

THE SHIFT SPACE FOR AN INFINITE ITERATED FUNCTION SYSTEM

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The aim of the paper is to define the shift space for an infinite iterated function systems (IIFS) and to describe the relation between this space and the attractor of the IIFS. We construct a canonical projection (which turns out to be continuous) from the shift space of an IIFS on its attractor and provide sufficient conditions for this function to be onto.

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1. INTRODUCTION

The shift (or code) space of an iterated function system (IFS for short) and the address of the points lying on the attractor of the IFS are very good tools to get a more precise description of the invariant dynamics of the IFS. The theory of fractal tops provides a useful mapping from an IFS attractor into the associated code space that may be applied to assign colors to the IFS attractor via a method introduced by M.F. Barnsley and J. Hutchinson (which they refer as colour-stealing) and to construct homeomorphisms between attractors (roughly speaking, if the symbolic dynamical systems associated with the tops of two IFSs are topologically conjugate, then the attractors of the IFSs are homeomorphic). Moreover, Barnsley [2] proved that if two hyperbolic IFS attractors are homeomorphic, then they have the same entropy.

In this paper we present a generalization of the notion of shift space associated with an IFS. More precisely, we define the shift space of an infinite iterated function systems (IIFS) and describe the relation between this space and the attractor of the IIFS. We construct a canonical projection (which turns out to be continuous) from the shift space of an IIFS on its attractor and provide sufficient conditions for this function to be onto.

2. PRELIMINARIES

In this section we present notation and the main results concerning the Hausdorff-Pompeiu semidistance and infinite iterated function systems.

Notation 2.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. By $C(X, Y)$ we denote the set of continuous functions from X to Y .

Definition 2.1. Let (X, d_X) and (Y, d_Y) be two metric spaces. A family of functions $(f_i)_{i \in I} \subseteq C(X, Y)$ is called bounded if the set $\bigcup_{i \in I} f_i(A)$ is bounded, for every bounded subset A of X .

Definition 2.2. Let (X, d) be a metric space. For a function $f : X \rightarrow X$, the Lipschitz constant associated with f is

$$\text{Lip}(f) = \sup_{x, y \in X; x \neq y} \frac{d(f(x), f(y))}{d(x, y)} \in [0, +\infty].$$

A function $f : X \rightarrow X$ is called Lipschitz if $\text{Lip}(f) < +\infty$ and is called contraction if $\text{Lip}(f) < 1$.

LEMMA 2.1. *For a metric space (X, d) and a function $f : X \rightarrow X$, we have*

$$\text{diam}(f(A)) \leq \text{Lip}(f) \text{diam}(A),$$

for all subsets A of X .

Notation 2.2. For a set X , $\mathcal{P}(X)$ denotes the set of all subsets of X . By $\mathcal{P}^*(X)$ we mean $\mathcal{P}(X) - \{\emptyset\}$ while, for a subset $A \in \mathcal{P}(X)$, by A^* we mean $A - \{\emptyset\}$.

For a metric space (X, d) , $\mathcal{K}(X)$ denotes the set of compact subsets of X and $\mathcal{B}(X)$ the set of bounded closed subsets of X .

Remark 2.1. We have

$$\mathcal{K}(X) \subseteq \mathcal{B}(X) \subseteq \mathcal{P}(X).$$

Definition 2.3. For a metric space (X, d) , we consider on $\mathcal{P}^*(X)$ the generalized Hausdorff-Pompeiu pseudometric $h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \rightarrow [0, +\infty]$ defined by

$$h(A, B) = \max(d(A, B), d(B, A)),$$

where

$$d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \left(\inf_{y \in B} d(x, y) \right).$$

Concerning the Hausdorff-Pompeiu semidistance we have the important properties below.

PROPOSITION 2.1 (see [1], [3] or [6]). *Given two metric spaces (X, d_X) and (Y, d_Y) , the assertions below hold.*

(1) *If H and K are two nonempty subsets of X , then $h(H, K) = h(\overline{H}, \overline{K})$.*

(2) *If $(H_i)_{i \in I}$ and $(K_i)_{i \in I}$ are two families of nonempty subsets of X , then*

$$h\left(\bigcup_{i \in I} H_i, \bigcup_{i \in I} K_i\right) = h\left(\overline{\bigcup_{i \in I} H_i}, \overline{\bigcup_{i \in I} K_i}\right) \leq \sup_{i \in I} h(H_i, K_i).$$

(3) *If H and K are two nonempty subsets of X , then*

$$h(f(K), f(H)) \leq \text{Lip}(f)h(K, H),$$

where f is a function from X to X .

