete lattices A and B, a complete join morphism is a mapping from A into B which preserves 0 and arbitrary joins. The tensor product $A \otimes B$ of complete lattices A and B is defined to be the set of all complete join morphisms from A into the dual of B, ordered by the pointwise partial order [10, p. 2]. This tensor product has the following properties: for complete lattices A, B and C and arbitrary posets M and N, we have (i) $A \otimes B \cong B \otimes A$; (ii) $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$; (iii) $A \otimes 2^M \cong A^M$; (iv) $2^M \otimes 2^N \cong 2^{M \times N}$.

The semilattice tensor product studied in the present paper does not satisfy properties (iii) and (iv). If $A = 2$ and M and N are the one element poset then both (iii) and (iv) assert that $2 \otimes 2 \cong 2$. But $2 \otimes 2$ consists of five elements, including four distinct generators. Property (i) holds for the semilattice tensor product. We have the following result for property (ii).

THEOREM 5.1. *Let A, B and C be finite distributive lattices. Then* $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.

PROOF. By Theorem 2.6, $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are distributive lattices. It follows from Theorem 3.6 that the join-irreducible elements of $(A \otimes B) \otimes C$ are of the form $(a \otimes b) \otimes c$ where a, b, c are join-irreducible in A, B, and C respectively. We have $(a_1 \otimes b_1) \otimes c_1 \leqslant (a_2 \otimes b_2) \otimes c_2$ if and only if $a_1 \leqslant a_2$, $b_1 \leqslant b_2$ and $c_1 \leqslant c_2$. Similarly the join-irreducible elements of $A \otimes (B \otimes C)$ are of the form $a \otimes (b \otimes c)$ where a, b, c are join-irreducible in A, B and C respectively, and $a_1 \otimes (b_1 \otimes c_1) \leqslant a_2 \otimes (b_2 \otimes c_2)$ if and only if $a_1 \leqslant a_2$, $b_1 \leqslant b_2$ and $c_1 \leqslant c_2$. Thus the poset of nonzero join-irreducible elements of $(A \otimes B) \otimes C$ is isomorphic with the poset of nonzero join-irreducible elements of $A \otimes (B \otimes C)$. Since $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are finite, it follows that $(A \otimes B) \otimes C$ is isomorphic with $A \otimes (B \otimes C)$.

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