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THE SEPARATION PRINCIPLE FOR IMPULSE CONTROL PROBLEMS

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ABSTRACT. In this paper, one shows that the combined problem of optimal impulse control and filtering, for a stochastic linear dynamic system observed via a noisy linear channel, can be reduced to two independent problems of impulse control and filtering, respectively.

1. Introduction. W. M. Wonham [8] showed that the combined problem of optimal control and filtering, for a stochastic linear dynamic system observed via a noisy linear channel, can be reduced to two independent problems of stochastic control and filtering, respectively. This result was improved by M. H. A. Davis [3] using the concept of Girsanov solutions of stochastic differential equations.

A. Bensoussan and J. L. Lions [1] proved that the same separation principle holds for stopping time problems.

In all cases, a nondegeneracy on the observation matrix is imposed. This assumption would rarely be met in practice.

In [5], we showed that the separation principle for stopping time problems holds even under degeneracy.

Let us also mention the work of J. Szpirglas and G. Mazziotto [7].

The object of this article is to prove that the combined problem of optimal impulse control and filtering, for a stochastic linear dynamic system observed via a noisy linear channel, can be reduced to two independent problems of impulse control and filtering, respectively. In general, the optimal impulse control depends parametrically on the intensity of channel noise; the result means, however, that channel noise plays qualitatively the same role as dynamic disturbances in determination of the feedback law.

2. Statement of the problem. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and T be a positive constant.

Given matrices $F(t)$, $G(t)$, $H(t)$, $0 \leq t \leq T$, such that

$$\begin{cases} F(\cdot), G(\cdot) \in C([0, T]; \mathbf{R}^N \times \mathbf{R}^N), \\ H(\cdot) \in C([0, T]; \mathbf{R}^N \times \mathbf{R}^P), \end{cases} \tag{2.1}$$

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we denote by $y^\circ(t)$ the solution of the linear Itô equation

$$\begin{cases} dy^\circ(t) = F(t)y^\circ(t)dt + G(t)dw(t), & 0 \leq t \leq T, \\ y^\circ(0) = x + \zeta, & x \in \mathbf{R}^N, \end{cases} \tag{2.2}$$

where $w(t)$ is a standard Wiener process in $\mathbf{R}^N$ and ζ is a Gaussian random variable with vanishing expectation and covariance matrix P_0; ζ is independent of the process $w(t)$, $0 \leq t \leq T$.

The current state of the system without control at the instant t is $y^\circ(t)$, but we cannot observe the system. The information is provided by the channel output $z^\circ(t)$ defined by

$$\begin{cases} dz^\circ(t) = H(t)y^\circ(t)dt + d\eta(t), & 0 \leq t \leq T, \\ z^\circ(0) = 0, \end{cases} \tag{2.3}$$

where $\eta(t)$ is a Wiener process in $\mathbf{R}^P$ independent of $w(t)$, with vanishing expectation and covariance matrix $R(t)$ such that

$$\begin{cases} R(\cdot) \in C([0, T]; \mathbf{R}^P \times \mathbf{R}^P), \\ R(t) \geq rI, \quad r > 0 \quad \forall t \in [0, T]. \end{cases} \tag{2.4}$$

We denote by $\mathfrak{Z}^t$, $0 \leq t \leq T$, the nondecreasing right continuous family of completed σ-algebras generating by the process $z^\circ(t)$.

An admissible impulse control ν is a set $\{\theta_1, \xi_1; \ldots; \theta_i, \xi_i; \ldots\}$ where $\{\theta_i\}_{i=1}^\infty$ is an increasing sequence of stopping times with respect to $\mathfrak{Z}^t$ convergent to T $(0 \leq \theta_i \leq \theta_{i+1} \leq T,\ [\theta_i \leq t] \in \mathfrak{Z}^t,\ \theta_i \to T)$ and $\{\xi_i\}_{i=1}^\infty$ is a sequence of random variables taking values in $\mathbf{R}_+^N$, adapted with respect to $\{\theta_i\}_{i=1}^\infty$ ($\xi_i\colon \Omega \to \mathbf{R}^N$, $\xi_i \geq 0$, $\mathfrak{Z}^{\theta_i}$-measurable).