THEOREM 2.1 (see [1], [3], [5] and [6]). *For a metric space (X, d) let $h : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \rightarrow [0, \infty]$ be the Hausdorff-Pompeiu semidistance. The assertions below hold.*

(1) *$(\mathcal{B}^*(X), h)$ and $(\mathcal{K}^*(X), h)$ are metric spaces and $(\mathcal{K}^*(X), h)$ is closed in $\mathcal{B}^*(X)$.*

(2) *If (X, d) is complete, then $(\mathcal{B}^*(X), h)$ and $(\mathcal{K}^*(X), h)$ are complete metric spaces.*

(3) *If (X, d) is compact, then $(\mathcal{K}^*(X), h)$ is compact and in this case $\mathcal{B}^*(X) = \mathcal{K}^*(X)$.*

(4) *If (X, d) is separable, then $(\mathcal{K}^*(X), h)$ is separable.*

Definition 2.4. An infinite iterated function system (IIFS for short) on X consists of a bounded family of contractions $(f_i)_{i \in I}$ on X such that

$$\sup_{i \in I} \text{Lip}(f_i) < 1.$$

It is denoted $S = (X, (f_i)_{i \in I})$. With an infinite iterated function system $S = (X, (f_i)_{i \in I})$ one can associate the function $F_S : \mathcal{B}^*(X) \rightarrow \mathcal{B}^*(X)$ defined as

$$F_S(B) = \overline{\bigcup_{i \in I} f_i(B)}, \quad B \in \mathcal{B}^*(X).$$

Remark 2.2. We note that F_S is a contraction and

$$\text{Lip}(F_S) \leq \sup_{i \in I} \text{Lip}(f_i).$$

Using the Banach's contraction theorem, one can prove the result below.

THEOREM 2.2 (see [6]). *Given a complete metric space (X, d) and an IIFS $S = (X, (f_i)_{i \in I})$ such that $c \stackrel{\text{def}}{=} \sup_{i \in I} \text{Lip}(f_i) < 1$, there exists a unique $A(S) \in \mathcal{B}^*(X)$ such that $F_S(A(S)) = A(S)$.*

Moreover, for any $H_0 \in \mathcal{B}^*(X)$ the sequence $(H_n)_{n \geq 0}$ defined by $H_{n+1} = F_S(H_n)$, $n \in \mathbb{N}$, is convergent to $A(S)$. As for the speed of convergence, we have

$$h(H_n, A(S)) \leq \frac{c^n}{1-c} h(H_0, H_1), \quad n \in \mathbb{N}.$$

Definition 2.5. The set $A(S)$ is called the attractor associated with S .

3. THE SHIFT SPACE FOR AN IIFS

In this section we introduce the shift (or the code space) of an IIFS, which generalizes the notion of the shift space for an IFS (see [1] or [6]).

Terminology and notation. \mathbb{N} denotes the natural numbers, $\mathbb{N}^* = \mathbb{N} - \{0\}$, and $\mathbb{N}_n^* = \{1, 2, \dots, n\}$, where $n \in \mathbb{N}^*$. Given two sets A and B , by B^A we mean the set of functions from A to B . By $\Lambda = \Lambda(B)$ we mean the set $B^{\mathbb{N}^*}$ while by $\Lambda_n = \Lambda_n(B)$ we mean the set $B^{\mathbb{N}_n^*}$. The elements of $\Lambda = \Lambda(B) = B^{\mathbb{N}^*}$ are written as words $\omega = \omega_1\omega_2\dots\omega_m\omega_{m+1}\dots$ while the elements of $\Lambda_n = \Lambda_n(B) = B^{\mathbb{N}_n^*}$ are written as words $\omega = \omega_1\omega_2\dots\omega_n$. Hence $\Lambda(B)$ is the set of infinite words with letters from the alphabet B and $\Lambda_n(B)$ is the set of words of length n with letters from the alphabet B . By $\Lambda^* = \Lambda^*(B)$ we denote the set of all finite words $\Lambda^* = \Lambda^*(B) = \bigcup_{n \in \mathbb{N}^*} \Lambda_n(B) \cup \{\lambda\}$, where by λ we mean the empty word. If $\omega = \omega_1\omega_2\dots\omega_m\omega_{m+1}\dots \in \Lambda(B)$ or if $\omega = \omega_1\omega_2\dots\omega_n \in \Lambda_n(B)$, where $m, n \in \mathbb{N}^*$, $n \geq m$, then the word $\omega_1\omega_2\dots\omega_m$ is denoted by $[\omega]_m$. For two words $\alpha \in \Lambda_n(B)$ and $\beta \in \Lambda_m(B)$ or $\beta \in \Lambda(B)$, by $\alpha\beta$ we mean the concatenation of the words α and β , i.e., $\alpha\beta = \alpha_1\alpha_2\dots\alpha_n\beta_1\beta_2\dots\beta_m$ and $\alpha\beta = \alpha_1\alpha_2\dots\alpha_n\beta_1\beta_2\dots\beta_m\beta_{m+1}\dots$, respectively.

