ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 18, nº 4 (1991), p. 525-539
<http://www.numdam.org/item?id=ASNSP_1991_4_18_4_525_0>

The Shape of the Free Surface of a Unilaterally Supported Elastic Body

PIERO VILLAGGIO

Abstract.

An elastic plate, loaded parallel to its middle plane, has a flat base resting, without friction, on a horizontal rigid plane. Under given loads it may occur that a part of the base leaves the supporting plane. The shape of the detached part of the base is, in general, not explicitly determinable. However, in some cases it is possible to find the exact profile of the free boundary and describe its regularity.

1. - Introduction

The prototype of a unilateral problem in two-dimensional linear elasticity is the following. An elastic thin disk occupies a domain $D \subset \mathbb{R}^2$. The boundary s of D consists of two parts Γ and Γ', the first of which is a rectilinear segment initially resting on a rigid plane support which may be taken as the x-axis of a Cartesian system of coordinates (Fig. 1).

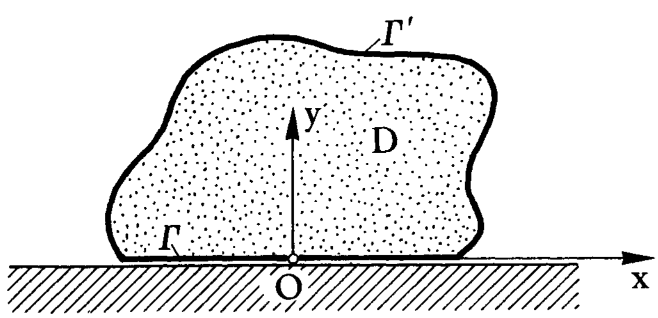

Fig. 1

Pervenuto alla Redazione il 30 Marzo 1991.

The supporting plane excludes displacements in the direction of the negative y-axis, whereas tangential displacements are allowed since contact is assumed to be smooth. Regarding exterior loads, there are only tractions applied on the part Γ' of the boundary, the body forces being assumed zero for simplicity. The tractions exerted on Γ' must satisfy certain conditions of compatibility able to ensure that D does not slide along the x-axis nor lose contact with it. If P_x and P_y are the components of the resultant of surface tractions, these conditions require that $P_x = 0$ and $P_y \leq 0$.

The existence of weak solutions to the problem is nowadays a classical result of the theory of variational inequalities. As Fichera [1972] and Lions and Stampacchia [1967] have proven with different techniques, there is a unique solution belonging to the Hilbert space $H^{1,2}(D)$ such that the displacement component v along the y-axis is greater than or equal to zero on Γ. It is thus natural to ask how smooth the solutions are. In the case of two-dimensional elasticity, Kinderlehrer [1981] proved the $C^{1,\alpha}$ continuity of solutions, but this result could not be extended to higher dimensions. Recently Schumann [1989] proved that the property also holds for an n-dimensional elastic body, but without determing the precise value of the Hölder exponent, $\alpha \in (0,1)$.

But, besides the question of regularity, the shape of the part of Γ which leaves the plane $y = 0$, whenever this happens, is still unknown. The explicit construction of these pieces of free boundary is, in general, extremely difficult even in relatively simple geometric situations. The case considered here is limiting, in the sense that D occupies the entire half-plane $y > 0$ and Γ is the whole x-axis. In addition, the loads have certain symmetries so as to give in advance some idea of the shape of the detached set. But, having constructed the solution in terms of elementary functions, the properties predicted by the abstract theory can then be checked with the formulae, which show, for instance, that solutions are of class $C^{1,\alpha}$ with $\alpha = \frac{1}{2}$ at the points where the elastic body leaves its support.

2. - A contact problem for a semi-infinite disk

Let V be an elastic disk of thickness $2h$, sufficiently small with respect to its other transversal dimensions, and let x, y, z indicate the axes of a Cartesian reference system such that the x, y-plane coincides with the middle plane of the disk, so that the faces are planes with the equations $z = \pm h$. The section of the disk through its middle plane is a region D of the x, y-plane, with boundary S. In this specific case, D is a half-plane and the x, y-axes can always be placed in such a way that it occupies the upper part $y > 0$ of the plane (Fig. 2).

