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THE SETTLING OF AN ARBITRARY NUMBER OF SPHERICAL PARTICLES ARRANGED ON THE CORNERS OF A REGULAR POLYGON IN A VISCOUS FLUID

by
Eric Robert Bixon

Dissertation submitted to the Faculty of the Graduate School
of the New Jersey Institute of Technology in partial fulfillment
of the requirements for the degree of
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APPROVAL SHEET

Title of Thesis: The Settling of an Arbitrary Number of Spherical Particles Arranged on the Corners of a Regular Polygon in a Viscous Fluid

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ABSTRACT

Title of Thesis: The Settling of an Arbitrary Number of Spherical Particles Arranged on the Corners of a Regular Polygon in a Viscous Fluid

Eric R. Bixon, Doctor of Engineering Science, 1983

Thesis directed by: Ernest N. Bart, Assistant Professor

The creeping motion equation has been solved for the case of planar arrays of spheres settling under the influence of gravity in a viscous fluid. The solution is a general solution which applies to an arbitrary number of spheres. All particles will lie at the corners of a regular polygon. Thus, two particles side by side, three particles in an equilateral triangular array, or four spheres in a square array will be special cases of the general solution.

The solution has been obtained by a unique application of the method of reflections. Only a first correction to the drag has been obtained which puts an additional constraint on the solution since the higher order terms have been neglected. As a result, the solution is most accurate when the spheres are far apart.

In order to verify the general solution for the case of two spheres, the result has been compared with the literature value which exists for the case of two spheres falling perpendicular to their line of centers. The solution obtained in this work for two spheres is in exact agreement with the literature solution for the two sphere case. The results of the general solution indicate that as the number of spheres in the array is increased, the terminal settling velocity increases rapidly.

ACKNOWLEDGMENTS

Grateful acknowledgment is hereby made to the members of the author's doctoral advisory board for their many valuable suggestions. Above all, the author wishes to acknowledge the continuous encouragement, advice, and help of Professor Ernest N. Bart who supervised this research.

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1. Background

The slow settling of spherical particles has been a subject of investigation for many years. The method of reflections is a technique which was first developed by Stokes[9] in 1845. It is a technique which is often used to solve the creeping motion equation and the equation of continuity for various boundary conditions. Application of the method involves "reflection" of the motion of a sphere from another boundary surface, and back again to the original sphere. Essentially, what the technique does is to find the influence of the other bounding surface on the motion of the sphere. The boundary surfaces which have been investigated in the literature include a plane wall, the inside of a square or cylindrical duct, and many others including another adjacent sphere. A number of solutions which use this technique have been obtained for the case of two identical spheres settling slowly in a viscous fluid. A complete review of the literature involving the method of reflections including a summary of the work that has been done involving the two sphere case is given in Happel and Brenner[5]. The two sphere situation has also been solved by Goldman, Cox, and Brenner[4] by using a system of bipolar coordinates.

The problem of a single sphere settling in a wedge space has been solved by Sano and Hasimoto[8]. The problem has also been solved in an unpublished work by Bart[1] which confirms the results of Sano and Hasimoto. In these solutions the effect of the wedge walls on the motion of the sphere is evaluated. In Bart's solution the method of reflections is used to satisfy a Dirichlet type boundary condition on the wedge wall. The solution of the single sphere in the wedge space is closely related to the solution obtained in the present work. Both solutions (The solution obtained in this work and the solution obtained for a sphere in a wedge by Bart) use the same mathematical expression for the second reflection solution. (Note that the same <u>general</u> form of the mathematical expression is used. The actual constants are different, of course, for each problem.)

Another related problem has been solved by Bart and Horwat[2]. This problem involves heat or mass transfer occuring between identical spheres arranged on the corners of a regular polygon. The solution satisfies Laplace's equation. The method of reflections is used to generate a solution which satisfies a Neumann type boundary condition on the plane(s) of symmetry.

In the present work a general solution has been obtained for the case of N spheres arranged on the corners of a regular polygon settling slowly in a viscous fluid. The solution satisfies the creeping motion equation and the equation of continuity. The method of reflections is used to generate a solution which obeys a Neumann type boundary condition on the plane(s) of symmetry.

2. Description of the Problem

A. Geometrical Description

(1) The system consists of N identical spheres settling at a velocity such that the fluid velocity field obeys the Creeping Motion equation.

(2) The spheres are arranged in a planar array such that the center of each sphere is located on the corners of a regular polygon. Thus, three spheres are on triangular centers; four spheres are on square centers, and so on. The center of each sphere is displaced a distance x_o from the centroid of the array. This situation is shown in Figure 2-1 for the case of three spheres.

(3) Let a cylindrical coordinate system (ρ, ϕ, z) be superimposed on the array so that the N spheres are falling parallel to the z axis, and the origin of the coordinate system is at the centroid of the array. Also, define the origin in such a way that the plane at $\phi = 0$, which is parallel to the z axis, divides one of the spheres in the array into two equal hemispheres.

(4) The dihedral angle formed by the intersection of two planes parallel to the z axis which pass through the origin and the centers of two adjacent spheres is $2\phi_o$, where $\phi_o = \pi/N$

(5) A plane of symmetry exists along the plane which bisects the dihedral angle at $\phi = \phi_o$. Thus, between any two adjacent spheres in the array there is a plane of symmetry. This situation is shown in Figure 2-2 for the case of two, three, and four spheres.

(6) A geometrical relationship exists between the sphere radius, a, and the distance from the origin to the sphere center, x_o. This distance is fixed by the fact that for a given sphere radius, the distance x_o can only be decreased until the point where the spheres touch. Such a situation is shown in Figure 2-3. The geometrical relationship is given by:

$$\sin \phi_o = (a/x_o) \qquad 2\text{-}1$$

where the x_o in Equation 2-1 is the minimum value for a fixed value of the sphere radius, a.

B. Statement of the Partial Differential Equations

If the spheres are settling very slowly such that the Reynolds Number is less than .1, then the Equation of Motion reduces to the Creeping Motion equation.

$$\mu\nabla^2 \bar{V} = \nabla P \qquad (2\text{-}2)$$

The velocity field must also satisfy the Equation of Continuity:

$$\nabla\cdot\bar{V} = 0 \qquad (2\text{-}3)$$

C. Statement of the Boundary Conditions

The boundary conditions that Equations (2-2) and (2-3) are to be solved with arise naturally from the symmetry of the problem. They may be summarized by noting that a minimum occurs in the velocity field on the surface of the plane(s) of symmetry, and that the net fluid velocity accross the plane(s) of symmetry is zero. These statements are expressed mathematically below:

$$(\partial\bar{V}/\partial\phi)_{\phi=\phi_o} = 0 \qquad (2\text{-}4)$$

and

$$V_\phi = 0 \quad \text{at } \phi = \phi_o \qquad (2\text{-}5)$$

It should be noted that Equation (2-4) is a vector equation representing the three scalar equations given below:

$$\partial V_\phi/\partial\phi = 0 \qquad \text{at } \phi=\phi_o \qquad (2\text{-}6)$$

$$\partial V_\rho/\partial\phi = 0 \qquad \text{at } \phi=\phi_o \qquad (2\text{-}7)$$

$$\partial V_z/\partial\phi = 0 \qquad \text{at } \phi=\phi_o \qquad (2\text{-}8)$$

FIGURE 2-1: Top and Side Views of Three Spheres Arranged on Triangular Centers Settling in a Viscous Fluid

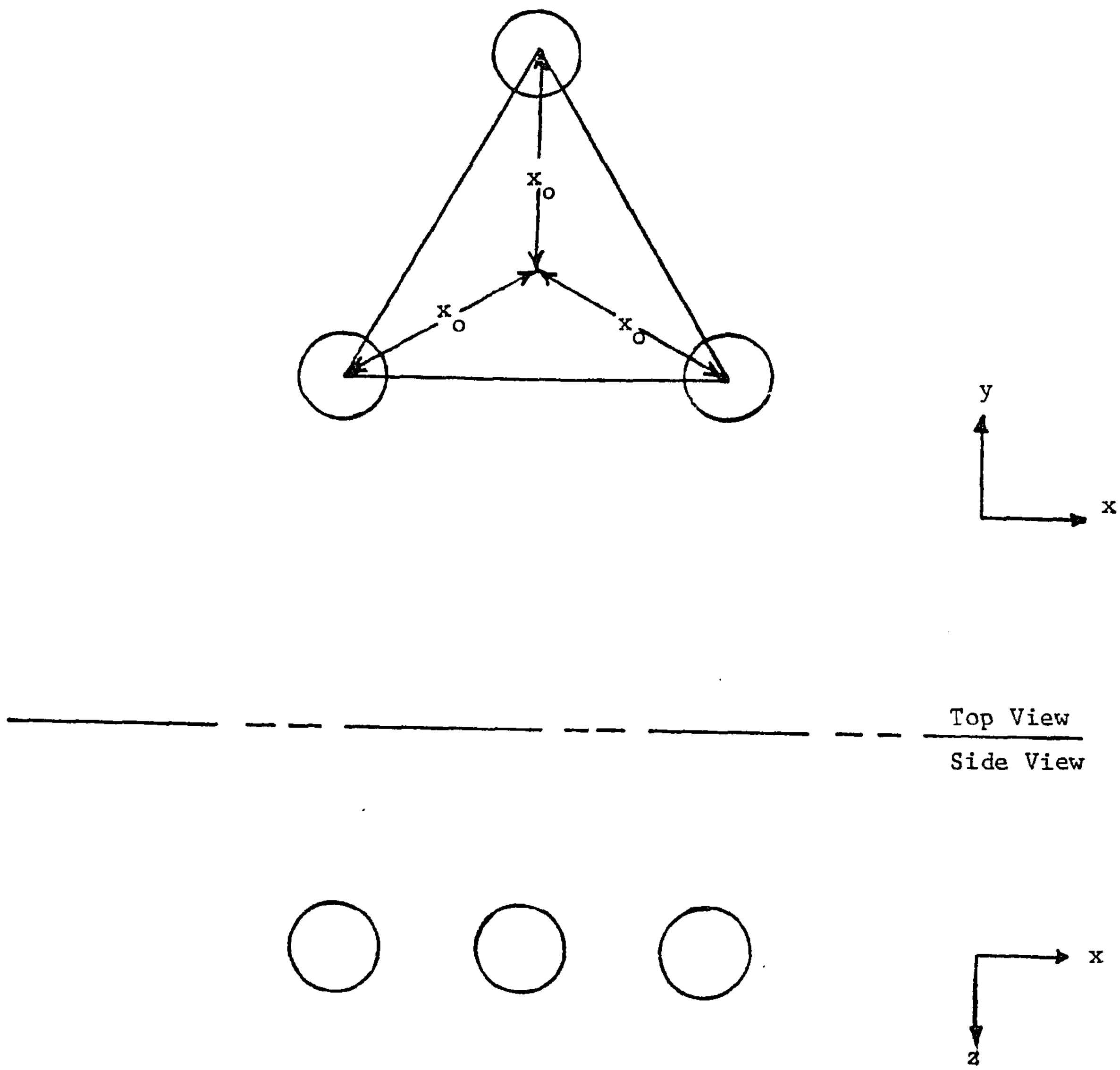
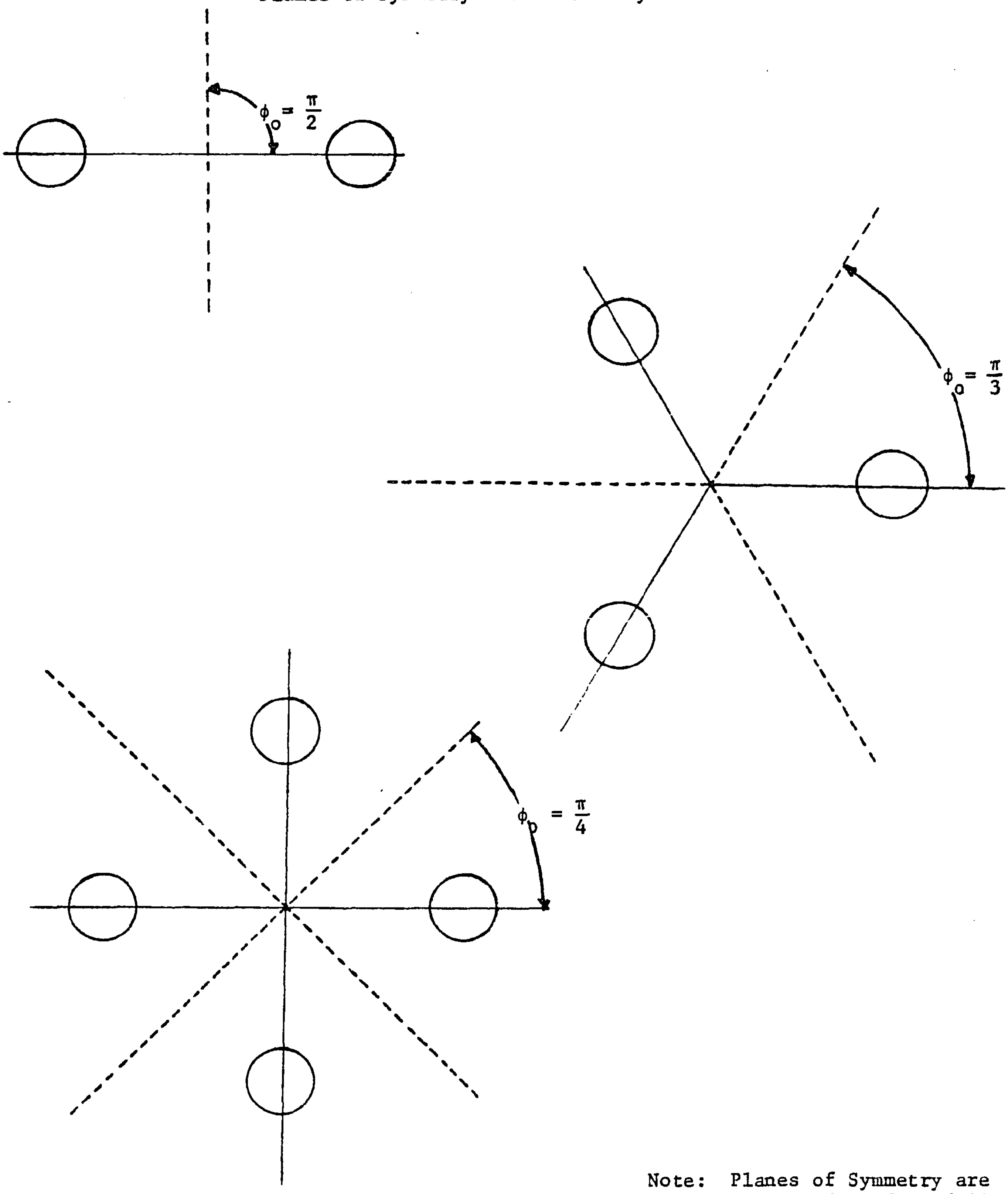


Figure 2-2: Planar Arrays of 2, 3, and 4, particles illustrating the Planes of Symmetry for Each Array



Note: Planes of Symmetry are indicated by dotted lines

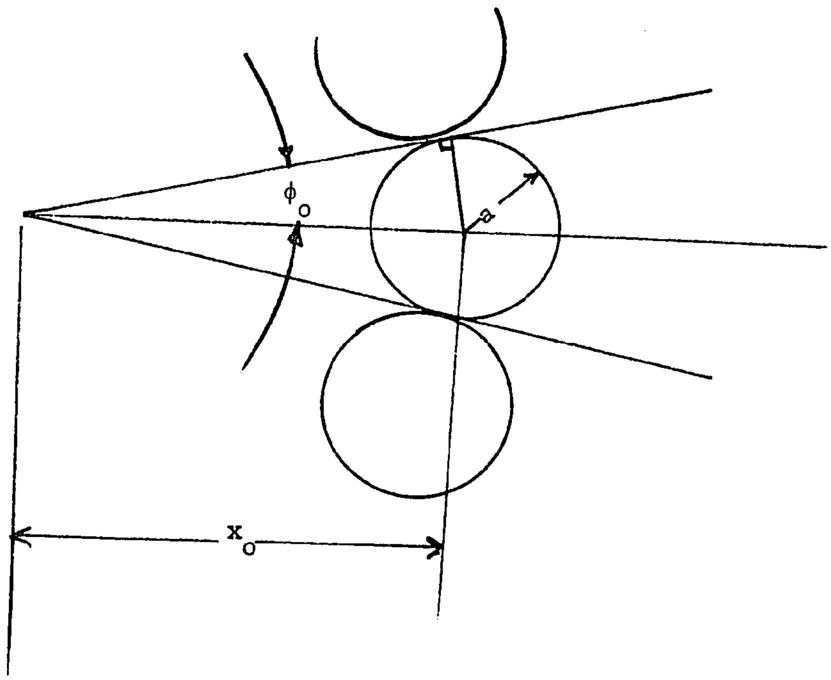


Figure 2-3: Sphere tangent to an adjacent sphere

$$\sin \phi_o = \frac{a}{x_o}$$

It can be argued on physical grounds that if Equation (2-5) is satisfied, then Equation (2-7) is automatically satisfied. The rationale for this statement follows:

(1) Assume that Equation (2-5) is satisfied and that the ϕ component of the velocity is zero on the plane $\phi = \phi_o$

(2) The ϕ component of the velocity is nonzero at points not on the plane $\phi = \phi_o$. That is, it is either positive or negative on all planes where $\phi \neq \phi_o$. (Far from the spheres, the fluid velocity will be zero on all points. This statement pertains to points in the fluid where the velocity is still non-zero)

(3) If statements (1) and (2) are true, then the component of velocity in the ϕ direction is either a minimum or a maximum with respect to ϕ along the plane $\phi = \phi_o$.