For a nonvoid set I , on $\Lambda = \Lambda(I) = (I)^{\mathbb{N}^*}$ we consider the metric

$$d_\Lambda(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1 - \delta_{\alpha_k}^{\beta_k}}{3^k},$$

where $\delta_x^y = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$.

Definition 3.1. The metric space $(\Lambda(I), d_\Lambda)$ is called the shift space associated with the IIFS $S = (X, (f_i)_{i \in I})$.

Remark 3.1. (i) The convergence in the metric space $(\Lambda(I), d_\Lambda)$ is convergence on components.

(ii) $(\Lambda(I), d_\Lambda)$ is a complete metric space.

More terminology and notation. For $i \in I$ let us consider the function $F_i : \Lambda(I) \rightarrow \Lambda(I)$ defined by $F_i(\omega) = i\omega$ for all $\omega \in \Lambda(I)$.

The continuous functions F_i are called the right shift functions. Let us note that

$$d_\Lambda(F_i(\alpha), F_i(\beta)) = \frac{d_\Lambda(\alpha, \beta)}{3},$$

for all $i \in I$, $\alpha, \beta \in \Lambda(I)$.

For $\omega = \omega_1\omega_2 \dots \omega_m \in \Lambda_m(I)$, consider $F_\omega = F_{\omega_1} \circ F_{\omega_2} \circ \dots \circ F_{\omega_m}$ and $\Lambda_\omega = F_\omega(\Lambda)$. We also consider $F_\lambda = \text{Id}$ and $\Lambda_\lambda = \Lambda$.

Remark 3.2. (i) We have $\Lambda(I) = \bigcup_{i \in I} F_i(\Lambda(I))$, so that $\Lambda(I)$ is the attractor of the IIFS $S = (\Lambda(I), (F_i)_{i \in I})$.

(ii) For a given $m \in \mathbb{N}^*$ we have

$$\Lambda = \bigcup_{\alpha \in \Lambda_m} \Lambda_\alpha$$

and

$$\Lambda_\omega = \bigcup_{\alpha \in \Lambda_m} \Lambda_{\omega\alpha}$$

for every $\omega \in \Lambda^*$.

Notation 3.1. Let (X, d) be a metric space, $S = (X, (f_i)_{i \in I})$ an IIFS on X and $A = A(S)$ its attractor. For $\omega = \omega_1\omega_2 \dots \omega_m \in \Lambda_m(I)$, we consider $f_\omega \stackrel{\text{not}}{=} f_{\omega_1} \circ f_{\omega_2} \circ \dots \circ f_{\omega_m}$ and, for a subset H of X , $H_\omega \stackrel{\text{not}}{=} f_\omega(H)$. In particular $A_\omega = f_\omega(A)$.

We also consider

$$f_\lambda = \text{Id} \quad \text{and} \quad A_\lambda = A.$$

Notation 3.2. For a contraction $f : X \rightarrow X$ we denote by e_f the fixed point of f . If $f = f_\omega$, we denote by e_{f_ω} (or by e_ω) the fixed point of the contraction $f = f_\omega$.

LEMMA 3.1. *Let (X, d) be a complete metric space and $m \in \mathbb{N}^*$. Let $f : X \rightarrow X$ be a contraction and H be a closed set such that $f(H) \subseteq H$. If e_f is the fixed point of f , then $e_f \in H$.*

Proof. Let $a_0 \in H$ and let $(a_n)_{n \in \mathbb{N}}$ be the sequence defined by $a_{n+1} = f(a_n)$, for all $n \in \mathbb{N}$. Then since $a_n \in H$ for all $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} a_n = e_f$$

on account of the fact that H is a closed set, we have $e_f \in H$. \square

4. THE MAIN RESULT

The following result describes the relation between the attractor of an IIFS and the shift space associated with it.

THEOREM 4.1. Let $\mathcal{S} = (X, (f_i)_{i \in I})$ be an IIFS, where (X, d) is a complete metric space, $A \stackrel{\text{not}}{=} A(\mathcal{S})$ the attractor of \mathcal{S} , and $c \stackrel{\text{not}}{=} \sup_{i \in I} \text{Lip}(f_i) < 1$.

Then the assertions below hold.

