THE SHAPE AND ENERGY OF A REVOLVING LIQUID MASS HELD TOGETHER BY SURFACE TENSION

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Summary

The shape of a drop of incompressible fluid held together by the action of surface tension and made to rotate about an axis is determined, the effect of gravity being neglected. Two distinct problems are investigated. In the first an isolated drop of liquid in the form of a surface of revolution is considered. At zero angular speed the drop is spherical and with increasing angular momentum the oblateness increases until a maximum angular speed is reached (where the total mechanical potential is a minimum) beyond which a new linear series of equilibrium forms emerges. A solution in the form of a toroid is also found.

In the second problem the drop is rotating together with a denser medium. It is shown that the drop tends to become a cylindrical thread with increasing angular speed and that there is a critical angular speed at which the angular momentum is a maximum and the total energy is a minimum.

I. Introduction

The equilibrium of a revolving isolated finite mass of fluid under the action of capillary force was considered by Lord Rayleigh (1914). He found a solution in which the liquid drop is a surface of revolution that meets the axis of the rotation provided that the angular speed is sufficiently small. At a certain critical speed this drop collapses onto its equator and, for a finite range of speeds, two equilibrium configurations are mathematically possible; for low speeds one meets the axis of the rotation and the other has the form of a toroid, but for higher speeds both meet the axis of the rotation. Lord Rayleigh used a graphical technique to determine the angular speed corresponding to collapse but did not point out the difficulties associated with reaching this stage, nor the non-uniqueness of the configurations as a function of the angular speed.

When a drop of liquid rotates rigidly with a denser surrounding medium the drop tends to become a cylindrical thread as the angular speed tends to infinity. Rosenthal (1962) obtained this result but failed to show that the angular momentum of the system has a maximum where the total energy is a minimum.

It is the purpose of the present paper to compute the potential and kinetic energies of the systems since, according to Orr (1906, 1907), Lyttleton (1953), and Pozharitiskii (1964), these quantities may be related to a discussion of the stability of rigid body rotation. Such a discussion appears in the following paper (present issue, pp. 837–44).

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II. Equilibrium Forms of Surfaces of Revolution

It is rather difficult to consider cases in which the liquid does not form a surface of revolution because the expression for the total curvature at a point on a general surface is very complicated. The possible existence of other configurations is suggested from an analogous problem to do with self-gravitating masses. For these it is known (Lichtenstein 1933) that for a rotating system the plane through the mass centre perpendicular to the axis of the rotation must be a plane of symmetry of an equilibrium form and that ellipsoids having three unequal axes are admissible provided that the angular momentum has the appropriate value. Of course not all these forms are stable to small disturbances. Poincaré (1902) was even able to show that "pear-shaped figures" existed which are not axially symmetrical around the longest axis.

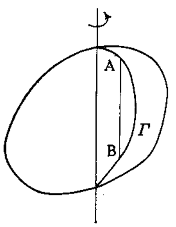


Fig. 1.—Typical rotating liquid mass showing cylindrical element AB parallel to the axis of rotation.

For a rotating liquid held together by the action of the capillary force we do not have such a simple theorem relating to symmetrical forms. At best we can argue as follows. Suppose that the mass of liquid is simply connected and that it is divided up into elementary cylindrical columns parallel to the axis of the rotation and with infinitesimal cross section. Let Γ be the curve which lies at the intersection of the liquid mass with a plane through the axis of the rotation and which contains the typical cylindrical element AB (see Fig. 1). Then it is apparent that the pressure differences at the boundary points A and B are equal, because the contributions from the centrifugal force are the same. Since we do not take into account variations in the coefficient of surface tension over the surface it follows that the total curvature of the surface at A and B must be equal. This, by itself, does not imply geometrical symmetry, although it does show that the rate of change of total curvature with arc length measured along the curve Γ must be zero at the point on Γ that is furthermost from the axis of the rotation. Whether or not this criterion can be used to limit more general configurations to a very much narrower class has still to be examined.

For a viscous fluid not acted upon by forces other than those in the interface it is known that no steady motion can exist unless the fluid rotates as a rigid body. So we shall assume that the angular speed ω is constant. It is then found convenient to make use of the cylindrical polar coordinates (r, θ, z) with the z axis as the axis of rotation and to take as the equation of the interface, say, $r = f(z)$.

