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THE SHADOW PRICE OF INFORMATION IN CONTINUOUS TIME DECISION PROBLEMS

by

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Abstract

We formulate a continuous time stochastic control problem and establish the existence of the shadow price of information. This shadow price is the Lagrange multiplier for the constraint that the control be adapted or predictable; it is a stochastic process of integrable variation; and, in one formulation, it is a martingale. The results are applied to problems of security investment, selling an asset, and economic growth. In the last application, it is shown that the existence of the shadow price of information implies the validity of the stochastic maximum principle.

Keywords: stochastic control, duality, martingales, stochastic maximum principle, economic growth, optimal portfolios.

1. INTRODUCTION

In contrast with deterministic problems, a characteristic feature of stochastic decision problems is that relevant information is acquired with the passage of time. In this paper we ask: what prices would one pay for acquiring information in advance? There are no natural units in which to measure information, so we assess the prices in an indirect fashion. We seek (state and time contingent) prices to be imposed on actions, which will be such that the expected cost (measured in the units of the utility function) is zero if the actions do not utilize advance information and such that the decision-maker will be indifferent at the margin between using advance information at the given expected cost and using only the information which is naturally available. We call this system of prices the shadow price of information.

The distinction between stochastic and deterministic problems is eradicated if one introduces this shadow price into the former. Specifically, the stochastic problem is reduced to a family of deterministic problems, problems indexed by the various "states of nature." In each of the deterministic problems there is complete knowledge of the future but also an additional linear term in the utility function. The coefficient of this term specifies the price to be paid at each date for using advance knowledge of the particular state of nature.

Rockafellar and Wets [21] first considered this issue and established the existence of the shadow price of information for discrete-time finite-horizon stochastic optimization problems. Dempster [10] and Flam [12] extended the result to discrete-time infinite-horizon problems. The purpose of this paper is to extend the result to the setting of continuous time. We also give several applications.

The heart of the paper is Section 3 (Section 2 is devoted to preliminaries). In that section we pose the general optimization problem and establish the existence of the shadow price of information (Theorem 3.4). The objective function of the problem is an extended-real-valued concave function, the domain of which is a space of bounded m-dimensional stochastic (control) processes. The incompleteness of information at the various dates is modeled by requiring the control process to be adapted to an exogenously given filtration (it is important to note that even in formulating the price to be paid for advance information, we are assuming that the information which would arrive in the future is exogenously given, i.e., independent of the choice of the control vector). More precisely, the adaptedness requirement is allowed to take one of two forms: the control process is constrained either to be predictable or to be optional. The Lagrange multiplier for this constraint is the shadow price of information. It is a stochastic process of integrable variation. The adding-up of the costs of using advance information at the various dates is done in the form of a Lebesgue-Stieltjes integral, the integrator being the shadow price.

Our proof is based on an adaptation to continuous time of the idea of Rockafellar-Wets [21], an idea more fully elaborated in Rockafellar-Wets [23]. Rockafellar and Wets [21] apply this idea by induction on the time set (which is made possible by dynamic programming). We, instead, apply the idea in a single step to the complete control problem, as Flam [12] has done for an infinite-horizon discrete-time problem. This forces us to assume that there is a solution of the problem at which the state constraints, if any, are "slack." We should also note here that Topcuoglu [25] has proven the existence of the shadow price of information in the discrete-time model by assuming the existence of an interior solution. This result is weaker than ours, since we allow for a (binding) constraint that the control process be a selection of a given adapted multifunction (specifying the control set at each date and state).

The applications are in Sections 4, 5 and 6. In Section 4 we consider the problem of choosing a portfolio of securities at each date of a continuous trading model. Harrison and Kreps [14] have shown that if markets are "frictionless" and this problem has a solution, then the security price process must be a martingale under a probability measure mutually absolutely continuous with the trader's subjective probability measure. It is easy to see that in this case the security price process scaled by the Radon-Nikodym derivative of the new measure is the shadow price of information. Thus the Harrison-Kreps result implies the existence of the shadow price of information (which can be interpreted as a first-order condition for optimal trading strategies). We extend this implication of the Harrison-Kreps result to cases in which there exist market frictions in the form of short-sales constraints. We remark parenthetically that the martingale property of the shadow price of information was emphasized by Pliska [16]. We should also note that, unlike Harrison-Kreps [14], we allow for trading strategies which are not "simple." In this we follow Harrison-Pliska [15]. However we also impose an important restriction on the Harrison-Kreps [14] model: we require the security prices to be of integrable variation.

In Section 5 we illustrate how the reduction to a family of deterministic problems may be used to establish other properties of a stochastic optimization problem (in this we also follow Rockafellar-Wets [22], [24]). We consider the one-sector economic growth model formulated by Foldes [13] (and also studied by Cox-Ingersoll-Ross [17] and Duffie-Huang [11]). Foldes establishes for this problem a martingale property of the marginal utility evaluated at the optimal consumption plan (Foldes [13], Theorem 6). This result is a particular instance of the stochastic maximum principle studied by Bismut in a series of papers, including [3], [4], and [5]. The martingale property or maximum principle is known to be valid for this problem only when the state constraint (nonnegativity of the capital stock) is not binding at the optimum. We show that, regardless of whether the state constraint is binding, if the shadow price of information exists then the stochastic maximum principle is valid. Essentially one can construct a coextremal for the stochastic problem by piecing together the coextremals for the various deterministic problems to which the stochastic problem is reduced. This provides what is hopefully a useful new perspective on the problem of verifying the stochastic maximum principle. However we do not obtain a stronger result on the validity of the maximum principle, because we assume that the state constraint is not binding in order to establish the existence of the shadow price of information.

Section 6 deals with a problem of optimally timing the sale of an asset when confronted with an exogenous price process. One reason for considering this problem is to demonstrate that the shadow price of information may exist in circumstances other than those covered by our main result. Section 7 contains a few concluding remarks on possible generalizations.

2. NOTATION

As well as establishing the notation to be used, we collect here some facts regarding the space of controls to be studied and the linear functionals on that space. These are obtained from Yosida-Hewitt [26] (hereinafter cited as YH). We will follow Dellacherie-Meyer [8, 9] (cited as DM) for stochastic process theory.

Denote by $(\Omega, F, \mathbb{P})$ a complete probability space, and let $(F_t)_{0 \leq t \leq T}$ be a filtration satisfying the "usual conditions" (DM, IV.48). Denote by N the class of evanescent sets in $[0,T] \times \Omega$. Let $M = (B \otimes F) \vee N$, where B is the Borel field of $[0,T]$. Take O (resp. P) to be the optional (resp. predictable) σ-field, also augmented by N. When we refer in the sequel to the measurable, optional or predictable σ-field, we always mean the respective augmented σ-field.

To avoid repetition of results valid for each of O and P, we allow S to represent a σ-field which can be taken to be either O or P. We also use the symbol "s" to identify the projection or dual projection of a process. These are the projections identified by the superscript "p" in the predictable case and "o" in the optional case by Dellacherie-Meyer. It is to be understood that if $S = P$ then s = p and if $S = O$ then s = o.

The controls and shadow prices will be m-dimensional processes $X = (X^1, \ldots, X^m)$ and $A = (A^1, \ldots, A^m)$. In Sections 5 and 6 we assume $m = 1$; otherwise $m < \infty$ is arbitrary. As usual we identify indistinguishable processes, i.e., processes $Y = (Y^1, \ldots, Y^m)$ and $Z = (Z^1, \ldots, Z^m)$ such that the set $\{(t,\omega) | Y_t(\omega) \neq Z_t(\omega)\}$ is evanescent.

A process X will be said to be bounded if there is some $\alpha < \infty$ such that the set $\{(t,\omega) | \max_{i=1,\ldots,m} |X_t^i(\omega)| > \alpha\}$ is evanescent. In this case, write $\|X\|_\infty$ for the infimum of such α.

Denote by $L_\infty(M)$ [resp. $L_\infty(S)$] the class of bounded, M-measurable (resp. S-measurable), m-dimensional processes. These are Banach spaces under the norm $\|\cdot\|_\infty$; see, e.g., YH, Theorem 2.2. We view $L_\infty(S)$ as a subspace of $L_\infty(M)$.

If Γ is a multifunction from $[0,T] \times \Omega$ to $\mathbb{R}^m$ (i.e. a function on $[0,T] \times \Omega$ whose values are subsets of $\mathbb{R}^m$) we write $L_\infty(M;\Gamma)$ [resp. $L_\infty(S;\Gamma)$] for the family of $X \in L_\infty(M)$ [resp. $X \in L_\infty(S)$] such that the set

$\{(t,\omega)|X_t(\omega) \notin \Gamma_t(\omega)\}$ is evanescent.

Given $X = (X^1,\ldots,X^m) \in L_\infty(M)$, denote by ${}^sX$ the vector $({}^sX^1,\ldots,{}^sX^m)$ where ${}^sX^i$ is the predictable (if s = p) or optional (if s = o) projection of X^i (DM, VI.43). We have ${}^sX \in L_\infty(S)$.

The class of norm-continuous linear functionals on $L_\infty(M)$ will be denoted by $L^*_\infty(M)$. According to YH, Theorem 2.3, these functionals can be identified with vectors $\pi = (\pi^1,\ldots,\pi^m)$ where each π^i is a finitely additive set function with domain M satisfying $\sup\{\pi^i(E)|E \in M\} < \infty$ and $\pi^i(E) = 0, \quad \forall\, E \in N$. The value of π at X is given by

$$\langle X,\pi\rangle = \sum_{i=1}^{m} \int_{[0,T]\times\Omega} X^i_t(\omega)d\pi^i.$$

Given $\pi = (\pi^1,\ldots,\pi^m) \in L^*_\infty(M)$, denote by $|\pi|$ the set function with values $|\pi|(E) = \sum_{i=1}^m[\pi^i_+(E) + \pi^i_-(E)]$, where π^i_+ and π^i_- are the nonnegative set functions such that $\pi^i = \pi^i_+ - \pi^i_-$ (YH, 1.12).

Each $\pi \in L^*_\infty(M)$ can be written uniquely as $\pi_c + \pi_p$ where $\pi_c = (\pi^1_c,\ldots,\pi^m_c) \in L^*_\infty(M)$ is a vector of countably additive measures and $\pi_p = (\pi^1_p,\ldots,\pi^m_p) \in L^*_\infty(M)$ is a vector of "purely finitely additive" set functions, (YH, Theorem 1.24).

If $\pi_p \in L^*_\infty(M)$ is purely finitely additive and μ is a finite, countably additive measure on $([0,T] \times \Omega, M)$, then there exists a decreasing sequence of sets $E_n \in M$ such that $\lim_{n\to\infty}|\mu(E_n)| = 0$ but $|\pi_p|([0,T] \times \Omega\setminus E_n) = 0$ for each n (YH, Theorem 1.22). Here the symbol "\" denotes, as usual, set-theoretic subtraction.