Now we define the sequence of diffusion processes with jumps, $\{y^n(t)\}_{n=1}^\infty$, $y^n(t) = y^n(t, \nu)$, $t \in [0, T]$, ν any admissible impulse control, by the stochastic equation

$$\begin{cases} dy^n(t) = F(t)y^n(t)dt + G(t)dw(t), & \theta_n \leq t \leq T, \\ y^n(t) = y^{n-1}(t) + 1_{\theta_n = t}\xi_n, & 0 \leq t \leq \theta_n.^1 \end{cases} \tag{2.5}$$

We have

$$y^n(t) = y^i(t) \quad \text{on } [0, \theta_n[\,, \forall i \geq n. \tag{2.6}$$

Defining

$$y(t, \nu) = \lim_{n\to\infty} y^n(t), \qquad 0 \leq t \leq T, \tag{2.7}$$

the process $y(t) = y(t, \nu)$, which is right continuous with left limits existing, satisfies the following stochastic equation:

$$\begin{cases} dy(t) = F(t)y(t)dt + G(t)dw(t) + \sum_{i=1}^\infty \xi_i\delta(t - \theta_i)dt, & 0 \leq t \leq T, \\ y(0) = x + \zeta, \end{cases} \tag{2.8}$$

where $\delta(t)$ is the Dirac measure.

[1] $1_{\theta_n = t}$ denotes the characteristic function of the set $\{\theta_n = t\}$.

The current state of the system with impulse control ν at the instant t is represented by $y(t)$, and

$$\hat{y}(t) = E\{y(t)/\mathfrak{X}^t\} \tag{2.9}$$

is the information state process; we also have $\hat{y}(0) = x$.

We call the impulse process $\beta(t)$ the solution of the equation

$$\begin{cases} d\beta(t) = F(t)\beta(t)dt + \sum_{i=1}^{\infty} \xi_i \delta(t - \theta_i)dt, & 0 < t < T, \\ \beta(0) = 0. \end{cases} \tag{2.10}$$

Clearly, $\beta(t) = \beta(t, \nu)$ is built in the same way as $y(t)$ by iteration. Notice, the process $\beta(t)$ is right continuous with left limits and adapted to the observation $\mathfrak{X}^t$. Thus, according to the equation (2.2), (2.8), (2.10) we deduce from (2.9)

$$\hat{y}(t) = E\{y^\circ(t)/\mathfrak{X}^t\} + \beta(t). \tag{2.11}$$

We introduce the process $\varepsilon(t)$, called the estimation error, given by

$$\varepsilon(t) = y^\circ(t) - E\{y^\circ(t)/\mathfrak{X}^t\}, \qquad 0 < t < T, \tag{2.12}$$

which is independent of $\mathfrak{X}^t$ and verifies

$$\varepsilon(t) = y(t) - \hat{y}(t), \qquad 0 < t < T. \tag{2.13}$$

We also define $\hat{w}(t)$ by

$$\begin{cases} d\hat{w}(t) = R^{-1/2}(t)H(t)\varepsilon(t)dt + R^{-1/2}(t)d\eta(t), & 0 < t \leqslant T, \\ \hat{w}(0) = 0 \end{cases} \tag{2.14}$$

which is a standard Wiener process and satisfies the martingale property

$$\hat{w}(t) = E\{\hat{w}(s)/\mathfrak{X}^t\}, \qquad 0 < t < s < T. \tag{2.15}$$

Then, the assertions (2.10), (2.11) and the R. E. Kalman-R. S. Bucy [4] theory show that $\hat{y}(t)$ is the solution of the following stochastic equation

$$\begin{cases} d\hat{y}(t) = F(t)\hat{y}(t)dt + P(t)H^*(t)R^{-1/2}(t)d\hat{w}(t) + \sum_{i=1}^{\infty} \xi_i\delta(t - \theta_i)dt, & 0 < t < T, \\ \hat{y}(0) = x, \end{cases} \tag{2.16}$$

where the matrix $P(t)$ is the unique solution of the Riccati equation

$$\begin{cases} P'(t) = FP + PF^* - PH^*R^{-1}HP + GG^*, & 0 < t < T,^2 \\ P(0) = P_0. \end{cases} \tag{2.17}$$

We also deduce that the estimation error $\varepsilon(t)$ is the unique solution of the Itô equation

$$\begin{cases} d\varepsilon(t) = (F - PH^*R^{-1}H)\varepsilon dt - PH^*R^{-1}d\eta + Gdw, & 0 < t < T, \\ \varepsilon(0) = \zeta. \end{cases} \tag{2.18}$$

[2] The prime (′) means time derivative and the star (*) denotes the transpose.

3. Optimal impulse control. Let $f(x, t)$ be a nonnegative, continuous and bounded function on $\mathbf{R}^N \times [0, T]$ taking values in $\mathbf{R}$,

$$f \in C_b(\mathbf{R}^N \times [0, T]), \qquad f(x, t) \geq 0 \; \forall x \in \mathbf{R}^N, t \in [0, T], \tag{3.1}$$

and let $k(\xi)$ be a continuous function from $\mathbf{R}^N_+$ into $\mathbf{R}$ such that

$$k \in C(\mathbf{R}^N_+), \qquad k(\xi) \geq k_0 > 0, \qquad k(\xi) \to \infty \quad \text{if } |\xi| \to \infty. \tag{3.2}$$

Now, for any admissible impulse control $\nu = \{\theta_1, \xi_1; \ldots; \theta_i, \xi_i; \ldots\}$ and $x \in \mathbf{R}^N$ we set

$$J_x(\nu) = E\left\{ \int_0^T f(y(t), t) e^{-\alpha t} \, dt + \sum_{i=1}^{\infty} k(\xi_i) 1_{\theta_i < T} e^{-\alpha\theta_i} \right\}, \tag{3.3}$$

where α is a real constant.