To begin we shall suppose that $r = f(z)$ is a single-valued function, for then one way of determining the equilibrium configuration at a given angular speed is to apply a variational principle to the liquid drop of density ρ_1 and its surroundings of density ρ_2. Lyttleton (1953) used Lagrange's equations of motion to show that for a dynamical system of particles rotating with constant angular velocity the positions of relative equilibrium are obtained by finding stationary values of the Lagrangian L of the motion. Thus, if L is calculated from the kinetic energy due to the rigid body rotation and from the energy associated with the capillary force in the interface then the problem becomes one of finding the function $f = f(z)$ for which

$$L = T_1 + T - U, \tag{1}$$

with

$$T = \tfrac{1}{2}\pi \int (\rho_1 - \rho_2)\omega^2 f^4 \,\mathrm{d}z \qquad \text{and} \qquad U = 4\pi \int \alpha f (1+f'^2)^{\frac{1}{2}} \,\mathrm{d}z, \tag{1a}$$

is stationary but subject to the restriction that the volume

$$V = 2\pi \int f^2 \,\mathrm{d}z \tag{2}$$

is known beforehand. Here T_1 is the kinetic energy that the system would have if the liquid drop had the same density as its surroundings and is independent of the shape of the interface although it depends on the angular speed, T is the kinetic energy of the drop minus the kinetic energy that the drop would have if it had the same density as its surroundings, and U is the energy stored in the interface. The coefficient of surface tension α is taken to be constant in time and over the whole interface, and the prime denotes differentiation with respect to z. Actually the quantity U may be interpreted as a potential energy, for work is done by the capillary force when the surface of separation undergoes an infinitesimal displacement which reduces the total surface area. It is also to be noted that T is negative when $\rho_1 < \rho_2$. The reason for choosing this derivation of the equation lies in the connection between the energy integral in (1a) and the stability criteria to be discussed in the following paper. This technique is also discussed by Landau and Lifshitz (1959), who used a variational principle to derive similar equilibrium conditions.

In reality the two media are separated by a narrow transitional layer, but this is so thin that it may be neglected as a first approximation. The existence of this layer and the consequent phenomena associated with it, which are neglected here, are discussed at length by Hirschfelder, Curtiss, and Bird (1954). The most important omissions concern the variation in the coefficient of surface tension due to surface contamination or the existence of a temperature gradient and the neglect of the gravitational forces. The latter omission is justified for small drops of fluid that are not too small for the transition layer to be dominant provided that the characteristic length g/ω^2 (where g is the acceleration due to gravity) is small and that the capillary length, defined by $\{2\alpha/(|\rho_1-\rho_2|g)\}^{\frac{1}{2}}$, is large compared with the dimensions of the drop.

Now the extremal problem can be solved by using a first integral of the Euler–Lagrange equation, namely

$$F - f'\,\partial F/\partial f' = \text{constant}, \qquad F = T - U - 2\pi\beta \int f^2 \,\mathrm{d}z, \tag{3}$$

because (1) has no term in z. This then leads to the differential equation

$$\pm(1+f'^2)^{-\frac{1}{2}} = \beta f/2\alpha + (\rho_1-\rho_2)\omega^2 f^3/8\alpha - A/f, \tag{4}$$

where $2\pi\beta$ is the usual Lagrange multiplier and A is a constant of integration that has the value zero if the liquid intersects the axis of the rotation.

Actually a solution to this problem exists for which $r = f(z)$ is not single valued. In this case we may imagine that the interface is divided into two parts, for example, $r = f_1(z)$ and $r = f_2(z)$, with $f_1(z) \geqslant f_2(z)$ for all z, and where the equal sign applies only at the point z_0 where $\mathrm{d}z/\mathrm{d}f_1 = \mathrm{d}z/\mathrm{d}f_2 = 0$. We now alter the equations for T, U, and V by changing the variable of integration from z to f since we are then dealing with single-valued functions and in this way we obtain equation (4) once again. The plus and minus signs in equation (4) thus refer respectively to the upper curve $f_1(z)$ and the lower curve $f_2(z)$. However, the constant A may now be different for each curve, but we shall follow Lord Rayleigh (1914) and examine only those configurations for which A is the same.

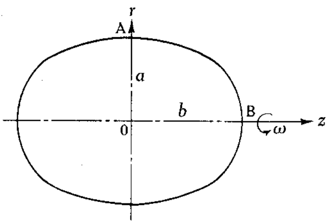


Fig. 2.—Plane section through the axis $0z$ of the liquid drop.

III. Case 1: Drops that Intersect the Axis of Rotation

Firstly we consider the case when the liquid meets the axis of rotation (see Fig. 2) so that we can write (4) in the form

$$\pm(1+f'^2)^{-\frac{1}{2}} = \beta f/2\alpha + \omega^{*2}f^3/d^3, \qquad \text{with} \qquad \omega^{*2} = (\rho_1-\rho_2)\omega^2 d^3/8\alpha, \tag{5}$$

where β is a constant to be determined from the condition that the enclosed volume is given the value V, and d is the mean radius of the drop defined by the relation

$$V = 4\pi d^3/3. \tag{6}$$

Clearly the quantity $|\omega^*|$ will serve as a nondimensional angular speed (note that $\omega^{*2} \leqslant 0$ whenever $\rho_1 < \rho_2$). Now the magnitude of the total curvature at a point on the surface is given by

$$J = \frac{1}{f}\frac{\mathrm{d}}{\mathrm{d}f}\left(f(1+f'^2)^{-\frac{1}{2}}\right) = \frac{\beta}{\alpha} + \frac{4\omega^{*2}f^2}{d^3} \tag{7}$$

and hence we can identify β as the pressure difference $\Delta p = p_1 - p_2$ on the axis of rotation at the poles.