Each countably additive measure π^i_c is called a $\mathbb{P}$-measure (DM, VI.64) since it vanishes on evanescent sets. As a functional on $L_\infty(M)$, each countably additive $\pi_c \in L^*_\infty(M)$ can be identified with a vector $A = (A^1,\ldots,A^m)$

of M-measurable integrable variation processes under the representation

$$\sum_{i=1}^{m} \int_{[0,T]\times\Omega} X_t^i(\omega)d\pi_c^i = \sum_{i=1}^{m} \mathbb{E}[\int_{[0,T]} X_t^i(\omega)dA_t^i(\omega)]. \quad (2.1)$$

See DM, VI.65 (use Remark VI.72(b) of DM to interpret condition VI.65.2). Here, and throughout the paper, the symbol "$\mathbb{E}$" denotes expectation under the probability measure $\mathbb{P}$.

Let IV(M) [resp. IV(S)] denote the family of vectors $A = (A^1,\ldots,A^m)$ where each A^i is an M-measurable [resp. S-measurable] integrable variation process. For $X \in L_\infty(M)$ and $A \in IV(M)$, write $\mathbb{E}[\int_0^T X_t dA_t]$ for the right-hand side of (2.1).

Given $A = (A^1,\ldots,A^m) \in IV(M)$, denote by A^s the vector $(A^{1s},\ldots,A^{ms})$ where A^{is} is the predictable (if s = p) or optional (if s = o) dual projection of A^i (see DM, VI.73). The defining characteristic of A^s is that for each $X \in L_\infty(M)$, $\mathbb{E}[\int_0^T X_t dA_t^s] = \mathbb{E}[\int_0^{Ts} X_t dA_t]$.

We will write as usual $L_\infty(\Omega,F,\mathbb{P})$ for the Banach space of (equivalence classes of) essentially bounded real-valued functions with the ess-sup norm. The dual space of $L_\infty(\Omega,F,\mathbb{P})$ is characterized in a way similar to that of $L_\infty(M)$.

3. THE SHADOW PRICE OF INFORMATION

3.1. Introduction

We consider an optimization problem of the form

(*) maximize $\Phi(X)$ subject to $X \in L_\infty(S)$

where the objective function Φ is a concave function from $L_\infty(M)$ to $\mathbb{R} \cup \{-\infty,+\infty\}$. We assume $-\infty < \sup\{\Phi(X) | X \in L_\infty(S)\} < \infty$. We do not discuss in this paper how the objective function is defined on the space of nonadapted processes. Instead we simply take the function to be given. In applications this definition of the function will be a delicate issue when the problem involves stochastic integrals.

For any $X \in L_\infty(M)$ the condition "$\Phi(X) = -\infty$" is interpreted to mean that X is an infeasible policy. Denote $\{X \in L_\infty(M) | \Phi(X) > -\infty\}$ by dom Φ. Let Γ be a multifunction from $[0,T] \times \Omega$ to $\mathbb{R}^m$ such that each set $\Gamma_t(\omega)$ is closed and convex. We assume dom $\Phi = L_\infty(M;\Gamma) \cap D$ for some set $D \subset L_\infty(M)$. As yet this is unrestrictive, since we could have $\Gamma_t(\omega) \equiv \mathbb{R}^m$ and $D = \text{dom}\,\Phi$.

We seek a shadow price for the nonanticipativity constraint that $X \in L_\infty(S)$. Specifically we seek a process $A \in IV(M)$ such that (i) if $\bar{X} \in L_\infty(S)$ then $\bar{X}$ solves (*) iff $\bar{X}$ solves

$$\text{maximize } \Phi(X) - \mathbb{E}\left[\int_0^T X_t dA_t\right] \text{ subject to } X \in L_\infty(M) \tag{3.1}$$

and (ii) $\mathbb{E}\left[\int_0^T X_t dA_t\right] = 0, \ \forall\ X \in L_\infty(S)$.

We remark that in the case $S = P$ the conditions (i) and (ii) imply the existence of a martingale M such that (i') if $\bar{X} \in L_\infty(P)$ then $\bar{X}$ solves (*) iff $\bar{X}$ solves

$$\text{maximize } \Phi(X) - \mathbb{E}\left[\int_0^T X_t dM_t\right] \text{ subject to } X \in L_\infty(O).$$

This will give us a martingale characterization of optimal security trading strategies in Section 4. To see that it is true, take $M = A^o$, the vector of optional dual projections of the components $A^1,\ldots,A^m$ of the shadow price A.

We have $\mathbb{E}[\int_0^T X_t dM_t] = \mathbb{E}[\int_0^T X_t dA_t]$ for each $X \in L_\infty(\mathcal{O})$, so (i) implies (i'). Since $L_\infty(\mathcal{P}) \subset L_\infty(\mathcal{O})$, the condition (ii) implies that $\mathbb{E}[\int_0^T X_t dM_t] = 0$ for each $X \in L_\infty(\mathcal{P})$, so the predictable dual projection (= predictable compensator) of M is the null process. Hence M is a martingale (DM, VI.73).

Conditions (i) and (ii) hold if the process A solves a certain problem dual to (*) and if the dual problem has the same value as the problem (*). The exact nature of the dual problem will be evident, to those familiar with Rockafellar [19], from the proof of Lemma 3.2. For the fact that, given the equality of the values of the problems, solutions of the dual have properties (i) and (ii), see Rockafellar [19, Corollary 15A]. Without relying explicitly on the machinery of Rockafellar, here we can express this relationship in the following convenient form.

LEMMA 3.1. Let $A \in IV(\mathcal{M})$. Then (i) and (ii) above hold if

$$\sup_{X \in L_\infty(\mathcal{S})} \Phi(X) = \sup_{X \in L_\infty(\mathcal{S}), Y \in L_\infty(\mathcal{M})} \{\Phi(X+Y) - \mathbb{E}[\int_0^T Y_t dA_t]\}. \tag{3.2}$$

Proof. Assume (3.2) holds. By assumption $\Phi(Z) > -\infty$ for some $Z \in L_\infty(\mathcal{S})$. If (ii) fails there is some $Y \in L_\infty(\mathcal{S})$ such that $-\mathbb{E}[\int_0^T Y_t dA_t] \geqslant 1$. Setting $Y_n = nY$ and $X_n = Z - Y_n$ with $n \uparrow \infty$ we see that the right-hand side of (3.2) is $+\infty$, contrary to the maintained assumption that the left-hand side is finite. Hence (ii) must be true.

It follows from (ii) that the value in braces in (3.2) is unchanged if one replaces X by X' = 0 and Y by Y' = X + Y. Therefore the equality in (3.2) means that the value of the problem (3.1) is the same as that of the problem (*). Using (ii) again it is evident from this that the solutions in $L_\infty(\mathcal{S})$ coincide, so (i) is established. []

3.2. A Generalized Shadow Price

A constraint qualification is needed in order to obtain a duality result. It would be unduly restrictive to adopt a condition that would directly yield a shadow price $A \in IV(M)$. A better result is obtained by first exhibiting a "generalized shadow price" $\pi \in L_\infty^*(M)$ and then using the Yosida-Hewitt decomposition of $L_\infty^*(M)$ to obtain the desired shadow price $A \in IV(M)$. In doing so, we follow directly in the footsteps of Rockafellar-Wets [21].

Constraint Qualification. There exists $\tilde{X} \in L_\infty(S)$, $\varepsilon > 0$ and $\alpha > -\infty$ such that $\Phi(\tilde{X}+Y) \geqslant \alpha$ for each $Y \in L_\infty(M)$ satisfying $\|Y\|_\infty < \varepsilon$.

LEMMA 3.2. Assume the Constraint Qualification holds. Then there exists $\pi \in L_\infty^*(M)$ such that

$$\sup_{X \in L_\infty(S)} \Phi(X) = \sup_{X \in L_\infty(S), Y \in L_\infty(M)} [\Phi(X+Y) - \langle Y,\pi\rangle] \tag{3.3}$$

Proof. For $X \in L_\infty(S)$ and $Y \in L_\infty(M)$ set $F(X,Y) = -\Phi(X+Y)$ and $\phi(Y) = \inf\{F(X,Y) | X \in L_\infty(S)\}$. In view of the Constraint Qualification, there is a norm neighborhood of the origin in $L_\infty(M)$ on which $\phi(Y) \leqslant -\Phi(\tilde{X}+Y) \leqslant -\alpha$. Hence from Theorems 16 and 17a of Rockafellar [19] there exists $\pi \in L_\infty^*(M)$ such that

$$\inf_{X \in L_\infty(S)} F(X,0) = \inf_{X \in L_\infty(S)} \inf_{Y \in L_\infty(M)} \{F(X,Y) + \langle Y,\pi\rangle\}.$$

Multiplying by -1, this is equation (3.3). []

3.3. Nonanticipativity of the Control Set

We view the problem (*) as being a control problem: the set $\Gamma_t(\omega)$ is the set of choices available for the control $X_t(\omega)$ at time t and state ω, and that set does not depend on the past choices which have been made. State constraints, if any, are modeled by restricting X to belong to the set $D \subset L_\infty(\mathcal{M})$. It is natural to assume that the set $\Gamma_t(\omega)$ evolves in a nonanticipative way (in fact it will often be the case in applications that $\Gamma_t(\omega)$ is independent of ω). Formally this means that the multifunction Γ is $\mathcal{S}$-measurable, in the usual sense that the set $\{(t,\omega) | \Gamma_t(\omega) \cap C \neq \emptyset\}$ belongs to $\mathcal{S}$, for each closed $C \subset \mathbb{R}^m$.

We will need the following characterization of measurability. The proof is adapted from Valadier's analysis of the conditional expectations of multifunctions (cf. Castaing-Valadier [6, Theorem VII.35]). Recall that we write ${}^sY$ for the vector of predictable (if s = p) or optional (if s = o) projections of the components $Y^1,\ldots,Y^m$ of Y.

LEMMA 3.3. Assume the closed, convex-valued multifunction Γ is $\mathcal{S}$-measurable. Then the operator $Y \mapsto {}^sY$ maps $L_\infty(\mathcal{M};\Gamma)$ onto $L_\infty(\mathcal{S};\Gamma)$.

Proof. Since the operator is a projection and $L_\infty(\mathcal{S};\Gamma) \subset L_\infty(\mathcal{M};\Gamma)$, it suffices to show that the image of $L_\infty(\mathcal{M};\Gamma)$ is contained in $L_\infty(\mathcal{S};\Gamma)$.

Let $Y \in L_\infty(\mathcal{M};\Gamma)$. Fix a version of ${}^sY$, and let $E = \{(t,\omega) | {}^sY_t(\omega) \notin \Gamma_t(\omega)\}$. It must be shown that E is evanescent.

