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Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario, Canada
L8S 4K1
E-mail: wangs@icarus.math.mcmaster.ca

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The set of automorphisms of $B(H)$ is topologically reflexive in $B(B(H))$

by

LAJOS MOLNÁR (Debrecen)

Abstract. The aim of this paper is to prove the statement announced in the title which can be reformulated in the following way. Let H be a separable infinite-dimensional Hilbert space and let $\Phi : B(H) \to B(H)$ be a continuous linear mapping with the property that for every $A \in B(H)$ there exists a sequence (Φ_n) of automorphisms of $B(H)$ (depending on A) such that $\Phi(A) = \lim_n \Phi_n(A)$. Then Φ is an automorphism. Moreover, a similar statement holds for the set of all surjective isometries of $B(H)$.

Introduction. If X is a Banach space, then we denote by $L(X)$ and $B(X)$ the algebras of all linear and bounded linear operators on X, respectively. $F(X)$ and $C(X)$ stand for the ideals of $B(X)$ consisting of all finite-rank and compact operators, respectively. A subset $\mathcal{E} \subset B(X)$ is called *topologically* [*algebraically*] *reflexive* if $T \in B(X)$ belongs to $\mathcal{E}$ whenever $Tx \in \overline{\mathcal{E}x}$ [$Tx \in \mathcal{E}x$] for all $x \in X$. This concept has proved very useful in the analysis of operator algebras.

The study of algebraic reflexivity of the subspace of derivations on operator algebras has been begun by Kadison [Kad2] and Larson and Sourour [LS] from a different point of view. Since then the problem of algebraic reflexivity of the sets of derivations and automorphisms has been investigated in full detail and the preliminary results have been improved significantly [Bre, BS1, BS2].

The notion of topological reflexivity is due to Loginov and Shul'man [LoS], although they defined it only for the case of subspaces. Nevertheless, surprisingly enough, from the two fundamental concepts of derivations and automorphisms, the problem of topological reflexivity has so far been

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treated only for the former one. Namely, Shul'man proved that the subspace of all derivations of a C^*-algebra is topologically reflexive [Shu, Corollary 2]. Our aim is to consider the same problem for automorphisms. Plainly, the topological reflexivity does not hold true for the automorphism group of any C^*-algebra. One can get an easy counterexample by considering the function algebra $C[a,b]$. In fact, choosing a sequence (φ_n) of homeomorphisms of $[a,b]$ onto itself which converges uniformly to a not injective function $\varphi : [a,b] \to [a,b]$, we obtain a continuous endomorphism Φ of $C[a,b]$ defined by $\Phi(f) = f \circ \varphi$ $(f \in C[a,b])$ which is the pointwise limit of a sequence of automorphisms but is not itself an automorphism. Consequently, a Shul'man-type general theorem cannot be expected for the set of automorphisms. However, we present a positive result for the case of the operator algebra $B(H)$ on the Hilbert space H equipped with the usual operator norm.

The starting point of our investigations is the result obtained by Brešar and Šemrl in [BS2]. They proved that, in our language, the group of automorphisms of $B(H)$ is algebraically reflexive in $L(B(H))$ provided that H is separable and infinite-dimensional. In Theorem 2 below we show that this group is topologically reflexive in $B(B(H))$, a result which can be considered stronger in some sense. Finally, we conclude the paper with a similar result for the set of all surjective isometries of $B(H)$.

Results. In what follows we need the concept of Jordan homomorphism. A linear mapping ϕ from an algebra A into another algebra B is called a *Jordan homomorphism* if $\phi(x^2) = \phi(x)^2$ $(x \in A)$. These mappings are extensively studied in ring theory and have important connection to the mathematical foundations of quantum theory. It is easy to see that every Jordan homomorphism $\phi : A \to B$ satisfies

$$\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x), \qquad \phi(xyx) = \phi(x)\phi(y)\phi(x), \tag{1}$$

for every $x, y \in A$ (see [Pal, 6.3.2 Lemma]).

Our key theorem that follows can be considered as an automatic surjectivity result for Jordan endomorphisms of $B(H)$ and hopefully has independent interest. It states that the inclusion of merely two extremal operators (one being rank-one and the other having dense range) in the range of a Jordan endomorphism Φ of $B(H)$ assures that Φ is automatically bijective. This result provides a unifying frame in which our reflexivity theorems will be easy to obtain.