When the angular speed is zero the drop is under the influence of the capillary force alone and so the liquid surface must be spherical; therefore, at small angular speeds ω^*, $\Delta p > 0$ and the drop has width $2a$ corresponding to the curve on the surface for which $f' = 0$. On combining this condition with equation (5) it appears that

$$\Delta p = 2\alpha(1-e)/a, \qquad \text{with} \qquad e = (\rho_1-\rho_2)\omega^2 a^3/8\alpha, \tag{8}$$

and so $\Delta p \geqslant 0$ as long as the parameter $e \leqslant 1$. However, the curvature K of a plane section through the axis of the drop is given by

$$K = -\frac{\mathrm{d}}{\mathrm{d}f}\left((1+f'^2)^{-\frac{1}{2}}\right) = -\frac{1-e}{a} - \frac{3ef^2}{a^3} \tag{9}$$

and this remains negative as long as $-\frac{1}{2} < e \leqslant 1$. Here the term $(1-e)/a$ can be interpreted as the curvature of the plane section at the poles.

If we now define a function $\phi(f)$ by the relation

$$\phi = \phi(f) = (1-e)f/a + e(f/a)^3 \tag{10}$$

then equation (5) takes on the form

$$\mathrm{d}z/\mathrm{d}f = -\phi(1-\phi^2)^{-\frac{1}{2}}, \tag{11}$$

since it is sufficient to take $\mathrm{d}z/\mathrm{d}f \leqslant 0$ when f is small. If $\Delta p \geqslant 0$ then, as f/a increases from zero, ϕ increases to its maximum value 1 and, at this point, $\mathrm{d}f/\mathrm{d}z = 0$. For further decrease in z we set

$$\mathrm{d}z/\mathrm{d}f = +\phi(1-\phi^2)^{-\frac{1}{2}} \tag{12}$$

and this leads to a closed section in which f has only the one turning value at $f/a = 1$.

If $e < 0$ then the density of the liquid is less than that of its surroundings. For $e < -\frac{1}{2}$, $\phi = 1$ has one positive zero less than a which means that equation (11) takes on complex values; for $e = -\frac{1}{2}$ the left-hand side of (11) has a first-order pole at $f/a = 1$ and, since the volume of the drop is finite, this leads to the conclusion that the length of the drop tends to infinity as $e \to -\frac{1}{2}+0$. If we now use equations (5) and (8) then this condition shows that

$$\omega^{*2} = e(a/d)^{-3} \to -\infty \qquad \text{as} \qquad e \to -\tfrac{1}{2}+0. \tag{13}$$

This result was obtained by Rosenthal (1962), who showed that the drop is always convex and that it becomes threadlike as the angular speed tends to infinity. If we look at the angular momentum H of the drop (see Figs. 3 and 4) then it is clear that it is not a monotonically increasing function of the angular speed because the moment of intertia of an infinitely thin thread must be zero. (A formal proof is given in equation (36).) It follows that the angular speed is the fundamental quantity to be considered in this case.

Table 1 makes clear the following points.

(1) There exists a surface of revolution corresponding to each angular speed.

(2) The maximum angular momentum occurs when the sum of the kinetic and potential energies is least and is given by $H_{\max} = 0\cdot6609\,\pi\{2(\rho_2-\rho_1)\alpha d^7\}^{\frac{1}{2}}$, where the angular speed $\omega = 1\cdot2851\{8\alpha/(\rho_2-\rho_1)d^3\}^{\frac{1}{2}}$ and $e = 0\cdot4859$.

(3) For each value of the angular momentum less than the maximum there are two possible surfaces of revolution.

(4) The angular speed increases monotonically and indefinitely as the parameter e decreases from 0 to $-\frac{1}{2}$ so that, for a given angular speed, there exists only one surface of revolution.

(5) The potential energy of the interface increases from the value $4\pi\alpha d^2$, when the drop is spherical, to infinity when the drop is an infinitely long thread; while the kinetic energy associated with the system increases from zero to infinity.