The preceeding rationale implies that Equation (2-6) is redundant and that the boundary conditions at $\phi = \phi_o$ will be satisfied by using only Equations (2-5), (2-7), and (2-8).

In order to fully describe the problem it is not enough to satisfy only those boundary conditions which exist on the plane of symmetry. It is also necessary to satisfy the boundary condition which exists on the surface of the sphere. Since the spheres are settling at a constant velocity, U, the additional requirement imposed on the solution is that the fluid velocity must be equal to U on the surface of the sphere. This last boundary condition may be expressed mathematically as:

$$\bar{V} = -\bar{k}U \qquad (2\text{-}9)$$

An illustration showing the boundary conditions which must be satisfied on the sphere surface and on the planes of symmetry is shown for the case of N = 6 spheres in Figure 2-4. Note that the boundary conditions must be satisfied simultaneously on all six spheres and on all six planes of symmetry.

The planes of symmetry in Figure 2-4 divide the velocity field into six cells. Since all the spheres are identical, the fluid velocity fields within each cell are also identical. Thus, the

FIGURE 2-4: Boundary Conditions Which Must Be Satisfied On the Sphere Surface and on the Planes of Symmetry

On the Planes of Symmetry:

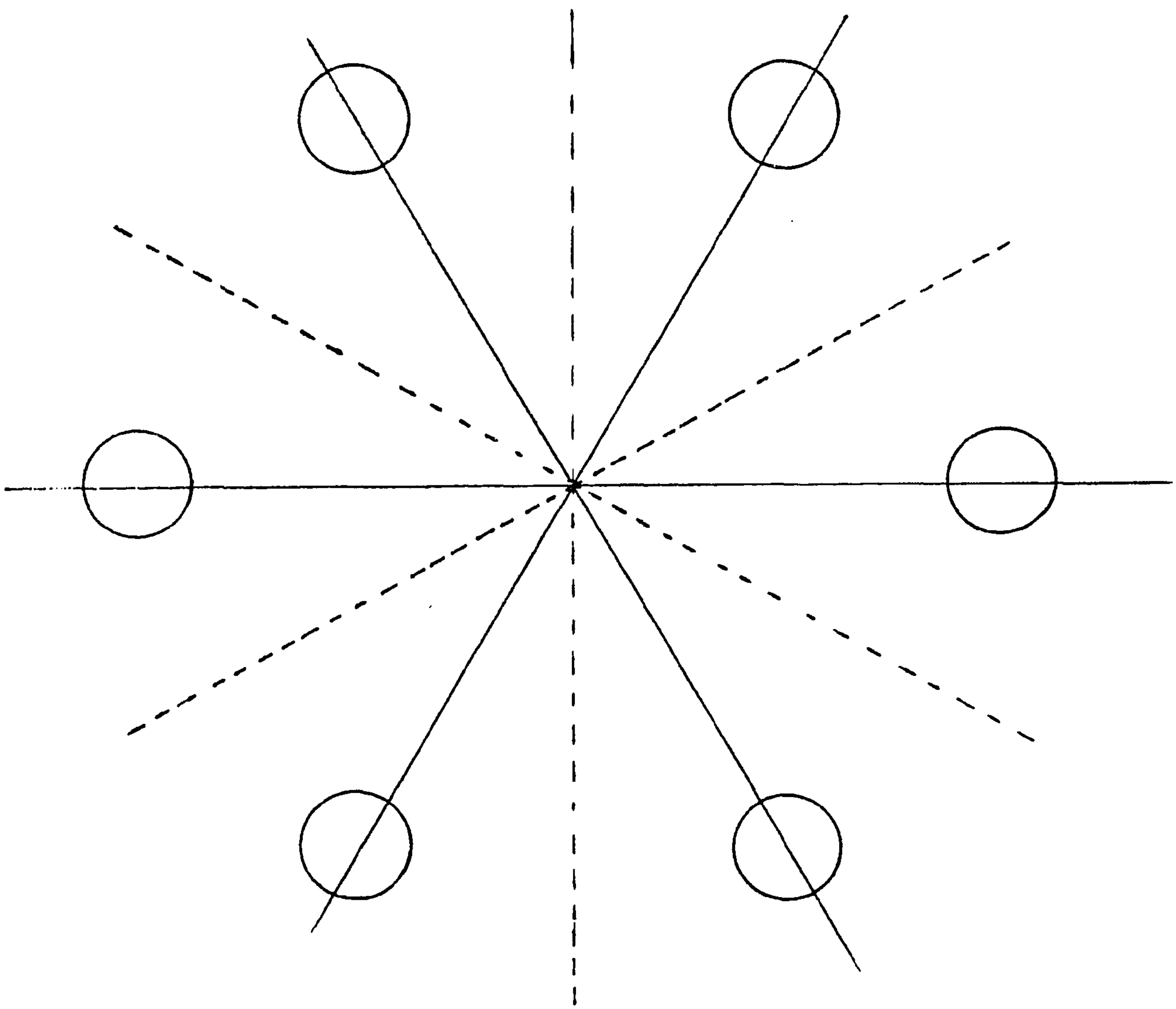
$$V_\phi = 0$$

$$\frac{\partial V_\rho}{\partial \phi} = 0$$

$$\frac{\partial V_z}{\partial \phi} = 0$$

On the Sphere Surface:

$$\bar{V} = -\bar{k}U$$



Note: Planes of Symmetry are indicated by dotted lines multisphere problem reduces down to solving the problem for a single sphere in the array with the appropriate boundary conditions.

D. Method of Solution

It is evident that no single coordinate system can simultaneously satisfy boundary conditions on the sphere surface and on the plane(s) of symmetry. For this reason the Method of Reflections is used (Happel and Brenner (5)).

The method of reflections uses the principle of superposition of solutions. The fluid velocity field and the drag force may be expressed as the sum of the reflected solutions:

$$\bar{V} = \bar{V}^{(1)} + \bar{V}^{(2)} + \bar{V}^{(3)} + \ldots + \bar{V}^{(n)} \qquad (2\text{-}10)$$

$$\bar{F} = \bar{F}^{(1)} + \bar{F}^{(2)} + \bar{F}^{(3)} + \ldots + \bar{F}^{(n)} \qquad (2\text{-}11)$$

When written in this form, the odd numbered solutions satisfy boundary conditions on the sphere surface, and the even numbered solutions satisfy boundary conditions on the plane(s) of symmetry.

Since the even numbered velocity fields satisfy boundary conditions on the plane(s) of symmetry, these solutions exist in the interior of the sphere (i.e., they satisfy boundary conditions which imply the absence of the sphere). As a result, the even numbered velocity fields make no contribution to the drag force, and Equation (2-11) becomes:

$$\bar{F} = \bar{F}^{(1)} + \bar{F}^{(3)} + \ldots + \bar{F}^{(2n + 1)} \qquad (2\text{-}12)$$

Only a second reflection will be obtained in this solution, therefore, Equations (2-10) and (2-12) become:

$$\bar{V} = \bar{V}^{(1)} + \bar{V}^{(2)} \qquad (2\text{-}13)$$

$$\bar{F} = \bar{F}^{(1)} + \bar{F}^{(3)} \qquad (2\text{-}14)$$

E. Mathematical forms of the First and Second Reflections

The first reflection solution, $\bar{V}^{(1)}$, is the well known Stokes solution for a single sphere settling in an unbounded medium, such that the fluid velocity field obeys the Creeping Motion Equation and the Equation of Continuity. This solution, when written for a cartesian coordinate system with origin at the sphere center, yields the following expressions for the fluid velocities in the x, y, and z directions:

$$V_x^{(1)} = \frac{3}{4}\, Ua\underline{x}z\left\{ \frac{a^2}{(\underline{x}^2 + y^2 + z^2)^{5/2}} - \frac{1}{(\underline{x}^2 + y^2 + z^2)^{3/2}} \right\} \quad (2\text{-}15)$$

$$V_y^{(1)} = \frac{3}{4}\, Uayz\left\{ \frac{a^2}{(\underline{x}^2 + y^2 + z^2)^{5/2}} - \frac{1}{(\underline{x}^2 + y^2 + z^2)^{3/2}} \right\} \quad (2\text{-}16)$$

$$V_z^{(1)} = \frac{1}{4}\, Ua\left\{ \frac{3a^2z^2}{(\underline{x}^2 + y^2 + z^2)^{5/2}} - \frac{3z^2 + a^2}{(\underline{x}^2 + y^2 + z^2)^{3/2}} - \frac{3}{(\underline{x}^2 + y^2 + z^2)^{1/2}} \right\} \quad (2\text{-}17$$

In order to obtain the second reflection it is necessary to find an independent solution to the Creeping Motion Equation and the Equation of Continuity. Such a solution has been shown to be(1):

$$V_x^{(2)} = \int_0^\infty \int_0^\infty \left(B + \frac{A\rho}{2}\cos\phi\right) K_{i\tau}(\lambda\rho)\, \cosh\tau\phi\, \sin\lambda z\, d\lambda\, d\tau \quad (2\text{-}18)$$

$$V_y^{(2)} = \int_0^\infty \int_0^\infty \left(C \sinh \tau\phi + \frac{A}{2}\sin\phi \cosh \tau\phi\right) K_{i\tau}(\lambda\rho)\, \sin \lambda z\, d\lambda\, d\tau \quad (2\text{-}19)$$

$$V_z^{(2)} = \int_0^{\infty}\int_0^{\infty} \{(B \cos\phi \cosh\tau\phi + C \sin\phi \sinh\tau\phi) \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \quad (C \cos\phi \cosh\tau\phi - B \sin\phi \sinh\tau\phi) \frac{\tau K_{i\tau}(\lambda\rho)}{\rho}$$

$$+ \frac{A\rho}{2} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} \cosh\tau\phi + A\, K_{i\tau}(\lambda\rho) \cosh\tau\phi \} \cos\lambda z \, d\lambda \, d\tau \qquad (2\text{-}20)$$

It can readily be shown that the above equations satisfy the creeping motion equation and the equation of continuity and are independent of the first reflection solution since the solution remains finite at the sphere centers. Note that in Equations (2-18) through (2-20), the coordinates ρ, ϕ, and z, are measured with respect to an origin which would be at the centroid of the array of spheres in this problem.

Equations (2-18) through (2-20) are integral transform expressions for the x, y, and z components of the fluid velocity. The integrations involving $\sin\lambda z \, d\lambda$ are Fourier transforms and the integrations involving $K_{i\tau}(\lambda\rho)$ are Lebedev[6] transforms.

The function $K_{i\tau}(\lambda\rho)$ is a K Bessel function of imaginary order $(i\tau)$, and real argument, $(\lambda\rho)$. It has the integral representation given below:

$$K_{i\tau}(\lambda\rho) = \int_0^{\infty} e^{-\lambda\rho \cosh t} \cos\tau t \, dt \qquad (2\text{-}21)$$

The three constants of integration given in Equations (2-18) through (2-20) (i.e., the constants A, B, and C) will be functions of ϕ_o, a, and x_o, and the dummy variables τ and λ, but will be independent of ρ, ϕ, and z.

In order to satisfy the boundary conditions on the plane(s) of symmetry both the first reflection (Stokes solution) and the second reflection (integral transform expressions) must be expressed relative to the origin located at the centroid of the array of spheres. This is accomplished by translating the Stokes solution a distance x_o from the sphere center to the centroid of the array.

The relationship between the coordinates of a point P, ($\underline{x}$, y, z), based on a sphere centered origin, and the coordinates of the same point P, (x, y, z), based on an origin located at the centroid of the array is a simple linear translation of the sphere centered origin given by the equation:

$$\underline{x} + x_o = x \quad (2\text{-}22)$$

or,

$$\underline{x} = x - x_o \quad (2\text{-}23)$$

Note that the 'y' and 'z' values remain the same with respect to both origins.

The square of the distance from the sphere centered origin to the point P is given by the Pythagorean Theorem:

$$r^2 = \underline{x}^2 + y^2 + z^2 \quad (2\text{-}24)$$

Combination of Equations (2-23) and (2-24) yields an expression for r^2 based on an origin located at the centroid of the array:

$$r^2 = (x - x_o)^2 + y^2 + z^2 \quad (2\text{-}25)$$

The relationship between a cylindrical coordinate system, and a rectangular coordinate system (both in relation to a common origin located at the centroid of the array) is given by the following equations:

$$x = \rho\cos\phi \quad (2\text{-}26)$$

$$y = \rho\sin\phi \quad (2\text{-}27)$$

$$z = z \quad (2\text{-}28)$$

Substitution of Equations (2-26) through (2-28) into Equation (2-25) yields:

$$r^2 = \rho^2\cos^2\phi - 2\rho\, x_o \cos\phi + x_o^2 + \rho^2\sin^2\phi + z^2 \quad (2\text{-}29)$$

Simplifying,

$$r^2 = \rho^2 - 2\rho\, x_o \cos\phi + x_o^2 + z^2 \quad (2\text{-}30)$$

Noting that the term r^2 appears implicitly in Equations (2-15) through (2-17), and substitution of Equations (2-23), (2-26) through (2-28), and (2-30) into Equations (2-15) through (2-17) yields the translated Stokes' solutions:

$$V_x^{(1)} = \frac{3}{4} Uaz\, (\rho\cos\phi - x_o)\left\{ \frac{a^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}} - \frac{1}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}} \right\} \quad (2\text{-}31)$$

$$V_y^{(1)} = \frac{3}{4} Uaz\, (\rho\sin\phi)\left\{ \frac{a^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}} - \frac{1}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}} \right\} \quad (2\text{-}32)$$

$$V_z^{(1)} = \frac{1}{4} U a \{ \frac{3a^2 z^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}} - \frac{3z^2 + a^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}} - \frac{3}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{1/2}} \}$$

(2-33)

3. Evaluation of Constants

Since the boundary conditions stated in Equations (2-5) through (2-8) are given in cylindrical coordinates, it is desirable to transform the expressions given for $\bar{V}^{(1)}$ and $\bar{V}^{(2)}$ from rectangular coordinates to cylindrical coordinates. The relationship between the velocity components in the two coordinate systems is given below:

$$V_\rho = \cos\phi \; V_x + \sin\phi \; V_y \tag{3-1}$$

$$V_\phi = -\sin\phi \; V_x + \cos\phi \; V_y \tag{3-2}$$

$$V_z = V_z \tag{3-3}$$

Application of Equations (3-1) and (3-2) to the expressions for V_x and V_y given in Equations (2-15), (2-16), (2-18), and (2-19) yields the following expressions:

$$V_\rho^{(1)} = \frac{3}{4} Uaz(\rho - x_o \cos\phi)\left\{\frac{a^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}} - \frac{1}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}}\right\} \tag{3-4}$$

$$V_\phi^{(1)} = \frac{3}{4} Uaz(x_o \sin\phi)\left\{\frac{a^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}} - \frac{1}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}}\right\} \tag{3-5}$$

$$V_\rho^{(2)} = \int_0^\infty \int_0^\infty (B \cos\phi \cosh\tau\phi + C \sin\phi \sinh\tau\phi + \frac{A\rho}{2} \cosh \tau\phi)\lambda K_{i\tau}(\lambda\rho)\sin\lambda z \, d\lambda d\tau$$

(3-6)

$$V_\phi^{(2)} = \int_0^\infty \int_0^\infty (C \cos \phi \sinh \tau\phi - B \sin \phi \cosh \tau\phi)\lambda K_{i\tau}(\lambda\rho) \sin \lambda z \, d\lambda d\tau$$

(3-7)

In order to satisfy the boundary conditions given in Equations (2-5), (2-7), and (2-8), it is necessary for the second reflection to cancel off the effect of the first reflection on the surface of the plane of symmetry. With this line of reasoning, Equations (2-5), (2-7), and (2-8) will be satisfied by the following system of equations:

$$V_\phi^{(2)} = - V_\phi^{(1)} \quad \text{at } \phi = \phi_o \qquad (3\text{-}8)$$

$$\frac{\partial V_\rho^{(2)}}{\partial\phi} = \frac{-\partial V_\rho^{(1)}}{\partial\phi} \quad \text{at } \phi = \phi_o \qquad (3\text{-}9)$$

$$\frac{\partial V_z^{(2)}}{\partial\phi} = \frac{-\partial V_z^{(1)}}{\partial\phi} \quad \text{at } \phi = \phi_o \qquad (3\text{-}10)$$

Substitution of Equations (3-4) and (3-6) into Equation (3-8) yields the following result:

$$\int_0^\infty \int_0^\infty (C \cos \phi_o \sinh \tau\phi_o - B \sin \phi_o \cosh \tau\phi_o)\, \lambda K_{i\tau}(\lambda\rho) \sin \lambda z \, d\lambda \, d\tau$$

$$= \frac{3}{4} U a x_o \sin \phi_o \left\{ \frac{z}{(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 + z^2)^{3/2}} - \frac{a^2 z}{(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 + z^2)^{5/2}} \right\} \quad (3\text{-}11)$$

Substitution of Equations (3-4) and (3-6) into the boundary condition given by Equation (3-9) and evaluation at $\phi = \phi_o$ yields the following result:

$$\int_0^\infty \int_0^\infty \{B(\tau \cos \phi_o \sinh \tau\phi_o - \sin \phi_o \cosh \tau\phi_o) + C(\tau \sin \phi_o \cosh \tau\phi_o + \cos \phi_o \sinh \tau\phi_o) + \frac{A\rho}{2} \tau \sinh \tau\phi_o\} \lambda K_{i\tau}(\lambda\rho) \sin \lambda z \, d\lambda \, d\tau$$

$$= \frac{3}{4} U a \frac{\partial}{\partial\phi} \left\{ (x_o \cos \phi - \rho) \left\{ \frac{a^2 z}{(\rho^2 - 2\rho x_o \cos \phi + x_o^2 + z^2)^{5/2}} - \frac{z}{(\rho^2 - 2\rho x_o \cos \phi + x_o^2 + z^2)^{3/2}} \right\} \right\}_{\phi=\phi_o} \quad (3\text{-}12)$$

Substitution of Equations (2-20) and (2-33) into the boundary condition given by Equation (3-10), and evaluation at $\phi = \phi_o$ yields the following result:

$$\int_0^\infty \int_0^\infty \Big(\{ C(\cos\phi_o \sinh\tau\phi_o + \tau\sin\phi_o \cosh\tau\phi_o)$$

$$+B(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o) \} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \{ C(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$- B(\cos\phi_o \sinh\tau\phi_1 + \tau\sin\phi_o \cosh\tau\phi_o) \} \frac{\tau K_{i\tau}(\lambda\rho)}{\rho}$$

$$+ \frac{A\rho}{2} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} \tau \sinh\tau\phi_o + A\tau K_{i\tau}(\lambda\rho) \sinh\tau\phi_o \Big) \cos\lambda z \, d\lambda \, d\tau$$

$$= \frac{1}{4} U a \frac{\partial}{\partial\phi} \Big(\frac{3}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{1/2}}$$

$$+ \frac{3z^2 + a^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}}$$

$$- \frac{3a^2 z^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}} \Big)_{\phi=\phi_o} \qquad (3\text{-}13)$$

Equations (3-11), (3-12), and (3-13) may be simplified by performing the following operations (Details are shown in Appendix A):

(1) Inversion of the Fourier Transforms

(2) Differentiation with respect to ϕ (as indicated in Equations (3-12) and (3-13))

(3) Separation of the constants A, B, and C into two parts as given by the equations below:

$$A = A_1 + A_2 \tag{3-14}$$

$$B = B_1 + B_2 \tag{3-15}$$

$$C = C_1 + C_2 \tag{3-16}$$

In Equations (3-14) through (3-16), the constants A_1, B_1, and C_1, satisfy the portion of the solution containing a, the sphere radius, raised to the first power; the constants A_2, B_2, and C_2, satisfy the portion of the solution containing a, the sphere radius, raised to the third power.

