MATHEMATICS OF COMPUTATION
VOLUME 39, NUMBER 159
JULY 1982, PAGES 249-254

The Sequence of Radii of the Apollonian Packing

By David W. Boyd

Abstract. We consider the distribution function $N(x)$ of the curvatures of the disks in the Apollonian packing of a curvilinear triangle. That is, $N(x)$ counts the number of disks in the packing whose curvatures do not exceed x. We show that $\log N(x)/\log x$ approaches the limit S as x tends to infinity, where S is the exponent of the packing.

A numerical fit of a curve of the form $y = An^s$ to the values of $N^-(1000n)$ for $n = 1, 2, \ldots, 6400$ produces the estimate $S \approx 1.305636$ which is consistent with the known bounds $1.300197 < S < 1.314534$.

1. Introduction. Let T be a curvilinear triangle bounded by three mutually tangent circles. The Apollonian or osculatory packing of T is a sequence of disks $\{D_n\}$ all contained in T and such that, for each n, D_n has the largest radius of all disks contained in $T \setminus (D_1 \cup \cdots \cup D_{n-1})$.

The *exponent* of the packing was defined by Melzak [7] to be

$$S = \inf\left\{t\colon \sum r_n^t < \infty\right\} = \sup\left\{t\colon \sum r_n^t = \infty\right\}, \tag{1}$$

where r_n denotes the radius of D_n. In [1], [2] we developed an algorithm which, for any real $\kappa > 0$, produces bounds $\lambda(\kappa) < S < \mu(\kappa)$ which converge to S as $\kappa \to \infty$. This produced the numerical bounds $1.300197 < S < 1.314534$. In [3], the methods of [1] were used to show that S is the Hausdorff dimension of the residual set of the packing.

In [8], Melzak described a computer experiment in which the first 19660 disks of the packing were generated. A curve of the form $f(n) = An^s$ was fitted to the computed function $\mathrm{Num}(n) = \#\{D_k\colon r_k \geqslant (1000n)^{-1}\}$, giving $s = 1.306951$ as a heuristic estimate for S. This experiment has been repeated by the author on a number of occasions. In our most recent computation, we generated 41,694,859 disks obtaining the estimate $S \approx 1.305636$. This is described more fully in Section 7 of this paper.

As Wilker [9] pointed out, the success of such experiments suggests that if $N(x) = \#\{n\colon r_n^{-1} \leqslant x\}$, then it may be true that

$$\lim_{x\to\infty} \log N(x)/\log x = S, \tag{2}$$

or equivalently that

$$\lim_{n\to\infty} \log r_n/\log n = -1/S. \tag{3}$$

Received November 10, 1981.
1980 *Mathematics Subject Classification.* Primary 52A45; Secondary 52-04.

As is well known, this does not follow from (1). What does follow from (1), without using any geometrical facts, is that

$$\limsup_{x\to\infty} \log N(x)/\log x = S \tag{4}$$

and

$$\limsup_{n\to\infty} \log r_n/\log n = -1/S. \tag{5}$$

But, if r_n is simply required to be a decreasing sequence of positive numbers, one can prescribe the lim inf of these expressions in an arbitrary way consistent with (4) or (5). Wilker [9, p. 122] gives such examples as well as an investigation of the relationships between the exponent and many other measures of the rate of convergence of numerical series.

The purpose of this paper is to show how the methods of [1] can be used to prove (2) and (3). Thus, for any $\varepsilon > 0$, we will show that there are constants $A(\varepsilon)$ and $B(\varepsilon)$ so that

$$A(\varepsilon)x^{S-\varepsilon} < N(x) < B(\varepsilon)x^{S+\varepsilon}. \tag{6}$$

We do not know whether or not $N(x)x^{-S}$ converges as $x \to \infty$. The experimental results described in Section 7 suggest that this may be false and that perhaps a relationship such as $N(x) \sim Ax^S(\log(x/B))^T$ might be more appropriate. It is not known, either, whether or not $\sum r_n^S = \infty$, and it does not appear that this can be answered by the methods of [1].

It is worth mentioning that my motivation for examining this question was a problem posed by Coxeter [5], which asks one to find the radius of the smallest circle into which disks of radius $1/n$ ($n = 1, 2, \ldots$) can all be packed. A rather elegant proof that the answer is $3/2$ would be to show that the disks in a certain Apollonian packing have radii satisfying $r_n^{-1} \leqslant 3n$. (See [4] for a more elementary solution.) The methods of this paper can be used to provide effective estimates of this sort, but the numerical details are considerable.