(1) For $m \in \mathbb{N}$ we have $A_{[\omega]_{m+1}} \subseteq A_{[\omega]_m}$ for all $\omega \in \Lambda = \Lambda(I)$ and

$$\lim_{m \rightarrow \infty} d(A_{[\omega]_m}) = 0.$$

More precisely,

$$d(\overline{A_{[\omega]_m}}) = d(A_{[\omega]_m}) \leq c^m d(A).$$

(2) If a_ω is defined by $\{a_\omega\} = \bigcap_{m \in \mathbb{N}^*} \overline{A_{[\omega]_m}}$, then

$$\lim_{m \rightarrow \infty} d(e_{[\omega]_m}, a_\omega) = 0.$$

(3) For every $a \in A$ and every $\omega \in \Lambda$ we have

$$\lim_{m \rightarrow \infty} f_{[\omega]_m}(a) = a_\omega.$$

(4) For every $\alpha \in \Lambda^*$ we have

$$A = A(\mathcal{S}) = \overline{\bigcup_{\omega \in \Lambda} \{a_\omega\}}$$

and

$$\overline{A_\alpha} = \overline{\bigcup_{\omega \in \Lambda} \{a_{\alpha\omega}\}}.$$

If $A = \bigcup_{i \in I} f_i(A)$, then

$$A = A(\mathcal{S}) = \bigcup_{\omega \in \Lambda} \{a_\omega\}.$$

(5) We have

$$\overline{\{e_{[\omega]_m} \mid \omega \in \Lambda \text{ and } m \in \mathbb{N}^*\}} = A.$$

(6) The function $\pi : \Lambda \rightarrow A$, defined by $\pi(\omega) = a_\omega$ for every $\omega \in \Lambda$, has the following properties:

(i) it is continuous;

(ii) $\overline{\pi(\Lambda)} = A$;

(iii) if $A = \bigcup_{i \in I} f_i(A)$, then π is onto.

(7) $\pi(F_i(\alpha)) = f_i(\pi(\alpha))$ for every $i \in I$ and $\alpha \in \Lambda$.

Proof. (1) We prove that $A_{[\omega]_{m+1}} \subseteq A_{[\omega]_m}$, for every $m \in \mathbb{N}$. Indeed, let us first note that $A_{[\omega]_1} = f_{\omega_1}(A) \subseteq A$. Moreover, for all $m \in \mathbb{N}$, $m \geq 2$, we have

$$A_{[\omega]_m} = f_{[\omega]_m}(A) = f_{[\omega]_{m-1}} \circ f_{\omega_m}(A) \subseteq f_{[\omega]_{m-1}}(A) = A_{[\omega]_{m-1}}.$$

The fact that $d(A_{[\omega]_m}) \leq c^m d(A)$ for all $m \in \mathbb{N}$, follows from Lemma 2.1.

(2) The fact that $\bigcap_{m \in \mathbb{N}^*} \overline{A_{[\omega]_m}}$ consists of one point follows from (1), since X is a complete metric space. Let

$$\bigcap_{m \in \mathbb{N}^*} \overline{A_{[\omega]_m}} = \{a_\omega\}.$$

Since $f_{[\omega]_m}$ is continuous for every $m \in \mathbb{N}^*$, we deduce that

$$f_{[\omega]_m}(\overline{A_{[\omega]_m}}) \subseteq \overline{f_{[\omega]_m}(A)} = \overline{A_{[\omega]_m}}$$

and it follows from Lemma 3.1 that $e_{[\omega]_m} \in \overline{A_{[\omega]_m}}$. Then

$$d(e_{[\omega]_m}, a_\omega) \leq d(\overline{A_{[\omega]_m}}) = d(A_{[\omega]_m}) \leq c^m d(A)$$

for every $m \in \mathbb{N}^*$ and, consequently,

$$\lim_{m \rightarrow \infty} d(e_{[\omega]_m}, a_\omega) = 0.$$

(3) Since

$$d(f_{[\omega]_m}(a), a_\omega) \leq d(\overline{A_{[\omega]_m}}) = d(A_{[\omega]_m}) \leq c^m d(A),$$

for every $m \in \mathbb{N}^*$ and

$$\lim_{m \rightarrow \infty} c^m = 0,$$

we deduce that for every $a \in A$ and every $\omega \in \Lambda$, we have

$$\lim_{m \rightarrow \infty} f_{[\omega]_m}(a) = a_\omega.$$

(4) On the one hand, it is obvious that

$$\overline{\bigcup_{\omega \in \Lambda} \{a_\omega\}} \subseteq A = A(S).$$

On the other hand, let us first note that

$$\overline{\bigcup_{\omega \in \Lambda_m} A_\omega} = A,$$

for every $m \in \mathbb{N}^*$. We prove this assertion by mathematical induction. For $m = 1$ it holds by Theorem 2.2. Now, let us suppose that the assertion is true for m , i.e.,