THEOREM 1. *Let H be a separable infinite-dimensional Hilbert space and $\Phi : B(H) \to B(H)$ a linear mapping. If Φ is a Jordan homomorphism whose range contains a rank-one operator and an operator with dense range, then Φ is either an automorphism or an antiautomorphism.*

For the proof we need the following two lemmas.

LEMMA 1. *Let H be as above. Then any Jordan homomorphism $\Phi : B(H) \to B(H)$ is continuous.*

Proof. We first show that there exists a projection (i.e. a selfadjoint idempotent) $P \in B(H)$ with infinite rank and corank for which the mapping $A \mapsto \Phi(PAP)$ is continuous. Suppose that $\Phi \neq 0$. Since the kernel of Φ is a Jordan ideal, by [FMS, Theorem 3] it is also an associative ideal. Let P be an infinite-dimensional projection. If $\Phi(P) = 0$, then using the ideal property of $\ker \Phi$, we easily see that $I \in \ker \Phi$, yielding $\ker \Phi = B(H)$, which contradicts $\Phi \neq 0$. Thus $\Phi(P) \neq 0$.

Now, let (P_n) be a sequence of pairwise orthogonal infinite-dimensional projections and assume, on the contrary, that for every $n \in \mathbb{N}$ there is an operator $A_n \in B(H)$ such that $\|A_n\| = 1$ and

$$\|\Phi(P_n A_n P_n)\| \geq n2^n \|\Phi(P_n)\|^2.$$

Define $A = \sum_n \frac{1}{2^n} P_n A_n P_n \in B(H)$. Then

$$\|\Phi(P_n)\|^2 \|\Phi(A)\| \geq \|\Phi(P_n A P_n)\| = \frac{1}{2^n} \|\Phi(P_n A_n P_n)\| \geq n \|\Phi(P_n)\|^2.$$

Since $\|\Phi(P_n)\| \neq 0$ and the inequality above holds for every $n \in \mathbb{N}$, we arrive at a contradiction.

So, let $P \in B(H)$ be a projection with infinite rank and corank for which $A \mapsto \Phi(PAP)$ is continuous. Write $P = \sum_{n=1}^{\infty} e_n \otimes e_n$, where (e_n) is an orthonormal sequence. Let (f_n) be a complete orthonormal sequence which extends (e_n). Consider the operators

$$T = \sum_n f_n \otimes e_n, \qquad S = \sum_n e_n \otimes f_n.$$

It follows that $TPS = I$ and $SPT = Q$ is a projection with infinite-dimensional range. Since

$$\Phi(AQ) = \Phi(TPSASPT) = \Phi(T)\Phi(P(SAS)P)\Phi(T)$$

and the mapping $A \mapsto \Phi(P(SAS)P)$ is obviously continuous, so is $A \mapsto \Phi(AQ)$, and similarly for $A \mapsto \Phi(QA)$. Therefore, if $Q' = I - Q$, then we have the continuity of the linear mappings

$$A \mapsto \Phi((QA)Q) = \Phi(QAQ),$$
$$A \mapsto \Phi((Q'A)Q) = \Phi(Q'AQ),$$
$$A \mapsto \Phi(Q(AQ')) = \Phi(QAQ').$$

Let $Q=\sum_{n=1}^{\infty} e'_n\otimes e'_n$ with some orthonormal sequence (e'_n). Extend (e'_n) by (f'_n) to a complete orthonormal sequence and define

$$R=\sum_n f'_n\otimes e'_n+\sum_n e'_n\otimes f'_n.$$

Plainly, $RQR=Q'$ and hence the mapping

$$A\mapsto \Phi(Q'AQ')=\Phi(RQRARQR)=\Phi(R)\Phi(Q(RAR)Q)\Phi(R)$$

is continuous. Finally, since

$$\Phi(A)=\Phi(QAQ)+\Phi(QAQ')+\Phi(Q'AQ)+\Phi(Q'AQ')\quad (A\in B(H)),$$

we obtain the continuity of Φ. ■

LEMMA 2. *Let H be as above. If $\Phi: B(H)\to B(H)$ is a Jordan homomorphism, then there exists an idempotent $E\in B(H)$ such that for any maximal family (P_n) of pairwise orthogonal rank-one projections, the sequence $(\sum_{k=1}^{n}\Phi(P_k))$ converges strongly to E. Moreover, E commutes with the range of Φ.*

Proof. Some steps of the argument below have been motivated by ideas from the proof of [PS, 2.2 Lemma].