For $b \in \mathbb{R}^m$, set

$$\delta^*(t,\omega,b) = \sup\{x \cdot b | x \in \Gamma_t(\omega)\}.$$

If $(t,\omega) \in E$, then by the Separating Hyperplane Theorem (Rockafellar [18], Corollary 11.4.2) there exists $b \in \mathbb{R}^m$ and an integer n such that

$$^{s}Y_t(\omega) \cdot b \geq \frac{1}{n} + \delta^*(t,\omega,b) \tag{3.4}$$

For each (t,ω) let $\sum_{nt}(\omega)$ denote the set of b for which (3.4) holds, and let $E_n = \{(t,\omega) | \sum_{nt}(\omega) \neq \emptyset\}$. Since $E \subset \bigcup_{n=1}^{\infty} E_n$, it suffices to show that each E_n is evanescent.

If Γ is $\mathcal{S}$-measurable then it admits a Castaing representation, i.e., a family $\{\gamma^1,\gamma^2,\ldots\}$ of $\mathcal{S}$-measurable functions such that $\Gamma_t(\omega) = \text{cl}\{\gamma_t^1(\omega),\gamma_t^2(\omega),\ldots\}$ for each (t,ω). See Castaing-Valadier [6, Proposition III.12 and Theorem III.8]. This implies that, for any $\mathcal{S}$-measurable function b,

$$\delta^*\big(t,\omega,b_t(\omega)\big) = \sup_i\{\gamma_t^i(\omega) \cdot b_t(\omega)\},$$

which is $\mathcal{S}$-measurable.

We will show that $\sum_n$ is $\mathcal{S}$-measurable. For this it suffices to show that the set $\{(t,\omega)|\sum_{nt}(\omega) \cap K \neq \emptyset\}$ belongs to $\mathcal{S}$ for each compact $K \subset \mathbb{R}^m$; see, e.g., Rockafellar [17].

Let K be compact. Note that the extended-real-valued function $\delta^*(t,\omega,\cdot)$ is lower-semicontinuous. Hence $\sum_{nt}(\omega) \cap K \neq \emptyset$ -- i.e., there exists $b \in K$ satisfying (3.4) -- iff

$$\sup\{^{s}Y_t(\omega)\cdot b - \delta^*(t,\omega,b)\,|\,b \in K\} \geq \frac{1}{n}.$$

Since the supremum in the left-hand side can be taken over a countable dense subset of K, the left-hand side is $\mathcal{S}$-measurable in (t,ω). This implies that $\sum_n$ is $\mathcal{S}$ -measurable.

Now by the results previously cited from Castaing-Valadier [6], there exists an $\mathcal{S}$-measurable function b_n from E_n to $\mathbb{R}^m$ such that $b_{nt}(\omega) \in \sum_{nt}(\omega)$ for each $(t,\omega) \in E_n$. Note that $E_n \in \mathcal{S}$, since $E_n = \{(t,\omega) | \sum_{nt}(\omega) \cap \mathbb{R}^m \neq \emptyset\}$. If $(t,\omega) \notin E_n$ set $b_{nt}(\omega) = 0$. Summarizing, we have

$$ {}^{s}Y_t(\omega) \cdot b_{nt}(\omega) \geq \frac{1}{n} + \delta^*(t,\omega,b_{nt}(\omega)) \tag{3.5} $$

for $(t,\omega) \in E_n$.

Since $Y \in L_\infty(\mathcal{M};\Gamma)$, it is certainly the case that

$$ Y_t(\omega) \cdot b_{nt}(\omega) \leq \delta^*(t,\omega,b_{nt}(\omega)) \tag{3.6} $$

except possibly on an evanescent set. It has already been observed that the right-hand side of (3.6) is $\mathcal{S}$-measurable. Since the $\mathcal{S}$-projection is isotone (DM, VI.43(a)), the right-hand side must majorize the $\mathcal{S}$-projection of the left. Using DM VI.43(b) and VI.44(e), this yields

$$ {}^{s}Y_t(\omega) \cdot b_{nt}(\omega) \leq \delta^*(t,\omega,b_{nt}(\omega)) \tag{3.7} $$

outside an evanescent set.

Comparing (3.5) and (3.7) we conclude that E_n is evanescent. Hence E is evanescent, and ${}^{s}Y \in L_\infty(\mathcal{S};\Gamma)$. [ ]

3.4. The Duality Theorem

We require two additional assumptions. One is a continuity condition for

Φ on dom Φ. The other is that the constraint that $X \in D$ be "slack" at some solution $\hat{X}$ of (*).

The continuity condition is relative to an otherwise arbitrary positive $\mathbb{P}$-measure μ.

<u>Continuity Assumption</u>. There exists a positive $\mathbb{P}$-measure μ such that the following holds: if $X \in \text{dom}\,\Phi$ and (X_n) is a $\|\cdot\|_\infty$-bounded sequence from dom Φ satisfying

$$\mu\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{(t,\omega) | X_{nt}(\omega) \neq X_t(\omega)\}\right) = 0,$$

then $\Phi(X) \leqslant \liminf_{n\to\infty} \Phi(X_n)$.

We remark that this would certainly be true if Φ were lower-semicontinuous on dom Φ with regard to the topology of the pseudo-norm

$$\|X\|_{\mu,p} \equiv \{\int_{[0,T]\times\Omega} \|X_t(\omega)\|^p d\mu\}^{1/p}$$

for any $p < \infty$. It would also suffice to have lower-semicontinuity in the Mackey topology of $L_\infty([0,T] \times \Omega, \ ,\mu;\mathbb{R}^m)$ for the pairing with $L_1([0,T] \times \Omega, \ ,\mu;\mathbb{R}^m)$. This is the assumption used by Bewley in his study of competitive equilibrium [2, Theorem 3], and our use of the Continuity Assumption will be the same.

The notation "int" is used to designate the $\|\cdot\|_\infty$ interior of a subset of $L_\infty(M)$.

THEOREM 3.4. Assume Γ is S-measurable and the Constraint Qualification and Continuity Assumption hold. Assume further that there is some solution $\hat{X}$ of (*) such that $\hat{X} \in \text{int } D$. Then there exists $A \in IV(M)$ such that (i) each solution $\bar{X}$ of (*) solves (3.1) and each solution of (3.1) belonging to $L_\infty(S)$ solves (*), and (ii) $\mathbb{E}[\int_0^T X_t dA_t] = 0$, $\forall X \in L_\infty(S)$.

COROLLARY 3.5. Assume Γ is S-measurable and the Constraint Qualification and Continuity Assumption hold. Let $\bar{X} \in L_\infty(S) \cap \text{int } D$. Then $\bar{X}$ solves (*) iff there exists $A \in IV(M)$ such that $\bar{X}$ solves (3.1) and such that $\mathbb{E}[\int_0^T X_t dA_t] = 0$, $\forall X \in L_\infty(S)$.

Proof of Corollary 3.5. The necessity is given by the theorem. On the other hand it is clear that if $\bar{X}$ solves (3.1) for any $A \in IV(M)$ satisfying $\mathbb{E}[\int_0^T X_t dA_t] = 0$, $\forall X \in L_\infty(S)$, then $\bar{X}$ solves (*). []

Proof of Theorem 3.4. Let π be as in Lemma 3.2. Reasoning as in the proof of Lemma 3.1, one deduces from equation (3.3) that $\langle Z,\pi\rangle = 0$, $\forall Z \in L_\infty(S)$.

Let B be the element of $IV(M)$ which represents π_c, and let $A = B - B^s$. It suffices to establish (3.2), and only the inequality

$$\sup_{X\in L_\infty(S)} \Phi(X) \geq \sup_{X\in L_\infty(S), Y\in L_\infty(M)} \{\Phi(X+Y) - \mathbb{E}[\int_0^T Y_t dA_t]\} \tag{3.8}$$

is not obvious. We prove (3.8) by means of two lemmas.

LEMMA A. Under the hypothesis of Theorem 3.4, if $X \in L_\infty(S)$ and $Y \in L_\infty(M)$ satisfy $X + Y \in \text{dom } \Phi$, then

$$\mathbb{E}[\int_0^T \hat{X}_t dB_t^s] \geq \mathbb{E}[\int_0^T (X+Y)_t dB_t^s] \tag{3.9}$$

Proof of Lemma A. Let $X \in L_\infty(S)$ and $Y \in L_\infty(M)$ satisfy $X + Y \in \text{dom } \Phi$. Since π is orthogonal to $L_\infty(S)$, we have

$$-\langle \hat{X}-X-{}^sY,\pi_p\rangle = \langle \hat{X}-X-{}^sY,\pi_c\rangle \equiv \mathbb{E}\{\int_0^T (\hat{X}-X-{}^sY)_t dB_t\}.$$

Moreover by the definition of the dual projection,

$$\mathbb{E}[\int_0^T (\hat{X}-X-{}^sY)_t dB_t] = \mathbb{E}[\int_0^T (\hat{X}-X-Y)_t dB_t^s].$$

Hence (3.9) is equivalent to

$$\langle \hat{X},\pi_p\rangle \leq \langle X+{}^sY,\pi_p\rangle. \tag{3.10}$$

Now fix $\varepsilon > 0$ such that $Z \in D$ whenever $\|\hat{X}-Z\|_\infty < \varepsilon$. Choose $0 < \lambda < 1$ such that $\lambda\|\hat{X}-X-Y\|_\infty < \varepsilon$. Since $\hat{X}$, $X+Y \in \text{dom } \Phi \subset L_\infty(M;\Gamma)$ and each set $\Gamma_t(\omega)$ is convex, we have $[1-\lambda]\hat{X} + \lambda X + \lambda Y \in L_\infty(S;\Gamma)$. The S-projection of this process is $[1-\lambda]\hat{X} + \lambda X + \lambda^s Y$, by the linearity of the projection. It follows from Lemma 3.3 that $[1-\lambda]\hat{X} + \lambda X + \lambda^s Y \in L_\infty(S;\Gamma)$.

Choose a decreasing sequence of sets $E_n \in M$ satisfying $\lim_{n\to\infty}\mu(E_n) = 0$, $\lim_{n\to\infty}|\pi_c|(E_n) = 0$, and $|\pi_p|\big([0,T) \times \Omega\backslash E_n\big) = 0$ for each n. Let $Z_n = \lambda(X+{}^sY-\hat{X})I_{E_n}$ where "I" denotes the indicator function. Since $\hat{X} + Z_n$ equals $\hat{X}$ on the complement of E_n and $[1-\lambda]\hat{X} + \lambda X + \lambda^s Y$ on E_n, we have $\hat{X} + Z_n \in L_\infty(M;\Gamma)$. Moreover $\|Z_n\|_\infty \leq \lambda\|X+{}^sY-\hat{X}\|_\infty$, and since the S-projection is isotone, $\|X+{}^sY-\hat{X}\|_\infty \leq \|X+Y-\hat{X}\|_\infty$. Hence $\|Z_n\|_\infty < \varepsilon$, and we conclude that

$\hat{X} + Z_n \in D$.