The simplified expressions containing the constants A_1, B_1, and C_1 then become:

$$\int_0^\infty \left(C_1 \cos \phi_o \sinh \tau\phi_o - B_1 \sin \phi_o \cosh \tau\phi_o \right) K_{i\tau}(\lambda\rho)\, d\tau$$

$$= \frac{3Ua}{2\pi} x_o \sin \phi_o K_o(Z) \tag{3-17}$$

$$\int_0^\infty \Big(B_1(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$+ C_1(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)$$

$$+ \frac{A_1\rho}{2}\,\tau\sinh\tau\phi_o\Big) K_{i\tau}(\lambda\rho)\,d\tau$$

$$= \frac{3Ua}{2\pi}\Big((x_o\cos\phi_o - \rho)\,\lambda^2\rho x_o \sin\phi_o \frac{K_1(Z)}{Z} + x_o\sin\phi_o K_o(Z)\Big) \qquad (3\text{-}18)$$

$$\int_0^\infty \Big(\{C_1(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)$$

$$+ B_1(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)\}\frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \{C_1(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$- B_1(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)\}\,\frac{\tau K_{i\tau}(\lambda\rho)}{\rho}$$

$$+ \frac{A_1\rho}{2}\;\frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}\tau\sinh\tau\phi_o + A_1\,K_{i\tau}(\lambda\rho)\,\tau\sinh\tau\phi_o\Big)\,d\tau$$

$$= \frac{3Ua}{2\pi}\Big(-2\lambda^2\rho x_o \sin\phi_o \frac{K_1(Z)}{Z} + \lambda^2\rho x_o \sin\phi_o K_o(Z)\Big) \qquad (3\text{-}19)$$

Note that the variable Z which appears in Equations (3-17) through (3-19) is given by the expression:

$$Z = \lambda(\rho^2 - 2\rho x_o \cos\phi_o + x_o^2)^{\frac{1}{2}} \qquad (3\text{-}20)$$

Equations (3-17), (3-18), and (3-19) can be manipulated through the use of identities and by inversion of the Lebedev transforms (Details of this process are given in Appendix B). This results in the following expressions for A_1, B_1, and C_1:

$$C_1 = -\frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o)\ \alpha_1 \qquad (3\text{-}21)$$

where $\alpha_1 = (\sin\phi_o \cos\phi_o \sinh\tau\pi)/(\cos^2\phi_o - \cosh^2\tau\phi_o)$

$$B_1 = \frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o)\ \alpha_2 \qquad (3\text{-}22)$$

where $\alpha_2 = \dfrac{\sin^2\phi_o \cosh\tau\phi_o \cosh\tau(\pi-\phi_o) - \cos^2\phi_o \sinh\tau\phi_o \sinh\tau(\pi-\phi_o)}{\cos^2\phi_o - \cosh^2\tau\phi_o}$

$$A_1 = -\frac{6Ua}{\pi^2} K_{i\tau}(\lambda x_o)\ \alpha_3 \qquad (3\text{-}23)$$

where $\alpha_3 = (\sinh\tau(\pi-\phi_o))/\sinh\tau\phi_o$

4. Evaluation of the Drag

Faxen's law is used to obtain the drag, $\bar{F}^{(3)}$:

$$\bar{F}^{(3)} = 6\pi\mu a \left(\bar{V}_o^{(2)} + \frac{a^2}{6} \left(\nabla^2 \bar{V}_o^{(2)}\right)\right) \tag{4-1}$$

where the subscript, o, indicates the functions are to be evaluated at the sphere center ($\underline{x} = y = z = 0$). When these conditions are substituted into Equations (2-18), (2-19), and (2-20), the expression for $\bar{V}_o^{(2)}$ becomes:

$$\bar{V}_o^{(2)} = \bar{k} \int_0^\infty \int_0^\infty \left(\left(B + \frac{Ax_o}{2}\right) \frac{\partial K_{i\tau}(\lambda x_o)}{\partial x_o} + \left(\frac{C\,\tau}{x_o} + A\right) K_{i\tau}(\lambda x_o) \right) d\lambda\, d\tau \tag{4-2}$$

If only the terms of the order of a to the first power are retained, Equations (4-1) and (4-2) may be combined and expressed in terms of the constants A_1, B_1, and C_1:

$$\bar{F}^{(3)} = 6\pi\mu a\bar{k} \int_0^\infty \int_0^\infty \left(\left(B_1 + \frac{A_1 x_o}{2}\right) \frac{\partial K_{i\tau}(\lambda x_o)}{\partial x_o} + \left(\frac{C_1 \tau}{x_o} + A_1\right) K_{i\tau}(\lambda x_o) \right) d\lambda\, d\tau \tag{4-3}$$

Substitution of the expressions for A_1, B_1, and C_1, given in Equations (3-21) through (3-23) into Equation (4-3), and integration with respect to λ yields (See Appendix D for the details of this operation):

$$\bar{F}^{(3)} = 6\pi\mu a U\bar{k} \left(-\frac{3}{8}\right) \frac{a}{x_o} \int_0^\infty \left(\alpha_2 + 2\tau\alpha_1 + 3\alpha_3\right) \frac{d\tau}{\cosh \tau\pi} \tag{4-4}$$

or,

$$\bar{F}^{(3)} = 6\pi\mu aU\bar{k} \left(\frac{a}{x_o} \right) f_1(\phi_o) \qquad (4\text{-}5)$$

In most cases it is necessary to perform the integration indicated in Equation (4-4) numerically in order to obtain the value of $f_1(\phi_o)$ defined in Equation (4-5). However, for the case of two spheres (i.e., $\phi_o = \pi/2$), Equation (4-4) may be integrated analytically (See Appendix D for details). The value of $f_1(\phi_o)$ for the case $\phi_o = \pi/2$ is (-3/8). This result is in exact agreement with the value given in Happel and Brenner.[5] The value of the function $f_1(\phi_o)$ for various values of ϕ_o is presented in Table 1.

The total drag force on the sphere is given by Equation (2-14) repeated, below:

$$\bar{F} = \bar{F}^{(1)} + \bar{F}^{(3)} \qquad (2\text{-}14)$$

The first reflection solution, $\bar{F}^{(1)}$, is the well known Stoke's law for the drag force on a single sphere:

$$\bar{F}^{(1)} = 6\pi\mu aU\bar{k} \qquad (4\text{-}6)$$

Substitution of Equations (4-5) and (4-6) into Equation (2-14) gives an expression for the total drag force on a sphere in the array:

$$\bar{F} = 6\pi\mu aU\bar{k} \left(1 + f_1(\phi_o)\left\{\frac{a}{x_o}\right\}\right) \qquad (4\text{-}7)$$

TABLE 1

VALUES OF THE COEFFICIENT $f_1(\phi_o)$ FOR VARIOUS VALUES OF N

N	$\phi_o = \pi/N$	$-f_1(\phi_o)$
2	$\pi/2$	.375
3	$\pi/3$	.866
4	$\pi/4$	1.436
5	$\pi/5$	2.065
10	$\pi/10$	5.794
100	$\pi/100$	112.939
1000	$\pi/1000$	1292.584

5. Evaluation of the Terminal Settling Velocity

The terminal settling velocity of a sphere settling in the array may be expressed in terms of the Stokes' velocity for a single identical sphere settling in the same type of fluid in an unbounded situation.

It should be noted that the drag force experienced by a sphere is independent of the presence of the other spheres and is equal to the difference between the force of gravity, and the buoyant force exerted by the fluid:

$$F = (\rho' - \rho_f) g \frac{4}{3} \pi a^3 \tag{5-1}$$

where,

ρ' = mean density of the sphere

and

ρ_f = density of the surrounding fluid

The drag force on a single sphere in an unbounded medium is given by Stokes' law:

$$F = 6\pi\mu a U_S \bar{k} \tag{5-2}$$

where the subscript S is used to denote the Stokes' velocity.

The drag force on a sphere in the array is given by Equation (4-7):

$$F = 6\pi\mu a U \bar{k} \; (1 + f_1(\phi_o) [\frac{a}{x_o}]) \tag{4-7}$$

Since the spheres are identical, the drag force on the single sphere and the sphere in the array are equal and Equations (5-2) and (4-7) may be set equal to one another .

$$6\pi\mu aU\bar{k}(1 + f_1(\phi_o)\{\frac{a}{x_o}\}) = 6\pi\mu aU_S\bar{k} \tag{5-3}$$

or,

$$\frac{U}{U_S} = \frac{1}{(1 + f_1(\phi_o)\{\frac{a}{x_o}\})} \tag{5-4}$$

Note that the ratio U/U_S given in Equation (5-4) is always greater than 1, since the presence of additional spheres increases the velocity of the array.

APPENDIX A-1

$$\int_0^\infty \int_0^\infty \left(C\cos\phi_o \sinh\tau\phi_o - B\sin\phi_o \cosh\tau\phi_o\right) \lambda K_{i\tau}(\lambda\rho)\, d\tau \sin\lambda z\, d\lambda =$$

$$\frac{3}{4} U\, a\, x_o \sin\phi_o \frac{z}{(\rho^2 - 2\rho x_o \cos\phi_o + x_o^2 + z^2)^{3/2}}$$

$$- \frac{a^2 z}{(\rho^2 - 2\rho x_o \cos\phi_o + x_o^2 + z^2)^{5/2}} = f(z) \qquad (3\text{-}11)$$

Examination of Equation (3-11) reveals that the inner integral is a function of λ alone, $g(\lambda)$:

$$f(z) = \int_0^\infty \left(\int_0^\infty \{C\cos\phi_o \sinh\tau\phi_o - B\sin\phi_o \cosh\tau\phi_o\}\lambda K_{i\tau}(\lambda\rho)\, d\tau\right) \sin\lambda z\, d\lambda$$

The functions $f(z)$ and $g(\lambda)$ may be regarded as a non symmetrical pair of inversion formulas for the Fourier sin transform, thus

$$f(z) = \int_0^\infty g(\lambda) \sin\lambda z\, d\lambda \qquad (A\text{-}1.1)$$

$$g(\lambda) = \frac{2}{\pi} \int_0^\infty f(z) \sin\lambda z\, dz \qquad (A\text{-}1.2)$$

Inversion of Equation (3-11) according to Equation (A-1.2) yields:

$$\int_0^\infty \left(C\cos\phi_o \sinh\tau\phi_o - B\sin\phi_o \cosh\tau\phi_o \right) \lambda K_{i\tau}(\lambda\rho)\, d\tau =$$

$$\frac{3\ U\ a\ x_o\ \sin\phi_o}{2\ \pi}\int_0^\infty\left(\frac{z}{(\rho^2 - 2\rho x_o\cos\phi_o + x_o^2 + z^2)^{3/2}}\right.$$

$$\left.-\frac{a^2 z}{(\rho^2 - 2\rho x_o\cos\phi_o + x_o^2 + z^2)^{5/2}}\right)\sin\lambda z\ dz \qquad \text{(A-1.3)}$$

The right hand side of Equation (A-1.3) is easily evaluated using the formula given below from the table of Fourier sin transforms given in Magnus and Oberhettinger[7] (page 414):

$f(x)$	$g(y) = (\frac{2}{\pi})^{\frac{1}{2}}\int_0^\infty f(x)\sin(xy)\,dx$
$x^{2m+1}(b^2 + x^2)^{-n-\frac{1}{2}}$	
$-1 \leq m < n$	$(-1)^{m+1}2^{\frac{1}{2}-n}b^{-n}\left(\Gamma(\frac{1}{2}+n)\right)^{-1}\frac{d^{2m+1}}{dy^{2m+1}}\left(y^n K_n(by)\right)$
$m,n, = 0,1,2$	

<u>Case 1: m=0 n=1</u>

$$f(x) = x/(b^2 + x^2)^{3/2}$$

$$g(y) = (-1)\ 2^{-\frac{1}{2}}\ b^{-1}\ \left(\Gamma(1.5)\right)^{-1}\frac{d}{dy}\left(yK_1(by)\right)$$

$$\frac{d}{dy}(yK_1(by)) = \frac{d}{d(by)} (byK_1(by)) = - by\ K_o(by)$$

$$\Gamma(1.5) = \sqrt{\pi}/2$$

$$g(y) = \frac{(-1)}{2^{\frac{1}{2}}\ (b\sqrt{\pi}\ /2)} (-by)\ K_o(by)$$

$$g(y) = (2/\pi)^{\frac{1}{2}}\ y\ K_o(by)$$

$$\text{or, } \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \frac{x}{(b^2 + x^2)^{3/2}} \sin xy\ dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} y\ K_o(by)$$

The following identities are useful in changing from the nomenclature of Magnus and Oberhettinger to the nomenclature used in this paper:

$$b^2 = \rho^2 - 2\rho x_o \cos \phi_o + x_o^2$$

$$y = \lambda$$

$$x = z$$

$$\int_0^{\infty} \frac{z}{(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 + z^2)^{3/2}} \sin \lambda z\ dz = \lambda K_o \left(\lambda \sqrt{\rho^2 - 2\ \rho x_o \cos \phi_o + \rho_o^2}\ \right)$$

(A-1.4)

Case 2: m=0 n=2

$$f(x) = x/(b^2 + x^2)^{5/2}$$

$$g(y) = (-1)2^{-3/2}\, b^{-2}\, (\Gamma(5/2))^{-1} \frac{d}{dy} (y^2 K_2(by))$$

$$\frac{d}{dy}(y^2 K_2(by)) = -(by)^2\, K_1(by)$$

$$\frac{d}{dy}(y^2 K_2(by)) = -\frac{1}{b}\frac{d}{d(by)}((by)^2\, K_2(by)) = \frac{-(by)^2}{b}\; K_1(by)$$

$$\Gamma(5/2) = \frac{3\pi^{\frac{1}{2}}}{4}$$

$$g(y) = \frac{(-1)}{2^{3/2}}\,\frac{1}{b^2}\,\frac{4}{3\pi^{\frac{1}{2}}}\,\frac{-(by)^2}{b}\; K_1(by)$$

$$g(y) = (2/\pi)^{\frac{1}{2}}\,\frac{1}{3}\,(y^2/b)\; K_1(by)$$

$$(2/\pi)^{\frac{1}{2}} \int_0^\infty \frac{x}{(b^2 + x^2)^{5/2}} \sin xy\, dy = (2/\pi)^{\frac{1}{2}}\,(y^2/3b)\; K_1\;(by)$$

or in the nomenclature of this paper:

$$\int_0^\infty \frac{z \sin \lambda z\;\; dz}{(\rho^2 - 2\rho x_o\; \cos\, \phi_o + x_o^2 + z^2)^{5/2}} = \frac{\lambda^3\; K_1(Z)}{3Z} \qquad \text{(A-1.5)}$$

where $Z = \lambda\sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o^2}$

Substitution of Equations (A-1.4) and (A-1.5) into Equation (A-1.3) yields:

$$\int_0^\infty \left(C \cos \phi_o \sinh \tau\phi_o - B \sin \phi_o \cosh \tau\phi_o\right) \lambda K_{i\tau}(\lambda\rho)\, d\tau =$$

$$\frac{3Ua}{2\pi} x_o \sin \phi_o \left(\lambda K_o(Z) - \frac{a^2\lambda^3 K_1(Z)}{3\ Z}\right) \qquad \text{(A-1.6)}$$