Call two idempotents $P,Q\in B(H)$ (mutually) *orthogonal* if $PQ=QP=0$. Observe that Φ maps idempotents to idempotents and preserves orthogonality between them. Indeed, if P,Q are mutually orthogonal idempotents, then by (1) we have $0=\Phi(PQ+QP)=\Phi(P)\Phi(Q)+\Phi(Q)\Phi(P)$, which readily implies the orthogonality of the idempotents $\Phi(P)$ and $\Phi(Q)$.

Now, let (P_n) be a maximal family of pairwise orthogonal rank-one projections in $B(H)$. Let $S_n=\sum_{k=1}^{n}\Phi(P_k)$ and define $E(P)$ as the idempotent having range $R=\overline{\operatorname{span}}\{\operatorname{rng} S_n : n\in\mathbb{N}\}$ and kernel $K=\bigcap_n \ker S_n$. To verify that $E(P)$ is well defined, we have to prove that $R\oplus K=H$.

We first show that $R\cap K=\{0\}$. Let (x_n) be a sequence in $\operatorname{span}\{\operatorname{rng} S_n : n\in\mathbb{N}\}$ which converges to an $r\in K$. From Lemma 1 we learn that Φ is bounded. Let M denote the norm of Φ. For every $\varepsilon>0$ there exists an $n_0\in\mathbb{N}$ such that $\|r-x_n\|<\varepsilon/M$ $(n\ge n_0)$. Since x_n is in the range of an idempotent S_k and r is in its kernel, it follows that $\|0-x_n\|<\varepsilon$ for every $n\ge n_0$. Thus $x_n\to 0$ and we have $r=0$.

We next prove that $R+K=H$. Let $h\in H$. For every $n\in\mathbb{N}$ we have $h=S_nh+(I-S_n)h$. Since (S_nh) is a bounded sequence in a Hilbert space, it has a weakly convergent subsequence, which we still denote by (S_nh). Since closed subspaces of H are weakly closed, the first term in the sum $h=w\text{-}\lim S_nh+w\text{-}\lim(I-S_n)h$ belongs to R. Moreover, $S_nS_m=S_mS_n=S_n$ for every $n\le m$. Since bounded operators on H are weakly continuous, we conclude that the second term above is in K.

So, $E(P)\in B(H)$ is the idempotent corresponding to the decomposition $R\oplus K=H$. It is easy to see that $S_nh\to E(P)h$ whenever $h\in\operatorname{span}\{\operatorname{rng} S_n : n\in\mathbb{N}\}$ or $h\in K$. By the Banach-Steinhaus theorem, (S_n) converges strongly to $E(P)$.

Let us now show that $E(P)$ is independent of the choice of (P_n). Let (Q_n) have the same properties as (P_n) has and denote by T_n the nth partial sum of the series $\sum_n\Phi(Q_n)$. Clearly, $\sum_{k=1}^{n}P_kQ_i+Q_i\sum_{k=1}^{n}P_k\to 2Q_i$ in the operator norm. By the continuity of Φ, $S_nT_k+T_kS_n\overset{n\to\infty}{\longrightarrow}2T_k$ also in the operator norm. Since $S_n\to E(P)$ strongly, it follows that $E(P)T_k+T_kE(P)=2T_k$ $(k\in\mathbb{N})$. We then conclude that $E(P)E(Q)+E(Q)E(P)=2E(Q)$. Similarly, $E(Q)E(P)+E(P)E(Q)=2E(P)$. Hence $E(P)=E(Q)$ as claimed. Set $E=E(P)$.