By (3.3) and the fact that $\hat{X}$ solves (*), we have, for each n,

$$\Phi(\hat{X}) \geq \Phi(\hat{X}+Z_n) - \langle Z_n,\pi\rangle.$$

By the Continuity Assumption, $\liminf_{n\to\infty}\Phi(\hat{X}+Z_n) \geq \Phi(\hat{X})$, so the above yields

$$0 \geq \limsup_{n\to\infty} - \langle Z_n,\pi\rangle. \tag{3.11}$$

By the construction of the sets E_n, we have $\lim_{n\to\infty}\langle Z_n,\pi_c\rangle = 0$ and $\langle Z_n,\pi_p\rangle = \lambda\langle X+{}^{S}Y-\hat{X},\pi_p\rangle$ for each n, so the right-hand side of (3.11) equals $\lambda\langle\hat{X}-X-{}^{S}Y,\pi_p\rangle$. Thus (3.11) implies (3.10), and the proof is complete. []

LEMMA B. Under the hypothesis of Theorem 3.4, if $X \in L_\infty(S)$ and $Y \in L_\infty(M)$ then

$$\Phi(\hat{X}) - \mathbb{E}\left[\int_0^T \hat{X}_t dB_t\right] \geq \Phi(X+Y) - \mathbb{E}\left[\int_0^T (X+Y)_t dB_t\right]. \tag{3.12}$$

<u>Proof of Lemma B</u>. Fix $X \in L_\infty(S)$ and $Y \in L_\infty(M)$. We may without loss of generality assume $X + Y \in \text{dom}\,\Phi$. Let ε, λ and the sets E_n be as in the proof of Lemma A. Let $W_n = \lambda(X+Y-\hat{X})I_{([0,T]\times\Omega)\setminus E_n}$. Then $\hat{X} + W_n$ equals $\hat{X}$ on E_n and $(1-\lambda)\hat{X} + \lambda(X+Y)$ on the complement of E_n, so the convexity of the sets $\Gamma_t(\omega)$ implies that $\hat{X} + W_n \in L_\infty(M;\Gamma)$. Also $\|W_n\| < \varepsilon$, so $\hat{X} + W_n \in D$. Now the Continuity Assumption implies that $\liminf_{n\to\infty}\Phi(\hat{X}+W_n) \geq \Phi\big((1-\lambda)\hat{X} + \lambda(X+Y)\big)$. From (3.3) and the fact that $\hat{X}$ solves (*) we have that, for each n, $\Phi(\hat{X}) + \langle W_n,\pi\rangle \geq \Phi(\hat{X}+W_n)$. Combining these inequalities with the concavity

of Φ gives

$$\Phi(\hat{X}) + \liminf_{n\to\infty}\langle W_n,\pi\rangle \geq (1-\lambda)\Phi(\hat{X}) + \lambda\Phi(X+Y) \tag{3.13}$$

The construction of the sets E_n is such that $\liminf_{n\to\infty}\langle W_n,\pi_c\rangle = \lambda\langle X+Y-\hat{X},\pi_c\rangle$ and $\langle W_n,\pi_p\rangle = 0$ for each n. Therefore (3.13) implies

$$\langle X+Y-\hat{X},\pi_c\rangle \geq \Phi(X+Y) - \Phi(\hat{X}),$$

from which (3.12) follows by virtue of the fact that B represents π_c. []

We now return to the proof of the theorem. In taking the supremum in the right-hand side of (3.8) it certainly suffices to consider X, Y such that $X + Y \in \operatorname{dom}\Phi$. In this case we have from the lemmas that

$$\Phi(X+Y) - \mathbb{E}\left[\int_0^T (X+Y)_t dA_t\right] \leq \Phi(\hat{X}) - \mathbb{E}\left[\int_0^T \hat{X}_t dA_t\right].$$

Since X, $\hat{X} \in L_\infty(S)$ and $A = B - B^s$, we have that $\mathbb{E}\left[\int_0^T X_t dA_t\right] = \mathbb{E}\left[\int_0^T \hat{X}_t dA_t\right] = 0$. Therefore

$$\sup_{X\in L_\infty(S),Y\in L_\infty(M)} \left\{\Phi(X+Y) - \mathbb{E}\left[\int_0^T Y_t dA_t\right]\right\} \leq \Phi(\hat{X}),$$

which implies (3.8). []

4. SECURITIES MARKETS AND OPTIMAL PORTFOLIOS

The model will be the same as in Harrison-Pliska [15], except that the security price process $Z = (Z^1,\ldots,Z^m)$ will be assumed to be of integrable variation; that is, $Z \in IV(\mathcal{O})$. For a trading strategy $X \in L_\infty(\mathcal{M})$, $\int_0^t X_s dZ_s$ is the capital gains earned through date t. Trading strategies are constrained to be predictable, reflecting the fact that price changes cannot be perfectly anticipated.

In the background is a riskless asset (a bond), the price of which is always one. Denote the holdings of this asset by B_t. Let w be the wealth of the trader at date zero. Then his wealth at date t is $w + \int_0^t X_s dZ_s$, and this must equal the value of his portfolio, i.e.,

$$w + \int_0^t X_s dZ_s = B_t + \sum_{i=1}^m X_t^i Z_t^i. \tag{4.1}$$

We will assume that B_t is not constrained, so one can regard (4.1) as being simply a definition of B_t.

Denote by u a concave, strictly increasing function from $\mathbb{R}$ to $\mathbb{R}$. Assume $u(y(\cdot))$ is $\mathbb{P}$-integrable whenever $y \in L_1(\Omega,\mathcal{F},\mathbb{P})$. Interpret $\mathbb{E}[u(\int_0^T X_t dZ_t)]$ as the utility obtained from the total capital gains, where $T < \infty$ is a fixed time horizon. Note that the bond B_t does not appear in this expression, since no capital gains can be earned by trading in an asset with constant price. Note also that one would ordinarily take utility as being defined over final wealth, $w + \int_0^T X_t dZ_t$, but since w is a scalar our formula follows by translation.

We are interested in the following type of optimization problem:

$$\text{maximize } \mathbb{E}\left[u\left(\int_0^T X_t \, dZ_t\right)\right] \quad \text{subject to} \quad X \in L_\infty(\mathcal{P}) \tag{4.2}$$

The outstanding result on this kind of problem is due to Harrison and Kreps [14, Theorems 1,2]. They allow the price process Z to have unbounded variation, provided $Z_t^i \in L_2(\Omega,\mathcal{F},\mathbb{P})$ for each i and t. However, they restrict the optimization problem to be over the class of simple processes X, so there is no difficulty in the definition of the stochastic integral when X is not predictable. Their result may be stated in the following fashion. Assume that $\lim_{n\to\infty}\mathbb{E}[u(y_n)] = \mathbb{E}[u(y)]$ whenever $\lim_{n\to\infty} y_n = y$ in $L_2(\Omega,\mathcal{F},\mathbb{P})$. Then there exists $\rho \in L_2(\Omega,\mathcal{F},\mathbb{P})$ such that $\rho > 0$ a.s. and such that the process A with values $A_t(\omega) = \rho(\omega)Z_t(\omega)$ satisfies (i) if $\bar{X} \in L_\infty(\mathcal{P})$ then $\bar{X}$ solves (4.2) iff $\bar{X}$ solves

$$\text{maximize } \mathbb{E}\left[u\left(\int_0^T X_t \, dZ_t\right) - \int_0^T X_t \, dA_t\right] \quad \text{subject to} \quad X \in L_\infty(\mathcal{M}) \tag{4.3}$$

and (ii) $\mathbb{E}[\int_0^T X_t \, dA_t] = 0, \ \forall \ X \in L_\infty(\mathcal{P})$. In other words, the shadow price of information exists and is of the form $A = \rho Z$.

Actually, instead of proving that if $\bar{X}$ solves (4.2) then $\bar{X}$ solves (4.3), Harrison and Kreps show that $\bar{y} \equiv \int_0^T \bar{X}_t \, dZ_t$ solves

$$\text{minimize } \mathbb{E}[\rho y] \quad \text{subject to} \quad y \in L_2(\Omega,\mathcal{F},\mathbb{P})$$

$$\text{and} \quad \mathbb{E}[u(y)] \geq \mathbb{E}\left[u\left(\int_0^T \bar{X}_t \, dZ_t\right)\right].$$

The Lagrangian for this, multiplied by -1, is

$$\lambda\mathbb{E}[u(y)] - \mathbb{E}[\rho y].$$

Dividing by λ and restricting y to the class $\{\int_0^T X_t dZ_t | X \in L_\infty(M)\}$, one obtains the problem (4.3).

For the theory of contingent claim valuation, to which the Harrison-Kreps paper is addressed, the important aspect of the above result is the condition (ii). It means that the linear functional $y \mapsto \mathbb{E}[\rho y]$ on $L_2(\Omega, F, \mathbb{P})$ is orthogonal to the class of capital gains $\int_0^T X_t dZ_t$ which can be realized by trading strategies $X \in L_\infty(P)$. From this it follows that the linear functional is a "rational" pricing scheme for contingent claims $y \in L_2(\Omega, F, \mathbb{P})$. Moreover one can write $\mathbb{E}[\int_0^T X_t dA_t] = \mathbb{E}_Q[\int_0^T X_t dZ_t]$ where "$\mathbb{E}_Q$" denotes integration by the measure Q defined by $\frac{dQ}{d\mathbb{P}} = \rho$. Thus (ii) implies that the price process Z is a martingale under a measure Q which is absolutely continuous with respect to $\mathbb{P}$.

Another interpretation of the martingale result is that the process (M_t) is a martingale under $\mathbb{P}$, where $M_t = \mathbb{E}[\rho | F_t] Z_t$. The maximization in (4.3) implies that $\bar{X}$ maximizes $\mathbb{E}\left[u(\int_0^T X_t dZ_t) - \int_0^T X_t dM_t\right]$ on $L^\infty(O)$.

The set-up implied by (4.2) is referred to as a "frictionless market," because there are no constraints on the trading strategies X (other than being bounded and predictable). For the balance of this section we shall use Corollary 3.5 to obtain results when market frictions are present in the form of restrictions on short sales. Using our earlier notation, set $\Gamma_t(\omega) = \mathbb{R}_+^m$ and $D = L_\infty(M)$. Define $\Phi: L_\infty(M) \to \mathbb{R} \cup \{-\infty, \infty\}$ by

$$\Phi(X) = \begin{cases} \mathbb{E}\left[u(\int_0^T X_t dZ_t)\right] & \text{if} \quad X \in L_\infty(M;\Gamma) \\ -\infty & \text{otherwise.} \end{cases}$$

Observe that dom $\Phi = L_\infty(M;\Gamma)$ by virtue of our assumptions about u and the fact that $\int_0^T X_t dZ_t \in L_1(\Omega, F, \mathbb{P})$ for all $X \in L_\infty(M)$. Note also that Φ is concave.