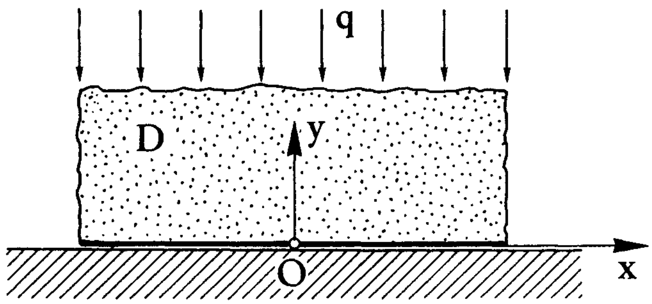

Fig. 2

The disk is made up of an elastic, homogeneous, isotropic material and is loaded by forces whose resultants act in the middle plane $z = 0$. If thickness $2h$ is thin, the average values of the components of displacement, strain, and stress, taken over the thickness of the plate, lead to a knowledge nearly as useful as that of the actual values of the quantities at each point. The averaged values of elastic displacements in the directions of the x, y-axes are denoted by u and v, respectively; the averaged strains are consequently

$$\varepsilon_x = \frac{\partial u}{\partial x}, \ \varepsilon_y = \frac{\partial v}{\partial y}, \ \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{2.1}$$

Upon the assumptions made on the nature of the material, stresses are related to strains by Lamé's equations

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \sigma\sigma_y) \quad , \\ \varepsilon_y &= \frac{1}{E}(\sigma_y - \sigma\sigma_x) \quad , \\ \varepsilon_{xy} &= \frac{1+\sigma}{E}\tau_{xy} \quad , \end{aligned} \tag{2.2}$$

where E is Young's modulus and σ Poisson's ratio. States of stresses such as these are termed "generalized plane stresses", according to a denomination first introduced by Filon (Love [1927, Art. 94]).

The half-plane D is bounded by the straight line $y = 0$, representing the trace of a rigid smooth wall preventing v-displacements in the negative direction of the y-axis but not those in the positive direction. Assuming that the entire half-plane is pushed against the wall by a uniform uniaxial stress $\sigma_y^0 = -q$, the state of stress in D is of the form

$$\sigma_y^0 = -q, \ \sigma_x^0 = \tau_{xy}^0 = 0, \tag{2.3}$$

and the associated elastic displacements are simply

$$u^0 = \frac{\sigma}{E} qx, \quad v^0 = -\frac{1}{E} qy. \tag{2.4}$$

This state of stresses and displacements clearly satisfy the condition of unilateral contact on the line $y = 0$, since here the v-displacement is zero and the normal stress σ_y^0 is purely compressive. Such a state is called "fundamental."

Now let Y be an additional point load applied at the origin in the positive direction of the y-axis. The effect of Y is that of perturbing the fundamental state and, in particular, of causing the detachment of the lower border of the elastic half-plane from its support in the neighborhood of the origin. The problem thus arises of determining the final distribution of stresses in the half-plane and of finding, if possible, the shape and the extent of the detached part of the boundary.

In order to answer the question, it is necessary to make some assumptions on how the lower boundary of D leaves the support in the vicinity of the origin, since it is not known in advance whether this rising occurs along a single interval of the x-axis or is fragmented into many small intervals. In the present situation, however, it is reasonable to conjecture that the interval is unique and, due to the symmetry of the loads, symmetrically placed with respect the y-axis. The end points of this interval, $-a \leq x \leq a$, are of course unknown and must be determined by the condition that the normal stress σ_y and the v-displacement vanish at them.

Another situation, in a certain sense complementary to that just described, occurs when the displaced state is generated, not by Y, but instead by a point force X applied at the origin in the direction of the positive x-axis. In this case again an explicit solution can be found upon the assumption that the rising boundary is made up of a single interval. In contrast to the previous case, however, this interval is no longer symmetric with respect to the origin but instead is placed completely to the right of the load, which is now operative at the left end of the interval. The length of this interval must be determined by requiring that the normal stress σ_y vanishes at the right end, but not at the left end, which coincides with the point of application of X.

3. - The case of a vertical load Y

The first of the two situations described above is the rising of the border of the half-plane $y = 0$ effected by an upwardly directed force Y applied at the origin (Fig. 3) superimposed onto the state $\sigma_y = -q$. Under the influence of Y, the boundary in the neighboorhood of the origin detaches from the x-axis along a certain unknown segment. Since the domain D is symmetric with respect to the y-axis, as are the loads, it is natural to expect that this segment admits the origin as its mid point and is representable as $-a \leq x \leq a$.

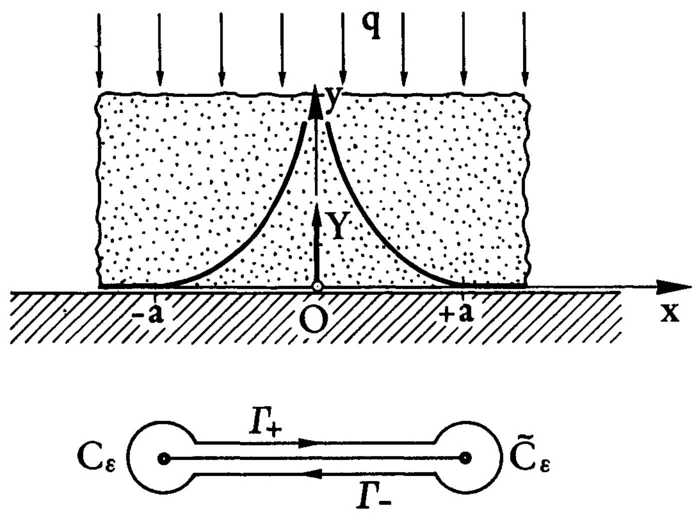

Fig. 3

In order to find the stresses and displacements associated to the loads, it is convenient to introduce complex coordinates