We prove that $E\Phi(A)=\Phi(A)E$ for every $A\in B(H)$. Let Q be a projection of arbitrary rank. Choose a maximal family (P_n) of pairwise orthogonal rank-one projections in such a way that for every $n\in\mathbb{N}$ either $P_nQ=QP_n=0$ or $P_nQ=QP_n=P_n$. In the first case $P_nQ+QP_n=0$ and hence $\Phi(P_n)\Phi(Q)+\Phi(Q)\Phi(P_n)=0$, while in the second case $\Phi(P_n)\Phi(Q)+\Phi(Q)\Phi(P_n)=2\Phi(P_n)$. Since $\Phi(P_n)$ and $\Phi(Q)$ are idempotents, an easy argument proves that $\Phi(P_n)\Phi(Q)=\Phi(Q)\Phi(P_n)$ $(n\in\mathbb{N})$ in both cases. We now have $E\Phi(Q)=\Phi(Q)E$. Since Q was an arbitrary projection, using the spectral theorem for selfadjoint operators and the continuity of Φ, we get the last assertion. ■

Now, we are in a position to prove our first theorem.

Proof of Theorem 1. By Lemma 1 and [FMS, Theorem 3], $\ker\Phi$ is a closed ideal. We intend to prove that $\ker\Phi=\{0\}$, that is, Φ is injective. A classical theorem of Calkin states that every nontrivial ideal of $B(H)$ is included in $C(H)$ and contains $F(H)$. Hence, supposing $\ker\Phi\ne\{0\}$, we have $\ker\Phi=C(H)$. We have already learnt that Φ maps idempotents to idempotents and preserves their orthogonality. It is well known that there exists an uncountable family I of infinite subsets of $\mathbb{N}$ with the property that any two different members of I have finite intersection. Therefore, we have an uncountable family of projections in $B(H)$ such that the product of any two of them is a finite-rank projection. Taking images under Φ, we see that $B(H)$ contains an uncountable family (P_α) of pairwise orthogonal nonzero idempotents. But this is a contradiction. To see this, for every α, pick a vector x_α from the range of P_α for which $\|x_\alpha\|\ge\|P_\alpha\|$. We then have

$$\|P_\alpha\|\cdot\|x_\alpha-x_\beta\|\ge\|P_\alpha x_\alpha-P_\alpha x_\beta\|=\|x_\alpha\|\ge\|P_\alpha\|$$

and thus $\|x_\alpha-x_\beta\|\ge 1$ whenever $\alpha\ne\beta$. But this is impossible in a separable metric space, and we get the injectivity of Φ.

We next assert that there is a rank-one idempotent whose image under Φ is also rank-one. Let $T = x \otimes y$ be a rank-one operator in the range of Φ. Suppose that $\langle x, y\rangle \neq 0$. In this case, multiplying T by an appropriate constant, we get a rank-one idempotent P in $\operatorname{rng}\Phi$ and one can easily verify that the idempotent $\Phi^{-1}(P)$ is also rank-one. Now, suppose that $\langle x, y\rangle = 0$. Then $T^2 = 0$. Let A be such that $\Phi(A) = T$. Consider a rank-two projection P with $PAP \neq 0$. We infer that $0 \neq \Phi(PAP) = \Phi(P)T\Phi(P)$. Since the operator $\Phi(P)T\Phi(P)$ is also rank-one, we can assume that it is square-zero. Consequently, we have a rank-one, square-zero operator S and an operator B with rank not greater than 2 for which $\Phi(B) = S$. Suppose that the square-zero operator B has rank two. Then there are independent vectors $\{x, x'\}$ and $\{y, y'\}$ such that

$$B = x \otimes y + x' \otimes y'.$$

Using the property that $B^2 = 0$, it is elementary to show that $\{x, x'\} \perp \{y, y'\}$. Let $\lambda, \mu \in \mathbb{C}$ be such that $x' - \lambda x = x_0 \perp x$ and $y' - \mu y = y_0 \perp y$. Let z, z' be orthogonal unit vectors such that $\{z, z'\} \perp \{x, x', y, y'\}$. Consider the operator

$$C = \frac{1}{\|x\|^2} z \otimes x + \frac{1}{\|x_0\|^2}(z' - \lambda z) \otimes x_0 + \frac{1}{\|y\|^2} y \otimes z + \frac{1}{\|y_0\|^2} y_0 \otimes (z' - \mu z).$$

One can easily check that $CBC = z \otimes z + z' \otimes z'$ and hence $\Phi(C)S\Phi(C) = \Phi(CBC) \neq 0$ is a rank-one operator which is the sum of two orthogonal nonzero idempotents. But this is a contradiction. Therefore, B is of the form $B = x \otimes y$ and defining

$$C = \frac{1}{\|x\|^2} z \otimes x + \frac{1}{\|y\|^2} y \otimes z$$

with a unit vector $z \perp \{x, y\}$, it follows just as above that $\Phi(z \otimes z) = \Phi(CBC)$ is a rank-one idempotent.