With these specifications problem (*) is the optimal portfolio problem (4.2) with the modification that short sales of the risky securities (indexed by $i = 1, \ldots, m$) are prohibited. Note, however, that one can still sell the bond short (i.e., one can have $B_t < 0$) and that there is no particular lower bound (such as zero) on overall wealth.

Here is the main result of this section. We do not impose the L_2-continuity of expected utility assumed by Harrison and Kreps.

THEOREM 4.1. Let $\bar{X} \in L_\infty(P)$. Then $\bar{X}$ solves (*) iff there exists a martingale $M \in IV(O)$ such that $\bar{X}$ maximizes $\Phi(X) - \mathbb{E}[\int_0^T X_t dM_t]$ over $X \in L_\infty(O)$.

Proof. In view of the remark immediately following (3.1), it suffices to apply Corollary 3.5. The only nontrivial hypotheses to check are the Constraint Qualification and the Continuity Assumption.

With regard to the former, one could choose $\tilde{X}_t^i(\omega) \equiv 1$ and $\varepsilon = 1$. If $\|X-\tilde{X}\|_\infty < 1$, then $X \geq 0$ and thus $\Phi(X) > -\infty$. Moreover $\int_0^T X_t dZ_t \geq -2\int_0^T |dZ_t|$, so, since u is increasing, $\mathbb{E}[u(\int_0^T X_t dZ_t)] \geq \mathbb{E}[u(-2\int_0^T |dZ_t|)] > -\infty$.

Finally, to verify the Continuity Assumption, let μ^i be the $\mathbb{P}$-measure defined by

$$\int_{[0,T]\times\Omega} X_t(\omega) d\mu^i = \mathbb{E}[\int_{[0,T]} X_t(\omega) |dZ_t^i|$$

for each bounded measurable X (see DM, VI.64), and set $\mu = \mu^1 + \ldots + \mu^m$. Suppose $r < \infty$, $X \in \text{dom}\,\Phi$, and (X_n) is a sequence from $\text{dom}\,\Phi$ satisfying $\|X\|_\infty < r$, $\|X_n\|_\infty < r$, and $\mu(\bigcap_{n=1}^\infty E_n) = 0$, where $E_n = \bigcup_{\ell=n}^\infty \{(t,\omega) | X_{\ell t}(\omega) \neq X_t(\omega)\}$. Then the random variables b_n defined by

$$b_n(\omega) = \sum_{i=1}^{m} \int_{\{\omega|(t,\omega) \in E_n\}} |dZ_t^i(\omega)|$$

satisfy $b_n \geq b_{n+1} \geq 0$, $\forall n$, and $\mathbb{E}[b_n] \equiv \mu(E_n) \downarrow 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} b_n(\omega) = 0$ a.s. Since

$$|\int_0^T X_{nt}(\omega)dZ_t(\omega) - \int_0^T X_t(\omega)dZ_t(\omega)| \leq rb_n(\omega) \text{ a.s.},$$

we have, almost surely,

$$\lim_{n\to\infty} \int_0^T X_{nt}(\omega)dZ_t(\omega) = \int_0^T X_t(\omega)dZ_t(\omega), \text{ and}$$

$$\int_0^T X_t(\omega)dZ_t(\omega) - rb_1(\omega) \leq \int_0^T X_{nt}(\omega)dZ_t(\omega) \leq \int_0^T X_t(\omega)dZ_t(\omega) + rb_1(\omega).$$

Therefore the continuity and monotonicity of $u(\cdot)$ allows us to apply the Lebesgue Convergence Theorem to conclude that $\Phi(X) = \lim_{n\to\infty}\Phi(X_n)$. []

The price process Z need not be a martingale under any equivalent probability measure in this situation, where short sales of the risky securities are prohibited. For example, the sample paths of Z could be decreasing. However it is simple to show that Z must be a supermartingale under an equivalent measure.

We conclude this section by remarking that our methods can be used to study security models with other kinds of market frictions. However, the duality theorem may not be valid if the market friction (constraint) is such that the optimum is not in the interior of D. This difficulty can arise, for example, if the terminal wealth $w + \int_0^T X_t dZ_t$ is required to be nonnegative; see Back [1] for the details.

5. THE MAXIMUM PRINCIPLE FOR ECONOMIC GROWTH

5.1. The Optimal Economic Growth Problem

This section addresses the one-sector optimal economic growth (or optimal savings) problem as formulated by Foldes [13]. The underlying assumption is that the horizon is infinite. However we will truncate at some finite horizon T and assign a value to the terminal capital stock. This truncation allows one to accomodate the assumption that there is an optimal solution with the capital stock being bounded away from zero.

A consumption plan will be a real-valued, nonnegative, optional process C. A boundedness restriction will also be imposed below. The initial capital stock is a positive constant K_0. The evolution of the capital stock is determined by the equation

$$dK_t = -C_t dt + K_t dW_t, \tag{5.1}$$

where the rate-of-return process W is an exogenously given, continuous semimartingale. By Ito's Lemma, the solution of (5.1) is

$$K_t = K_0 \exp(Z_t) - \exp(Z_t)\int_0^T \exp(-Z_s) C_s ds \tag{5.2}$$

where $Z = W - \frac{1}{2}\langle W,W\rangle$.

Let $u\colon [0,T] \times \Omega \times \mathbb{R}_+ \to \mathbb{R}$ and $v\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$. The following assumptions will be maintained throughout Section 5.

A. For each $(t,\omega) \in [0,T] \times \Omega$, the function $u_t(\omega,\cdot)$ is concave, continuous and monotone $\left(\text{i.e., } u_t(\omega,d) \geq u_t(\omega,c) \text{ if } d > c\right)$ on $\mathbb{R}_+$. For each $c \in \mathbb{R}_+$ the function $(t,\omega) \to u_t(\omega,c)$ is $\mathcal{O}$-measurable and $dt \otimes d\mathbb{P}$-integrable.

B. For each $\omega \in \Omega$, the function $v(\omega,\cdot)$ is concave, continuous and monotone on $\mathbb{R}_+$. For each $k \in \mathbb{R}_+$, the function $v(\cdot,k)$ is $\mathcal{F}$-measurable and $\mathbb{P}$-integrable.

For future convenience extend each of $u_t(\omega,\cdot)$ and $v(\omega,\cdot)$ to all of $\mathbb{R}$ by setting $u_t(\omega,c) = v(\omega,k) = -\infty$ if $c, k < 0$.

We remark that, in view of Corollary 3.1 and Theorem 4 of Rockafellar [17], the function v is a "normal concave integrand." This means in particular that the function $v(\cdot,k(\cdot))$ is $\mathcal{F}$-measurable when $k(\cdot)$ is $\mathcal{F}$-measurable. The function u is a normal concave integrand by the same reasoning, in this case with respect to the optional σ-field $\mathcal{O}$. Since normality with respect to $\mathcal{O}$ implies normality with respect to $\mathcal{M}$, we have that the process $(t,\omega) \to u_t(\omega,C_t(\omega))$ is optional when C is optional and measurable when C is measurable.

The problem is to choose a consumption plan to maximize

$$\mathbb{E}\left[\int_0^T u_t(\omega,C_t(\omega))\,dt + v(\omega,K_T(\omega))\right], \tag{5.3}$$

where K_T is defined by (5.2) and subject to the restriction that the processes C and K be nonnegative.

Note that (5.3) is well-defined in $\mathbb{R} \cup \{+\infty\}$ whenever C and K are nonnegative, by virtue of the monotonicity of u_t and v and the integrability of $u_\bullet(\cdot,0)$ and $v(\cdot,0)$.

5.2. Relation to the Problem of Section 3

We will model the economic growth problem as being the problem of choosing a process $X \in L_\infty(\mathcal{O})$, where m = 1. Given $X \in L_\infty(\mathcal{M})$, define H by

$$H_t = K_0 - \int_0^t X_s ds. \tag{5.4}$$

Let $\Gamma_t(\omega) \equiv \mathbb{R}_+$ and denote by D the class of $X \in L_\infty(\mathcal{M})$ such that H_T defined by (5.4) is nonnegative almost surely. If $X \in L_\infty(\mathcal{M};\Gamma) \cap D$ let $\Phi(X)$ denote the value in (5.3) obtained by setting

$$C_t = X_t \exp(Z_t) \quad \text{and} \quad K_T = H_T \exp(Z_T).$$

Notice that, in view of (5.2), if X is nonnegative then the nonnegativity of the process K is equivalent to the condition $X \in D$. Therefore the constraints of the economic growth problem are satisfied iff X is optional and belongs to $L_\infty(\mathcal{M};\Gamma) \cap D$. If $X \in L_\infty(\mathcal{M}) \backslash L_\infty(\mathcal{M};\Gamma) \cap D$, set $\Phi(X) = -\infty$. We assume $\sup\{\Phi(X) | X \in L_\infty(\mathcal{O})\} < \infty$. The inequality $\sup\{\Phi(X) | X \in L_\infty(\mathcal{O})\} > -\infty$ follows from Assumptions A and B (use X = 0).

We take as the economic growth problem the problem (*), where $\mathcal{S} = \mathcal{O}$. The study of X rather than C (i.e., analysis in "reduced units") follows Foldes [13], though the boundedness restriction is new.

For future convenience define $L_t(\omega,x) = u_t(\omega,\exp(Z_t(\omega))x)$ and $\ell(\omega,h) = v(\omega,\exp(Z_T(\omega))h)$.

It is worthwhile to note here that under the assumptions already introduced, the function $(t,\omega) \to L_t(\omega,X_t(\omega))$ is $dt\otimes d\mathbb{P}$-integrable and the function $\omega \to \ell(\omega,h(\omega))$ is $\mathbb{P}$-integrable whenever $X \in L_\infty(\mathcal{M})$, $h \in L_\infty(\Omega,\mathcal{F},\mathbb{P})$, $X_t(\omega) \geq 0$ $dt\otimes d\mathbb{P}$-a.e., and $h(\omega) \geq 0$ $\mathbb{P}$-a.s. To see this, note that Assumptions

A and B yield integrability when $X = 0$, $h = 0$. The assumption $\sup\{\Phi(X) \mid X \in L_\infty(\mathcal{O})\} < \infty$ implies integrability when $X = K_0/2T$ and $H = K_0/2$. Using the monotonicity and concavity, these facts imply integrability for all nonnegative bounded processes X and functions h.

5.3. The Maximum Principle and the Martingale Property

The maximum principle for this problem can be stated as follows.