$$\overline{\bigcup_{\omega \in \Lambda_m} A_\omega} = A.$$

We will show that

$$\overline{\bigcup_{\omega \in \Lambda_{m+1}} A_\omega} = A.$$

Indeed, for $a \in A$ there exists a sequence $(a_i)_i$ of elements from A_{ω_i} , where $\omega_i \in \Lambda_m$, such that $\lim_i a_i = a$. Since $A_{\omega_i} = f_{\omega_i}(A) = f_{\omega_i}\left(\overline{\bigcup_{j \in I} A_j}\right) \subseteq \overline{\bigcup_{j \in I} f_{\omega_i}(A_j)}$ for a given i , there exist $j_i \in I$ and $b_i \in (A_{j_i})_{\omega_i}$ such that $d(a_i, b_i) < \frac{1}{i}$. Then $\lim_i b_i = a$, and $b_i \in \Lambda_{m+1}$, so $a \in \overline{\bigcup_{\omega \in \Lambda_{m+1}} A_\omega}$.

Now, we are able to prove that

$$A \subseteq \overline{\bigcup_{\omega \in \Lambda} \{a_\omega\}}.$$

Indeed, for $a \in A$, since $A = \overline{\bigcup_{\omega \in \Lambda_m} A_\omega}$, there exist $\omega_m \in \Lambda_m$ and $a_m \in A_{\omega_m}$ such that $d(a_m, a) < \frac{1}{m}$. For $\alpha \in \Lambda$ we have $a_{\omega_m \alpha} \in \overline{A_{\omega_m}}$ and, consequently, $d(a_m, a_{\omega_m \alpha}) \leq d(\overline{A_{[\omega]_m}}) = d(A_{[\omega]_m}) \leq c^m d(A)$. Therefore, $\lim_m a_{\omega_m \alpha} = a$, so $a \in \overline{\bigcup_{\omega \in \Lambda} \{a_\omega\}}$.

In a similar manner, one can prove that

$$\overline{A_\alpha} = \overline{\bigcup_{\omega \in \Lambda} \{a_{\alpha \omega}\}},$$

for every $\alpha \in \Lambda^*$.

If $A = \bigcup_{i \in I} f_i(A)$, then

$$A_\alpha = f_\alpha(A) = f_\alpha\left(\bigcup_{i \in I} f_i(A)\right) = \bigcup_{i \in I} A_{\alpha i},$$

for every $\alpha \in \Lambda^*$. Therefore, for $a \in A$ there exists a sequence $(\omega^m)_m$ with the following properties:

- (i) $\omega^m \in \Lambda_m$;
- (ii) $[\omega^{m+1}]_m = \omega^m$, and
- (iii) $a \in A_{\omega^m}$, for all m .

If $\omega^1 = \omega_1$ and ω_m is defined by $\omega^m = \omega^{m-1} \omega_m$, then $a = a_\omega$, where $\omega = \omega_1 \omega_2 \dots \omega_m \omega_{m+1} \dots \in \Lambda$, so $a \in \bigcup_{\omega \in \Lambda} \{a_\omega\}$. Hence

$$A \subseteq \bigcup_{\omega \in \Lambda} \{a_\omega\}.$$

Since

$$\bigcup_{\omega \in \Lambda} \{a_\omega\} \subseteq \overline{\bigcup_{\omega \in \Lambda} \{a_\omega\}} = A,$$

we deduce that

$$\bigcup_{\omega \in \Lambda} \{a_\omega\} = A.$$

(5) This follows from (4) and (2).

6) (i) Let ω be a fixed element of Λ and $a \stackrel{\text{not}}{=} \pi(\omega) = a_\omega$. For $\varepsilon > 0$, since $\lim_{m \rightarrow \infty} d(A_{[\omega]_m}) = 0$, there exists $m \in \mathbb{N}^*$ such that

$$\overline{A_{[\omega]_m}} = \overline{\pi(\{\alpha \in \Lambda \mid [\omega]_m = [\alpha]_m\})} \subseteq B_X(a, \varepsilon).$$

Since

$$B_\Lambda(\omega, 1/3^m) \subseteq \{\alpha \in \Lambda \mid [\omega]_m = [\alpha]_m\},$$

we deduce that

$$B_\Lambda(\omega, 1/3^m) \subseteq \Lambda_{[\omega]_m} = \{\alpha \in \Lambda \mid [\omega]_m = [\alpha]_m\} \subseteq \pi^{-1}(\overline{A_{[\omega]_m}}) \subseteq \pi^{-1}(B_X(a, \varepsilon)),$$

i.e., $\pi(B_\Lambda(\omega, 1/3^m)) \subseteq B_X(a, \varepsilon)$ and, consequently, π is continuous.