We now claim that $\Phi_{|F(H)}$ is either a homomorphism or an antihomomorphism. In fact, this follows from a classical theorem of Jacobson and Rickart [JR, Theorem 8] stating that every (additive) Jordan homomorphism on a locally matrix ring is the sum of an (additive) homomorphism and an (additive) antihomomorphism. Using this theorem we have additive functions $\Psi_1, \Psi_2 : F(H) \to B(H)$ such that $\Phi_{|F(H)} = \Psi_1 + \Psi_2$, Ψ_1 is a homomorphism and Ψ_2 is an antihomomorphism. Let $P \in F(H)$ be an idempotent for which $\Phi(P)$ is rank-one. Since $\Phi(P) = \Psi_1(P) + \Psi_2(P)$ and the terms of this sum are also idempotents, we see that either $\Psi_1(P) = 0$ or $\Psi_2(P) = 0$. By the simplicity of the ring $F(H)$ this shows that either $\Psi_1 = 0$ or $\Psi_2 = 0$. Plainly, this is equivalent to what has been stated above. In what follows, with no loss of generality, we can assume that $\Phi_{|F(H)}$ is a homomorphism.

Now, let $y, z \in H$ be such that $\Phi(y \otimes y)z \neq 0$. Define a linear operator T on H by

$$Tx = \Phi(x \otimes y)z \quad (x \in H).$$

Then T is bounded and, by the multiplicativity of Φ on $F(H)$, it is very easy to see that $TA = \Phi(A)T$ $(A \in F(H))$. If $Tx = 0$, then $TAx = \Phi(A)Tx = 0$ for every $A \in F(H)$. Obviously, this implies $x = 0$ and hence T is injective.

We claim that T is surjective as well. To show this, we first prove that the idempotent E given in Lemma 2 is the identity on H. In fact, since E commutes with the range of Φ, the mapping

$$\Psi : A \mapsto \Phi(A)(I - E)$$

is a Jordan homomorphism. Moreover, as can be easily verified, Ψ vanishes on every finite-rank projection and hence on the whole $F(H)$. Examining the kernel of Ψ and applying the argument used in the first part of the proof in connection with the cardinality of any set of pairwise orthogonal nonzero idempotents in $B(H)$, we have $(I - E)\Phi(A) = \Phi(A)(I - E) = 0$ for every $A \in B(H)$. Taking A with $\Phi(A)$ having dense range, we infer that $E = I$.

We are now in a position to show that the range of T is dense. Let (P_n) be a maximal family of pairwise orthogonal rank-one projections. We know that Φ maps rank-one operators into rank-one operators. Indeed, this follows from the fact that there exists a rank-one idempotent whose image under Φ is also rank-one and from the assumption that $\Phi_{|F(H)}$ is a homomorphism. Since this implies that every $\Phi(P_n)$ has rank one and the series $\sum_n \Phi(P_n)$ converges strongly to I, we have vector sequences (e_n) and (f_n) in H for which $f_n \otimes e_n = \Phi(P_n)$ and

$$\sum_n \langle x, e_n\rangle\langle f_n, y\rangle = \langle x, y\rangle \quad (x, y \in H).$$

This immediately implies that the subspace generated by $\{f_n : n \in \mathbb{N}\}$ is dense. But every f_n is in $\operatorname{rng} T$. Indeed, since $TP_n = f_n \otimes e_n T = f_n \otimes T^* e_n$ and, by the injectivity of T, we have $TP_n \neq 0$, it easily follows that $f_n \in \operatorname{rng} T$.