$$z = x + iy, \quad \overline{z} = x - iy, \tag{3.1}$$

and to express the stress components in terms of a biharmonic function (the Airy stress function)

$$F = z\overline{f}_1(\overline{z}) + \overline{z}f_1(z) + f_2(z) + \overline{f}_2(\overline{z}), \tag{3.2}$$

where $f_1(z)$, $f_2(z)$ are two analytic functions called "complex potentials". The stresses are related to the complex potentials by the formulae

$$\begin{aligned} \sigma_x + \sigma_y &= \nabla^2 F = 4\left(f_1'(z) + \overline{f}_1'(\overline{z})\right), \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 4\,\frac{\partial^2 F}{\partial z^2} = 4\left(\overline{z}\overline{f}_1''(\overline{z}) + f_2''(z)\right), \end{aligned} \tag{3.3}$$

while the displacements have the complex expression

$$2G(u + iv) = 2\left(\kappa f_1(z) - z\overline{f}_1'(\overline{z}) - \overline{f}_2'(\overline{z})\right), \tag{3.4}$$

where $G = \dfrac{E}{2(1+\sigma)}$ is the tangential modulus of elasticity and $\kappa = \dfrac{3-\sigma}{1+\sigma}$ is a numerical constant depending solely on Poisson's ratio. Summing (3.3) and (3.4) immediately yields the combination

$$\sigma_y + i\tau_{xy} = 2\left(f_1'(z) + \overline{f}_1'(\overline{z}) + \overline{z}f_1''(z) + f_2''(z)\right), \tag{3.5}$$

or, alternatively,

$$\sigma_y - i\tau_{xy} = 2\left(f_1'(z) + \overline{f}_1'(\overline{z}) + z\overline{f}_1''(\overline{z}) + \overline{f}_2''(\overline{z})\right). \tag{3.6}$$

Another useful form for stresses and displacements has been suggested by Muskhelishvili [1953] and consists in introducing the auxiliary function

$$f_3''(z) = \overline{f}_2''(z) + \overline{f}_1'(z) + z\overline{f}_1''(z),$$

so that (3.4) and (3.6) can be written as

$$2G(u+iv) = 2\left[\kappa f_1(z) - f_3'(\overline{z}) - (z-\overline{z})\overline{f}_1'(\overline{z})\right], \tag{3.7}$$

$$\sigma_y - i\tau_{xy} = 2\left[f_1'(z) + f_3''(\overline{z}) + (z-\overline{z})\overline{f}_1''(\overline{z})\right]. \tag{3.8}$$

These stresses and displacements must satisfy the boundary conditions of the problem, which are of mixed type since they require

$$\begin{aligned} &\sigma_y - i\tau_{xy} = 0 \qquad \text{for} \qquad y=0\ ,\ |x|<a, \\ &\tau_{xy} = v = 0 \qquad \text{for} \qquad y=0\ ,\ |x|>a. \end{aligned} \tag{3.9}$$

In addition, the inequalities $v \geq 0$ for $|x| < a$, and $\sigma_y \leq 0$ for $|x| > a$ must be imposed.

In order to find the functions $f_1(z)$ and $f_3(z)$ it is convenient to observe that they are also solutions to the elastic problem of an infinite plane with a straight cut of length $2a$, subjected to a uniform pressure $\sigma_y^0 = -q$ at infinity and to a pair of forces Y and $-Y$ applied at the mid point of the upper and lower edge. In other words, the solution does not change by removing the rigid smooth support $y = 0$ and at the same time prolonging D with its loads into its mirror image with respect to the x-axis.

After this symmetrization the solution is reduced to finding the stresses in an infinite plane with a straight cut: a problem which can be treated with a method first introduced by Muskhelishvili [1953]. The idea of the procedure is based on the condition that, having extended the domain to the entire plane, if z lies in the half-plane $y > 0$, then $\overline{z}$ lies in the half-plane $y < 0$, and viceversa. Let now $(\sigma_y - i\tau_{xy})^+$ be the stress vector on the upper edge and $(\sigma_y - i\tau_{xy})^-$ that on the lower edge; thus the boundary conditions on both edges can be written as

$$\text{(3.10)} \qquad \begin{cases} (\sigma_y - i\tau_{xy})^+ = 2(f_1'(x))^+ + 2(f_3''(x))^- = -p^+(x), \\ (\sigma_y - i\tau_{xy})^- = 2(f_1'(x))^- + 2(f_3''(x))^+ = -p^-(x), \end{cases}$$