Eliminating the higher order terms:

$$\int_0^\infty \left(C_1 \cos \phi_o \sinh \tau\phi_o - B_1 \sin \phi_o \cosh \tau\phi_o\right) K_{i\tau}(\lambda\rho)\, d\tau =$$

$$\frac{3Ua}{2\pi} x_o \sin \phi_o K_o(Z) \qquad \text{(3-17)}$$

APPENDIX A-2

$$\int_0^\infty \int_0^\infty \Big(B(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$+ C(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)$$

$$+ \frac{A\rho}{2} \tau\sinh\tau\phi_o\Big) \lambda K_{i\tau}(\lambda\rho) \sin\lambda z \, d\lambda \, d\tau =$$

$$\frac{3Ua}{4} \frac{\partial}{\partial\phi} \Big((x_o \cos\phi - \rho)\Big(\frac{a^2 z}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}}$$

$$- \frac{z}{(\rho^2 - 2\rho\, x_o \cos\phi_o + x_o^2 + z^2)^{3/2}}\Big)\Big)_{\phi=\phi_o} \qquad (3\text{-}12)$$

Inversion of the Fourier sin transform yields:

$$\int_0^\infty \Big(B(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$+ C(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o) + \frac{A\rho}{2}\tau\sinh\tau\phi_o\Big) \lambda K_{i\tau}(\lambda\rho\ ) \, d\tau$$

$$= \frac{3Ua}{2\pi} \frac{\partial}{\partial\phi}\Big(\int_0^\infty (x_o \cos\phi - \rho)\Big(\frac{a^2 z^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}}$$

$$- \frac{z}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}}\Big)\sin\lambda z \, dz\Big)_{\phi=\phi_o} \qquad (A\text{-}2.1)$$

$$\int_0^\infty \Big(B(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$+ C(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)$$

$$+ \frac{A\rho}{2} \tau\sinh\tau\phi_o\Big) \lambda K_{i\tau}(\lambda\rho)\, d\tau =$$

$$\frac{3Ua}{2\pi} \frac{\partial}{\partial\phi}\Big((x_o \cos\phi - \rho)\Big(\frac{(a^2 \lambda^3)K_1(Z)}{3\,Z} - \lambda K_o(Z)\Big)\Big)_{\phi=\phi_o}$$

Eliminate higher order terms:

$$\int_0^\infty \Big(B_1(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$+ C_1(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o) + \frac{A_1\rho}{2} \tau\sinh\tau\phi_o\Big) K_{i\tau}(\lambda\rho)\, d\tau$$

$$= \frac{3Ua}{2\pi} \frac{\partial}{\partial\phi} \Big((x_o \cos\phi - \rho)(-K_o(Z))\Big)$$

$$= \frac{3Ua}{2\pi} \Big((x_o \cos\phi - \rho)\frac{\partial}{\partial\phi}(-K_o(Z)) - K_o(Z)(-x_o \sin\phi)\Big)_{\phi=\phi_o} \qquad \text{(A-2.2)}$$

$$\frac{\partial}{\partial\phi} K_o(Z) = \frac{\partial K_o(Z)}{\partial Z} \frac{\partial Z}{\partial\phi}$$

$$\frac{\partial Z}{\partial\phi} = \frac{\partial}{\partial\phi}(\lambda\sqrt{\rho^2 - 2\rho x_o\cos\phi + x_o^2})$$

$$\frac{\partial Z}{\partial \phi} = \frac{\lambda^2 \rho x_o \sin \phi}{\lambda \sqrt{\rho^2 - 2\rho x_o \cos \phi + x_o^2}}$$

$$\frac{\partial K_o(Z)}{\partial Z} = - K_1(Z)$$

$$\frac{\partial}{\partial \phi} K_o(Z) = - \frac{\lambda^2 \rho \; x_o \sin \phi}{Z} K_1(Z) \qquad \text{(A-2.3)}$$

Substitution of Equation (A-2.3) into Equation (A 2.2) yields:

$$\int_0^\infty \Big(B_1(\tau \cos \phi_o \sinh \tau\phi_o - \sin \phi_o \cosh \tau\phi_o)$$
$$+ C_1(\tau \sin \phi_o \cosh \tau\phi_o + \cos \phi_o \sinh \tau\phi_o)$$
$$+ \frac{A_1 \rho}{2} \tau \sinh \tau\phi_o\Big) K_{i\tau}(\lambda\rho) \, d\tau$$
$$= \frac{3Ua}{2\pi} \left((x_o \cos \phi_o - \rho) \frac{\lambda^2 \rho x_o \sin\phi_o}{Z} K_1(Z) + x_o \sin \phi_o K_o(Z)\right)$$

(3-13)

APPENDIX A-3

Invert the Fourier Transform given in Equation (3-13):

$$\int_0^\infty \Big(\Big[C(\tau \sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)$$

$$+ B(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)\Big]\frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \Big[C(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$- B(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)\Big]\frac{\tau K_{i\tau}(\lambda\rho)}{\rho}$$

$$+ \frac{A\rho}{2} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} \tau \sinh\tau\phi_o + A\tau K_{i\tau}(\lambda\rho) \sinh\tau\phi_o\Big) d\tau =$$

$$\frac{Ua}{2\pi}\frac{\partial}{\partial\phi}\Big(\int_0^\infty \frac{3}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{1/2}}$$

$$+ \frac{3z^2 + a^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}}$$

$$- \frac{3a^2 z^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}} \Big)_{\phi=\phi_o} \cos\lambda z \, dz \qquad \text{(A-3.1)}$$

Note that for this inversion, the following pair of inversion formulae for the Fourier Cosine transform have been used:

$$f(z) = \int_0^\infty g(\lambda) \cos \lambda z \, d\lambda \qquad \text{(A-3.2)}$$

$$g(\lambda) = \frac{2}{\pi} \int_0^\infty f(z) \sin \lambda z \, dz \qquad \text{(A-3.3)}$$

Use is made of the formulae in Magnus and Oberhettinger (pg. 400)

$f(z)$	$g(y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(x) \cos xy \, dx$
$(b^2 + x^2)^{-\nu-\frac{1}{2}}$	$2^{\frac{1}{2}-\nu} \left(\frac{y}{b}\right)^\nu \left(\Gamma(\tfrac{1}{2} + \nu)\right)^{-1} K_\nu(by)$
Real $\nu > -\frac{1}{2}$	

<u>Case 1: $\nu = 0$</u>

$$g(y) = 2^{\frac{1}{2}} \left(\Gamma\left(\tfrac{1}{2}\right)\right)^{-1} K_o(by) \qquad \Gamma(.5) = \sqrt{\pi}$$

$$g(y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K_o(by)$$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\cos xy \, dx}{(b^2 + x^2)^{\frac{1}{2}}} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} K_o(by)$$

$$\int_0^{\infty} \frac{1}{(b^2 + x^2)^{\frac{1}{2}}} \cos xy \, dx = K_o(by)$$

$$\int_0^{\infty} \frac{1}{(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 + z^2)^{1/2}} \cos \lambda z \, dz = K_o(Z) \qquad \text{(A-3.4)}$$

Using the same general formula from Magnus and Oberhettinger:

<u>Case 2: $\nu = 1$</u>

$$f(x) = \frac{1}{(b^2 + x^2)^{3/2}}$$

$$g(y) = 2^{-\frac{1}{2}} (y/b) \left(\Gamma(1.5)\right)^{-1} K_1(by) \qquad \Gamma(1.5) = (\sqrt{\pi}/2)$$

$$g(y) = (1/\sqrt{2}) \; (y/b) \; (2/\sqrt{\pi}) K_1(by)$$

$$g(y) = (2/\pi)^{.5} \; (y/b) \; K_1(by)$$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \frac{1}{(b^2 + x^2)^{3/2}} \cos xy \, dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (y/b) \; K_1(by)$$

$$\int_0^{\infty} \frac{1}{(\rho^2 - 2\rho \, x_o \cos \phi_o + x_o^2 + z^2)^{3/2}} \cos \lambda z \, dz = \frac{\lambda^2 \; K_1(Z)}{Z} \qquad \text{(A-3.5)}$$

Use is made of the Fourier cosine table in Ditkin and Prudnikov(3), page 173:

$f(t)$	$g(y) = \int_0^\infty f(t) \cos yt \, dt$
$t^{2n}(b^2+t^2)^{-\nu-\frac{1}{2}}$	$(-1)^n\sqrt{\pi}\,(2b)^{-\nu}\left(\Gamma(\tfrac{1}{2}+\nu)\right)^{-1}\dfrac{d^{2n}}{du^{2n}}\left(u^\nu K_\nu(bu)\right)$

Case: n=1, ν=2

$$f(t) = \frac{t^2}{(b^2+t^2)^{5/2}} \qquad\qquad \Gamma(5/2) = 1.5(\sqrt{\pi}/2)$$

$$g(y) = \frac{(-1)\sqrt{\pi}}{(2b)^2}\left(\Gamma(5/2)\right)^{-1}\frac{d^2}{du^2}\left(u^2 K_2(bu)\right)$$

$$\frac{d^2}{du^2}\left(u^2K_2(bu)\right) = \frac{d^2}{d(ub)^2}\left((ub)^2 K_2(bu)\right) = \frac{d^2}{dZ^2}\left(Z^2K_2(Z)\right)$$

Magnus and Oberhettinger(7), pg. 67:

$$\left(\frac{1}{Z}\frac{d}{dZ}\right)^m\left(Z^\nu K_\nu(Z)\right) = (-1)^m Z^{\nu-m} K_{\nu-m}(Z) \qquad (A\text{-}3.6)$$

Lebedev(6), page 110:

$$\frac{d}{dZ}\left(Z^\nu K_\nu(Z)\right) = -Z^\nu K_{\nu-1}(Z) \qquad (A\text{-}3.7)$$

For m = 2 and ν = 2, Equation (A-3.6) becomes:

$$\left(\frac{1}{Z}\frac{d}{dZ}\right)^2 (Z^2K_2(Z)) = (-1)^2 Z^0 K_o(Z) \qquad \text{(A-3.8)}$$

Also,

$$\left(\frac{1}{Z}\frac{d}{dZ}\right)^2 = \frac{1}{Z}\frac{d}{dZ}\left(\frac{1}{Z}\frac{d}{dZ}\right)$$

$$= \frac{1}{Z}\left(\frac{1}{Z}\frac{d^2}{dZ^2} + \frac{d}{dZ}(-Z^{-2})\right)$$

$$\left(\frac{1}{Z}\frac{d}{dZ}\right)^2 = \frac{1}{Z^2}\frac{d^2}{dZ^2} - \frac{1}{Z^3}\frac{d}{dZ} \qquad \text{(A-3.9)}$$

For ν = 2 in Equation (A-3.7):

$$\frac{d}{dZ}(Z^2K_2(Z)) = -Z^2 K_1(Z) \qquad \text{(A-3.10)}$$

From Equation (A-3.9),

$$\left(\frac{1}{Z}\frac{d}{dZ}\right)^2 (Z^2K_2(Z)) = \frac{1}{Z^2}\frac{d^2}{dZ^2}(Z^2K_2(Z)) - \frac{1}{Z^3}\frac{d}{dZ}(Z^2K_2(Z)) \qquad \text{(A-3.11)}$$

Combination of Equations (A-3.8) and (A-3.11) yields:

$$\frac{1}{Z^2} \frac{d^2}{dZ^2} (Z^2 K_2(Z)) - \frac{1}{Z^3} \frac{d}{dZ} (Z^2 K_2(Z)) = K_o(Z) \qquad \text{(A-3.12)}$$

Substitution of Equation (A-3.10) into Equation (A-3.12) yields:

$$\frac{d^2}{dZ^2} (Z^2 K_2(Z)) = Z^2 K_o(Z) - ZK_1(Z) \qquad \text{(A-3.13)}$$

These identities must be substituted into the expression for g(y):

$$g(y) = \frac{(-1)\sqrt{\pi}}{(2b)^2} \left(\Gamma(5/2)\right)^{-1} \frac{d^2}{du^2} (u^2 K_2(bu))$$

$$= \frac{(-1)\sqrt{\pi}}{(2b)^2} \frac{2}{(1.5)\sqrt{\pi}} \left(Z^2 K_o(Z) - Z\ K_1(z)\right)$$

$$= \frac{(-1)}{3b^2} (Z^2 K_o(Z) - Z\ K_1(Z))$$

Substitution for b in terms of the nomenclature of this paper:

$$g(y) = \frac{\lambda^2}{3Z^2} \left(Z^2 K_o(Z) - Z\ K_1(Z)\right)$$

$$g(y) = \int_0^\infty \frac{z^2 \cos \lambda z\ dz}{(\rho^2 - 2\rho\ x_o \cos \phi_o + x_o^2 + z^2)^{5/2}}$$

$$\int_0^\infty \frac{z^2 \cos \lambda z\ dz}{(\rho^2 - 2\rho\ x_o \cos \phi_o + x_o^2 + z^2)^{5/2}} = \frac{\lambda^2\ K_1(Z)}{3Z} - \frac{\lambda^2\ K_o(Z)}{3} \qquad \text{(A-3.14)}$$

$$\frac{d^2}{du^2}(uK_1(bu)) = b\,\frac{d^2}{d(bu)^2}(ubK_1(bu)) = \sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o^2}\;\frac{d^2}{dZ^2}(ZK_1(Z)) \quad .$$

Evaluation of the identity given in Equation (A-3.7) for the case of $\nu = 1$ yields:

$$\frac{d}{dZ}(ZK_1(Z)) = -Z^1\, K_o\,(Z)$$

Therefore,

$$\frac{d^2}{dZ^2}(ZK_1(Z)) = \frac{d}{dZ}(-ZK_o(Z)) = -Z\,\frac{dK_o(Z)}{dZ} - K_o(Z)$$

$$\frac{d^2}{du^2}(uK_1(bu)) = \sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o^2}\;\left(-Z\,\frac{dK_o(Z)}{dZ} - K_o(Z)\right)$$

but $\dfrac{dK_o(Z)}{dZ} = -\,K_1(Z)$

therefore,

$$\frac{d^2}{du^2}(uK_1(bu)) = \sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o^2}\;\left(Z\,K_1(Z) - K_o(Z)\right) \qquad \text{(A-3.15)}$$

Again, use the formula from Ditkin and Prudnikov, pg 173 .

<u>Case: $n = 1$, $\nu = 1$</u>

$$f(t) = t^2/(b^2 + t^2)^{-3/2}$$

$$g(y) = (-1)^1 \sqrt{\pi}\,(2b)^{-1} \left(\Gamma(\tfrac{1}{2} + 1)\right)^{-1} \frac{d^2}{du^2} (u^1 K_1(bu))$$

$$\Gamma(1.5) = \sqrt{\pi}/2$$

$$g(y) = \frac{(-1)\sqrt{\pi}}{2b} \frac{2}{\sqrt{\pi}} \left(\sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o^2}\,(ZK_1(Z) - K_o(Z))\right)$$

Note that the nomenclature is somewhat mixed in this expression, however, it should be noted that "b" from Ditkin and Prudnikov is the same as $\sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o^2}$. Therefore, the above expression reduces to:

$$g(y) = K_o(Z) - Z\,K_1(Z)$$

and the following identity has been shown:

$$\int_0^\infty \frac{z^2 \cos\lambda z\, dz}{(\rho^2 - 2\rho x_o \cos\phi_o + x_o^2 + z^2)^{3/2}} = K_o(Z) - Z\,K_1(Z) \qquad \text{(A-3.16)}$$

Summary

$$\int_0^\infty \frac{\cos \lambda z \; dz}{(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 + z^2)^{1/2}} = K_o(Z) \qquad \text{(A-3.4)}$$

$$\int_0^\infty \frac{\cos \lambda z \; dz}{(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 + z^2)^{3/2}} = \frac{\lambda^2 \; K_1(Z)}{Z} \qquad \text{(A-3.5)}$$

$$\int_0^\infty \frac{z^2 \cos \lambda z \; dz}{(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 + z^2)^{5/2}} = \frac{\lambda^2 \; K_1(Z)}{3Z} - \frac{\lambda^2 \; K_o(Z)}{3} \qquad \text{(A-3.14)}$$

$$\int_0^\infty \frac{z^2 \cos \lambda z \; dz}{(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 + z^2)^{3/2}} = K_o(Z) - Z \; K_1(Z) \qquad \text{(A-3.16)}$$

Substitution of Equations (A-3.4), (A-3.5), (A-3.14) and (A-3.16) into the right hand side of Equation (A-3.1) yields:

$$\frac{Ua}{2\pi}\frac{\partial}{\partial\phi}\Bigg(\int_0^\infty \frac{3}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{1/2}}$$

$$+ \frac{3\,z^2 + a^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{3/2}}$$

$$- \frac{3a^2 z^2}{(\rho^2 - 2\rho x_o \cos\phi + x_o^2 + z^2)^{5/2}}\Bigg)_{\phi=\phi_o} \cos\lambda z \, dz$$

$$= \frac{Ua}{2\pi}\frac{\partial}{\partial\phi}\Bigg(3K_o(Z) + 3K_o(Z) - 3ZK_1(Z) + \frac{a^2\lambda^2 K_1(Z)}{Z}$$

$$- 3a^2\left(\frac{\lambda^2 K_1(Z)}{3Z} - \frac{\lambda^2 K_o(Z)}{3}\right)\Bigg)_{\phi=\phi_o}$$

$$= \frac{Ua}{2\pi}\frac{\partial}{\partial\phi}\left(6K_o(Z) - 3ZK_1(Z) + a^2\lambda\, K_o(Z)\right)_{\phi=\phi_o} \qquad \text{(A-3.17)}$$