(ii) and (iii), i.e., the fact that $\pi(\Lambda) = A$ and the fact that π is onto, in the case when $A = \bigcup_{i \in I} f_i(A)$, both follow from (3).

(7) Let us note that for $\alpha \in \Lambda_m$ and $\omega = F_i(\alpha) = i\alpha \in \Lambda_{m+1}$, where $i \in I$, we have

$$A_\omega = A_{F_i(\alpha)} = f_{F_i(\alpha)}(A) = f_{i\alpha}(A) = (f_i \circ f_\alpha)(A) = f_i(A_\alpha).$$

For $\alpha \in \Lambda$ and $i \in I$, $(\pi \circ F_i)(\alpha) = a_{F_i(\alpha)}$ is the unique element of the set $\bigcap_{m \in \mathbb{N}^*} \overline{A_{[F_i(\alpha)]_m}}$. Hence, in order to prove that $(\pi \circ F_i)(\alpha) = (f_i \circ \pi)(\alpha)$, it is enough to check that $f_i(\pi(\alpha)) \in \overline{A_{[F_i(\alpha)]_m}}$, for every $m \in \mathbb{N}^*$. We have

$$\{a_\alpha\} = \{\pi(\alpha)\} = \bigcap_{m \in \mathbb{N}^*} \overline{A_{[\alpha]_m}}.$$

Therefore, for $m = 1$ we get $\pi(\alpha) \in A = \overline{A}$ and, consequently,

$$f_i(\pi(\alpha)) \in f_i(A) = A_i = A_{[F_i(\alpha)]_1} \subseteq \overline{A_{[F_i(\alpha)]_1}}.$$

In general, from $\pi(\alpha) \in \overline{A_{[\alpha]_{m-1}}}$ we get

$$f_i(\pi(\alpha)) \in f_i(\overline{A_{[\alpha]_{m-1}}}) \subseteq \overline{f_i(A_{[\alpha]_{m-1}})} = \overline{A_{F_i([\alpha]_{m-1})}} = \overline{A_{[F_i(\alpha)]_m}}.$$

This completes the proof. \square

Definition 4.1. The function $\pi : \Lambda \rightarrow A$ from Theorem 4.1 (6) is called the canonical projection from the shift space on the attractor of the IIFS.

5. SUFFICIENT CONDITIONS FOR THE CANONICAL PROJECTION FROM THE SHIFT SPACE ON THE ATTRACTOR OF THE IIFS, TO BE ONTO

In this section we provide sufficient conditions for π to be onto.

PROPOSITION 5.1. *Let (X, d) be a complete metric space, $S = (X, (f_i)_{i \in I})$ an IIFS and $A = A(S)$ its attractor. Then $A = \bigcup_{i \in I} f_i(A)$ if and only if the canonical projection $\pi : \Lambda \rightarrow A$ is onto. In such a case*

$$A_\alpha = \bigcup_{\omega \in \Lambda} \{a_{\alpha\omega}\} = \bigcup_{\omega \in \Lambda_m} A_{\alpha\omega},$$

for every $\alpha \in \Lambda^*$ and every $m \in \mathbb{N}^*$.

Proof. It was proved in Theorem 4.1 that if $A = \bigcup_{i \in I} f_i(A)$, then π is onto. If π is onto then $\pi(\Lambda) = A$. Hence, by Theorem 4.1 (7), $f_i(A) = f_i(\pi(\Lambda)) = \pi(F_i(\Lambda))$, so

$$\bigcup_{i \in I} f_i(A) = \bigcup_{i \in I} \pi(F_i(\Lambda)) = \pi\left(\bigcup_{i \in I} F_i(\Lambda)\right) = \pi(\Lambda) = A.$$