2. A Result from the Theory of Numerical Series. Since we will be using (4) here, we give a proof. The result is also proved in [9]. The proof of (5) then follows by observing that $n \mapsto 1/r_n$ and $x \mapsto N(x)$ are essentially inverse functions.

To prove (4), one observes that, if $t > S$, then $\sum r_n^t = A(t) < \infty$. Thus

$$x^{-t}N(x) \leqslant \sum_{n \leqslant N(x)} r_n^t \leqslant A(t).$$

Taking logarithms and letting $x \to \infty$ and then $t \downarrow S$ shows that

$$\limsup \log N(x)/\log x \leqslant S.$$

On the other hand, if the inequality were strict, then partial summation would show that $\sum r_n^t$ converges for some $t < S$ contrary to the definition of S.

To prove (2) then, we need only prove that

$$\liminf_{x\to\infty} \log N(x)/\log x \geqslant S. \tag{7}$$

3. A Basic Inequality. We follow the notation of [1], [2], [3] to which we refer the reader for more details. As usual, if a disk D has radius r, then we call $k = 1/r$ the curvature of D. Let $T(a, b, c)$ be the curvilinear triangle bounded by three mutually externally tangent circles with curvatures a, b, c satisfying $0 \leq a \leq b \leq c$ and $b > 0$. The Apollonian packing of $T(a, b, c)$ will be denoted by $P(a, b, c)$. We can index these disks in a consistent way by a parameter α which is a vector of arbitrary length (including 0) with components 1, 2, or 3. We denote the curvature of the αth disk by $k(\alpha; a, b, c)$. The proof of Lemma 1 of [1] then gives

$$bk(\alpha; 0, 1, 1) \leq k(\alpha; a, b, c) \leq (a + c)k(\alpha; 0, 1, 1). \tag{8}$$

4. The Necklace Decomposition. As in [1], we decompose $T(a, b, c)$ into the disjoint union of an infinite number of disks and curvilinear triangles. This decomposition appears to have been first used in [6] in a proof that the Apollonian packing is complete.

Let A_0, B_0, C_0 denote the sides of $T(a, b, c)$ and let C_n be the disk tangent to A_0, B_0, C_{n-1} for $n = 1, 2, \ldots$. Similarly, define A_n and B_n, (so $A_1 = B_1 = C_1$). Let $c_n = g_n(a, b, c)$ be the curvature of C_n, so that $g_n(b, c, a)$ and $g_n(c, a, b)$ are the curvatures of A_n and B_n, respectively.

For notational convenience, write

$$\begin{aligned}\sum_{S_3} F(a, b, c) = F(a, b, c) + F(a, c, b) + F(b, a, c) + F(b, c, a)\\ + F(c, a, b) + F(c, b, a),\end{aligned} \tag{9}$$

and use a similar notation for unions. Then $T(a, b, c)$ may be decomposed as follows:

$$T(a, b, c) = C_1 \cup \bigcup_{n=2}^{\infty} (A_n \cup B_n \cup C_n) \cup \bigcup_{S_3} \bigcup_{n=1}^{\infty} T(a, c_n, c_{n+1}), \tag{10}$$

where, in the last term, it should be understood that c_n denotes the *function* $g_n(a, b, c)$.

In particular, then

$$P(a, b, c) \supset \bigcup_{S_3} \bigcup_{n=1}^{\infty} P(a, c_n, c_{n+1}). \tag{11}$$

Now, define

$$N(a, b, c; x) = \#\{k(\alpha; a, b, c) \leq x\}. \tag{12}$$

Then (11) gives the inequality

$$N(a, b, c; x) \geq \sum_{S_3} \sum_{n=1}^{\infty} N(a, c_n, c_{n+1}; x). \tag{13}$$

5. A Partial Iteration of (13). We shall use (13) together with (8) to extract information about $N(x) = N(0, 1, 1; x)$. As (8) shows, it suffices to consider the particular choice $(a, b, c) = (0, 1, 1)$ in order to prove (4) for all (a, b, c).

From (8), we deduce that

$$N(a, b, c; x) \geq N(0, 1, 1; x/(a + c)) = N(x/(a + c)), \tag{14}$$

since each α for which $k(\alpha; 0, 1, 1) \leq x/(a + c)$ has $k(\alpha; a, b, c) \leq x$.