where $p^+(x)$ and $p^-(x)$ represent the stress vectors at each edge. In the case under consideration these tractions are zero everywhere except in a small region near the origin, say $-\varepsilon \leq x \leq \varepsilon$, where p^+ and p^- have a constant value p_0 such that

$$\text{(3.11)} \qquad \lim_{\varepsilon \to 0} 2\varepsilon p_0 = Y.$$

With these preliminaries understood, the remaining procedure is classical. The equations (3.10), after summation and subtraction, become

$$\text{(3.12)} \qquad \begin{cases} 2\Big[f_1'(x) + f_3''(x)\Big]^+ + 2\Big[f_1'(x) + f_3''(x)\Big]^- = -2(p^+(x) + p^-(x)), \\ 2\Big[f_1'(x) - f_3''(x)\Big]^- - 2\Big[f_1'(x) - f_3''(x)\Big]^- = 0. \end{cases}$$

These represent two boundary value problems of Riemann-Hilbert type, the solutions of which are expressible in the forms

$$\text{(3.13)} \qquad 2\Big[f_1'(z) - f_3''(z)\Big] = 2\Big[f_1'(\infty) - f_3''(\infty)\Big],$$

$$\text{(3.14)} \qquad \begin{aligned} 2\Big[f_1'(z) + f_3''(z)\Big] &= -\frac{1}{\pi i \sqrt{z^2 a^2}} \int_{-a}^{+a} \frac{\sqrt{x^2 - a^2}(p^+(x) + p^-(x))}{x - z} dx \\ &\quad + \frac{a_0 + a_1 z}{\sqrt{z^2 - a^2}}, \end{aligned}$$

where $f_1'(\infty)$, $f_3''(\infty)$ denote the values of $f_1'(z)$, $f_3''(z)$ at infinity and a_0, a_1 are complex constants to be determined by the conditions at infinity.

The integral in the latter equation can be evaluated in finite form. Recalling that $p^+(x) = p^-(x) = p_0$ have support on $-\varepsilon < x < \varepsilon$, and using integral tables (cf. Gröbner and Hofreiter [1966]), it is easy to find that

$$\int_{-a}^{+a} \frac{\sqrt{x^2 - a^2}(p^+(x) + p^-(x))}{x - z} dx = 2p_0 i \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{a^2 - x^2}}{x - z} dx$$

$$= 2p_0 i \left\{ \Big[\sqrt{a^2 - x^2}\Big]_{-\varepsilon}^{+\varepsilon} - z \Big[\arcsin \frac{x}{a}\Big]_{-\varepsilon}^{+\varepsilon} - \sqrt{z^2 - a^2} \Big[\arcsin \frac{a^2 - zx}{a(z - x)}\Big]_{-\varepsilon}^{+\varepsilon} \right\}.$$

Hence, with recourse to the addition formulae of the function arcsinx (cf. Gradshteyn and Rizhyk [1965, 1.6]) and taking the limit for $\varepsilon \to 0$, it follows that

$$\lim_{\varepsilon\to 0} 2p_0 i \int_{-\varepsilon}^{+\varepsilon} \frac{\sqrt{a^2-x^2}}{x-z} dx = \lim_{\varepsilon\to 0}\left(-\frac{2p_0 i a\varepsilon}{z}\right) = -\frac{iaY}{z}.$$

To determine the constants it is sufficient to recall that, at infinity, the only active stress component is $\sigma_y(\infty) = \sigma_y^0(\infty) = -q$; consequently the complex potentials have the values

$$f_1'(\infty) = B = -\frac{q}{8},\quad f_2''(\infty) = B' = -\frac{q}{4},\quad f_3''(\infty) = B + \overline{B}' = -\frac{3}{8}q,$$

where the symbols B and B' used here to denote fundamental stresses were introduced by Muskhelishvili. To find the other coefficients a_0, a_1 in (3.14) it is expedient to choose the branch of the multivalued function $\sqrt{z^2-a^2}$ such that $z^{-1}\sqrt{z^2-a^2} \to 1$ as $|z| \to \infty$ and to put

$$\left(\sqrt{x^2-a^2}\right)^+ = -\left(\sqrt{x^2-a^2}\right)^- = i\sqrt{a^2-x^2} \qquad \text{for } |x| < a,$$

and $\sqrt{x^2-a^2} = +\sqrt{x^2-a^2}$ for $x > a$, $\sqrt{x^2-a^2} = -\sqrt{x^2-a^2}$ for $x < -a$ (cf. England [1971, §3.10]. It thus follows that a_1 is simply given by

$$a_1 = 2\left[f_1'(\infty) + f_3''(\infty)\right] = 2\left(2B + \overline{B}'\right) = -q.$$