Take the derivative with respect to ϕ of Equation (A-3.17)

$$\frac{\partial}{\partial\phi} = \frac{\partial}{\partial Z}\frac{\partial Z}{\partial\phi}$$

From Lebedev,

$$\frac{d}{dZ}(Z^{\nu}K_{\nu}(Z)) = -Z^{\nu}K_{\nu-1}(Z)$$

which for = 1 becomes:

$$\frac{d}{dZ}\ (ZK_1(Z)) = -\ Z\ K_o(Z)$$

Also,

$$\frac{d}{dZ}\ K_o(Z) = -\ K_1(Z)$$

$$\frac{\partial Z}{\partial \phi} = \frac{\partial}{\partial \phi}\ (\ \lambda\sqrt{\rho^2 - 2\rho x_o\ \cos\ \phi\ + x_o^2}\ )$$

$$= \lambda(\tfrac{1}{2})(\rho^2 - 2\rho x_o \cos\ \phi + x_o^2\ )^{-\frac{1}{2}}\ 2\rho x_o\ \sin\ \phi$$

$$\frac{\partial Z}{\partial \phi} = \frac{\lambda^2 \rho x_o\ \sin\ \phi}{Z}$$

$$\frac{\partial}{\partial \phi}\{K_o(Z)\} = \frac{\partial}{\partial \phi}\frac{dK_o(Z)}{dZ} = \lambda^2 \rho x_o \sin\ \phi\ (-K_1(Z)/Z) \qquad \text{(A-3.18)}$$

$$\frac{\partial}{\partial \phi}\ \{ZK_1(Z)\} = \frac{\partial Z}{\partial \phi}\frac{d}{dZ}\ (ZK_1(Z))$$

$$= \frac{\lambda^2 \rho x_o\ \sin}{Z}\ (-ZK_o(Z))$$

$$\frac{\partial}{\partial \phi}\{ZK_1(Z)\} = -\lambda^2 \rho x_o\ \sin\ \phi\ K_o(Z) \qquad \text{(A-3.19)}$$

Substitution of Equation (A-3.18) and (A-3.19) into Equation (A-3.17) yields:

$$\int_0^\infty \Big(\{ C\,(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)$$

$$+ B\,(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)\} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \{ C\,(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$- B\,(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o)\} \frac{\tau K_{i\tau}(\lambda\rho)}{\rho}$$

$$+ \frac{A\rho}{2} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} \tau \sinh\tau\phi_o + AK_{i\tau}(\lambda\rho)\, \tau\sinh\tau\phi_o \Big) d\tau =$$

$$\frac{Ua}{2\pi} \left(-6\lambda^2\rho x_o \sin\phi_o \frac{K_1(Z)}{Z} + 3\lambda^2\rho x_o \sin\phi_o K_o(Z) - a^2\lambda^2\lambda^2\rho x_o \sin\phi_o \frac{K_1(Z)}{Z} \right)$$

(A-3.20)

Separate the constants A, B, and C into two parts as given by Equations (3-14) through (3-16) and drop the higher order term from the right hand side of Equation (A-3.20). The result is Equation (3-19).

APPENDIX B

Equation (3-17) may be subtracted from (3-18) to yield:

$$\int_0^\infty \Big((B_1 \cos\phi_o \sinh\tau\phi_o + C_1 \sin\phi_o \cosh\tau\phi_o)\, \tau K_{i\tau}(\lambda\rho)$$

$$+ \frac{A_1\rho}{2}\, \tau K_{i\tau}(\lambda\rho) \sinh\tau\phi_o \Big)\, d\tau =$$

$$\frac{3Ua}{2\pi} \Big((x_o \cos\phi_o - \rho)\, \lambda^2 \rho x_o \sin\phi_o \frac{K_1(Z)}{Z} \Big) \qquad \text{(B-1)}$$

The derivative of Equation (B-1) with respect to ρ is:

$$\int_0^\infty \Big((B_1 \cos\phi_o \sinh\tau\phi_o + C_1 \sin\phi_o \cosh\tau\phi_o)\, \tau \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \frac{A_1}{2} \Big(K_{i\tau}(\lambda\rho) + \frac{\rho\,\partial K_{i\tau}(\lambda\rho)}{\partial\rho}\Big)\, \tau \sinh\tau\phi_o \Big)\, d\tau =$$

$$\frac{3Ua}{2\pi} \Big((x_o \cos\phi_o - 2\rho)\, \lambda^2 x_o \sin\phi_o \frac{K_1(Z)}{Z}$$

$$+ (x_o \cos\phi_o - \rho)\lambda^2 \rho x_o \sin\phi_o \frac{\partial}{\partial\rho} \Big(\frac{K_1(Z)}{Z} \Big)\Big) \qquad \text{(B-2)}$$

where $\frac{\partial}{\partial\rho} \Big(\frac{K_1(Z)}{Z} \Big) = \frac{\partial}{\partial Z} \Big(\frac{K_1(Z)}{Z} \Big) \frac{\partial Z}{\partial\rho}$

$$\frac{\partial}{\partial\rho} \Big(\lambda\sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o^2}\Big) = \frac{\partial Z}{\partial\rho}$$

$$\frac{\partial Z}{\partial \rho} = \tfrac{1}{2}\,(\lambda)\,(\rho^2 - 2\rho\, x_o \cos\phi_o + x_o^2\,)^{-\frac{1}{2}}\,(2\rho - 2x_o \cos\phi_o)$$

$$\frac{\partial Z}{\partial \rho} = \frac{\lambda^2\,(\rho - x_o \cos\phi_o)}{Z}$$

Evaluate $\frac{\partial}{\partial Z}\left(\frac{K_1(Z)}{Z}\right)$

$$\frac{d}{dZ}(Z^{-\nu}K_\nu(Z)) = -\,Z^{-\nu}\,K_{\nu+1}(Z) \qquad \text{(B-3)}$$

$$\frac{d}{dZ}\,(K_1(Z)/Z) = -\,K_2(Z)/Z$$

$$\frac{\partial}{\partial \rho}\,(K_1(Z)/Z) = \frac{\lambda^2(x_o \cos\phi_o - \rho)\,K_2(Z)}{Z^2} \qquad \text{(B-4)}$$

Substitution of Equation (B-4) into Equation (B-2) and combining similar terms leads to:

$$\int_0^\infty \Big(\,(B_1 \cos\phi_o \sinh\tau\phi_o + C_1 \sin\phi_o \cosh\tau\phi_o)\,\tau\,\frac{\partial K_{i\tau}(\lambda\rho)}{\partial \rho}$$

$$+\frac{A_1}{2}\,\Big(K_{i\tau}(\lambda\rho) + \rho\,\frac{\partial K_{i\tau}(\lambda\rho)}{\partial \rho}\Big)\,\tau\sinh\tau\phi_o\Big)\,d\tau$$

$$= \frac{3Ua}{2\pi} \Big((x_o \cos \phi_o - 2\rho) \lambda^2 x_o \sin \phi_o \frac{K_1(Z)}{Z}$$

$$+ (x_o \cos \phi_o - \rho) \lambda^2 \rho x_o \sin \phi_o \frac{(\lambda^2 (x_o \cos \phi_o - \rho) K_2(Z))}{Z^2} \Big)$$

Rearrangement of this expression yields:

$$\int_0^\infty \Big((B_1 \cos \phi_o \sinh \tau\phi_o + C_1 \sin \phi_o \cosh \tau\phi_o) \tau \frac{\partial K_{i\tau}(\lambda\rho)}{\partial \rho}$$

$$+ \frac{A_1}{2} \left(K_{i\tau}(\lambda\rho) + \rho \frac{\partial K_{i\tau}(\lambda\rho)}{\partial \rho} \right) \tau \sinh \tau\phi_o \Big) d\tau =$$

$$= \frac{3Ua}{2\pi} \Big((x_o \cos \phi_o - 2\rho) \lambda^2 x_o \sin \phi_o \frac{K_1(Z)}{Z}$$

$$+ (x_o^2 - x_o^2 \sin^2 \phi_o - 2\rho x_o \cos \phi_o + \rho^2) \lambda^4 \rho x_o \sin \phi_o \frac{K_2(Z)}{Z^2} \Big)$$

(B-5)

Subtract Equation (B-5) from Equation (3-19)

$$\int_0^\infty \Big(\{ C_1(\tau \sin \phi_o \cosh \tau\phi_o + \cos \phi_o \sinh \tau\phi_o)$$

$$+ B_1(\tau\cos \phi_o \sinh \tau\phi_o - \sin \phi_o \cosh \tau\phi_o) \} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \{ C_1(\tau \cos \phi_o \sinh \tau\phi_o - \sin \phi_o \cosh \tau\phi_o)$$

$$- B_1(\tau \sin \phi_o \cosh \tau\phi_o + \cos \phi_o \sinh \tau\phi_o) \} \frac{\tau K_{i\tau}(\lambda\rho)}{\rho}$$

$$+ \frac{A_1\rho}{2} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} \tau \sinh \tau\phi_o + A_1 K_{i\tau}(\lambda\rho) \tau \sinh \tau\phi_o\Big) d\tau$$

$$- \int_0^\infty \Big((B_1 \cos \phi_o \sinh \tau\phi_o + C_1 \sin \phi_0 \cosh \tau\phi_o) \frac{\tau\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$- \frac{A_1}{2} (K_{i\tau}(\lambda\rho) + \rho \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}) \tau \sinh \tau\phi_o \Big) d\tau$$

$$= \frac{3Ua}{2\pi} \Big(\lambda^4\rho x_o^3 \sin^3\phi_o \frac{K_2(Z)}{Z^2} - \lambda^2\rho x_o \sin \phi_o K_2(Z)$$

$$- \lambda^2 x_o^2 \sin \phi_o \cos \phi_o \frac{K_1(Z)}{Z} + \lambda^2 x_o \rho \sin \phi_o K_o(Z)\Big)$$

but $\lambda^2 \rho x_o \sin\phi_o \, (K_o(Z) - K_2(Z)) = \lambda^2 \rho x_o \sin\phi_o \, (-2K_1(Z)/Z)$

since, $K_o(Z) - K_2(Z) = -2K_1(Z)/Z$

therefore,

$$\int_0^\infty \Big(\{ C_1 \cos\phi_o \sinh\tau\phi_o - B_1 \sin\phi_o \cosh\tau\phi_o \} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \{ C_1(\tau\cos\phi_o \sinh\tau\phi_o - \sin\phi_o \cosh\tau\phi_o)$$

$$- B_1(\tau\sin\phi_o \cosh\tau\phi_o + \cos\phi_o \sinh\tau\phi_o) \} \frac{\tau K_{i\tau}(\lambda\rho)}{\rho}$$

$$+ \frac{A_1}{2} K_{i\tau}(\lambda\rho)\, \tau \sinh\tau\phi_o \Big) \, d\tau$$

$$= \frac{3Ua}{2\pi} \Big(\lambda^4 \rho x_o^3 \sin^3\phi_o \, (K_2(Z)/Z^2) - 2\lambda^2 \rho x_o \sin\phi_o \, (K_1(Z)/Z)$$

$$- \lambda^2 x_o^2 \sin\phi_o \cos\phi_o \, (K_1(Z)/Z) \Big) \qquad \text{(B-6)}$$

From Equation (B-6), subtract Equation (B-1) divided by ρ:

$$\int_0^\infty \Big((C_1 \cos \phi_o \sinh \tau\phi_o - B_1 \sin \phi_o \cosh \tau\phi_o) \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

$$+ \{ C_1(\tau \cos \phi_o \sinh \tau\phi_o - 2 \sin \phi_o \cosh \tau\phi_o)$$

$$- B_1(\tau \sin \phi_o \cosh \tau\phi_o + 2 \cos \phi_o \sinh \tau\phi_o) \} \frac{\tau K_{i\tau}(\lambda\rho)}{\rho} \Big) d\tau$$

$$= \frac{3Ua}{2\pi} \Big(\lambda^4 \rho x_o^3 \sin^3 \phi_o (K_2(Z)/Z^2) - \lambda^2 \rho x_o \sin \phi_o (K_1(Z)/Z)$$

$$- 2 \lambda^2 x_o^2 \sin \phi_o \cos \phi_o (K_1(Z)/Z) \Big) \qquad \text{(B-7)}$$

The derivative of Equation (3-17) with respect to ρ is:

$$\int_0^\infty (C_1 \cos \phi_o \sinh \tau\phi_o - B_1 \sin \phi_o \cosh \tau\phi_o) \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} d\tau$$

$$= \frac{3Ua}{2\pi} x_o \sin \phi_o \frac{\partial}{\partial\rho} (K_o(Z)) \qquad \text{(B-8)}$$

where $\frac{\partial}{\partial\rho} K_o(Z) = \frac{\partial Z}{\partial\rho} \frac{dK_o(Z)}{dZ}$

and $$\frac{\partial Z}{\partial\rho} = \frac{\lambda^2(\rho - x_o \cos \phi_o)}{Z} \qquad \text{(B-9)}$$

$$(dK_o(Z)/dZ) = -K_1(Z) \tag{B-10}$$

Substitution of Equations (B-9) and (B-10) into Equation (B-8) yields:

$$\int_0^\infty \{C_1 \cos \phi_o \sinh \tau\phi_o - B_1 \sin \phi_o \cosh \tau\phi_o\} \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} d\tau$$

$$= \frac{3Ua}{2\pi} x_o \sin \phi_o \lambda^2 (x_o \cos \phi_o - \rho) \frac{K_1(Z)}{Z} \tag{B-11}$$

Substitution of Equation (B-11) into Equation (B-7) and rearrangement of the result yields:

$$\int_0^\infty \Big((C_1 \cos \phi_o \sinh \tau\phi_o - B_1 \sin \phi_o \cosh \tau\phi_o) \frac{\tau^2 K_{i\tau}(\lambda\rho)}{\rho}$$

$$- 2(C_1 \sin \phi_o \cosh \tau\phi_o + B_1 \cos \phi_o \sinh \tau\phi_o) \frac{\tau K_{i\tau}(\lambda\rho)}{\rho} \Big) d\tau$$

$$= \frac{3Ua}{2\pi} \Big(\lambda^4 \rho x_o^3 \sin^3\phi_o \frac{K_2(Z)}{Z^2} - 3\lambda^2 x_o^2 \sin \phi_o \cos \phi_o \frac{K_1(Z)}{Z}\Big) \tag{B-12}$$

Substitution of Equation (C-21) from Appendix C into Equation (B-12) results in:

$$\int_0^\infty \Big(\{ C_1 \cos\phi_o \sinh\tau\phi_o - B_1 \sin\phi_o \cosh\tau\phi_o \} \frac{\tau^2 K_{i\tau}(\lambda\rho)}{\rho}$$

$$- 2\,(C_1 \sin\phi_o \cosh\tau\phi_o + B_1 \cos\phi_o \sinh\tau\phi_o) \frac{K_{i\tau}(\lambda\rho)}{\rho} \Big)\, d\tau$$

$$= \frac{3Ua}{2\pi} \Big(\frac{2x_o \sin\phi_o}{\pi} \int_0^\infty \frac{\tau^2 K_{i\tau}(\lambda\rho)}{\rho} K_{i\tau}(\lambda x_o) \cosh\tau(\pi - \phi_o)\, d\tau$$

$$+ \lambda^2 x_o^2 \sin\phi_o \cos\phi_o \frac{K_1(Z)}{Z} - 3\lambda^2 x_o^2 \sin\phi_o \cos\phi_o \frac{K_1(Z)}{Z} \Big)$$

(B-13)

A basic identity for cylindrical functions (See Lebedev) is:

$$K_o(Z) = \frac{2}{\pi}\int_0^\infty K_{i\tau}(\lambda\rho)\, K_{i\tau}(\lambda x_o) \cosh\tau(\pi - \phi_o)\, d\tau \qquad \text{(B-14)}$$

Substitution of Equation (B-14) into Equation (3-17) results in:

$$\int_0^\infty (C_1 \cos\phi_o \sinh\tau\phi_o - B_1 \sin\phi_o \cosh\tau\phi_o)\, K_{i\tau}(\lambda\rho)\, d\tau$$

$$= \frac{3Ua}{2\pi} x_o \sin\phi_o \Big(\frac{2}{\pi} \int_0^\infty K_{i\tau}(\lambda x_o)\, K_{i\tau}(\lambda\rho) \cosh\tau(\pi - \phi_o)\, d\tau \Big)$$

(B-15)

Equation (B-15) will be satisfied if

$$C_1 \cos\phi_o \sinh\tau\phi_o - B_1 \sin\phi_o \cosh\tau\phi_o$$

$$= \frac{3Ua}{\pi^2} x_o \sin\phi_o K_{i\tau}(\lambda\rho) \cosh\tau(\pi - \phi_o) \qquad \text{(B-16)}$$

Substitution of Equation (B-16) into Equation (B-13) yields:

$$\int_0^\infty \Big((\frac{3Ua}{\pi^2} x_o \sin\phi_o K_{i\tau}(\lambda x_o) \cosh\tau(\pi - \phi_o)) \frac{\tau^2 K_{i\tau}(\lambda\rho)}{\rho}$$

$$- 2(C_1 \sin\phi_o \cosh\tau\phi_o + B_1 \sin\phi_o \sinh\tau\phi_o) \frac{\tau K_{i\tau}(\lambda\rho)}{\rho} \Big) d\tau$$

$$= \int_0^\infty \Big(\frac{3Ua}{\pi^2} x_o \sin\phi_o K_{i\tau}(\lambda x_o) \cosh\tau(\pi - \phi_o) \Big) \frac{\tau^2 K_{i\tau}(\lambda\rho)}{\rho} d\tau$$

$$- \frac{3Ua}{\pi} \lambda^2 x_o^2 \sin\phi_o \cos\phi_o \frac{K_1(Z)}{Z} \qquad \text{(B-17)}$$