We remark that any admissible impulse control ν is adapted to the information state $\hat{y}(t)$ and not to the current state $y(t)$.

Our purpose is to characterize the optimal cost

$$u_0(x) = \inf\{ J_x(\nu) / \nu \text{ admissible impulse control} \} \tag{3.4}$$

and to obtain a separation principle for an eventual optimal admissible impulse control.

Let M be the operator

$$[M\phi](x) = \inf\{ k(\xi) + \phi(x + \xi) / \xi \in \mathbf{R}^N_+ \} \tag{3.5}$$

and $u(x, t)$ be an arbitrary function satisfying

$$u \in C_b(\mathbf{R}^N \times [0, T]), \qquad u \leq Mu \quad \text{in } \mathbf{R}^N \times [0, T]. \tag{3.6}$$

The admissible impulse control $\nu = \nu_x$ associated to the function u is defined as follows. First we select a function $\xi(x, t)$ verifying

$$\begin{cases} \xi: \mathbf{R}^N \times [0, T] \to \mathbf{R}^N_+, \text{ Borel measurable and bounded such that} \\ [Mu](x, t) = k(\xi(x, t)) + u(x + \xi(x, t), t) \quad \forall x \in \mathbf{R}^N, t \in [0, T]. \end{cases} \tag{3.7}$$

Next, define $\tilde{\theta}^\circ = 0$ and $\hat{y}^\circ(t)$ by

$$\begin{cases} d\hat{y}^\circ(t) = F(t)\hat{y}^\circ(t) dt + P(t) H^*(t) R^{-1/2}(t) d\hat{w}(t), & 0 \leq t \leq T, \\ \hat{y}^\circ(0) = x. \end{cases} \tag{3.8}$$

We define $\nu = \{\theta_1, \xi_1; \ldots; \theta_i, \xi_i; \ldots\}$ by the formulas

$$\tilde{\theta}^{i+1} = \inf\{ t \in [\tilde{\theta}^i, T] / u(\hat{y}^i(t), t) = [Mu](\hat{y}^i(t), t) \}, \qquad i = 0, 1, \ldots, \tag{3.9}$$

$$\theta_i = \begin{cases} \tilde{\theta}^i & \text{if } \tilde{\theta}^i < T, i = 1, 2, \ldots, \\ T & \text{otherwise,} \end{cases} \tag{3.10}$$

$$\xi_i = \xi(\hat{y}^{i-1}(\theta_i), \theta_i), \qquad i = 1, 2, \ldots, \tag{3.11}$$

$$\begin{cases} d\hat{y}^i(t) = F(t)\hat{y}^i(t) dt + P(t) H^*(t) R^{-1/2}(t) d\hat{w}(t), \\ \qquad\qquad \theta_i \leq t \leq T, \qquad i = 1, 2, \ldots, \\ \hat{y}^i(t) = \hat{y}^{i-1}(t) + 1_{\theta_i = t}\xi_i, \qquad 0 \leq t \leq \theta_i. \end{cases} \tag{3.12}$$

Clearly, if there exists a function u verifying (3.6) whose associated admissible impulse control ν is optimal, the separation principle is established. Notice, the fact that ν is optimal shows automatically that $\theta_i \to T$. Moreover, $\theta_i = T$ for all $i > n(\omega)$ almost surely.

Let $A(t)$ be the second order differential operator corresponding to the Itô equation (3.8),

$$A(t) = -\sum_{i,j=1}^{N} a_{ij}(t)\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{N} (F(t)x)_i \frac{\partial}{\partial x_i} + \alpha, \tag{3.13}$$

where

$$\left[a_{ij}(t)\right]_{ij} = \tfrac{1}{2}P(t)H^*(t)R^{-1}(t)H(t)P(t). \tag{3.14}$$

We remark that $A(t)$ is usually degenerate. W. M. Wonham [8], M. H. A. Davis [3], A. Bensoussan and J. L. Lions [1] supposed that the matrices $P(t)$ and $H(t)$ are nonsingular.

We set

$$l(x, t) = E\{f(x + \varepsilon(t), t)\} \quad \forall x \in \mathbf{R}^N, t \in [0, T], \tag{3.15}$$

where $\varepsilon(t)$ is given by (2.18).