The other coefficient a_0 must instead be determined by the condition of single-valuedness of displacements when the point z describes a generic closed curve C surrounding the cut. In particular, the curve C may be taken as constituted by the circle $C_\varepsilon : |z+a| = \varepsilon$; the segment $\Gamma_+ : -a+\varepsilon \le x \le a-\varepsilon$, $y = 0$, $\arg(z+a) = 0$, $\arg(z-a) = \pi$; the circle $\tilde{C}_\varepsilon : |z-a| = \varepsilon$, and the segment $\Gamma_- : a-\varepsilon \ge x \ge -a+\varepsilon$, $y = 0$, $\arg(z-a) = \pi$, $\arg(z+a) = 2\pi$, as pictured in Figure 3 (cf. Weinberger [1965, §62]). Since the integrals on C_ε and $\tilde{C}_\varepsilon$ vanish as $\varepsilon \to 0$, only the integrals on Γ_+ and Γ_- are left, so that the above condition becomes

$$\begin{aligned} G\oint_C \left[u'(z) + iv'(z)\right] dz &= G\int_{-a}^{a}\left[(u'+iv')^+ - (u'+iv')^-\right] dx \\ &= \int_{-a}^{a}\left[(\kappa f_1'(x))^+ - (f_3''(x))^- - (\kappa f_1'(x))^- + (f_3''(x))^+\right] dx = 0 \end{aligned} \tag{3.15}$$

To evaluate the latter integral it is necessary to solve (3.13) and (3.14) with respect to $f_1(z)$ and $f_3(z)$, namely

$$4f_1'(z) = +\frac{aY}{\pi z\sqrt{z^2-a^2}} + 2\left[f_1'(\infty) - f_3''(\infty)\right] + \frac{a_0 + a_1 z}{\sqrt{z^2-a^2}},$$

$$4f_3''(z) = +\frac{aY}{\pi z\sqrt{z^2-a^2}} - 2\left[f_1'(\infty) - f_3''(\infty)\right] + \frac{a_0 + a_1 z}{\sqrt{z^2-a^2}},$$

which, after substitution of $f_1'(\infty)$, $f_3''(\infty)$, a_1 with the values obtained before, become

$$4f_1'(z) = +\frac{aY}{\pi z\sqrt{z^2-a^2}} + \frac{q}{2} + \frac{a_0 - qz}{\sqrt{z^2-a^2}}, \tag{3.16}$$

$$4f_3''(z) = +\frac{aY}{\pi z\sqrt{z^2-a^2}} - \frac{q}{2} + \frac{a_0 - qz}{\sqrt{z^2-a^2}}. \tag{3.17}$$

Hence, considering the values of these functions on the upper and lower edge of the cut, it is easy to arrive at

$$\begin{aligned}&(\kappa f_1'(x))^+ - (f_3''(x))^-\\ &= -\left(\frac{\kappa+1}{4}\right)\frac{iaY}{\pi x\sqrt{a^2-x^2}} + \frac{\kappa+1}{8}q - i\left(\frac{\kappa+1}{4}\right)\frac{a_0 - qx}{\sqrt{a^2-x^2}},\end{aligned} \tag{3.18}$$

$$\begin{aligned}&(\kappa f_1'(x))^- - (f_3''(x))^+\\ &= +\left(\frac{\kappa+1}{4}\right)\frac{iaY}{\pi x\sqrt{a^2-x^2}} + \frac{\kappa+1}{8}q + i\left(\frac{\kappa+1}{4}\right)\frac{a_0 - qx}{\sqrt{a^2-x^2}}.\end{aligned} \tag{3.19}$$

Evaluating the integral (3.15) then yields $a_0 = 0$.

Having thus found the solution, the formulae (3.8) permit the stresses along the x-axis to be obtained for $|x| > a$:

$$\sigma_y - i\tau_{xy} = \begin{cases} \dfrac{aY}{\pi x\sqrt{x^2-a^2}} - \dfrac{qx}{\sqrt{x^2-a^2}} & \text{for } x > a,\\[2ex] \dfrac{aY}{\pi x\sqrt{x^2-a^2}} + \dfrac{qx}{\sqrt{x^2-a^2}} & \text{for } x < -a,\end{cases} \tag{3.20}$$

in agreement with a result quoted by Parker [1981, page 41].

The displacements on the upper side of the interval $|x| < a$ are instead given by integration of (3.18) with the end conditions $u(0) = 0$, $v(a) = v(-a) = 0$:

$$\begin{aligned}&G(u+iv)^+ = (\kappa f_1(x))^+ - (f_3'(x))^-\\ &= \frac{\kappa+1}{4}\left[\frac{-iaY}{\pi}\left(\frac{1}{a}\ln\frac{|x|}{a+\sqrt{a^2-x^2}}\right) + \frac{qx}{2} - iq\sqrt{a^2-x^2}\right].\end{aligned} \tag{3.21}$$

It is worth remarking that v^+ is logarithmically singular at the origin where Y acts.