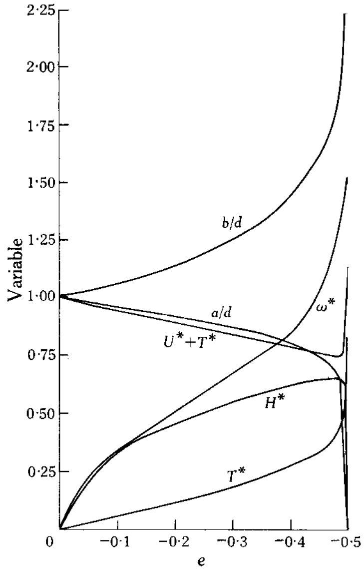


Fig. 3.—Changes in the physical variables with the parameter e for a drop rotating in a denser medium. Here $\omega^* = \omega\{(\rho_2-\rho_1)d^3/8\alpha\}^{\frac{1}{2}}$.

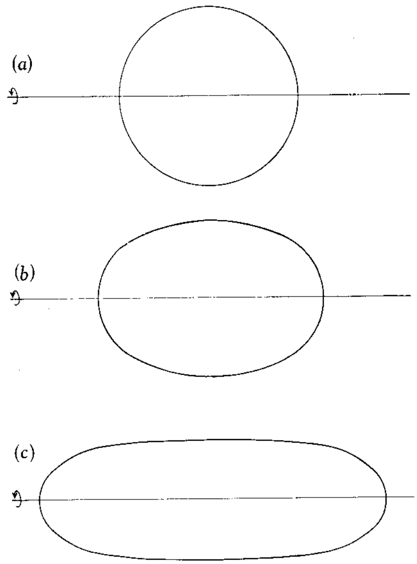


Fig. 4.—Shapes of the drop rotating at angular speeds of (a) 0, (b) 0·6713, and (c) 1·2851. The last configuration corresponds to the form of bifurcation of the linear series.

For $0 \leqslant e < 1$, $\phi(f)$ has only one real zero and so the drop is again convex; however, for $e > 1$ this function has two possible zeros at $f/a = \{(e-1)/e\}^{\frac{1}{2}}$ and $f/a = 1$. In the latter case equilibrium forms exist which are partly concave and for

Table 1

PHYSICAL QUANTITIES CORRESPONDING TO DROP OF LIQUID THAT ROTATES WITH DENSER MEDIUM

e	$\omega\{(\rho_2-\rho_1)d^3/8\alpha\}^{\frac{1}{2}}$	$H/\pi\{2(\rho_2-\rho_1)\alpha d^7\}^{\frac{1}{2}}$	$T/4\pi\alpha d^2$	$(U+T)/4\pi\alpha d^2$	a/d	b/d
0	0	0	0	1	1	1
−0·1	0·3338	0·3316	−0·0553	0·9466	0·9646	1·0723
−0·2	0·5038	0·4605	−0·1160	0·8931	0·9237	1·1604
−0·3	0·6713	0·5528	−0·1856	0·8394	0·8731	1·2771
−0·4	0·8813	0·6230	−0·2745	0·7857	0·8016	1·4619
−0·4859	1·2851	0·6609	−0·4247	0·7478	0·6651	1·9261
−0·5	∞	0	$-\infty$	∞	0	∞

these shapes the angular speed is such that the pressure difference at the poles is negative. It follows that there is then a tendency for the liquid to collapse onto its equator.

If we now replace the variables f by f/a and z by z/a then the volume of the drop becomes

$$V = -2\pi a^3 \int_0^1 f^2\, \mathrm{d}z/\mathrm{d}f\, \mathrm{d}f \tag{14}$$

and so, on using the expression (11) for $\mathrm{d}z/\mathrm{d}f$ and integrating in a suitable manner, we find that the above equation reduces to the form

$$V = 2\pi a^3\{1-(1-e)b/a\}/3e, \tag{15}$$

where

$$b/a = \int_0^1 \phi(1-\phi^2)^{-\frac{1}{2}}\, \mathrm{d}f \tag{16}$$

and $b = \mathrm{OB}$ (Fig. 2) is half the distance from pole to pole. If we express the volume in terms of the mean radius and use the definition of angular speed ω^* given in (5) then (15) becomes

$$2\omega^{*2} = 1-(1-e)b/a \tag{17}$$

and so it is seen that collapse corresponds to $2\omega^{*2} = 1$, where

$$\omega = \{16\pi\alpha/3(\rho_1-\rho_2)V\}^{\frac{1}{2}}. \tag{18}$$

This verifies that the action of surface tension is to try to prevent collapse and that smaller drops are less prone to this kind of "instability". Again, for $e > 1$ we have $\mathrm{d}z/\mathrm{d}f = 0$ only where $f = 0$ or $f = \{(e-1)/e\}^{\frac{1}{2}}$ and so if we allow e to increase from one then the point of collapse must occur before e has a value that allows the interface to have a second point at which $\mathrm{d}f/\mathrm{d}z = 0$. That is, there can be no positive root of the equation $\phi = -1$. A simple calculation shows that this critical value must be less than four. In fact it is the positive root of the transcendental equation

$$\int_0^1 \phi(1-\phi^2)^{-\frac{1}{2}}\, \mathrm{d}f = 0 \tag{19}$$

and has the value $2\cdot3291$. Of course, whether collapse can be realized in practice has still to be investigated. Lord Rayleigh (1914) did not consider this question at all but used a graphical technique to estimate the value of the parameter e at collapse. He obtained the value $2\cdot4$.