We introduce the following quasi-variational inequality. Find $u(x, t)$ such that

$$\begin{cases} u \in C_b(\mathbf{R}^N \times [0, T]), \qquad u(x, T) = 0 \ \forall x \in \mathbf{R}^N, \\ -\dfrac{\partial u}{\partial t} + A(t)u \leq l \quad \text{in } \mathfrak{D}'(\mathbf{R}^N \times [0, T]), u \leq Mu \text{ in } \mathbf{R}^N \times [0, T], \\ -\dfrac{\partial u}{\partial t} + A(t)u = l \quad \text{in } \mathfrak{D}'([u < Mu]).^3 \end{cases} \tag{3.16}$$

We have the

SEPARATION PRINCIPLE THEOREM. *Let the assumptions* (2.1), (2.4), (3.1), (3.2) *hold. Then there exists one and only one solution u of the quasi-variational inequality* (3.16). *Moreover the admissible impulse control ν defined by* (3.7)–(3.12), *associated to the function u given by* (3.16), *is optimal* [*i.e.*, $u_0(x) = J_x(\nu_x)$].

PROOF. First, using a general result in [6] applied to a degenerate operator $-\partial/\partial t + A(t)$, we deduce that there exists a solution of problem (3.16).

In order to prove the uniqueness, we denote by $z(s) = z_{xt}(s, \omega)$, $0 \leq t \leq s \leq T$, $x \in \mathbf{R}^N$, $\omega \in \Omega$, the diffusion associated to the operator $-\partial/\partial t + A(t)$, i.e.,

$$\begin{cases} dz(s) = F(s)z(s)ds + P(s)H^*(s)R^{-1/2}(s)d\hat{w}(s), \qquad t \leq s \leq T, \\ z(t) = x. \end{cases} \tag{3.17}$$

Now let $u(x, t)$ be an arbitrary solution of (3.16). We set $\theta = \theta_{xt}(\omega)$, $0 \leq t \leq T$, $x \in \mathbf{R}^N$, $\omega \in \Omega$, the first exit time of process $z(s)$ from $[u < Mu]$, i.e.,

$$\theta = \inf\{s \in [t, T] / u(z(s), s) = [Mu](z(s), s)\}. \tag{3.18}$$

[3] C_b denotes the space of continuous and bounded functions, and $\mathfrak{D}'$ is the space of distributions.

Then, using the fact that the coefficients of the second order terms of operator $A(t)$ are constant and that $u(x, t)$ is continuous, we establish by convolution techniques the following Itô formulas for each $x \in \mathbf{R}^N$, $t \in [0, T]$:

$$u(x, t) \leq E\left\{\int_t^{T\wedge\tau} l(z(s), s)e^{-\alpha s}ds + u(z(T\wedge\tau), T\wedge\tau)e^{-\alpha(T\wedge\tau)}\right\}$$

$$\forall\tau \geq t \text{ stopping time}, \tag{3.19}$$

$$u(x, t) = E\left\{\int_t^{\theta} l(z(s), s)e^{-\alpha s}ds + u(z(\theta), \theta)e^{-\alpha\theta}\right\}. \tag{3.20}$$

Therefore, as in [6], the properties (3.19), (3.20) imply the uniqueness of the solution u.

Next, from (3.7)–(3.12) and (3.20), we deduce

$$u(x, 0) = E\left\{\int_0^T l(\hat{y}(t), t)e^{-\alpha t}dt + \sum_{i=1}^{\infty} k(\xi_i)1_{\theta_i<T}e^{-\alpha\theta_i}\right\}, \tag{3.21}$$

and from (2.13), (3.15) we have

$$E\left\{\int_0^T l(\hat{y}(t), t)e^{-\alpha t}\, dt\right\} = E\left\{\int_0^T f(y(t), t)e^{-\alpha t}\, dt\right\}; \tag{3.22}$$

hence

$$u(x, 0) = J_x(\nu), \qquad \nu \text{ associated to } u. \tag{3.23}$$

Similarly, using (3.19), we obtain

$$u(x, 0) = \inf\{J_x(\nu)/\nu \text{ admissible impulse control}\}. \tag{3.24}$$

Then, (3.23) and (3.24) give

$$u(x, 0) = u_0(x), \quad \text{optimal cost (3.4)}, \tag{3.25}$$

and the theorem is proved. □

REMARK 1. If the function $f(x, t)$ is Lipschitz continuous, so is the function $u(x, t)$. In this case, u is also the maximum solution of a classical quasi-variational inequality introduced by A. Bensoussan and J. L. Lions [2]. □

REMARK 2. This result can be extended for a function $k(\xi, x, t)$ instead of $k(\xi)$ appearing in the definition of cost (3.3). Clearly, we can replace the condition $\xi \in \mathbf{R}_+^N$ by $\xi \in \Lambda$, where Λ is a closed subset of $\mathbf{R}^N$. □

REMARK 3. Using the technique presented in this paper, we can improve the result obtained in [5]. □

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