So far the magnitude of a is still unknown and must be determined by requiring that the normal stress $\sigma_y(\pm a)$ vanish at the ends of the free boundary. This condition is not immediate, since the stresses are singular at these points; it is thus necessary to suppose that $\lim_{x \to a^+} \sqrt{x-a}\ \sigma_y(x) = \lim_{x \to a^-} \sqrt{x+a}\ \sigma_y(x) = 0$, which yields

$$a = \frac{Y}{\pi q} \tag{3.22}$$

as the half-length of the boundary.

It remains to be verified that the stress $\sigma_y(x)$ is purely compressive for $|x| > a$. But this property is evident, since, by virtue of (3.22), $\sigma_y(x)$ has the expression

$$\sigma_y(x) = -\frac{q}{\sqrt{x^2 - a^2}}\left(a - \frac{a^2}{x}\right) \quad \text{for } |x| > a,$$

which is everywhere negative. As to the proof of the other contraint, that $v^+(x)$ is non-negative for $|x| < a$, this too is almost trivial since the formula (3.18), after replacement of $\frac{Y}{\pi}$ by qa, gives

$$G(v'(x))^+ = -\left(\frac{\kappa+1}{4}\right)\frac{q}{\sqrt{a^2 - x^2}}\left(\frac{a^2}{x} - x\right),$$

proving that $v^+(x)$ is strictly increasing for $-a < x < 0$, strictly decreasing for $0 < x < a$, and vanishing at the ends. A qualitative picture of the profile of the free boundary is shown by the heavy line in Figure 3. As a final remark, the equation for $v^+(x)$ confirms the properties, stated by Kinderlehrer [1981], that, in plane elasticity, the elastic body leaves the unilateral obstacle maintaining continuity of normal displacements and of their tangential derivatives. This property is also known as Barenblatt's conjecture (Fichera [1972]), but, in this case, the result is rendered more precise because displacements are exactly of class $C^{1,\alpha}$ with $\alpha = \frac{1}{2}$.

4. - The case of a horizontal load X

The solution is more difficult when the additional load is a force X acting in the positive direction of the x-axis (Fig. 4) since now stresses and strains are no longer symmetric with respect the y-axis and therefore the interval of detachement is not necessarily centered at the origin.

Hower, it is possible to exploit the artifice of reflection with respecy to the x-axis since the solution is the same as that of an infinite elastic plane with a rectilinear cut of length $2a$ loaded by two equal forces X applied at each edge of the cut. Furthermore, there are good reasons to assume that, in analogy to what happens for a single half-plane under tangential tractions over a portion of its boundary (cf. Love [1927, Art. 152]), the material tends to abandon the support on the side towards which the loads act. As a consequence the edge will rise to the right of the point of application of X, which in turn implies that, if the support of the free boundary is the interval $-a < x < a$, then X may be regarded as concentrated at $x = -a$ (Fig. 4).

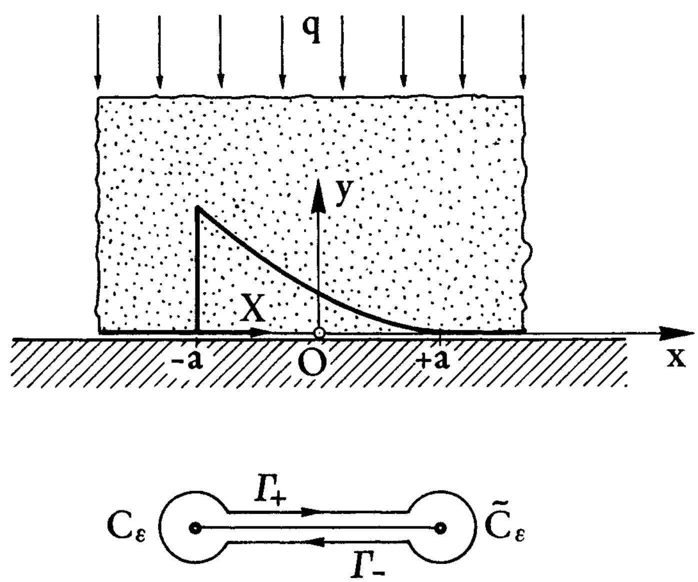

Fig. 4

In order to find the stresses induced in the plane by the forces X, the latter are replaced by two constant distributions of tangential tractions $s^+ = -s^- = s_0$ such that

$$\lim_{\varepsilon \to 0} 2\varepsilon s_0 = X. \tag{4.1}$$

Here again the solution is fully defined by two analytic functions, still denoted by $f_1(z)$ and $f_2(z)$, which are solutions to the Riemann-Hilbert problem

$$\begin{cases} 2\Big[f_1'(x) + f_3''(x)\Big]^+ + 2\Big[f_1'(x) + f_3''(x)\Big]^- = 0, \\ 2\Big[f_1'(x) - f_3''(x)\Big]^+ - 2\Big[f_1'(x) - f_3''(x)\Big]^- = -2i(s^+(x) - s^-(x)), \end{cases} \tag{4.2}$$