If we look at equation (17) (also Fig. 5) then it is clear that $2\omega^{*2} = 1$ for one other value of the parameter e, namely, $e = 1$, which corresponds to the case where the pressure reversal just begins. This shows that ω^* does not increase monotonically with the parameter e as one might have suspected from the fact that the ratio b/a diminishes monotonically with e. This latter statement comes from the inequality

$$\frac{\mathrm{d}(b/a)}{\mathrm{d}e} = -\int_0^1 \frac{f(1-f^2)}{(1-\phi^2)^{3/2}}\, \mathrm{d}f < 0. \tag{20}$$

Actually ω^* has one maximum at $\omega^* = \omega_{\max} = 0\cdot7540$ where $e = 1\cdot6608$. Thus in order to make the liquid collapse onto its equator it is necessary to increase the angular speed of the drop up to its maximum value $\omega_{\max}$ and then to decrease it to the value $0\cdot7071$. At this stage the drop ought to have collapsed. Clearly it is necessary that changes in the angular speed be very gradual so that no instabilities are introduced from the source of angular momentum.

Now it might appear as if an increase in the angular speed followed by a decrease, as described above, would ultimately lead back to the spherical form, but this is not so if we can ensure that the angular momentum of the liquid increases throughout the change. Figure 6 shows the variations in the typical drop geometry with angular speed.

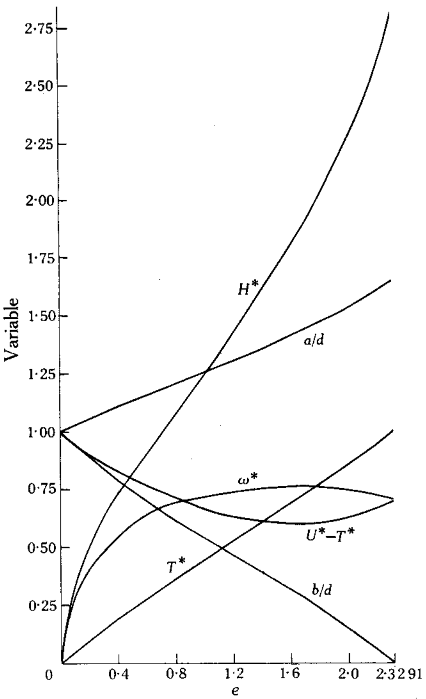


Fig. 5.—Changes in the physical variables with the parameter e for an isolated drop.

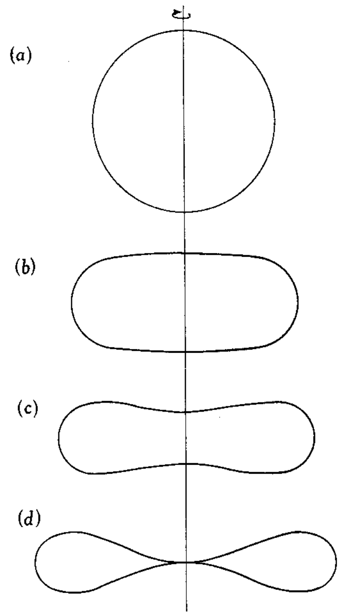


Fig. 6.—Shapes of isolated drops rotating at angular speeds of (a) 0, (b) $0\cdot7071$, (c) $0\cdot7540$, and (d) $0\cdot7071$, which are respectively the drop at rest, the last of the convex forms, the stage beyond which the drop becomes unstable, and the collapse configuration.

Table 2 makes clear the following points.

(1) There exists a surface of revolution for each value of the angular momentum $H \leqslant H_{\max} = 2\cdot8506\,\pi\{2(\rho_1-\rho_2)\alpha d^7\}^{\frac{1}{2}}$.

(2) The maximum angular speed occurs when the difference between the potential and kinetic energies is least and is given by $\omega_{\max} = 0\cdot7540\{8\alpha/(\rho_1-\rho_2)d^3\}^{\frac{1}{2}}$ when $e = 1\cdot6608$.

(3) For each value of the angular speed less than the maximum but greater than $0\cdot25\pi\{(\rho_1-\rho_2)d^3/\alpha\}^{\frac{1}{2}}$ two possible surfaces of revolution exist.