Simplifying,

$$-2 \int_0^\infty \Big((C_1 \sin\phi_o \cosh\tau\phi_o + B_1 \cos\phi_o \sinh\tau\phi_o) \frac{\tau K_{i\tau}(\lambda\rho)}{\rho} \Big) d\tau$$

$$= - \frac{3Ua}{\pi} \lambda^2 x_o^2 \sin\phi_o \cos\phi_o \frac{K_1(Z)}{Z} \qquad \text{(B-18)}$$

Substitution of Equation (C-22) into Equation (B-18) yields:

$$-2\int_0^\infty \left((C_1 \sin \phi_o \cosh \tau\phi_o + B_1 \cos \phi_o \sinh \tau\phi_o) \frac{\tau K_{i\tau}(\lambda\rho)}{\rho}\right) d\tau$$

$$= - \frac{3Ua}{\pi} \lambda^2 x_o^2 \sin \phi_o \cos \phi_o \frac{2}{\pi\lambda^2 \rho x_o \sin \phi_o} \int_0^\infty K_{i\tau}(\lambda\rho) K_{i\tau}(\lambda x_o) \tau \sinh \tau(\pi - \phi_o) \, d\tau$$

(B-19)

or

$$- \int_0^\infty (C_1 \sin \phi_o \cosh \tau\phi_o + B_1 \cos \phi_o \sinh \tau\phi_o) \frac{\tau K_{i\tau}(\lambda\rho)}{\rho} d\tau$$

$$= - \frac{3Ua}{\pi^2} x_o \cos \phi_o \int_0^\infty (K_{i\tau}(\lambda x_o) \sinh \tau(\pi - \phi_o) \frac{\tau K_{i\tau}(\lambda\rho)}{\rho} d\tau$$

(B-20)

Equation (B-20) will be satisfied if

$$C_1 \sin \phi_o \cosh \tau\phi_o + B_1 \cos \phi_o \sinh \tau\phi_o =$$

$$\frac{3Ua}{\pi^2} x_o \cos \phi_o K_{i\tau}(\lambda x_o) \sinh \tau(\pi - \phi_o) \qquad \text{(B-21)}$$

Substitution of Equation (B-21) into Equation (B-1) yields:

$$\int_0^\infty \left(\frac{3Ua}{\pi^2} x_o \cos\phi_o K_{i\tau}(\lambda x_o)\tau\sinh\tau(\pi - \phi_o) K_{i\tau}(\lambda\rho) + \frac{A_1\rho}{2} K_{i\tau}(\lambda\rho)\tau\sinh\tau\phi_o\right) d\tau$$

$$= \frac{3Ua}{2\pi} \left((x_o \cos\phi_o - \rho)\lambda^2\rho x_o \sin\phi_o \frac{K_1(Z)}{Z}\right) \qquad \text{(B-22)}$$

Substitution of Equation (B-19) into Equation (B-22) results in:

$$\int_0^\infty \left(\frac{3Ua}{\pi^2} x_o \cos\phi_o K_{i\tau}(\lambda x_o) \sinh\tau(\pi - \phi_o) \tau K_{i\tau}(\lambda\rho)\right.$$

$$\left. + \frac{A_1\rho}{2} \tau K_{i\tau}(\lambda\rho) \sinh\tau\phi_o\right) d\tau$$

$$= \frac{3Ua}{2\pi} \left((x_o \cos\phi_o - \rho)\lambda^2\rho\, x_o \sin\phi_o \frac{2}{\pi\lambda^2\rho x_o \sin\phi_o}\right)$$

$$\cdot \left(\int_0^\infty K_{i\tau}(\lambda\rho) K_{i\tau}(\lambda x_o) \tau \sinh\tau(\pi - \phi_o) d\tau \right. \qquad \text{(B-23)}$$

Simplifying Equation (B-23):

$$\int_0^\infty \frac{A_1\rho}{2} \tau \sinh\tau\phi_o K_{i\tau}(\lambda\rho) d\tau$$

$$= -\int_0^\infty \frac{3Ua}{\pi^2} \rho K_{i\tau}(\lambda x_o) K_{i\tau}(\lambda\rho) \tau \sinh\tau(\pi - \phi_o) d\tau \qquad \text{(B-24)}$$

Equation (B-24) will be satisfied if

$$A_1 = -\frac{6Ua}{\pi^2}\frac{\sinh \tau(\pi - \phi_o)}{\sinh \tau\phi_o} K_{i\tau}(\lambda x_o) \tag{B-25}$$

The expressions for B_1 and C_1 may be obtained by simultaneous solution of Equations (B-16) and (B-21).

Using Kramer's rule, C_1 may be written in terms of determinants:

$$C_1 = \frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o) \frac{\begin{vmatrix} \sin\phi_o \cosh\tau(\pi - \phi_o) & -\sin\phi_o \cosh\tau\phi_o \\ \cos\phi_o \sinh\tau(\pi - \phi_o) & \cos\phi_o \sinh\tau\phi_o \end{vmatrix}}{\begin{vmatrix} \cos\phi_o \sinh\tau\phi_o & -\sin\phi_o \cosh\tau\phi_o \\ \sin\phi_o \cosh\tau\phi_o & \cos\phi_o \sinh\tau\phi_o \end{vmatrix}} \tag{B-26}$$

Expanding the determinants in the numerator and the denominator of Equation (B-26):

$$C_1 = \frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o) \cdot \left(\frac{\sin\phi_o \cos\phi_o \sinh\tau\phi_o \cosh\tau(\pi-\phi_o) + \sin\phi_o \cosh\tau\phi_o \cos\phi_o \sinh\tau(\pi-\phi_o)}{\cos^2\phi_o \sinh^2\tau\phi_o + \sin^2\phi_o \cosh^2\tau\phi_o} \right) \tag{B-27}$$

In a similar fashion, the constant B_1 may be solved for explicitly using Kramer's rule and written in terms of determinants:

$$B_1 = \frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o) \frac{\begin{vmatrix} \cos\phi_o \sinh\tau\phi_o & \sin\phi_o \cosh\tau(\pi-\phi_o) \\ \sin\phi_o \cosh\tau\phi_o & \cos\phi_o \sinh\tau(\pi-\phi_o) \end{vmatrix}}{\begin{vmatrix} \cos\phi_o \sinh\tau\phi_o & -\sin\phi_o \cosh\tau\phi_o \\ \sin\phi_o \cosh\tau\phi_o & \cos\phi_o \sinh\tau\phi_o \end{vmatrix}}$$

(B-28)

Expanding the determinants in the numerator and the denominator of Equation (B-28):

$$B_1 = \frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o) \cdot$$

$$\left(\frac{\cos^2\phi_o \sinh\tau\phi_o \sinh\tau(\pi-\phi_o) - \sin^2\phi_o \cosh\tau\phi_o \cosh\tau(\pi-\phi_o)}{\cos^2\phi_o \sinh^2\tau\phi_o + \sin^2\phi_o \cosh^2\tau\phi_o} \right)$$

(B-29)

The numerator of Equation (B-27) can be simplified:

$$\sinh\tau(\pi-\phi_o) \cosh\tau\phi_o + \cosh\tau(\pi-\phi_o) \sinh\tau\phi_o =$$

$$(\sinh\tau\pi \cosh\tau\phi_o - \cosh\tau\pi \sinh\tau\phi_o)\cosh\tau\phi_o$$

$$+ (\cosh \tau\phi_o \cosh \tau\pi - \sinh \tau\pi \sinh \tau\phi_o) \sinh \tau\phi_o$$

$$= \cosh^2 \tau\phi_o \sinh \tau\pi - \sinh \tau\phi_o \cosh \tau\phi_o \cosh \tau\pi$$

$$+ \sinh \tau\phi_o \cosh \tau\phi_o \cosh \tau\pi - \sinh^2 \tau\phi_o \sinh \tau\pi$$

$$= (\cosh^2 \tau\phi_o - \sinh^2 \tau\phi_o) \sinh \tau\pi$$

$$= \sinh \tau\pi$$

Therefore,

$$C_1 = \frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o) \sin \phi_o \cos \phi_o \sinh \tau\pi / (\cos^2\phi_o \sinh^2\tau\phi_o + \sin^2\phi_o \cosh^2\tau\phi_o)$$

(B-30)

Note that it is also possible to simplify the denominator of B_1 and C_1 for a more concise expression:

$$\cos^2 \phi_o \sinh^2\tau\phi_o + \sin^2\phi_o \cosh^2 \tau\phi_o =$$

$$\cos^2 \phi_o \sinh^2 \tau\phi_o + (1 - \cos^2 \phi_o) \cosh^2\tau\phi_o$$

$$\cos^2 \phi_o \sinh^2 \tau\phi_o + \sin^2 \phi_o \cosh^2 \tau\phi_o$$

$$= \cos^2 \phi_o (\sinh^2 \tau\phi_o - \cosh^2 \tau\phi_o) + \cosh^2 \tau\phi_o$$

$$\cos^2 \phi_o \sinh^2 \tau\phi_o + \sin^2 \phi_o \cosh^2 \tau\phi_o = - \cos^2\phi_o + \cosh^2 \tau\phi_o \tag{B-31}$$

Substitution of Equation (B-31) into Equations (B-30) and (B-29) yields:

$$C_1 = \frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o) \sin \phi_o \cos \phi_o \sinh \tau\pi/(\cosh^2 \tau\phi_o - \cos^2 \phi_o) \tag{B-32}$$

$$B_1 = \frac{3Ua}{\pi^2} x_o K_{i\tau}(\lambda x_o) \cdot \left(\frac{\cos^2\phi_o \sinh\tau\phi_o \sinh\tau(\pi-\phi_o) - \sin^2\phi_o \cosh\tau\phi_o \cosh\tau(\pi-\phi_o)}{(\cosh^2\tau\phi_o - \cos^2\phi_o)} \right) \tag{B-33}$$

APPENDIX C DERIVATION OF IDENTITIES

The following identity may be found in Lebedev

$$K_o(Z) = \frac{2}{\pi} \int_0^\infty K_{i\tau}(\lambda x_o)\ K_{i\tau}(\lambda\rho)\ \cosh\ \tau(\pi - \phi_o)\ d\tau \qquad \text{(C-1)}$$

where, $Z = \lambda\sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o^2}$

$$\frac{\partial\{K_o(Z)\}}{\partial\rho} = \frac{\partial Z}{\partial\rho}\frac{dK_o(Z)}{dZ} \qquad \text{(C-2)}$$

$$\frac{\partial Z}{\partial\rho} = \frac{\lambda^2(\rho - x_o \cos\phi_o)}{Z} \qquad \text{(B-9)}$$

$$\frac{dK_o(Z)}{dZ} = -K_1(Z)$$

$$\frac{\partial K_o(Z)}{\partial\rho} = -\lambda^2(\rho - x_o \cos\phi_o)\frac{K_1(Z)}{Z} \qquad \text{(C-3)}$$

Take the derivative of Equation (C-1) with respect to ρ:

$$\frac{\partial K_o(Z)}{\partial\rho} = \frac{2}{\pi}\int_0^\infty K_{i\tau}(\lambda x_o)\frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}\ \cosh\ \tau(\pi - \phi_o)\ d\tau \qquad \text{(C-4)}$$

Substitute Equation (C-3) into Equation (C-4):

$$\lambda^2(\rho - x_o \cos\phi_o)\frac{K_1(Z)}{Z} = -\frac{2}{\pi}\int_0^\infty K_{i\tau}(\lambda x_o)\frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}\cosh\tau(\pi-\phi_o)\,d\tau \tag{C-5}$$

Take the derivative with respect to ρ of Equation (C-5).
The left hand side becomes:

$$\frac{\partial}{\partial\rho}\left(\lambda^2(\rho - x_o\cos\phi_o)\frac{K_1(Z)}{Z}\right) = \lambda^2\left((\rho - x_o\cos\phi_o)\frac{\partial}{\partial\rho}\left(\frac{K_1(Z)}{Z}\right) + \frac{K_1(Z)}{Z}\right) \tag{C-6}$$

$$\frac{\partial}{\partial\rho}\left(\frac{K_1(Z)}{Z}\right) = \frac{\lambda^2(\rho - x_o\cos\phi_o)}{Z}\frac{d}{dZ}(K_1(Z)/Z) \tag{C-7}$$

$$\frac{d}{dZ}(Z^{-1}K_1(Z)) = -Z^{-1}K_2(Z) \tag{C-8}$$

Substitution of Equation (C-8) into (C-7) results in:

$$\frac{\partial}{\partial\rho}(K_1(Z)/Z) = \frac{\lambda^2(\rho - x_o\cos\phi_o)}{Z}(-K_2(Z)/Z) \tag{C-9}$$

Combination of Equations (C-6) and (C-9) results in:

$$\frac{\partial}{\partial\rho}\{\lambda^2(\rho - x_o \cos\phi_o)\frac{K_1(Z)}{Z}\} =$$

$$\lambda^2 \left((\rho - x_o \cos\phi_o)\frac{\lambda^2 (\rho - x_o \cos\phi_o)}{Z}\left(\frac{-K_2(Z)}{Z}\right) + \frac{K_1(Z)}{Z}\right)$$

(C-10)

The derivative with respect to ρ of Equation (C-5) is:

$$\frac{\partial}{\partial\rho}\left(\lambda^2(\rho - x_o \cos\phi_o)\frac{K_1(Z)}{Z}\right) = \frac{-2}{\pi}\int_0^\infty K_{i\tau}(\lambda x_o)\frac{\partial^2 K_{i\tau}(\lambda\rho)}{\partial\rho^2}\cosh\tau(\pi-\phi_o)\,d\tau$$

(C-11)

Substitution of Equation (C-10) into (C-11) results in:

$$\lambda^2\left(\lambda^2(\rho - x_o \cos\phi_o)^2(-K_2(Z)/Z^2) + K_1(Z)/Z\right) =$$

$$-\frac{2}{\pi}\int_0^\infty K_{i\tau}(\lambda x_o)\frac{\partial^2 K_{i\tau}(\lambda\rho)}{\partial\rho^2}\cosh\tau(\pi - \phi_o)\,d\tau$$

(C-12)

The left hand side of Equation (C-12) may be simplified:

$$\lambda^2(\rho - x_o \cos\phi_o)^2 = \lambda^2(\rho^2 - 2\rho x_o \cos\phi_o + x_o^2\cos^2\phi_o)$$

$$= \lambda^2(\rho^2 - 2\rho x_o \cos\phi_o + x_o^2(1 - \sin^2\phi_o))$$

$$\lambda^2(\rho - x_o \cos \phi_o)^2 = \lambda^2(\rho^2 - 2\rho x_o \cos \phi_o + x_o^2 - x_o^2 \sin^2 \phi_o)$$

$$\lambda^2(\rho - x_o \cos \phi_o)^2 = Z^2 - \lambda^2 x_o^2 \sin^2 \phi_o \qquad \text{(C-13)}$$

From Lebedev (page 110):

$$K_2(Z) = K_o(Z) + 2K_1(Z)/Z \qquad \text{(C-14)}$$

Substitution of Equations (C-13) and (C-14) into Equation (C-12) results in:

$$\lambda^2\left(\{\lambda^2 x_o^2 \sin^2 \phi_o K_2(Z)/Z^2\} - K_o(Z) - K_1(Z)/Z \right) =$$

$$-\frac{2}{\pi} \int_0^\infty K_{i\tau}(\lambda x_o) \frac{\partial^2 K_{i\tau}(\lambda\rho)}{\partial\rho^2} \cosh \tau(\pi - \phi_o)\, d\tau \qquad \text{(C-15)}$$

Substitution of Equation (C-1) into (C-15) results in:

$$\lambda^2\{\lambda^2 \, x_o^2 \, \sin^2\phi_o \, \frac{K_2(Z)}{Z^2} - \frac{K_1(Z)}{Z}\} =$$

$$\frac{2}{\pi}\int_0^\infty \left(\lambda^2 K_{i\tau}(\lambda\rho) - \frac{\partial^2 K_{i\tau}(\lambda\rho)}{\partial\rho^2}\right) K_{i\tau}(\lambda x_o) \cosh \tau(\pi - \phi_o) \, d\tau$$

(C-16)

Bessel's Equation of order $i\tau$ may be written

$$\frac{\partial^2 K_{i\tau}(\lambda\rho)}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} + \left(\frac{\tau^2}{\rho^2} - \lambda^2\right)K_{i\tau}(\lambda\rho) = 0$$

(C-17)

Rearrangement of Equation (C-17) yields:

$$\lambda^2 K_{i\tau}(\lambda\rho) - \frac{\partial^2 K_{i\tau}(\lambda\rho)}{\partial\rho^2} = \frac{\tau^2}{\rho^2} K_{i\tau}(\lambda\rho) + \frac{1}{\rho}\frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho}$$

(C-18)

Substitution of Equation (C-18) into (C-16) and multiplication by ρ yields:

$$\lambda^2(\lambda^2 \rho x_o^2 \sin^2\phi_o \frac{K_2(Z)}{Z^2} - \rho \frac{K_1(Z)}{Z}) =$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{\tau^2}{\rho} K_{i\tau}(\lambda\rho) + \frac{\partial K_{i\tau}(\lambda\rho)}{\partial\rho} \right) K_{i\tau}(\lambda x_o) \cosh \tau(\pi - \phi_o)\, d\tau$$