where, in contrast to equation (3.12), the non-homogeneous term appears in the second equation. From the general form of solutions to the Riemann-Hilbert problem, it turns out that the combinations $f_1'(z) - f_3''(z)$ and $f_1'(z) + f_3''(z)$ have the espressions

$$2\left[f_1'(z) - f_3''(z)\right] = 2\left[f_1'(\infty) - f_3''(\infty)\right] - \frac{1}{\pi}\int_{-a}^{a} \frac{(s^+(x) - s^-(x))}{x - z}dx, \tag{4.3}$$

$$2\left[f_1'(z) + f_3''(z)\right] = \frac{a_0 + a_1 z}{\sqrt{z^2 - a^2}}, \tag{4.4}$$

where again $f_1'(\infty) = B = -\frac{q}{8}$, $f_3''(\infty) = B + \overline{B}' = -\frac{3}{8}q$, and a_0, a_1 are new constants. To determine these constants it is first necessary to recall that

$$\int_{-a}^{a} \frac{(s^+(x) - s^-(x))}{x - z} = 2s_0 \int_{-a}^{a} \frac{dx}{x - z} = -2s_0 \ln \frac{z + a - 2\varepsilon}{z + a},$$

since now the support of $s^+(x) = -s^-(x)$ is a small interval contiguous to the point $x = -a$. In consequence, the limiting condition (4.1) yields

$$\lim_{\varepsilon \to 0}\left(-2s_0 \ln \frac{z + a - 2\varepsilon}{z + a}\right) = \frac{X}{z + a}.$$

Thus, knowing the values of $f_1'(z)$ and $f_3''(z)$ at infinity, it follows that $a_1 = -q$ as before. Hence $f_1'(z)$ and $f_3''(z)$ can be written as

$$4f_1'(z) = -\frac{X}{\pi(z + a)} + \frac{q}{2} + \frac{a_0 - qz}{\sqrt{z^2 - a^2}}, \tag{4.5}$$

$$4f_3''(z) = +\frac{X}{\pi(z + a)} - \frac{q}{2} + \frac{a_0 - qz}{\sqrt{z^2 - a^2}}, \tag{4.6}$$

and it is again easy to derive the displacements on both edges of the cut

$$\begin{aligned}(4.7)\quad & (\kappa f_1'(x))^+ - (f_3''(x))^- \\ & = -\left(\frac{\kappa + 1}{4}\right)\frac{X}{\pi(x + a)} + \left(\frac{\kappa + 1}{4}\right)\frac{q}{2} - i\left(\frac{\kappa + 1}{4}\right)\frac{a_0 - qx}{\sqrt{a^2 - x^2}},\end{aligned}$$

$$\begin{aligned}(4.8)\quad & (\kappa f_1'(x))^- - (f_3''(x))^+ \\ & = -\left(\frac{\kappa + 1}{4}\right)\frac{X}{\pi(x + a)} + \left(\frac{\kappa + 1}{4}\right)\frac{q}{2} + i\left(\frac{\kappa + 1}{4}\right)\frac{a_0 - qx}{\sqrt{a^2 - x^2}}.\end{aligned}$$

The remaining constant a_0 is determined by requiring the displacements to be single valued around the cut. Here again the simplest way to evaluate the integral of $u' + iv'$ along the cut is to choose the same contour as before and let $\varepsilon \to 0$. But now the functions $f_1'(z)$ and $f_3''(z)$ have a simple pole at $z = -a$, and their residues are

$$\operatorname*{Res}_{z=-a}\left[f_1'(z)\right] = -\operatorname*{Res}_{z=-a}\left[f_3''(z)\right] = +\operatorname*{Res}_{\bar{z}=-a}\left[\overline{f}_3''(\bar{z})\right] = -\frac{X}{\pi}.$$

Thus

$$\begin{aligned} &G\oint_C \left[u'(z) + iv'(z)\right] dz \\ &= G\oint_{C_\varepsilon}\left[u'(z) + iv'(z)\right] dz + G\int_{-a}^{a}\left[(u' + iv')^+ - (u' + iv')^-\right] dx \\ &= G\oint_{C_\varepsilon}\left[\kappa f_1'(z)dz - f_3''(\bar{z})d\bar{z}\right] + G\int_{-a}^{a}\Big[(\kappa f_1'(x))^+ \\ &\qquad - (f_3''(x))^- - (\kappa f_1'(x))^- + (f_3''(x))^+\Big] dx = 0. \end{aligned} \tag{4.9}$$

The first integral is easily evaluated through the residues at $z = \bar{z} = -a$:

$$G\oint_{C_\varepsilon}\left[\kappa f_1'(t)dt - f_3''(\bar{t})d\bar{t}\right] = -2\pi i G\,\frac{(\kappa - 1)}{\pi}\,X.$$