(4) The angular momentum increases monotonically to the finite value H_{max} as e increases from 0 to 2·3291 so that for a given angular momentum less than this only one surface of revolution can exist.

(5) The potential energy of the interface increases from the value $4\pi\alpha d^2$, when the drop is spherical, to the finite value $1\cdot7238\times4\pi\alpha d^2$ when the drop collapses; while the kinetic energy associated with the drop increases from 0 to $1\cdot0079\times4\pi\alpha d^2$.

TABLE 2

PHYSICAL QUANTITIES CORRESPONDING TO ISOLATED DROP

e	$\omega\{(\rho_1-\rho_2)d^3/8\alpha\}^{\frac{1}{2}}$	$H/\pi\{2(\rho_1-\rho_2)\alpha d^7\}^{\frac{1}{2}}$	$T/4\pi\alpha d^2$	$(U-T)/4\pi\alpha d^2$	a/d	b/d
0	0	0	0	1	1	1
0·5	0·5810	0·8173	0·2374	0·7948	1·1399	0·7404
1·0	0·7071	1·2591	0·4452	0·6677	1·2599	0·5433
1·5	0·7514	1·7130	0·6436	0·6044	1·3850	0·3580
1·6608	0·7540	1·8780	0·7080	0·5998	1·4295	0·2965
2·0	0·7424	2·2905	0·8503	0·6246	1·5367	0·1573
2·3291	0·7071	2·8506	1·0079	0·7159	1·6701	0

In order to interpret the occurrence of a maximum angular speed (see Pozharitiskii (1964), who related this to the stability of the liquid) we shall begin by relating the kinetic and potential energies to the angular speed and the corresponding angular momentum. We define the energies as fractions of the potential energy that the drop would have at rest, i.e. in units of $4\pi\alpha d^2$. With this definition the nondimensional kinetic and potential energies become respectively

$$T^* = (a/d)^2 \int_0^1 ef^4\phi(1-\phi^2)^{-\frac{1}{2}}\,df \tag{21}$$

and

$$U^* = (a/d)^2 \int_0^1 f(1-\phi^2)^{-\frac{1}{2}}\,df. \tag{22}$$

The asterisk indicates that the quantity is expressed as a fraction of $4\pi\alpha d^2$.

If we start with the identity

$$ef^4\phi = f-f(1-\phi^2)-(1-e)f^2\phi \tag{23}$$

and then combine it with equations (14), (16), (21), and (22), we get

$$T^*-U^* = \frac{2(e-1)}{3(a/d)}-(a/d)^2\int_0^1 f(1-\phi^2)^{\frac{1}{2}}\,df, \tag{24}$$

which, on integrating the last term by parts, leads to

$$5T^*-2U^* = 2(e-1)/(a/d). \tag{25}$$

This is a rather surprising result in view of the complexity of the defining relations and it has been used as a check on the numerical calculations.

If we now make T^* and U^* the subjects of equations (24) and (25) then we find that

$$T^* = \frac{2(e-1)}{9(a/d)} + \tfrac{2}{3}(a/d)^2 \int_0^1 f(1-\phi^2)^{\frac{1}{2}}\, \mathrm{d}f \tag{26}$$

and

$$U^* = -\frac{4(e-1)}{9(a/d)} + \tfrac{5}{3}(a/d)^2 \int_0^1 f(1-\phi^2)^{\frac{1}{2}}\, \mathrm{d}f, \tag{27}$$

and since

$$\left|\int_0^1 f(1-\phi^2)\, \mathrm{d}f\right| < \tfrac{1}{2} \qquad \text{for all} \qquad e \geqslant -\tfrac{1}{2} \tag{28}$$

it follows that $U^* \to +\infty$ and $T^* \to -\infty$ as $e \to -\frac{1}{2}+0$, while they tend to finite values as $e \to 2{\cdot}3291$. These results are shown in Tables 1 and 2.

The next step is to examine the condition under which the angular speed has a stationary value with respect to the parameter e. To investigate this situation we can differentiate (24) with respect to e and then, after a considerable amount of algebra which involves using the identity (25), we find that

$$\mathrm{d}(U^*-T^*)/\mathrm{d}(\omega^{*2}) = -T^*/\omega^{*2} \leqslant 0. \tag{29}$$

The quantity U^*-T^* is known as the total mechanical potential and this has an absolute minimum where the angular speed is greatest. Lyttleton (1953) showed that this criterion corresponds to "secular stability". This is discussed in the following paper.