Maximum Principle. There exists a martingale $(p_t)_{0 \leq t \leq T}$ such that $\bar{X}$ solves (*) iff

$$
\begin{aligned}
dt \otimes d\mathbb{P}\text{-a.e.}, \quad & L_t\big(\omega, \bar{X}_t(\omega)\big) - p_t(\omega)\bar{X}_t(\omega) \geq \\
& L_t(\omega, x) - p_t(\omega)x, \ \forall\, x \in \mathbb{R}, \text{ and}
\end{aligned} \tag{5.5}
$$

$$
\begin{aligned}
\text{a.s.}, \quad & \ell\big(\omega, \bar{H}_T(\omega)\big) - p_T(\omega)\bar{H}_T(\omega) \geq \\
& \ell(\omega, h) - p_T(\omega)h, \ \forall\, h \in \mathbb{R}
\end{aligned} \tag{5.6}
$$

In the above we are of course taking $\bar{H}_T = K_0 - \int_0^T \bar{X}_t dt$. Conditions (5.9) and (5.10) are Bismut's [3] coextremality conditions (4.9) and (4.10) specialized to this problem. We can take (p_t) to be a martingale because the state variable H_t enters the objective function and constraints only at the time $t = T$; therefore the dual variable $\dot{p}$, in Bismut's notation, must be the null process. See Bismut's equation (1.12). Note also that in equation (5.6) the minus sign in front of p_T, which does not appear in Bismut's equation (4.10), reflects the fact that we have $\dot{H}_t = -X_t$ rather than $\dot{H}_t = X_t$.

It is clear from (5.5) that if $u_t(\omega,\cdot)$ is differentiable on $\mathbb{R}_+$ then, $dt\otimes d\mathbb{P}$-a.e.,

$$\exp\big(Z_t(w)\big)u_t'\big(\omega,\exp\big(Z_t(w)\big)\bar{X}_t(\omega)\big) \leq p_t(\omega), \text{ and}$$

$$\bar{X}_t(\omega)\big[\exp\big(Z_t(w)\big)u_t'\big(\omega,\exp\big(Z_t(w)\big)\bar{X}_t(\omega)\big) - p_t(\omega)\big] = 0. \tag{5.7}$$

If $\bar{X}_t(\omega) > 0$ on a set of full $dt\otimes d\mathbb{P}$ measure, then (5.7) implies that the "reduced" utility process $\big(\exp(Z_t)u_t'\big)$ equals the martingale (p_t), $dt\otimes d\mathbb{P}$-a.e. This result was obtained by Foldes [13, Theorem 6], under the assumption that $\bar{H}_T$ is bounded away from zero. (Foldes also showed that one can choose modifications to obtain $\mathbb{P}$-a.s. equality for each t, i.e., that the process $\big(\exp(Z_t)u_t'\big)$ is a martingale).

We will give a different proof that the maximum principle is valid when there is a solution of (*) such that $\bar{H}_T$ is bounded away from zero. This proof will utilize the shadow price of information. First we show that, regardless of whether the state constraint $H_T \geq 0$ is binding, if the shadow price of information exists, then the maximum principle is valid.

5.4. The Shadow Price of Information and the Maximum Principle

We will assume in this subsection that $A \in IV(\mathcal{M})$ is the shadow price of information for the problem (*), i.e., that A satisfies conditions (i) and (ii) of Theorem 3.4.

Consider for fixed ω the problem

$$\text{maximize } \int_0^T L_t(\omega,x_t)dt + \ell(\omega,h_T) - \int_0^T x_t dA_t(\omega)$$

$$\text{subject to } x \in L_\infty([0,T],dt) \tag{5.8}$$

where $h_T = K_0 - \int_0^T x_t dt$.

Let $\Xi(\omega)$ denote the set of $p \in \mathbb{R}$ such that

$$\sup_x\{\int_0^T L_t(\omega,x_t)dt + \ell(\omega,K_0 - \int_0^T x_t dt) - \int_0^T x_t dA_t(\omega)\}$$

$$= \sup_{b,x}\{\int_0^T L_t(\omega,x_t)dt + \ell(\omega,b + K_0 - \int_0^T x_t dt) - \int_0^T x_t dA_t(\omega) - pb\},$$

where the suprema are taken over $x \in L_\infty([0,T],dt)$ and $b \in \mathbb{R}$.

The problem (5.8) is a deterministic version of the economic growth problem, one obtained by inserting the shadow price of information and assuming the state ω is known. The maximum principle for this deterministic problem coincides with the existence of a Lagrange multiplier for the constraint $h_T = K_0 - \int_0^T x_t dt$. The set $\Xi(\omega)$ consists precisely of those multipliers.

Consider the class of $p \in L_1(\Omega,F,\mathbb{P})$ such that the following holds:

$$\sup_X \mathbb{E}[\int_0^T L_t(\omega,X_t(\omega))dt + \ell(\omega,K_0 - \int_0^T X_t(\omega)dt)$$

$$- \int_0^T X_t(\omega)dA_t(\omega)] \tag{5.9}$$

$$= \sup_{b,X}\mathbb{E}[\int_0^T L_t(\omega,X_t(\omega))dt + \ell(\omega,b(\omega) + K_0 - \int_0^T X_t(\omega)dt)$$

$$- \int_0^T X_t(\omega)dA_t(\omega) - p(\omega)b(\omega)],$$

where in this case the suprema are taken over $X \in L_\infty(M)$ and $b \in L_\infty(\Omega,F,\mathbb{P})$. Though we have not verified the truth of this assertion, it seems that such p should be the integrable selections of Ξ. Regardless, we will show that such p exist and determine coextremals for the economic growth problem.

THEOREM 5.1. Assume the shadow price of information A exists. Then the Maximum Principle is valid. In fact one can take $p_t = \mathbb{E}[p|F_t]$ for each t, where $p \in L_1(\Omega,F,\mathbb{P})$ satisfies (5.9).

<u>Proof</u>. For $X \in L_\infty(M)$ and $b \in L_\infty(\Omega,F,\mathbb{P})$ set

$$F(X,b) = -\mathbb{E}[\int_0^T L_t(X_t)dt + \ell(b+K_0 - \int_0^T X_t dt) - \int_0^T X_t dA_t].$$

The functional F is clearly convex in (X,b). Setting $\tilde{X} = 0$, we have

$$F(\tilde{X},b) \leq -\mathbb{E}\{\int_0^T L_t(0)dt + \ell(0)\} < \infty$$

for all $b \in L_\infty(\Omega,F,\mathbb{P})$ such that ess $\sup_\omega |b(\omega)| \leq K_0$, by virtue of the monotonicity of ℓ. Therefore it follows from Theorems 18(a), 17(a), and 16 and equation (4.2) of Rockafellar [19] that there exists $\sigma \in L_\infty^*(\Omega,F,\mathbb{P})$ such that

$$\inf\{F(X,0)|X \in L_\infty(M)\}$$

$$= \inf\{F(X,b) + \langle b,\sigma\rangle | \ X \in L_\infty(M), \ b \in L_\infty(\Omega,F,\mathbb{P})\}. \qquad (5.10)$$

Let $\sigma = \sigma_c + \sigma_p$ be the Yosida-Hewitt decomposition of σ. We will now show that

$$\inf\{F(X,0)\,|\,X \in L_\infty(M)\}$$

$$= \inf\{F(X,b) + \langle b,\sigma_c\rangle\,|\,X \in L_\infty(M),\ b \in L_\infty(\Omega,F,\mathbb{P})\}. \tag{5.11}$$

It is evident that the left-hand side is no smaller than the right. To establish the reverse inequality we will use equation (5.10).

Consider an arbitrary $X \in L_\infty(M)$, $b \in L_\infty(\Omega,F,\mathbb{P})$. Select a decreasing sequence of sets $E_n \in F$ such that $\lim_{n\to\infty}\mathbb{P}(E_n) = 0$, $\lim_{n\to\infty}\sigma_c(E_n) = 0$, and $\sigma_p(\Omega\backslash E_n) = 0$ for each n. Let $X_n = XI_{([0,T]\times\Omega)\backslash([0,T]\times E_n)}$ and $b_n = bI_{\Omega\backslash E_n}$. Then

$$F(X_n,b_n) + \langle b_n,\sigma\rangle = F(X_n,b_n) + \langle b_n,\sigma_c\rangle,\ \forall\ n, \tag{5.12}$$

$$\lim_{n\to\infty}\langle b_n,\sigma_c\rangle = \langle b,\sigma_c\rangle, \text{ and} \tag{5.13}$$

$$\lim_{n\to\infty}F(X_n,b_n) = F(X,b). \tag{5.14}$$

Only (5.14) requires comment. Note that we have by assumption $F(X,b) \in \mathbb{R}$ unless $\int_0^T L_t(X_t) = -\infty$ or $\ell(b + K_0 - \int_0^T X_t) = -\infty$ on a set of positive $\mathbb{P}$-measure. Thus if $F(X,b) \notin \mathbb{R}$ then eventually $F(X_n,b_n) = F(X,b) = -\infty$. On the other hand if $F(X,b) \in \mathbb{R}$ then the functions $\int_0^T L_t(X_t)$ and $\ell(b + K_0 - \int_0^T X_t)$ are $\mathbb{P}$-integrable. The same is clearly true of the functions $\int_0^T L_t(X_{nt})$ and $\ell(b_n + K_0 - \int_0^T X_{nt})$. Hence in this case (5.14) follows from the continuity of the indefinite integral.

Collecting (5.12), (5.13) and (5.14) we see that it certainly must be true that

$$F(X,b) + \langle b,\sigma_c\rangle \geq \inf\{F(X,b) + \langle b,\sigma\rangle | X \in L_\infty(\mathcal{M}),\ b \in L_\infty(\Omega,\mathcal{F},\mathbb{P})\}.$$

Since the pair (X,b) was chosen arbitrarily, it follows that the right-hand side of (5.10) majorizes that of (5.11). This completes the proof of (5.11).

Now let p be the Radon-Nikodym derivative $\frac{d\sigma_c}{d\mathbb{P}}$. Multiplying both sides of (5.11) by minus one, we have exactly equation (5.9).

Let (p_t) be an r.c.l.l. version of the process $(E[p|\mathcal{F}_t])$. It remains to show that $\bar{X}$ solves (*) iff (5.5) and (5.6) hold.