We next prove that T is in fact surjective. Let $y \in H$ and (x_n) a sequence in H such that $Tx_n \to y$. Since $TA = \Phi(A)T$ $(A \in F(H))$, we find that (TAx_n) is convergent for every finite-rank operator A. Thus, for every $u \in H$ and for a fixed nonzero $v \in H$ the sequences $(\langle x_n, u\rangle Tv) = ((Tv \otimes u)(x_n))$ and, consequently, $(\langle x_n, u\rangle)$ are convergent. Plainly, this shows that (x_n) converges weakly to some $x \in H$. By the weak continuity of T, we have $Tx = y$ and this proves the surjectivity of T.

It is now apparent that $\Phi(A) = TAT^{-1}$ $(A \in F(H))$. In particular, the range of Φ contains every finite-rank operator. Following the proof of a well-known theorem of Herstein [Her] given in [Pal, Lemma 6.3.2, Lemma 6.3.6

and Theorem 6.3.7] stating that every (additive) Jordan homomorphism of an algebra onto a prime algebra is either a homomorphism or an antihomomorphism, one can verify that $F(H) \subset \operatorname{rng} \Phi$ implies that Φ is either a homomorphism or an antihomomorphism. If Φ is an antihomomorphism, then for every $A, B \in F(H)$ we have $TAT^{-1}TBT^{-1} = T(BA)T^{-1}$, i.e. $AB = BA$. Therefore, Φ is a homomorphism and one can check that $TA = \Phi(A)T$ and hence $\Phi(A) = TAT^{-1}$ for every $A \in B(H)$. This completes the proof of the theorem. ∎

Remark. One could be interested in the question of whether only one operator in the range of a Jordan homomorphism can be enough to imply a similar automatic surjectivity result. The answer is easily seen to be affirmative. Indeed, consider a rank-one operator $x \otimes y$ with $\langle x, y\rangle = 2$. If $A = I - x \otimes y$ is in the range of a Jordan homomorphism of $B(H)$, then its square $A^2 = I$ also belongs to it and hence the same is true for $x \otimes y$. Our theorem above now applies.

THEOREM 2. *Let H be a separable infinite-dimensional Hilbert space. Then the set of all automorphisms of $B(H)$ is topologically reflexive in $B(B(H))$.*

Proof. We use a quite standard argument. Let $\Phi : B(H) \to B(H)$ be a bounded linear operator with the property that for every $A \in B(H)$ there is a sequence (Φ_n) of automorphisms (depending on A) such that $\Phi(A) = \lim_n \Phi_n(A)$. Clearly, Φ maps idempotents to idempotents.

We assert that Φ is a Jordan homomorphism. If P, Q are orthogonal idempotents, then $P+Q$ and hence $\Phi(P+Q)$ are also idempotents. From $\Phi(P+Q)^2 = \Phi(P+Q)$ we infer that $\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P) = 0$. This shows that Φ preserves orthogonality between idempotents. Let $P_1, \ldots, P_n$ be pairwise orthogonal projections and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. We compute

$$\Phi\Big(\sum_{k=1}^{n} \lambda_k P_k\Big)^2 = \Big(\sum_{k=1}^{n} \lambda_k \Phi(P_k)\Big)^2 = \sum_{k=1}^{n} \lambda_k^2 \Phi(P_k) = \Phi\Big(\Big(\sum_{k=1}^{n} \lambda_k P_k\Big)^2\Big).$$

Using the continuity of Φ and the spectral theorem for selfadjoint operators, we obtain $\Phi(A^2) = \Phi(A)^2$ for every selfadjoint $A \in B(H)$. Linearizing this equality, i.e. replacing A by $A+B$ (B is also selfadjoint), we have $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$. Then it follows that

$$\begin{aligned}\Phi((A+iB)^2) &= \Phi(A^2) - \Phi(B^2) + i\Phi(AB+BA)\\ &= \Phi(A)^2 - \Phi(B)^2 + i(\Phi(A)\Phi(B) + \Phi(B)\Phi(A))\\ &= (\Phi(A) + i\Phi(B))^2,\end{aligned}$$

which implies that Φ is a Jordan homomorphism. Since the limit of a convergent sequence of rank-one idempotents is a rank-one idempotent and obviously $\Phi(I) = I$, the conditions of Theorem 1 are fulfilled. Consequently, Φ is either an automorphism or an antiautomorphism. If Φ is an antiautomorphism, then for a unilateral shift $U \in B(H)$ we infer $I = \Phi(I) = \Phi(U)\Phi(U^*)$, which means that $\Phi(U)$ has a right inverse. But $\Phi(U)$ is the limit of a sequence of operators all similar to U. Therefore, neither the elements of this sequence nor its limit $\Phi(U)$ have right inverses. This contradiction shows that Φ is an automorphism and the proof is complete. ∎