For $\alpha \in \Lambda^*$ we have

$$\begin{aligned} A_\alpha &= f_\alpha(A) = f_\alpha(\pi(\Lambda)) = \pi(F_\alpha(\Lambda)) = \\ &= \pi\left(F_\alpha\left(\bigcup_{\omega \in \Lambda} \{\omega\}\right)\right) = \pi\left(\bigcup_{\omega \in \Lambda} \{\alpha\omega\}\right) = \bigcup_{\omega \in \Lambda} \{a_{\alpha\omega}\} \end{aligned}$$

and

$$\begin{aligned} A_\alpha &= \pi(F_\alpha(\Lambda)) = \pi\left(F_\alpha\left(\bigcup_{\omega \in \Lambda_m} \Lambda_\omega\right)\right) = \pi\left(\bigcup_{\omega \in \Lambda_m} F_\alpha(\Lambda_\omega)\right) = \\ &= \pi\left(\bigcup_{\omega \in \Lambda_m} F_{\alpha\omega}(\Lambda)\right) = \bigcup_{\omega \in \Lambda_m} \pi(F_{\alpha\omega}(\Lambda)) = \bigcup_{\omega \in \Lambda_m} f_{\alpha\omega}(\pi(\Lambda)) = \\ &= \bigcup_{\omega \in \Lambda_m} f_{\alpha\omega}(A) = \bigcup_{\omega \in \Lambda_m} A_{\alpha\omega}. \quad \square \end{aligned}$$

PROPOSITION 5.2. *Let (X, d) be a complete metric space, $S = (X, (f_i)_{i \in I})$ an IIFS and $A = A(S)$ its attractor. If $\overline{A_\omega} = \bigcup_{i \in I} \overline{f_\omega(A_i)}$ for every $\omega \in \Lambda^*$, then the canonical projection $\pi : \Lambda \rightarrow A$ is onto. In such a case*

$$\overline{A_\omega} = A_\omega = \bigcup_{\alpha \in \Lambda} \{a_{\omega\alpha}\} = \bigcup_{\alpha \in \Lambda_m} A_{\omega\alpha},$$

for every $\omega \in \Lambda^*$.

Proof. For $a \in A$, since $A = \bigcup_{i \in I} \overline{A_i}$ and $\overline{A_\omega} = \bigcup_{i \in I} \overline{f_\omega(A_i)} = \bigcup_{i \in I} \overline{A_{\omega i}}$, for every $\omega \in \Lambda^*$, there exists a sequence $(\omega^m)_m$ with the following properties:

- (i) $\omega^m \in \Lambda_m$;
- (ii) $[\omega^{m+1}]_m = \omega^m$, and
- (iii) $a \in \overline{A_{\omega^m}}$, for all m .

If $\omega^1 = \omega_1$ and ω_m is defined by $\omega^m = \omega^{m-1}\omega_m$, then $a = a_\omega \in \pi(\Lambda)$, where $\omega = \omega_1\omega_2 \dots \omega_m\omega_{m+1} \dots \in \Lambda$. Hence $A = \pi(\Lambda)$, i.e., π is onto.

In a similar manner, if $a \in \overline{A_\alpha}$, where $\alpha \in \Lambda^*$, there exists a sequence $(\omega^m)_m$ with the following properties:

- (i) $\omega^m \in \Lambda_m$;
- (ii) $[\omega^{m+1}]_m = \omega^m$, and
- (iii) $a \in \overline{A_{\alpha\omega^m}}$, for all m .

If $\omega^1 = \omega_1$ and ω_m is defined by $\omega^m = \omega^{m-1}\omega_m$, then $a = a_\omega \in \pi(\Lambda_\alpha) = A_\alpha$, where $\omega = \alpha\omega_1\omega_2 \dots \omega_m\omega_{m+1} \dots \in \Lambda$. Hence $\overline{A_\omega} = A_\omega$ for every $\omega \in \Lambda^*$. \square

6. TWO RESULTS CONCERNING THE STRUCTURE OF $\overline{A_\omega}$

We start with

THEOREM 6.1. *Let (X, d) be a complete metric space, $S = (X, (f_i)_{i \in I})$ an IIFS and $A = A(S)$ its attractor. Assume that for every $\omega \in \Lambda^*$ there*

exist $\varepsilon_\omega > 0$ and $n_\omega \in \mathbb{N}^*$ such that for any different subscripts $i_1, i_2, \dots, i_{n_\omega}$ and any $x_1 \in A_{\omega i_1}, x_2 \in A_{\omega i_2}, \dots, x_{n_\omega} \in A_{\omega i_{n_\omega}}$, we have

$$\max\{d(x_i, x_j) \mid i, j \in \{1, 2, \dots, n_\omega\}\} \geq \varepsilon_\omega.$$

Hence $\overline{A_\omega} = \bigcup_{i \in I} \overline{A_{\omega i}}$ for every $\omega \in \Lambda^*$.