Assume $\bar{X}$ solves (*). Since A is the shadow price of information the left-hand side of (5.9) must be $\Phi(\bar{X})$, which, using the fact that

$$\mathbb{E}[p\bar{H}_T] = \mathbb{E}\left[p(K_0 - \int_0^T \bar{X}_t dt)\right] = K_0\mathbb{E}[p] - \mathbb{E}[\int_0^T p\bar{X}_t dt],$$

can be written as

$$\mathbb{E}\left[\int_0^T\{L_t(\bar{X}_t) - p\bar{X}_t\}dt\right] + \mathbb{E}\left[\ell(\bar{H}_T) - p\bar{H}_T\right] + K_0\mathbb{E}[p]. \tag{5.15}$$

Now given any $X \in L_\infty(\mathcal{O})$ and $H_T \in L_\infty(\Omega,\mathcal{F},\mathbb{P})$, we obtain from (5.9), by setting $b = H_T - K_0 + \int_0^T X_t dt$, that (5.15) majorizes

$$\mathbb{E}\left[\int_0^T\{L_t(X_t) - pX_t\}dt\right] + \mathbb{E}\left[\ell(H_T) - pH_T\right] + K_0\mathbb{E}[p].$$

It must be therefore be that the expression

$$\mathbb{E}\left[\int_0^T \{L_t(X_t) - pX_t\}dt\right] \tag{5.16}$$

is maximized on $L_\infty(\mathcal{O})$ at $\bar{X}$ and the expression

$$E\left[\ell(H_T) - pH_T\right] \tag{5.17}$$

is maximized on $L_\infty(\Omega,\mathcal{F},\mathbb{P})$ at $\bar{H}_T$.

If $X \in L_\infty(\mathcal{O})$ then the value of (5.16) is unchanged when p is replaced by p_t. Consider the function f on $[0,T] \times \Omega \times \mathbb{R}$ with values

$$f(t,\omega,x) = L_t(\omega,x) - p_t(\omega)x$$

$$\equiv u_t\left(\omega,\exp(Z_t(\omega))x\right) - p_t(\omega)x.$$

Since the process (p_t) is optional it follows from Corollary (3.1) and Theorem 4 of Rockafellar [17] that f is an $\mathcal{O}$-normal concave integrand. Clearly the maximization of (5.16) on $L_\infty(\mathcal{O})$ implies maximization on $L_\infty\left([0,T] \times \Omega,\mathcal{O},dt\otimes d\mathbb{P}\right)$. Using Theorem 3A of Rockafellar [21], we deduce from this that (5.5) must hold.

The same reasoning establishes that the maximization of (5.17) on $L_\infty(\Omega,\mathcal{F},\mathbb{P})$ implies (5.6).

Now, for the converse, assume that $\bar{X}$ satisfies (5.5) and (5.6). Then by adding (5.5) with (5.6) and rearranging terms we have, for $H_T = K_0 - \int_0^T X_t dt$,

$$\Phi(\bar{X}) - \Phi(X) \geqslant \mathbb{E}\left[\int_0^T p_t(\bar{X}_t - X_t)dt + p_T(\bar{H}_T - H_T)\right]$$

$$= \mathbb{E}\left[\int_0^T p_t(\bar{X}_t - X_t)dt\right] + \mathbb{E}\left[p_T\int_0^T (X_t - \bar{X}_t)dt\right]$$

$$= \int_0^T \mathbb{E}[p_t(\bar{X}_t - X_t)]dt + \int_0^T \mathbb{E}[p_T(X_t - \bar{X}_t)]dt$$

which is zero since $p_t = \mathbb{E}[p_T | F_t]$ and each of X_t and $\bar{X}_t$ are F_t-measurable. Hence $\bar{X}$ solves (*). []

5.5. Existence of the Shadow Price of Information

THEOREM 5.2. Assume there exists a solution $\hat{X}$ of (*) satisfying ess $\sup_\omega \int_0^T \hat{X}_t(\omega)dt < K_0$. Then there exists $A \in IV(M)$ satisfying conditions (i) and (ii) of Theorem 3.4.

COROLLARY 5.3. Under the hypothesis of Theorem 5.2, the Maximum Principle is valid.

Proof of Theorem 5.2. We must verify the hypothesis of Theorem 3.4. It is clear that F is concave, Γ is $\mathcal{O}$-measurable and $\hat{X} \in \text{int } \mathbb{D}$. We have assumed that $-\infty < \sup\{\Phi(X) | X \in L_\infty(\mathcal{O})\} < \infty$. It remains to verify the Constraint Qualification and Continuity Assumption.

Let $\tilde{X}_t(\omega) = K_0/2T$ for each (t,ω) and set $\varepsilon = K_0/2T$. If $X \in L_\infty(M)$ satisfies $\|X - \tilde{X}\|_\infty < \varepsilon$, then $X \geq 0$ and $H_T \equiv K_0 - \int_0^T X_t dt \geq 0$. Hence the monotonicity of L_t and ℓ yields

$$\Phi(X) \geq \mathbb{E}[\int_0^T L_t(0)dt + \ell(0)] > -\infty.$$

For the $\mathbb{P}$-measure μ in the Continuity Assumption, take $\mu = dt \otimes d\mathbb{P}$. Consider any $X \in \text{dom } \Phi$ and bounded sequence (X_n) from dom Φ satisfying $\mu(\bigcap_{m=1}^\infty \bigcup_{n=m}^\infty \{(t,\omega) | X_{nt}(\omega) \neq X_t(\omega)\}) = 0$.

Let r be a constant such that $\|X\|_\infty \leq r$ and $\|X_n\|_\infty \leq r$ for each n. Set $E_m = \bigcup_{n=m}^{\infty} \{(t,\omega)|X_{mt}(\omega) \neq X_t(\omega)\}$. Then

$$|\mathbb{E}[\int_0^T \{L_t(\omega, X_{mt}(\omega)) - L_t(\omega, X_t(\omega))\}dt]|$$

$$\leq \int_{E_m} \{L_t(\omega, r) - L_t(\omega, 0)\}d\mu,$$

which converges to zero by the continuity of the indefinite integral.

For each ω we have

$$0 \leq |\int_0^T X_{nt}(\omega)dt - \int_0^T X_t(\omega)dt| \leq r\int_{\{t|(t,\omega)\in E_n\}} dt.$$

The random variables $r\int_{\{t|(t,\cdot)\in E_n\}} dt$ are monotone decreasing and converge in expectation to zero; hence they converge a.s. to zero. Now, given the continuity and monotonicity of $\ell(\omega,\cdot)$, we can use the Lebesgue Convergence Theorem to conclude that

$$\lim_{n\to\infty} \mathbb{E}[\ell(K_0 - \int_0^T X_{nt} dt)] = \mathbb{E}[\ell(K_0 - \int_0^T X_t dt)].$$

This completes the verification of the Continuity Assumption. []

6. SELLING A RESOURCE

Our main result, Theorem 3.4, assumes the existence of a solution to the optimization problem; in fact it assumes the problem is solvable by some $\hat{X}$ in the $\|\cdot\|_\infty$-interior of the constraint set D. Here we will present some examples in which there may be no solution, yet the existence of the shadow price of information can be established.

Let λ be a positive constant and let (Z_t) be a one-dimensional, bounded adapted process.

We will take the dimension m of the control processes to equal one. Set $\Gamma_t(\omega) \equiv [0,\lambda]$ and denote by D the class of $X \in L_\infty(M)$ such that X is right-continuous and increasing. For $X \in D \cap L_\infty(M;\Gamma)$ set $\Phi(X) = \mathbb{E}[\int_0^T Z_t dX_t]$, and for $X \in L_\infty(M)\backslash D \cap L_\infty(M;\Gamma)$, set $\Phi(X) = -\infty$. The optimization problem is the problem (*), where $S = P$.

The interpretation is that the decision-maker has the quantity λ of goods to sell before time T, and X_t is the amount sold through time t. The price at time t is Z_t, and the total revenue is $\int_0^T Z_t dX_t$. We assume there is some (infinitesimal) time lag between the decision to sell and the actual transaction, the price being determined only at the later event. This inability to perfectly anticipate the price is modeled by restricting X to be predictable. We seek to determine the significance of this restriction, i.e., to find the shadow price of information.

The hypothesis of Theorem 3.4 will not be satisfied even when the problem has a solution, because the set D has no interior in the $\|\cdot\|_\infty$ topology. An instance in which the problem has no solution is the following: let τ be a totally inaccessible stopping time, and assume $Z_t = 0$ for $t < \tau$ and $Z_t = T-t$ for $t \geqslant \tau$.

To define the shadow price of information without reference to a solution of the problem, one can use the condition (3.2). This is equivalent to having $\mathbb{E}[\int_0^T X_t dA_t] = 0$, $\forall\, X \in L_\infty(P)$ and

$$\sup_{X\in L_\infty(P)} \Phi(X) = \sup_{X\in L_\infty(M)} \left\{\Phi(X) - \mathbb{E}[\int_0^T X_t dA_t]\right\}.$$

In keeping with the remark following (3.1) we will here establish, in two particular cases, the existence of a martingale M satisfying

$$\sup_{X\in L_\infty(\mathcal{P})} \Phi(X) = \sup_{X\in L_\infty(\mathcal{O})} \left\{\Phi(X) - \mathbb{E}[\int_0^T X_t dM_t]\right\}. \tag{6.1}$$

PROPOSITION 6.1. If the process (Z_t) is right-continuous with left-limits, and if $Z_{T-} = Z_T$ a.s., then (6.1) holds with $M = 0$.

<u>Proof</u>. Let $X \in L_\infty(\mathcal{O})$ be an arbitrary strategy for which $\Phi(X) > -\infty$. Let $\varepsilon > 0$ be arbitrary. We need to show there exists some $\hat{X} \in L_\infty(\mathcal{P})$ such that $\Phi(\hat{X}) + \varepsilon \geqslant \Phi(X)$.

Decompose X in the form of DM, VI.53:

$$X_t = X_t^c + \sum_n H_n I_{\{S_n \leqslant t\}},$$

where the S_n are (predictable if X is predictable) stopping times with disjoint graphs, H_n is a nonnegative random variable measurable with respect to $\mathcal{F}_{S_n}$ ($\mathcal{F}_{S_n-}$ if S_n is predictable), and X^c is continuous. Each of the stopping times S_n takes values only in $[0,T] \cup \{\infty\}$. Then $\int_{[0,T]} Z_t dX_t$ has the decomposition

$$\int_{[0,T]} Z_t dX_t = \int_{[0,T]} Z_t dX_t^c + \sum_n H_n Z_{S_n} I_{\{S_n < \infty\}}. \tag{6.2}$$

It suffices to show how to construct $\hat{X}$ from X.