Remark. It seems natural to ask what happens in the finite-dimensional case. In this case any nonzero Jordan homomorphism is injective and thus surjective as well. Thus, Theorem 1 remains valid. This is not true for Theorem 2. In fact, its proof shows that if we have a linear mapping on a finite-dimensional $B(H)$ which can be approximated at every operator by a sequence of automorphisms, then this mapping is either an automorphism or an antiautomorphism. However, nothing more can be stated as shown by the example of the mapping $A \mapsto A^T$ (transpose of A with respect to a fixed complete orthonormal system). This antiautomorphism has the above mentioned property of approximation since in the finite-dimensional case A and A^T are similar for every $A \in B(H)$.

OPEN PROBLEM. In connection with Theorem 2 and the result of Brešar and Šemrl [BS2] mentioned in the introduction we conjecture that the topological reflexivity of the automorphism group holds true also in $L(B(H))$, i.e. the approximated mappings should not have been assumed to be continuous.

In our last theorem we apply our key result to get the topological reflexivity of another very important set of transformations on $B(H)$.

THEOREM 3. *Let H be a separable infinite-dimensional Hilbert space. Then the set of all surjective linear isometries of $B(H)$ is topologically reflexive in $B(B(H))$.*

Proof. It is a folklore result that, by Kadison's fundamental theorem on the structure of surjective linear isometries of a unital C^*-algebra [Kad1, Theorem 7], every surjective isometry of $B(H)$ is of the form either $A \mapsto UAV$ or $A \mapsto UA^TV$ with some unitaries U, V.

Let $\Phi : B(H) \to B(H)$ be a linear mapping with the property that for every $A \in B(H)$ there exists a sequence (Φ_n) of surjective isometries such that $\Phi(A) = \lim_n \Phi_n(A)$. Plainly, Φ is an isometry. Our aim is to show that Φ is surjective. Since surjective isometries of $B(H)$ map unitaries to unitaries, so does Φ. Of course, we may suppose that $\Phi(I) = I$. Then the well-known Russo–Dye theorem [RD, Corollary 2] on the structure of the unitary group preserving mappings assures that Φ is a Jordan *-homomorphism. Since Φ

is easily seen to preserve the rank-one operators, Theorem 1 applies again to yield the surjectivity of Φ. ∎

Remark. We give a further example in order to emphasize how the topological reflexivity of the sets of all automorphisms as well as surjective isometries of $B(H)$ should be considered exceptional even among the cases represented by some "nice" operator algebras. In fact, we feel that, to some extent, this property characterizes $B(H)$ among its subalgebras.

Consider the C^*-algebra $C(H)$. Let (e_n) be a fixed complete orthonormal sequence in H. Choose unitary operators U_n such that

$$U_n e_k = e_{k+1} \quad (n \in \mathbb{N},\ 1 \le k \le n).$$

If U denotes the unilateral shift corresponding to the sequence (e_n), then it is obvious that $U_n e_k \stackrel{n\to\infty}{\longrightarrow} U e_k$ $(k \in \mathbb{N})$. The Banach–Steinhaus theorem shows that (U_n) converges strongly to U. Let

$$\Phi(A) = UAU^*, \quad \Phi_n(A) = U_n A U_n^* \quad (n \in \mathbb{N}).$$

Clearly, Φ_n is a *-automorphism and hence a surjective isometry of $C(H)$ $(n \in \mathbb{N})$. Moreover, $\Phi_n(A) \to \Phi(A)$ for every rank-one operator A and consequently, by the Banach–Steinhaus theorem again, for every $A \in C(H)$. However, Φ is not surjective.

OPEN PROBLEM. Does there exist a proper C^*-subalgebra of $B(H)$ which contains every finite-rank operator and has the property that the set of all its automorphisms or surjective isometries is topologically reflexive?

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Institute of Mathematics
Lajos Kossuth University
P.O. Box 12
4010 Debrecen, Hungary
E-mail: molnarl@math.klte.hu

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