In order to understand the physics of this problem more clearly it is found useful to determine the angular momentum of the liquid and how it varies with the parameter e. Since the drop rotates rigidly in an equilibrium configuration, we can express the angular momentum H in terms of the kinetic energy by means of the relation

$$H = 8\pi T^* \alpha d^2/\omega\,; \tag{30}$$

this can then be expressed in units of $\pi\{2(\rho_1-\rho_2)\alpha d^7\}^{\frac{1}{2}}$, in which case we may write

$$H^* = H/\pi\{2(\rho_1-\rho_2)\alpha d^7\}^{\frac{1}{2}} = 2T^*/\omega^*. \tag{31}$$

On differentiating this equation with respect to e we obtain

$$\omega^* \frac{\mathrm{d}H^*}{\mathrm{d}e} = -\frac{T^*}{\omega^{*2}}\frac{\mathrm{d}\omega^{*2}}{\mathrm{d}e} + \frac{2\mathrm{d}T^*}{\mathrm{d}e} \tag{32}$$

and then by applying (29) we have

$$\omega^* = \mathrm{d}(U^*+T^*)/\mathrm{d}H^* \tag{33}$$

and

$$H^* = -\,\mathrm{d}(U^*-T^*)/\mathrm{d}\omega^*. \tag{34}$$

The configuration of the drop and hence the potential and kinetic energies depend only on a single parameter and so, at various stages in the above derivation, we regard them as functions of either the parameter e, the angular speed, or the angular momentum, as the case may be. Equation (33) shows that the angular momentum increases with the "total energy" U^*+T^* whilst equation (34) verifies that the angular speed is a maximum when the total mechanical potential U^*-T^* is a minimum.

When the outer medium is the denser it is clear that there is a value of e for which the angular momentum is a maximum. This result is indicated by the fact that at high angular speeds the interface is approximated by a cylindrical surface with small corrections at the ends, and so the angular momentum is given by

$$H \simeq \tfrac{1}{2}Ma^2\omega\,, \tag{35}$$

where M is the mass of the drop of liquid and is finite. If we now apply condition (13) then we see that $H \to 0$ as $|\,\omega\,| \to \infty$. The existence of a maximum follows from the fact that H is a continuous function of e and that it is zero when $\omega = 0$. A formal proof of this can be obtained by using the fact that $e \to -\frac{1}{2}$ as $|\,\omega^*\,| \to \infty$ and then applying this condition in equations (26) and (27). Thus

$$H^* = \frac{2T^*}{\omega^*} = \frac{4(e-1)}{9\omega^*(a/d)} + \frac{4(a/d)^2}{3\omega^*}\int_0^1 f(1-\phi^2)^{\frac{1}{2}}\,\mathrm{d}f \to 0 \qquad \text{as} \qquad |\,\omega^*\,| \to \infty\,, \tag{36}$$

because of the inequality (28). Similarly

$$U^*/\omega^* \to 0 \qquad \text{as} \qquad |\,\omega^*\,| \to \infty\,. \tag{37}$$

IV. Case 2: Drops of Liquid in the Form of an Annulus

When the liquid does not meet the axis of rotation the constant of integration A that appears in equation (4) has to be retained. Hence, if the liquid is to form a closed surface it is necessary that $f' = 0$ for two positive values of f, namely $f = a_0$ and $f = a$, with $a > a_0$, say. The differential equation describing the surface then becomes

$$\pm f(1+f'^2)^{-\frac{1}{2}} = \omega^{*2}(f^2-a_0^2)(f^2-a^2)/d^3 + (f^2-aa_0)/(a-a_0)\,, \tag{38}$$

where the plus and minus signs have the same meaning as in equation (4). If we again replace f by f/a and z by z/a and then define a function $w(f)$ by the relation

$$w = w(f) = e(f^2-1)(f^2-a_1^2)+(f^2-a_1)/(1-a_1)\,, \tag{39}$$

where $a_1 = a_0/a$, equation (38) takes on the alternative form

$$\mathrm{d}z/\mathrm{d}f = -w(f^2-w^2)^{-\frac{1}{2}}\,, \tag{40}$$

so that we can get a closed drop only if the parameters e and a_1 are so related that

$$I(e,a_1) \equiv \int_{a_1}^1 w(f^2-w^2)^{-\frac{1}{2}}\,\mathrm{d}f = 0\,. \tag{41}$$

Actually these parameters must satisfy the inequality

$$e(1-a_1)(1+a_1)^2 < 4\,, \tag{42}$$

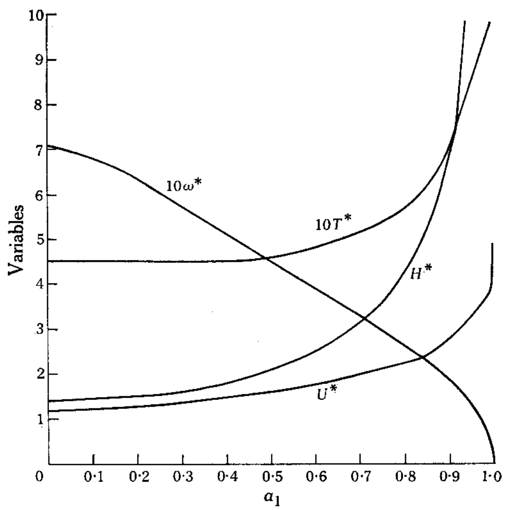
which is the condition that $\mathrm{d}z/\mathrm{d}f$ remain finite in the open interval $a_1 < f < 1$. This result serves as a guide to the numerical computation of the roots of the transcendental equation (41).