Let $\hat{X}^c = X^c$, so $\mathbb{E}[\int_{[0,T]} Z_t dX_t^c] = \mathbb{E}[\int_{[0,T]} Z_t d\hat{X}_t^c]$. If S_n is predictable and $H_n \in \mathcal{F}_{S_n-}$, then set $\hat{S}_n = S_n$ and $\hat{H}_n = H_n$. For the remaining case we shall also take $\hat{H}_n = H_n$, but for $\hat{S}_n$ we shall take

$$\hat{S}_n = \begin{cases} S_n + \frac{1}{i} & \text{if} \quad S_n + \frac{1}{i} \leqslant T \\ \infty & \text{if} \quad S_n + \frac{1}{i} > T, \end{cases}$$

where i is chosen large enough so that

$$\mathbb{E}[H_n Z_{\hat{S}_n} I_{\{\hat{S}_n<\infty\}}] + \varepsilon/2^n \geqslant \mathbb{E}[H_n Z_{S_n} I_{\{S_n<\infty\}}]. \tag{6.3}$$

Note for any i that $\hat{H}_n \in F_{\hat{S}_n-}$ and $\hat{S}_n$ is predictable. To see that $\hat{S}_n$ can be chosen so that (6.3) holds, our assumption that $S_n \neq T$, as well as the right-continuity of Z, imply $Z_{\hat{S}_n} I_{\{\hat{S}_n<\infty\}} \to Z_{S_n} I_{\{S_n<\infty\}}$ as $i \to \infty$. Since H_n and Z are bounded, this means that $\mathbb{E}[H_n Z_{\hat{S}_n} I_{\{\hat{S}_n<\infty\}}] \to \mathbb{E}[H_n Z_{S_n} I_{\{S_n<\infty\}}]$ as $i \to \infty$. Summing (6.3) over n now completes the proof. [ ]

PROPOSITION 6.2. Suppose S, U, and V are stopping times with U predictable, V totally inaccessible, and $S = U \wedge V$. Suppose $Z = \Delta W$, where $W_t = I_{\{S\leqslant t\}}$. Then $\hat{X}_t = \lambda I_{\{U\geqslant t\}}$ is the optimal selling strategy and (6.1) holds with $M = W - \tilde{W}$, where $\tilde{W}$ is the predictable compensator of W.

<u>Proof</u>. Let $X \in L_\infty(\mathcal{O})$ be an arbitrary process such that $\Phi(X) > -\infty$. Clearly

$$\Phi(X) = \mathbb{E}\Big[\int_{[0,T]} Z_t dX_t\Big] = \mathbb{E}\big[\Delta X_S I_{\{S\leqslant T\}}\big]$$

$$= \mathbb{E}\Big[\int_{[0,T]} \Delta X_t dW_t\Big]. \tag{6.4}$$

Moreover, if also $X \in L_\infty(\mathcal{P})$, then

$$\mathbb{E}[\Delta X_S I_{\{S\leq T\}}] = \mathbb{E}[\Delta X_U I_{\{U\leq T\}}|U<V]P(U<V) + \mathbb{E}[\Delta X_V I_{\{V\leq T\}}|V<U]P(V<U)$$

$$= \mathbb{E}[\Delta X_U I_{\{U\leq T\}}|U<V]P(U<V),$$

since $P(\Delta X_V>0) = 0$. Consequently, for arbitrary $X \in L_\infty(\mathcal{P})$ with $\Phi(X) > -\infty$,

$$\Phi(X) \leq \lambda\mathbb{E}[I_{\{U\leq T\}}|U<V]P(U<V) = \Phi(\hat{X}),$$

and so $\hat{X}$ is the optimal selling strategy.

With $M = W - \tilde{W}$ it follows from (6.4) that for arbitrary $X \in L_\infty(\mathcal{O})$ with $\Phi(X) > -\infty$,

$$\Phi(X) - \mathbb{E}[\int_{[0,T]} X_t dM_t] = \mathbb{E}[\int_{[0,T]} \Delta X_t dW_t] - \mathbb{E}[\int_{[0,T]} \Delta X_t dM_t] - \mathbb{E}[\int_{[0,T]} X_{t-} dM_t]$$

$$= \mathbb{E}[\int_{[0,T]} \Delta X_t d\tilde{W}_t] - \mathbb{E}[\int_{[0,t]} X_{t-} dM_t]$$

$$= \mathbb{E}[\int_{[0,T]} \Delta X_t d\tilde{W}_t]\text{m}$$

where the last equality is because the stochastic integral of the predictable process X_- with respect to the martingale M is a martingale. Now by DM VI.76, $\Delta\tilde{W}_U = \mathbb{E}[\Delta W_U | F_{U-}] = P(U<V|F_{U-})$ and $\tilde{W}$ is continuous outside the graph of U. Thus for arbitrary $X \in L_\infty(\mathcal{O})$ with $\Phi(X) > -\infty$ one has

$$\Phi(X) - \mathbb{E}\Big[\int_{[0,T]} X_t dM_t\Big] = \mathbb{E}\big[\Delta X_U \Delta\tilde{W}_U I_{\{U\leq T\}}\big]$$

$$= \mathbb{E}\big[\Delta X_U P(U<V)\,|\,F_{U-})I_{\{U\leq T\}}\big]$$

$$\leq \lambda P(U<V,\ U\leq T) = \Phi(\hat{X}) - \mathbb{E}\Big[\int_{[0,T]} \hat{X}_t dM_t\Big],$$

and so $\hat{X}$ maximizes $\Phi(X) - \mathbb{E}\big[\int_{[0,T]} X_t dM_t\big]$. This implies (6.1). []

In view of the preceding result one may be tempted to conjecture that if $Z = \Delta W$, where W is any adapted right-continuous process with left limits, of integrable variation, and with $W_0 = 0$, then (6.1) holds with $M = W - \tilde{W}$. However, Proposition 6.2 does not extend to the case of several jumps, as is shown by the following simple three-period example.

Suppose $\Omega = \{\omega_1,\omega_2\}$, $F_0 = \{\phi,\Omega\}$, $F_1 = F_2 = \{\phi,\Omega,\omega_1,\omega_2\}$, $P(\omega_1) = P(\omega_2) = \frac{1}{2}$, $W_0 = 0$, $W_1 = 1$, $W_2(\omega_2) = 1$, and $W_2(\omega_2) = 3$. Note that W is predictable, so $\tilde{W} = W$ and $M = 0$. By considering the adapted process $\Delta X_1(\omega_1) = \Delta X_2(\omega_2) = \lambda$ and $\Delta X_1(\omega_2) = \Delta X_2(\omega_1) = 0$ it follows that the right hand side of (6.1) is at least as large as $\mathbb{E}[Z_1\Delta X_1 + Z_2\Delta X_2] = (3/2)\lambda$. For the left hand side of (6.1) one should consider predictable processes X of the form $\Delta X_1 = c$, where c is a constant with $0 \leq c \leq \lambda$, and $\Delta X_2 = \lambda - c$. For any such X one has $\mathbb{E}[Z_1\Delta X_1+Z_2\Delta X_2] = \lambda$.

7. CONCLUDING REMARKS

We have analyzed continuous time problems in this paper by means of a fairly direct translation of techniques developed for discrete-time problems. It would in fact be more precise to say that we have used techniques developed for single-stage problems (see Rockafellar-Wets [23]). In this translation, predictable or optional projections play the role of conditional expectations, evanescent sets play the role of null sets in defining the L_∞ space, and integrable variation processes play the role of integrable functions in representing the countably additive parts of dual variables in the Yosida-Hewitt [26] decomposition.

In making this translation we have avoided one issue specific to the continuous time setting, namely the choice of definition of the stochastic integral when the integrand is not predictable. This imposed a significant constraint in our applications. In the portfolio choice problem we assumed that the security prices were of integrable variation in order to obtain an unambiguous definition of capital gains. It would have been much better to have allowed for general semimartingales (in particular for geometric Brownian motions) as in Harrison-Pliska [15]. Our economic growth problem is also quite special in that the stochastic differential equation for the capital stock can be explicitly solved and the solution is meaningful for any measurable control (consumption) process. To broaden the range of applications of our theory, it will be necessary to deal with this issue. One possibility of which we are aware is the use of the compensated stochastic integral (DM, VIII.32). However this might render trivial the comparison of optional and measurable controls, since, as Yor has shown, any measurable process has the same compensated stochastic integral as its optional projection (DM, VIII.35).

The accomodation of more general processes will also likely require a modification of our theory. For example it seems unlikely that one would obtain a shadow price of integrable variation in the portfolio choice problem if the security prices were not of integrable variation. Since a dual variable in $L_\infty^*(M)$ exists under quite mild assumptions, it is possible that one might find a dual stochastic process outside of $IV(M)$ by relaxing the Continuity Assumption.

A closely related concern is that our shadow price might be less useful if stochastic integrals are defined only in a probabilistic rather than a pathwise sense. Certainly our discussion in the introduction of reducing a stochastic problem to a family of deterministic problems would not apply if the state of the system were undefined pathwise. However the existence of the shadow price could still be of interest, as in the portfolio choice problem where, when the shadow price is of the form $A = \rho Z$, the factor ρ defines a rational pricing scheme for contingent claims.

In our applications we were also constrained by our assumption that there be an optimal solution belonging to the interior of D. This assumption is not of mere technical convenience. The shadow price may not exist when the assumption is not satisfied (see Back [1]). Even in deterministic problems the maximum principle must be modified in the presence of state constraints (by allowing for jumps in the co-extremals -- equivalently, for purely finitely additive measures; see the papers of Rockafellar referenced in [19]). We obtained a modified shadow price, $\pi \in L_\infty^*(M)$, when the state constraints are binding; however it would be useful to know whether a better modification is possible. One reasonable conjecture is that one could obtain a functional of the form $X \mapsto \mathbb{E}[\int_0^T X_t(\omega)\sigma(\omega,dt)]$, where $\sigma(\omega,\cdot)$ is a finitely additive measure on $[0,T]$. We think that a proof of this would require the use of dynamic programming as in Rockafellar-Wets [21].

Finally we were somewhat constrained in the study of the economic growth problem by our requirement that controls be bounded. There may be some control problems for which is it unreasonable to assume that the controls belong to L_∞, though even when this is apparently the case, an appropriate choice of units (e.g., "discounting") can sometimes justify the L_∞ formulation. For other problems, our results can be applied in the following way.

Assume the space of controls is some space $L(S) \supset L_\infty(S)$. Assume $\bar{X}$ maximizes Φ on $L(S)$. Define a functional ξ on $L_\infty(M)$ by setting $\xi(Y) = \Phi(\bar{X}+Y)$. Theorem 3.4 can be applied to the problem of maximizing ξ on $L_\infty(S)$. Under the hypothesis of that theorem, there must exist $A \in IV(M)$ which is orthogonal to $L_\infty(S)$ and which has the property that

$$\Phi(\bar{X}) \geq \Phi(\bar{X}+Y) - \mathbb{E}[\int_0^T Y_t dA_t]$$

for every $Y \in L_\infty(M)$. Thus we still obtain a shadow price of sorts for the bounded use of advance information.

A stronger result would be that

$$\Phi(\bar{X}) \geq \Phi(X+Y) - \mathbb{E}[\int_0^T Y_t dA_t]$$

for every $X \in L(S)$ and $Y \in L_\infty(M)$. This does not follow from Theorem 3.4, but we conjecture that it can be proven under very similar assumptions, since the duality argument depends only upon the perturbations (the variable Y in equation (3.3)) being bounded and not upon the boundedness of the controls. We also conjecture that this result could be used to prove Foldes' result on the maximum principle without the additional boundedness restriction appearing in Section 5.

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