(C-19)

Substitution of Equation (C-5) into Equation (C-19) yields:

$$\lambda^2(\lambda^2 \rho x_o^2 \sin^2\phi_o \frac{K_2(Z)}{Z^2} - \rho \frac{K_1(Z)}{Z}) =$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{\tau^2}{\rho} K_{i\tau}(\lambda\rho) K_{i\tau}(\lambda x_o) \cosh \tau(\pi - \phi_o)\, d\tau - \lambda^2(\rho - x_o \cos \phi_o) \frac{K_1(Z)}{Z}$$

(C-20)

Multiplication of Equation (C-20) by $x_o \sin \phi_o$ and rearrangement yields:

$$\lambda^2 \rho x_o^3 \sin^3 \phi_o \frac{K_2(Z)}{Z^2} = \frac{2x_o \sin \phi_o}{\pi} \int_0^\infty \frac{\tau^2 K_{i\tau}(\lambda\rho)}{\rho} K_{i\tau}(\lambda x_o) \cosh \tau(\pi - \phi_o)\, d\tau$$

$$+ \lambda^2 x_o^2 \sin \phi_o \cos \phi_o \frac{K_1(Z)}{Z} \qquad \text{(C-21)}$$

Starting with Equation (C-1) another useful identity may be derived. Take the derivative with respect to ϕ_o of both sides:

$$\frac{\partial K_o(Z)}{\partial \phi_o} = -\frac{2}{\pi}\int_0^\infty K_{i\tau}(\lambda\rho)\, K_{i\tau}(\lambda x_o)\ \tau \sinh\ \tau(\pi - \phi_o)\ d\tau$$

$$\frac{\partial K_o(Z)}{\partial \phi_o} = \frac{dK_o(Z)}{dZ}\ \frac{\partial Z}{\partial \phi_o}$$

$$\frac{\partial Z}{\partial \phi_o} = \frac{\partial}{\partial \phi_o}(\lambda\sqrt{\rho^2 - 2\rho x_o \cos\phi_o + x_o})$$

$$= \tfrac{1}{2}\,(2\rho x_o \sin\phi_o)(\rho^2 - 2\rho x_o \cos\phi_o + x_o^2)^{-\frac{1}{2}}$$

$$= \frac{\lambda\rho x_o \sin\phi_o}{(\rho^2 - 2\rho x_o \cos\phi_o + x_o^2)^{\frac{1}{2}}}$$

$$\frac{\partial Z}{\partial \phi_o} = \frac{\lambda^2\rho x_o \sin\phi_o}{Z}$$

$$\frac{dK_o(Z)}{dZ} = -K_1(Z)$$

$$\frac{\partial K_o(Z)}{\partial \phi_o} = -K_1(Z)\,\frac{\lambda^2\rho x_o \sin\phi_o}{Z}$$

$$\frac{\partial K_o(Z)}{\partial \phi_o} = \frac{-2}{\pi}\int_0^\infty K_{i\tau}(\lambda\rho)\ K_{i\tau}(\lambda x_o)\ \tau \sinh \tau(\pi - \phi_o)\ d\tau$$

Therefore,

$$\lambda^2 \rho x_o \sin \phi_o \frac{K_1(Z)}{Z} =$$

$$\frac{2}{\pi}\int_0^\infty K_{i\tau}(\lambda\rho)\ K_{i\tau}(\lambda x_o)\ \tau \sinh \tau(\pi - \phi_o)\ d\tau$$

(C-22)

APPENDIX D - ANALYTIC SOLUTION OF THE PROBLEM FOR THE TWO SPHERE CASE

$$F^{(3)} = 6\pi\mu a\bar{k} \int_0^\infty \int_0^\infty \Big((B_1 + \frac{A_1 x_o}{2}) \frac{\partial K_{i\tau}(\lambda x_o)}{\partial x_o} + $$

$$+ (\frac{C_1 \tau}{x_o} + A_1) K_{i\tau}(\lambda x_o) \Big) \; d\lambda \; d\tau \qquad (4\text{-}3)$$

Substitution of the expressions for A_1, B_1, and C_1 into Equation (4-3) results in:

$$F^{(3)} = 6\pi\mu a\bar{k} \int_0^\infty \int_0^\infty \Big(\{ \frac{3Ua}{\pi^2} (\alpha_2 - \alpha_3) \; x_o \frac{\partial K_{i\tau}(\lambda x_o)}{\partial x_o} K_{i\tau}(\lambda x_o) \}$$

$$+ \{ \frac{3Ua}{\pi^2} (-\tau\alpha_1 - 2\alpha_3) \; K^2_{i\tau}(\lambda x_o) \Big) \; d\lambda \; d\tau \qquad (D\text{-}1)$$

Since α_2, α_3, and α_1, are not functions of λ, the integrations with respect to λ may be carried out by evaluating the following two integrals:

$$\int_0^\infty K^2_{i\tau}(\lambda x_o) \; d\lambda \qquad (D\text{-}2)$$

$$\int_0^\infty x_o \frac{\partial K_{i\tau}(\lambda x_o)}{\partial x_o} K_{i\tau}(\lambda x_o)\, d\lambda \qquad \text{(D-3)}$$

The following identity is available from Magnus, Oberhettinger, and Soni (page 96):

$$K_\nu(Z)\, K_\mu(Z) = 2 \int_0^\infty K_{\nu\pm\mu}(2Z\cosh t)\, \cosh\, ((\mu \mp \nu)t)\, dt$$

$$\text{Re } Z > 0 \qquad \text{(D-4)}$$

Substitution of Equation (D-4) for the case $\nu=\mu=i\tau$, into the integral given in (D-2) yields:

$$2 \int_0^\infty \int_0^\infty K_{2i\tau}(2\lambda x_o \cosh t)\, dt\, d\lambda = \int_0^\infty K_{i\tau}^2(\lambda x_o)\, d\lambda \qquad \text{(D-5)}$$

Again, from Magnus, Oberhettinger, and Soni:

$$\int_0^\infty t^{\mu-1} K_\nu(at)\, dt = 2^{\mu-2}\, a^{-\mu}\Gamma(\tfrac{1}{2}\mu + \tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\nu) \qquad \text{(D-6)}$$

$$\text{Re } (\mu \pm \nu) > 0$$

Substitution of Equation (D-6) (for the case $a = 2x_o \cosh t$, $\nu = i\tau$, $t = \lambda$, and $\mu = 1$) into Equation (D-5) results in:

$$\int_0^\infty K_{i\tau}^2(\lambda x_o) = 2 \int_0^\infty \frac{1}{2} \frac{1}{2x_o \cosh t} \; (\tfrac{1}{2} + i\tau) \; (\tfrac{1}{2} - i\tau) \; dt \qquad \text{(D-7)}$$

but

$$\Gamma(\tfrac{1}{2} + Z) \; \Gamma(\tfrac{1}{2} - Z) = \frac{\pi}{\cos \pi Z} \qquad \text{(D-8)}$$

Substitution of Equation (D-8) into (D-7) results in:

$$\int_0^\infty K_{i\tau}^2(\lambda x_o) \; d\lambda = 2 \int_0^\infty \frac{1}{4x_o \cosh t} \; \frac{\pi}{\cos i\tau\pi} \; dt$$

(D-9)

$$= \frac{\pi}{2x_o \cosh \tau\pi} \int_0^\infty \frac{dt}{\cosh t}$$

$$= \frac{\pi}{2x_o \cosh \tau\pi} \; (\pi/2)$$

$$\int_0^\infty K_{i\tau}^2(\lambda x_o) \; d\lambda = \frac{\pi^2}{4x_o \cosh \tau\pi} \qquad \text{(D-10)}$$

Differentiation of both sides of Equation (D-10) with respect to x_o yields an identity for the integral given in Equation (D-3):

$$\int_0^{\infty} 2K_{i\tau}(\lambda x_o) \frac{\partial K_{i\tau}(\lambda x_o)}{\partial x_o} d\lambda = \frac{-\pi^2}{4x_o^2 \cosh \tau\pi} \tag{D-11}$$

or

$$\int_0^{\infty} x_o K_{i\tau}(\lambda x_o) \frac{\partial K_{i\tau}(\lambda x_o)}{\partial x_o} d\lambda = \frac{-\pi^2}{8x_o \cosh \tau\pi} \tag{D-12}$$

Substitution of Equations (D-10) and (D-12) into Equation (D-1) results in:

$$F^{(3)} = 6\pi\mu a\bar{k} \int_0^{\infty} \Big(\{\frac{3Ua}{\pi^2} (\alpha_2 - \alpha_3)\}\frac{-\pi^2}{8x_o \cosh \tau\pi}$$

$$+ \{ \frac{3Ua}{\pi^2} (-\tau\alpha_1 - \alpha_3)\}\frac{\pi^2}{4x_o \cosh \tau\pi} \Big) d\tau \tag{D-13}$$

Simplifying, and combining similar terms results in:

$$F^{(3)} = 6\pi\mu a\bar{k}\{\frac{-3}{8} \frac{a}{x_o}\} \int_0^{\infty} (\alpha_2 + 2\tau\alpha_1 + 3\alpha_3)\frac{d\tau}{\cosh \tau\pi} \tag{4-4}$$

The values α_1, α_2, and α_3, are defined in Equations (3-21), (3-22), and (3-23). For the value $\phi_o = (\pi/2)$, these constants become:

$$\alpha_1 = 0 \tag{D-14}$$

$$\alpha_2 = -1 \tag{D-15}$$

$$\alpha_3 = 1 \tag{D-16}$$

Substitution of Equations (D-14) through (D-16) into Equation (4-4) yields:

$$F^{(3)} = 6\pi\mu a\bar{k} \left(\frac{-3}{8}\right)\left(\frac{a}{x_o}\right) \int_0^\infty \frac{(-1 + 3)}{\cosh \tau\pi} d\tau \tag{D-17}$$

but,

$$\int_0^\infty \frac{d\tau}{\cosh \tau\pi} = \frac{1}{2} \tag{D-18}$$

Substitution of Equation (D-18) into Equation (D-17) results in:

$$F^{(3)} = 6\pi\mu a\bar{k} \left(\frac{-3}{8} \left\{ \frac{a}{x_o} \right\} \right) \tag{D-19}$$

APPENDIX E: NUMERICAL INTEGRATION PROCEDURE

Degree of Accuracy Obtainable

The degree of accuracy which is obtained in the numerical evaluation of an infinite integral depends, in general, on three factors:

(1) The degree to which the numerical quadrature formula approximates the contours of the actual function during the integration process. For example, if the area to be integrated is actually a trapezoid, then either the midpoint rule, or the trapezoidal rule will give an exact value for the area under the curve. If the function to be integrated is not linear, then the curvilinear sections of the curve must be approximated by straight lines. In this case the midpoint or trapezoidal rule would only give an approximation to the area.

(2) The Slit Width. Once an approximating quadrature formula is selected, the degree of accuracy obtained using the particular procedure is simply a function of the slit width. (This assumes that the number of iterations is held constant. Note that the slit width is simply the value of delta x, which is being used to represent the differential, dx. In general the smaller the slit width, the better is the approximation obtained by using the particular quadrature formula.

(3) The number of iterations. The accuracy of the numerical integration procedure depends on the number of iterations which are performed. This is true, strictly speaking, only when the value of the integrand is large enough to make a significant contribution to the area. Since the value of the function (or integrand) must

FIGURE E-1: LISTING OF THE COMPUTER PROGRAM

```
      PROGRAM CHRIS (INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT)
C
C   THIS PROGRAM CALCULATES THE AREA UNDER A CURVE FROM
C   ZERO TO INFINITY BY USING THE MIDPOINT RULE.
C
      L=1.0
      PI=3.141592
      EN=2
      FE=PI/EN
      ZE=PI-FE
      SUM=0.
      VALUE=0.
      X=0.
      DELTAX=.0125
C
      PRINT 900,EN
  900 FORMAT (5X,*THE NUMBER OF SPHERES IN THE ARRAY IS *,F7.0,/)
      VALP=VALUE*(3.0/8.0)
      PRINT 899
  899 FORMAT (10X,*X COORD*,16X,*F ONE *,15X,*VALUE OF FUNC*,//)
   10 FORMAT ( 3(E20.8,3X))
   17 DO 57,I=1,500
      X=X+DELTAX/2.0
      A=SIN(FE)*COS(FE)*SINH(X*PI)
      B=(COS(FE)**2)-(COSH(X*FE)**2)
      C=COSH(X*FE)*(COSH(X*ZE))*(SIN(FE)**2)
      D=SINH(X*FE)*SINH(X*ZE)*(COS(FE)**2)
      E=SINH(X*ZE)/SINH(X*FE)
      F=COSH(X*PI)
C
```

FIGURE E-1 (Continued)

```
      ALFA1=A/B
      ALFA2=(C-D)/B
      ALFA3=E
C
      SUM=(ALFA2+2.0*X*ALFA1 +3.0*ALFA3)/F
      VALUE=VALUE+SUM*DELTAX
      X=X+DELTAX/2.0
      VALP=VALUE*(3.0/8.0)
   57 CONTINUE
      PRINT 10,X,VALP,SUM
      L=L+1
      IF (L-37) 17,17,77
C
   77 CONTINUE
C
      PRINT 696
  696 FORMAT (5X,/////////,//)
      STOP
      END
```

tend towards zero as x approaches infinity, there exists a definite value of x for which further number of iterations does not increase the accuracy of the resulting answer. When this occurs, the value of x which corresponds to "infinity" has been reached for all practical purposes.

Comparison of the Results Obtained Numerically with the Analytical Results for the Case of Two Spheres.

In order to evaluate the accuracy of the numerical integration procedure the most logical place to start is for the case of two spheres, since for this case the analytical solution is already known. A computer program was designed perform the numerical integration using the midpoint rule. A listing of the program is shown as Figure E-1. Note that the variable of integration used prior to this in the body of the text (i.e., τ) has been replaced by x.

The program is basically very simple. It calculates the value of the integrand at the midpoint of the interval. It then assumes that the area under the curve for that interval may be approximated by a rectangle of width delta x, and of height the value of the function (or integrand) at the midpoint of the interval. Initially a value of delta x equal to .0125 was chosen. After the computer has performed 500 such iterations, it then prints out the x coordinate which has been reached, the value of the function $f_1(\phi_0)$ which has been calculated so far, and the value of the function (or integrand) at the particular x coordinate which has been reached. A typical print-out for the case of two spheres is shown in Figure E-2. These print-outs are referred to as "integration profiles" since they show the process of the integration as it proceeds along the x coordinate.

FIGURE E-2: Integration Profile for Two Spheres

THE NUMBER OF SPHERES IN THE ARRAY IS 2.

X COORD	F ONE	VALUE OF FUNC
.62500000E+01	.3750000 E+00	.12112537E-07
.12500000E+02	.3750000 E+00	.359[illegible]37E-16
.18750000E+02	.37500002E+00	.10[illegible]03E-24
.25000000E+02	.37500002E+00	.31[illegible]8784E-33
.31250000E+02	.37500002E+00	.94151292E-42
.37500000E+02	.37500002E+00	.2795[illegible]38E-50
.43750000E+02	.37500002E+00	.8[illegible]66E-59
.50000000E+02	.37500002E+00	.24[illegible]E-67
.56250000E+02	.37500002E+00	.73184219E-76
.62500000E+02	.37500002E+00	.21[illegible]75E-84
.68750000E+02	.37500002E+00	.64[illegible]772E-93
.75000000E+02	.37500002E+00	.191[illegible]469-101
.81250000E+02	.37500002E+00	.56886419-110
.87500000E+02	.37500002E+00	.16691040-118
.93750000E+02	.37500002E+00	.501[illegible]18-127
.10000000E+03	.37500002E+00	.1489[illegible]-135
.10625000E+03	.37500002E+00	.44[illegible]-144
.11250000E+03	.37500002E+00	.1[illegible]480-152
.11875000E+03	.37500002E+00	.38984[illegible]01-161
.12500000E+03	.37500002E+00	.115[illegible]-169
.13125000E+03	.37500002E+00	.34[illegible]00-178
.13750000E+03	.37500002E+00	.10[illegible]04-186
.14375000E+03	.37500002E+00	.30[illegible]60-195
.15000000E+03	.37500002E+00	.8947[illegible]72-204
.15625000E+03	.37500002E+00	.26716654-212
.16250000E+03	.37500002E+00	.79328611-221
.16875000E+03	.37500002E+00	.2355470[illegible]-229
.17500000E+03	.37500002E+00	.6[illegible]39[illegible]68-238
.18125000E+03	.37500002E+00	.20766974-246
.18750000E+03	.37500002E+00	.61662482-255
.19375000E+03	.37500002E+00	.18[illegible]09176-263
.20000000E+03	.37500002E+00	.54364648-272
.20625000E+03	.37500002E+00	.16142261-280
.21250000E+03	.37500002E+00	.47930522-289
.21875000E+03	.37500002E+00	0.
.22500000E+03	.37500002E+00	0.
.23125000E+03	.37500002E+00	0.

Note that in Figure E-2,the integration profile for the case of two spheres, the value of the integrand (i.e, the third column entitled "VALUE OF FUNC") decreases very rapidly. This indicates that only a small number of iterations are required before a significant result for the value of $f_1(\phi_o)$ is obtained. This is due to the fact that since the functional value becomes very small as x increases, the contribution of the "tail" of the function to the actual area under the curve becomes negligible very quickly. This can be seen by noting that after the first five hundred iterations the value of $f_1(\phi_o)$ is already the same as after 17,500 iterations.