By a simple differentiation we find that

$$\frac{\partial I(e,a_1)}{\partial e} = -\int_{a_1}^{1} \frac{f^2(1-f^2)(f^2-a_1^2)}{(f^2-w^2)^{3/2}}\,\mathrm{d}f \leqslant 0, \tag{43}$$

and so it follows that to each value of $a_1 > 0$ there corresponds one positive value for the parameter e. It should be mentioned that if a_1 is set equal to zero then we obtain the differential equation that corresponds to case 1; so we see that a_1 is zero for all shapes that meet the axis of the rotation and that a_1 is positive for the ring-shaped surfaces, vanishing only at the point of collapse. Also the volume of the closed drop is

$$V = -2\pi a^3 \int_{a_1}^{1} f^2 w(f^2-w^2)^{-\frac{1}{2}}\,\mathrm{d}f, \tag{44}$$



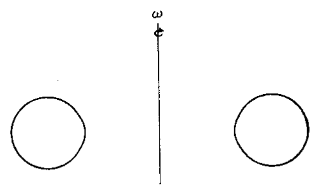


Fig. 7 (*left*).—Changes in the physical variables with the parameter a_1 for a toroidal drop.

Fig. 8 (*above*).—Typical toroidal drop at an angular speed $\omega^* = 0\cdot4460$ where $a_1 = \frac{1}{2}$.

where, for a given e, we choose a_1 to satisfy (41). Then, the kinetic and potential energies will be given by

$$T^* = (a/d)^2 \int_{a_1}^{1} ef^4 w(f^2-w^2)^{-\frac{1}{2}}\,\mathrm{d}f \tag{45}$$

and

$$U^* = (a/d)^2 \int_{a_1}^{1} f^2(f^2-w^2)^{-\frac{1}{2}}\,\mathrm{d}f, \tag{46}$$

and, in a manner analogous to case 1, we can prove that

$$5T^*-2U^* = 2\{e(1+a_1^2)-(1-a_1)^{-1}\}/(a/d) \tag{47}$$

and hence that

$$\omega^* = \mathrm{d}(U^*+T^*)/\mathrm{d}H^* \tag{48}$$

and

$$H^* = -\,\mathrm{d}(U^*-T^*)/d\omega^*. \tag{49}$$

Now it happens that $I(e, a_1)$ is a small quantity for a wide range of values of e and hence that it is not practical to solve equation (41) directly. Instead it is better to use the formulae for the kinetic and potential energies which are analogous to those given in equations (26) and (27) and then to find the value of e for which these equations are consistent with (45) and (46). In this case the relevant physical quantities are given in Table 3 and their variations with a_1 are illustrated in Figure 7.

Lord Rayleigh (1914) assumed that the toroidal form tends to a circular cross section as $a_1 \to 1-0$ and used a perturbation method to determine the shape at small angular speeds. The validity of this assumption is confirmed by the calculations and is illustrated in Figure 8.

TABLE 3

PHYSICAL QUANTITIES CORRESPONDING TO TOROIDAL DROP

a_1	e	$\omega\{(\rho_1-\rho_2)d^3/8\alpha\}^{\frac{1}{2}}$	$H/\pi\{2(\rho_1-\rho_2)\alpha d^7\}^{\frac{1}{2}}$	$T/4\pi\alpha d^2$	$U/4\pi\alpha d^2$
0·90	2·81	0·20	7·43	0·65	2·62
0·70	1·10	0·32	2·94	0·47	1·90
0·50	0·95	0·45	2·06	0·46	1·63
0·30	0·98	0·57	1·62	0·46	1·39
0·10	1·11	0·69	1·45	0·46	1·23
0·05	1·00	0·69	1·29	0·46	1·14
0·01	0·90	0·70	1·25	0·44	1·11

So far we have examined the shapes of rigidly rotating isolated drops corresponding to surfaces of revolution; however, it is most likely that other forms exist at angular speeds greater than ω_{max}. Two possibilities present themselves: either there are equilibrium forms that are not surfaces of revolution or else the fluid inside the drop ceases to rotate as a rigid body, in which case no steady-state configuration exists. A complete study of these problems is very difficult and is not attempted here.

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