Moreover the second integral can be computed by using (4.7), (4.8):

$$\begin{aligned} &G\int_{-a}^{a}\left[(\kappa f_1'(x))^+ - (f_3''(x))^- - (\kappa f_1'(x))^- + (f_3''(x))^+\right] dx \\ &= -2Gi\,\frac{\kappa + 1}{4}\int_{-a}^{a}\frac{a_0 - qx}{\sqrt{a^2 - x^2}}\,dx = 2Gi\,\frac{\kappa + 1}{4}\,\pi a_0. \end{aligned}$$

It thus follows that the constant a_0 has the value

$$a_0 = \frac{\kappa - 1}{\kappa + 1}\,\frac{X}{\pi},$$

and the final values of the complex potentials are

$$4f_1'(z) = +\frac{\kappa - 1}{\kappa + 1}\,\frac{X}{\pi\sqrt{z^2 - a^2}} + \frac{q}{2} - \frac{X}{\pi(z + a)} - \frac{qz}{\sqrt{z^2 - a^2}}, \tag{4.10}$$

$$4f_3''(z) = +\frac{\kappa-1}{\kappa+1}\,\frac{X}{\pi\sqrt{z^2-a^2}} - \frac{q}{2} + \frac{X}{\pi(z+a)} - \frac{qz}{\sqrt{z^2-a^2}}. \tag{4.11}$$

From these expressions the stresses for $y = 0$, $|x| > a$ can be immediately obtained by applying (3.8), that is

$$\sigma_y(x) - i\tau_{xy}(x) = \begin{cases} \dfrac{\kappa-1}{\kappa+1}\,\dfrac{X}{\pi\sqrt{x^2-a^2}} - \dfrac{qx}{\sqrt{x^2-a^2}} & \text{for } x > a, \\[2ex] -\dfrac{\kappa-1}{\kappa+1}\,\dfrac{X}{\pi\sqrt{x^2-a^2}} + \dfrac{qx}{\sqrt{x^2-a^2}} & \text{for } x < -a. \end{cases} \tag{4.12}$$

As for the displacements on the interval $|x| < a$, they must be determined by integrating the function

$$\begin{aligned} G\Big(u'(x) + iv'(x)\Big)^{+} &= \Big(\kappa f_1'(x)\Big)^{+} - \Big(f_3''(x)\Big)^{-} \\ &= -i(\kappa-1)\,\frac{X}{4\pi\sqrt{a^2-x^2}} + \frac{\kappa+1}{8}\,q - \frac{\kappa+1}{4\pi(x+a)} + i(\kappa+1)\,\frac{qx}{4\sqrt{a^2-x^2}} \end{aligned} \tag{4.13}$$

and by applying the boundary conditions. These conditions are two: the first is $v(a) = 0$, expressing the fact that the end point $x = a$ rests in contact with the plane $y = 0$; the second is not mandatory because tangential displacement u is defined within a constant rigid motion, which may be determined by putting, for instance, $u(0) = 0$. Thus (4.13), after integration, yields

$$\begin{aligned} G\Big(u(x) + iv(x)\Big)^{+} = \frac{\kappa+1}{4}\Bigg[&-i\,\frac{\kappa-1}{\kappa+1}\,\frac{X}{\pi}\left(\arcsin\frac{x}{a} - \frac{\pi}{2}\right) + \frac{qx}{2} \\ &-\frac{X}{\pi}\ln\frac{x+a}{a} - iq\sqrt{a^2-x^2}\Bigg]. \end{aligned} \tag{4.14}$$

The still unknown length a is then determined by the condition $\lim_{x\to a^+}\sqrt{x-a}\;\sigma_y(x) = 0$, whence

$$a = \frac{\kappa-1}{\kappa-1}\,\frac{X}{\pi q}. \tag{4.15}$$

Since $\dfrac{\kappa-1}{\kappa+1} < 1$, the length of the detached boundary is less than before.

It remains to be verified that the normal stress $\sigma_y(x)$ is nowhere tensile for $|x| > a$. But this property is clearly valid, since $\sigma_y(x)$, as a consequence of (4.15), has the form

$$\sigma_y(x) = \begin{cases} -\dfrac{q}{\sqrt{x^2-a^2}}(x-a) & \text{for } x > a, \\[2ex] \dfrac{q}{\sqrt{x^2-a^2}}(x-a) & \text{for } x < -a; \end{cases}$$

as to the normal displacement $v^+(x)$, this is given by (4.14):

$$Gv^+(x) = -\frac{\kappa + 1}{4}\, q \left[a\left(\arcsin\frac{x}{a} - \frac{\pi}{2}\right) + \sqrt{a^2 - x^2}\right],$$

which is everywhere positive in the interval $|x| < a$. In contrast to the case of a simple normal load Y, however, the free boundary leaves its support abrutlpy at the point $x = -a$, where the tangential force X acts, it instead returns smoothly to make contact at the other end $x = a$, confirming Barenblatt's conjecture (Fig. 4). Finally, the displacements at $x = a$ are of class $C^{1,\alpha}$ with $\alpha = \frac{1}{2}$.

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