Note that the final value reached for the function $f_1(\phi_o)$ is .37500002. Since the value obtained analytically is actually .375, the error brought about by using a numerical quadrature procedure is about 2×10^{-8}.

Calculation of the function $f_1(\phi_o)$ for other values of N

The largest value of N for which the function $f_1(\phi_o)$ may be evaluated by this procedure is based on the limitation of the computer and the accuracy desired in computing $f_1(\phi_o)$. In computing the value of the integrand for use in the numerical integration procedure, one of the terms which appears is cosh $\tau\pi$. As τ becomes very large, this number approaches infinity. Since the largest possible number which can be processed on the computer system where this work was done is 10^{322}, this is the maximum number which can be used in the computation of $f_1(\phi_o)$.

The value of 10^{322} for cosh $\tau\pi$ corresponds to carrying out the integration to a value of about 225 for τ. Thus all the integration profiles (for various values of N) were carried out to this value of τ. In order to determine the largest value of N which could be reliably handled by this technique, computer print-outs (similar to that shown in Figure E-2) were studied to determine the effect of the magnitude of the integrand on the value of the function $f_1(\phi_o)$.

The following conclusions were reached by studying a number of these integration profiles for various values of N:

(1) Generally, when the functional value of the integrand reaches a value of from 10^{-6} to 10^{-9}, the value of the function $f_1(\phi_o)$ remains unchanged to eight significant figures.

(2) A similar rule-of-thumb was deduced for the case of five significant figures in the value of the function $f_1(\phi_o)$. In order to obtain five significant figures in $f_1(\phi_o)$, the value of the integrand must be in the range of from 10^{-3} to 10^{-6}.

A typical print-out for an integration profile which exhibits these general tendencies is shown for the case of N=50 spheres in Figure E-3.

In order to determine where to terminate this procedure (i.e., determine how large a value of N can be handled by this technique), it is useful to look at some borderline cases. The integration profiles for N=100 and N=125 spheres are shown in Figures E-4 and E-5, respectively. Note that in the integration profile for 100 spheres, the value of $f_1(\phi_o)$ appears to be changing in the sixth significant figure at the

FIGURE E-3: Integration Profile for 50 Spheres

THE NUMBER OF SPHERES IN THE ARRAY IS 50.

X COORD	F ONE	VALUE OF FUNC
.62500000E+01	.48709541E+02	.19334709E+01
.12500000E+02	.50314569E+02	.26799312E-01
.18750000E+02	.49942981E+02	-.24980174E+00
.25000000E+02	.49157397E+02	-.23075124E+00
.31250000E+02	.48897617E+02	-.16072190E+00
.37500000E+02	.48596264E+02	-.99617219E-01
.43750000E+02	.48415572E+02	-.57766802E-01
.50000000E+02	.48312989E+02	-.32066088E-01
.56250000E+02	.48256880E+02	-.17258245E-01
.62500000E+02	.48227006E+02	-.90782943E-02
.68750000E+02	.48211420E+02	-.46923297E-02
.75000000E+02	.48203416E+02	-.23920397E-02
.81250000E+02	.48199356E+02	-.12059014E-02
.87500000E+02	.48197318E+02	-.60239922E-03
.93750000E+02	.48196304E+02	-.29863650E-03
.10000000E+03	.48195802E+02	-.14709423E-03
.10625000E+03	.48195556E+02	-.72051582E-04
.11250000E+03	.48195436E+02	-.35124701E-04
.11875000E+03	.48195377E+02	-.17050978E-04
.12500000E+03	.48195349E+02	-.82467601E-05
.13125000E+03	.48195335E+02	-.39754754E-05
.13750000E+03	.48195328E+02	-.19106088E-05
.14375000E+03	.48195325E+02	-.91600111E-06
.15000000E+03	.48195324E+02	-.43806154E-06
.15625000E+03	.48195323E+02	-.20904000E-06
.16250000E+03	.48195323E+02	-.99554568E-07
.16875000E+03	.48195322E+02	-.47326300E-07
.17500000E+03	.48195322E+02	-.22460368E-07
.18125000E+03	.48195322E+02	-.10642905E-07
.18750000E+03	.48195322E+02	-.50359610E-08
.19375000E+03	.48195322E+02	-.23797306E-08
.20000000E+03	.48195322E+02	-.11231464E-08
.20625000E+03	.48195322E+02	-.52947315E-09
.21250000E+03	.48195322E+02	-.24933480E-09
.21875000E+03	.48195322E+02	-.11729583E-09
.22500000E+03	.48195322E+02	-.55127671E-10
.23125000E+03	.48195322E+02	-.25886207E-10

FIGURE E-4: Integration Profile for 100 Spheres

THE NUMBER OF SPHERES IN THE ARRAY IS 100.

X COORD	F ONE	VALUE OF FUNC
.62500000E+01	.10543066E+03	.67142745E+01
.12500000E+02	.11408220E+03	.188[illegible]E+01
.18750000E+02	.11664727E+03	.53564[illegible]80E+00
.25000000E+02	.11720727E+03	.2022812[illegible]E-01
.31250000E+02	.11697628E+03	-.18434182E+00
.37500000E+02	.11644740E+03	-.25181[illegible]7E+00
.43750000E+02	.1158444[illegible]E+03	-.25580[illegible]07E+00
.50000000E+02	.11526995E+03	-.23161192E+00
.56250000E+02	.11476676E+03	-.19705726E+00
.62500000E+02	.11434735E+03	-.16114811E+00
.68750000E+02	.11400919E+03	-.12816[illegible]5E+00
.75000000E+02	.1137430[illegible]E+03	-.99840[illegible]44E-01
.81250000E+02	.11353738E+03	-.765197[illegible]3E-01
.87500000E+02	.11338078E+03	-.57884674E-01
.93750000E+02	.11326294E+03	-.43119199E-01
.10000000E+03	.11317515E+03	-.32128[illegible]96E-01
.10625000E+03	.11311027E+03	-.23646[illegible]78E-01
.11250000E+03	.11306268E+03	-.17[illegible]0499E-01
.11875000E+03	.1130279[illegible]E+03	-.12571256E-01
.12500000E+03	.11300280E+03	-.90948692E-02
.13125000E+03	.1129846[illegible]E+03	-.65511[illegible]14E-02
.13750000E+03	.1129715[illegible]E+03	-.4700[illegible]20E-02
.14375000E+03	.11296220E+03	-.33614059E-02
.15000000E+03	.11295[illegible]E+03	-.2396[illegible]73E-02
.15625000E+03	.11295076E+03	-.170[illegible]278E-02
.16250000E+03	.11294738E+03	-.12080154E-02
.168[illegible]000E+03	.11294494E+03	-.8546[illegible]07E-03
.17500000E+03	.11294330E+03	-.60[illegible]44[illegible]5E-03
.18125000E+03	.11294211E+03	-.425[illegible]5E-03
.18750000E+03	.11294127E+03	-.2991518[illegible]E-03
.19375000E+03	.11294068E+03	-.210106[illegible]5E-03
.20000000E+03	.112940[illegible]E+03	-.14734650E-03
.20625000E+03	.11293997E+03	-.10319072E-03
.21250000E+03	.11293977E+03	-.72174496E-04
.21875000E+03	.11293963E+03	-.50420482E-04
.22500000E+03	.11293953E+03	-.35183942E-04
.23125000E+03	.11293946E+03	-.24526026E-04

FIGURE E-5: Integration Profile for 125 Spheres

THE NUMBER OF SPHERES IN THE ARRAY IS 125.

X COORD	F ONE	VALUE OF FUNC
.62500000E+01	.13392614E+03	.92175196E+01
.12500000E+02	.14647371E+03	.30108902E+01
.18750000E+02	.15107358E+03	.11712[illegible]E+01
.25000000E+02	.15274515E+03	.39047[illegible]E+00
.31250000E+02	.15317262E+03	.19411[illegible]E-01
.37500000E+02	.15238313E+03	-.15833[illegible]E+00
.43750000E+02	.15210718E+03	-.23564[illegible]E+00
.50000000E+02	.15192017E+03	-.25870808E+00
.56250000E+02	.15131695E+03	-.25261707E+00
.62500000E+02	.15074751E+03	-.23171227E+00
.68750000E+02	.15023587E+03	-.20435272E+00
.75000000E+02	.14975094E+03	-.17538105E+00
.81250000E+02	.14941301E+03	-.14750315E+00
.87500000E+02	.14903765E+03	-.12213744E+00
.93750000E+02	.14883817E+03	-.99864706E-01
.10000000E+03	.14862711E+03	-.80811570E-01
.10625000E+03	.14845705E+03	-.64827013E-01
.11250000E+03	.14832113E+03	-.51620091E-01
.11875000E+03	.14821324E+03	-.40841853E-01
.12500000E+03	.14812811E+03	-.32134760E-01
.13125000E+03	.14806129E+03	-.25160711E-01
.13750000E+03	.14800508E+03	-.19615304E-01
.14375000E+03	.14796846E+03	-.15233466E-01
.15000000E+03	.14793696E+03	-.11789899E-01
.15625000E+03	.14791262E+03	-.90966214E-02
.16250000E+03	.14789387E+03	-.69990473E-02
.16875000E+03	.14787946E+03	-.53715479E-02
.17500000E+03	.14786841E+03	-.41130141E-02
.18125000E+03	.14785996E+03	-.31427732E-02
.18750000E+03	.14785350E+03	-.23967208E-02
.19375000E+03	.14784859E+03	-.18245561E-02
.20000000E+03	.14784485E+03	-.13867130E-02
.20625000E+03	.14784201E+03	-.10523464E-02
.21250000E+03	.14783986E+03	-.79748270E-03
.21875000E+03	.14783822E+03	-.60355508E-03
.22500000E+03	.14783699E+03	-.45623029E-03
.23125000E+03	.14783606E+03	-.34447494E-03

"end" of the integration profile. This is even more evident for the case of N=125 spheres shown in Figure E-5. For this reason the value of N=100 spheres was chosen as the largest value of N which could be reliably handled by this procedure. A summary of the values of $f_1(\phi_o)$ for various values of N as obtained by this procedure is given in Table E-1.

Calculation of the function $f_1(\phi_o)$ for large values of N

Substitution of the expressions for α_1, α_2, and α_3 given in Equations (3-21) through (3-23) into Equation (4-4) results in:

$$\bar{F}^{(3)} = 6\pi\mu aU\bar{k} \left(\frac{-3}{8} \right) \left\{ \frac{a}{x_o} \right\} \int_0^\infty \{ (\sin^2\phi_o \cosh \tau\phi_o \cosh \tau(\pi - \phi_o)$$

$$- \cos^2 \phi_o \sinh \tau\phi_o \sinh \tau(\pi - \phi_o)$$

$$+ 2\tau \sin \phi_o \cos \phi_o \sinh \tau\pi)(1/(\cos^2 \phi_o - \cosh^2 \tau\phi_o))$$

$$+ 3 \sinh \tau(\pi - \phi_o)/\sinh \tau\phi_o \} \left\{ \frac{d\tau}{\cosh \tau\pi} \right\} \qquad \text{(E-1)}$$

This may be expressed as

$$\bar{F}^{(3)} = 6\pi\mu aU\bar{k} \left(\frac{-3}{8} \frac{a}{x_o} \right) \int_0^\infty h(\tau)\, d\tau \qquad \text{(E-2)}$$

where the function $h(\tau)$ defined implicitly by Equations (E-1) and (E-2),

TABLE E-1

VALUES OF THE FUNCTION $f_1(\phi_o)$ FOR VALUES OF N UP TO 100

N	$f_1(\phi_o)$
2	.3750
3	.8660
4	1.436
5	2.065
6	2.741
10	5.794
15	10.145
25	19.960
30	25.258
35	30.756
40	36.425
50	48.195
60	60.446
75	79.553
100	112.939

is just the integrand of Equation (E-1).

By definition,

$$\cosh \tau\phi_o = (e^{\tau\phi_o} + e^{-\tau\phi_o})/2$$

$$\sinh \tau\phi_o = (e^{\tau\phi_o} - e^{-\tau\phi_o})/2$$

For large values of τ:

$$\cosh \tau\phi_o \simeq e^{\tau\phi_o}/2 \qquad \text{(E-3)}$$

$$\sinh \tau\phi_o \simeq e^{\tau\phi_o}/2 \qquad \text{(E-4)}$$

Substitution of the relationships given by Equations (E-3) and (E-4) into Equation (E-1), and combination of similar terms results in the following expression for the function $h(\tau)$, for large values of τ:

$$h(\tau) = \frac{2 + 4\,\tau \sin\phi_o \cos\phi_o}{4\cos^2\phi_o - e^{2\tau\phi_o}} \qquad \text{(E-5)}$$

The computer program listed in Figure E-1 was extended so that for values of τ greater than 225, the integrand of Equation (E-1) gets replaced by the expression given in Equation (E-5). Thus, the computer can now integrate to as large a value of τ as is necessary to get the desired accuracy in the function $f_1(\phi_o)$ for larger values of N.

A summary of the values of $f_1(\phi_o)$ for various values of N as obtained by this procedure is given in Table E-2. Note that the entries in Tables (E-1) and (E-2) have been combined in Table 1 in Chapter 4 of the main body of the text.

Numerical Evaluation of the function $f_1(\phi_o)$ Using Various Formulae

A number of numerical quadrature formulae were evaluated as possible alternatives to the midpoint rule. The results obtained using the midpoint rule, Simpson's Rule, and a six-point Newton-Coates formula were compared at a number of different slit widths for the case of N=2 spheres. These results are summarized in Table E-3. As can be seen from the results, for an equal number of iterations and slit width, the best performing quadrature procedure is the simple mid-point rule.

TABLE E-2

VALUES OF THE FUNCTION $f_1(\phi_o)$ FOR LARGER VALUES OF N

N	$f_1(\phi_o)$
300	416.334
500	740.990
1000	1292.584

TABLE E-3

COMPARISON OF THE RESULTS OBTAINED USING VARIOUS NUMERICAL INTEGRATION FORMULAE FOR THE EVALUATION OF THE FUNCTION $f_1(\phi_o)$ FOR THE CASE OF N = 2 SPHERES

Type of Quadrature Method Used	Slit Width	Value of $f_1(\phi_o)$
Simpson's Rule	.0125	.375000016742165
Simpson's Rule	.00125	.375000016739470
Simpson's Rule	.000125	.375000016717381
Six Point Newton-Coates	.0125	.37500001674239
Six Point Newton-Coates	.00125	.37500001674154
Six Point Newton-Coates	.000125	.37500001673591
Midpoint Rule	.0125	.375000016741783
Midpoint Rule	.00125	.375000016735898
Midpoint Rule	.000125	.375000016687387

ADDENDUM

In a recent verbal communication with Professor T. Greenstein, a member of the doctoral committee formed for this dissertation, Professor Greenstein verified the results which were obtained for $f_1(\phi_0)$ in this dissertation for the cases of three, four, five and six spheres. It should be noted that the solutions obtained by Dr. Greenstein were obtained by using a technique different from the one used in this dissertation.

Nomenclature

a	sphere radius
A	One of the three constants of integration in the Fourier Lebedev transform solutions (or second reflection solution)
b	a constant from the nomenclature of Ditkin and Prudnikov. This constant appears in Appendix A.
B	A second constant of integration appearing in the second reflection solution
C	A third constant of integration appearing in the second reflection solution
$f_1(\phi_o)$	Coefficient which estimates the first order effect on the drag force due to the presence of the other spheres
$\bar{F}$	drag force
$\bar{k}$	Unit vector in the z direction
$K_{i\tau}(\lambda\rho)$	A modified Bessel function of the second kind of real argument, $\lambda\rho$, and imaginary order, $i\tau$. This real function has the integral representation given below:

$$K_{i\tau}(\lambda\rho) = \int_0^\infty e^{-\lambda\rho \cosh t} \cos \tau t \, dt$$

N	Number of spheres in the array
r, θ, ϕ	Spherical coordinates of a point in space with respect to an origin at a sphere center. (This origin is displaced from the apex located origin by a distance x_o measured in the x direction.)
U	Velocity of a sphere in the array
U_S	Velocity of a single sphere settling under conditions where Stokes' Law holds. The Stoke's velocity
x_o	Distance of the sphere center from the wedge apex measured in the x direction. (For an array of particles this is the distance between the sphere centers and the centroid of the array.)
x, y, z	Cartesian coordinates of a point in space with respect to an origin located at the centroid of the array
$\underline{x}, y, z$	Cartesian coordinates of a point in space with respect to an origin located at the sphere center. (This origin is displaced from the origin which is located at the centroid of the array by the distance x_o.

Z — A combination of variables which appears as the argument of K Bessel functions ($K_o(Z)$, $K_1(Z)$, $K_2(Z)$), and as a term in the numerator and denominator of many expressions in the Appendices. Mathematically, Z is given by:

$$Z = \lambda\sqrt{\rho^2 - 2\rho x_o \cos \phi_o + x_o^2}$$

$\alpha_1, \alpha_2, \alpha_3$ — These three constants represent the trigonometric and hyperbolic functions portion of the constants A_1, B_1, and C_1.

λ — A dummy variable. Also, this is the separation constant in the transform solution to the Creeping Motion Equation in cylindrical coordinates.

ρ, ϕ, z — Cylindrical coordinates of a point in space with respect to an origin located at the centroid of an array.

τ — A dummy variable. Also, this is a separation constant which appears in the transform solution to the Creeping Motion Equation in cylindrical coordinates.

ϕ_o — One half the dihedral angle formed by the intersection of two planes which pass through the origin (located at the centroid of the array) and the centers of two adjacent spheres. Also ϕ_o is numerically equal to π/N.

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