Introducing a parameter $\kappa > 0$ as in [1], we deduce from (13) and (14) that

$$(15)\quad N(a,b,c;x) \geqslant \sum_{S_3}\Bigg(\sum_{c_n<\kappa} N(a,c_n,c_{n+1};x) + \sum_{c_n\geqslant\kappa} N(x/(a+c_{n+1}))\Bigg).$$

We will now iterate (15) until we reach a level at which $c_n \geqslant \kappa$ for all n so that no terms of the form $N(a, c_n, c_{n+1};\, x)$ will appear. The most straightforward way to describe this is to introduce operators $D(\kappa;\, a, b, c)$ defined recursively by

$$(16)\qquad D(\kappa;a,b,c) = \sum_{S_3}\sum_{c_n<\kappa} D(\kappa;a,c_n,c_{n+1}) + \sum_{S_3}\sum_{c_n\geqslant\kappa} E(a+c_{n+1}),$$

where $E(d)$ denotes the dilation operator defined by

$$(17)\qquad (E(d)N)(x) = N(x/d).$$

The argument of Lemma 4 of [1] shows that, after at most p steps, where $4^{p+1}b \geqslant \kappa$, the recursion (16) leads to an expression of $D(\kappa;\, a, b, c)$ as a sum of dilation operators.

Let us denote $D(\kappa;\, 0, 1, 1)$ by $D(\kappa)$. Then, from (15), (16), and (17), we deduce that, for any $\kappa > 0$,

$$(18)\qquad N(x) \geqslant (D(\kappa)N)(x).$$

Since all of the terms in the series for $D(\kappa)N$ are positive, there is no problem with convergence.

6. The Main Result. We now are in a position to prove the following

THEOREM. *Let $N(x)$ denote the number of disks in the Apollonian packing of $T(a, b, c)$ which have curvatures at most x. Let S be the exponent of the packing. Then*

$$(19)\qquad \lim_{x\to\infty} \log N(x)/\log x = S.$$

Proof. We need only prove (19) for $(a, b, c) = (0, 1, 1)$, as observed above. The operator $D(\kappa)$ of Section 5 may be explicitly expressed in terms of dilation operators as

$$(20)\qquad D(\kappa) = \sum_{m=1}^{\infty} E(d_m),$$

say, where d_m is a certain sequence of real numbers (actually integers) satisfying $d_m > 1$ for all m. We may assume that $\{d_m\}$ has been arranged in nondecreasing order.

Define

$$(21)\qquad g(\kappa;t) = \sum_{m=1}^{\infty} d_m^{-t}.$$

By the results of [1], the series converges for $t > 1/2$ and defines the function $g_m(\kappa;\, 0, 1, 1, t)$ (for $\delta_m^2 \geqslant \kappa$) referred to in Theorem 1 of [1]. Thus, there is a number $\lambda(\kappa)$ satisfying

$$(22)\qquad g(\kappa;\lambda(\kappa)) = 1$$

and furthermore

$$(23)\qquad \lim_{\kappa\to\infty} \lambda(\kappa) = S.$$

Now, let $t < \lambda(\kappa)$ so that $g(\kappa; t) > 1$, and let M be chosen so that

$$\sum_{m=1}^{M} d_m^{-t} > 1. \tag{24}$$

From (18) and (20), we have

$$N(x) > \sum_{m=1}^{M} N(x/d_m). \tag{25}$$

We claim that there is a constant $A = A(t)$ so that $N(x) \geqslant Ax^t$. To prove this, we first remark that there is certainly an x_0 for which $N(x_0) > 0$, since $N(x) \to \infty$. Let $x_1 = d_M x_0$, and define

$$A = \min_{x_0 \leqslant x \leqslant x_1} N(x)x^{-t}, \tag{26}$$

so certainly $N(x) \geqslant Ax^t$ for $x_0 \leqslant x \leqslant x_1$. Now define $x_n = d_1^{n-1}x_1$ for $n = 2, 3, \ldots$. Suppose that we have shown that $N(x) \geqslant Ax^t$ for $x_0 \leqslant x \leqslant x_{n-1}$, for some $n \geqslant 2$. Let $x_{n-1} \leqslant x \leqslant x_n$. Then, for $k = 1, \ldots, M$, we have $x/d_k \geqslant x_1/d_M = x_0$, while $x/d_k \leqslant x_n/d_1 = x_{n-1}$. Thus, applying (24) and (25), we have

$$N(x) \geqslant \sum_{m=1}^{M} A(x/d_m)^t > Ax^t \quad \text{if } x_{n-1} \leqslant x \leqslant x_n. \tag{27}$$

This completes an inductive proof that $N(x) \geqslant Ax^t$ for all $x \geqslant x_0$.

From this, we deduce that

$$\liminf_{x \to \infty} \log N(x)/\log x \geqslant t. \tag{28}$$

Since $t < \lambda(\kappa)$ is arbitrary and since $\lambda(\kappa) \to S$ as $\kappa \to \infty$, (7) follows from (28). Combined with (4), this proves the Theorem.

7. Experimental Estimates of S. The method used by Melzak [8] to estimate S was to compute $N(1000n)$ for $n = 1, \ldots, 20$ and to fit a curve of the form An^s to these values by least squares. The theorem of Section 6 lends some support to this technique but it does not rigorously justify it since the theorem is, after all, an asymptotic result.

In our adaptation of the method of [8], we choose initial curvatures a, b, c so that all of a, b, c, and $d = a + b + c + 2\sqrt{ab + bc + ca}$ are integers. (d is the curvature of the circle touching the sides of $T(a, b, c)$.) Then all of the curvatures in the packing are integers and can be generated by *linear* recurrence relations, see, e.g., [2]. Thus no square roots need be taken, and we remain in the realm of integer arithmetic.

We compute $y(n) = N^-(1000n)$ for $n = 1, \ldots, K$, where $N^-(x)$ denotes $\#\{n: r_n^{-1} < x\}$. Clearly $N^-(x)$ has the same asymptotic behavior as $N(x)$. For example, if $K = 6400$, the computation of these values took 6 minutes on an Amdahl 470 V/8 computer. One can then fit a curve An^s to $y(n)$ for $n = 1, \ldots, K$, by a variety of methods. The next table shows the values of A and s obtained by fitting $\log y(n)$ to $\log(An^s)$ for $n = 1, \ldots, K$, using linear least squares. The column "disks" gives $N^-(1000K)$, the total number of disks generated. In all cases $(a, b, c, d) = (0, 1, 4, 9)$.

TABLE

K	A	s	disks
200	446.063	1.306194	451,730
400	446.487	1.305935	1,116,800
800	446.678	1.305841	2,759,717
1600	446.769	1.305801	6,822,351
3200	446.898	1.305749	16,867,636
6400	446.992	1.305717	41,694,859

This data suggests that $S \approx 1.3057$. The fact that the fitted s decrease with K indicates that a curve of the form $Ax^s(\log(x/B))^t$ might be more appropriate, where $B = \sqrt{ab + bc + ca}$, say, is a factor included to preserve the scale invariance.

Another reasonable criterion would be to choose A and s to minimize the sum $\Sigma(y(n) - An^s)^2$. This leads to a pair of equations linear in A and thus to a nonlinear equation for s. Solving numerically gives the values $A = 447.285$ and $s = 1.305636$. The fit of An^s to $y(n)$ is better in a number of ways than that obtained by the method described above: the maximum difference $|y(n) - An^s|$ is smaller and the sign of $y(n) - An^s$ changes more often. For example, if $A_1 = 446.992$ and $s_1 = 1.3057169$, then $y(n) - A_1 n^{s_1}$ is negative for all n in the range $5652 \leqslant n \leqslant 6400$, while if $A_2 = 447.285$ and $s_2 = 1.305636$, then $y(n) - A_2 n^{s_2}$ changes sign over 100 times in this interval.

If one fits $\log y(n)$ to $\log(An^s)$ using only the values of n which satisfy $3200 < n \leqslant 6400$, then one obtains $A = 447.622$ and $s = 1.305548$, again suggesting that the values given in the above table overestimate S.

In summary, it appears that S is roughly 1.3056, with the last digit being somewhat questionable.

Acknowledgement. This work was supported in part by the NSERC and the Isaac Walton Killam Foundation.

Department of Mathematics
The University of British Columbia
Vancouver, B.C., Canada V6T 1Y4

1. D. W. Boyd, "The disk packing constant," *Aequationes Math.*, v. 7, 1972, pp. 182–193.

2. D. W. Boyd, "Improved bounds for the disk packing constant," *Aequationes Math.*, v. 9, 1973, pp. 99–106.

3. D. W. Boyd, "The residual set dimension of the Apollonian packing," *Mathematika*, v. 20, 1973, pp. 170–174.

4. D. W. Boyd, "Solution to problem P. 276," *Canad. Math. Bull.*, v. 23, 1980, pp. 251–253.

5. H. S. M. Coxeter, "Problem P. 276," *Canad. Math. Bull.*, v. 22, 1979, p. 248.

6. E. Kasner & F. Supnick, "The Apollonian packing of circles," *Proc. Nat. Acad. Sci. U.S.A.*, v. 29, 1943, pp. 378–384.

7. Z. A. Melzak, "Infinite packings of disks," *Canad. J. Math.*, v. 18, 1966, pp. 838–852.

8. Z. A. Melzak, "On the solid-packing constant for circles," *Math. Comp.*, v. 23, 1969, pp. 169–172.

9. J. B. Wilker, "Sizing up a solid packing," *Period. Math. Hungar.* (2), v. 8, 1977, pp. 117–134.