Proof. Let us consider a fixed $\omega \in \Lambda^*$. For $a \in \overline{A_\omega}$, since $\overline{A_\omega} = \overline{f_\omega(A)} = \overline{f_\omega\left(\bigcup_{i \in I} f_i(A)\right)} = \overline{f_\omega\left(\bigcup_{i \in I} f_i(A)\right)} = \bigcup_{i \in I} \overline{A_{\omega i}}$, there exists a sequence $(x_n)_n$ with the following properties:

- (i) for every $n \in \mathbb{N}^*$ there exists $i_n \in I$ such that $x_n \in A_{\omega i_n}$, and
- (ii) $\lim_{n \rightarrow \infty} x_n = a$.

We can suppose that $d(x_n, a) < \frac{\varepsilon_\omega}{2}$, for every $n \in \mathbb{N}^*$. Hence $d(x_n, x_m) < \varepsilon_\omega$, for every $n, m \in \mathbb{N}^*$. Since for any different subscripts $i_1, i_2, \dots, i_{n_\omega}$ and any $x_1 \in A_{\omega i_1}, x_2 \in A_{\omega i_2}, \dots, x_{n_\omega} \in A_{\omega i_{n_\omega}}$, we have $\max\{d(x_i, x_j) \mid i, j \in \{1, 2, \dots, n_\omega\}\} \geq \varepsilon_\omega$, the set $\{i_1, i_2, \dots, i_{n_\omega}, \dots\}$ is finite. So, there exists $i^* \in I$ such that a is the limit of a sequence whose elements are from $A_{\omega i^*}$. Therefore, $a \in \overline{A_{\omega i^*}} \subseteq \bigcup_{i \in I} \overline{A_{\omega i}}$. Consequently, $\overline{A_\omega} \subseteq \bigcup_{i \in I} \overline{A_{\omega i}}$. Since it is obvious that $A_{\omega i} \subseteq \overline{A_\omega}$ for all $i \in I$, we deduce that

$$\overline{A_\omega} = \bigcup_{i \in I} \overline{A_{\omega i}}. \quad \square$$

Definition 6.1. A function $f : X \rightarrow X$, where is (X, d) a metric space, is called bi-Lipschitz if there exist $\alpha, \beta \in (0, \infty)$ such that $\alpha \leq \beta$ and

$$\alpha d(x, y) \leq d(f(x), f(y)) \leq \beta d(x, y)$$

for every $x, y \in X$. A family of functions $(f_i)_{i \in I}$, where $f_i : X \rightarrow X$, is called bi-Lipschitz if there exist $\alpha, \beta \in (0, \infty)$ such that $\alpha \leq \beta$ and

$$\alpha d(x, y) \leq d(f_i(x), f_i(y)) \leq \beta d(x, y)$$

for every $x, y \in X$ and every $i \in I$.

THEOREM 6.2. *Let (X, d) be a complete metric space, $S = (X, (f_i)_{i \in I})$ an IIFS such that the family $(f_i)_{i \in I}$ is bi-Lipschitz and $A = A(S)$ the attractor of S . Assume that there exist $\varepsilon > 0$ and $n \in \mathbb{N}^*$ such that for any different subscripts i_1, i_2, \dots, i_n and any $y_1 \in A_{i_1}, y_2 \in A_{i_2}, \dots, y_n \in A_{i_n}$, we have*

$$\max\{d(y_k, y_l) \mid k, l \in \{1, 2, \dots, n\}\} \geq \varepsilon.$$

Then $\overline{A_\omega} = \bigcup_{i \in I} \overline{A_{\omega i}}$ for every $\omega \in \Lambda^*$.

Proof. Let us consider $\alpha, \beta \in (0, \infty)$ such that $\alpha \leq \beta$ and

$$\alpha d(x, y) \leq d(f_i(x), f_i(y)) \leq \beta d(x, y)$$

for any $x, y \in X$ and $i \in I$. Then for $\omega \in \Lambda^*$ we have

$$\alpha^{|\omega|}d(x, y) \leq d(f_\omega(x), f_\omega(y)) \leq \beta^{|\omega|}d(x, y),$$

for every $x, y \in X$ and every $i \in I$.

Let us consider different subscripts i_1, i_2, \dots, i_n and $x_1 \in A_{\omega_{i_1}}, x_2 \in A_{\omega_{i_2}}, \dots, x_n \in A_{\omega_{i_n}}$. Then there exist $y_1 \in A_{i_1}, y_2 \in A_{i_2}, \dots, y_n \in A_{i_n}$ such that $x_1 = f_\omega(y_1), x_2 = f_\omega(y_2), \dots, x_n = f_\omega(y_n)$, for which

$$\begin{aligned} \max\{d(x_k, x_l) \mid k, l \in \{1, 2, \dots, n\}\} &= \max\{d(f_\omega(y_k), f_\omega(y_l)) \mid k, l \in \{1, 2, \dots, n\}\} \geq \\ &\geq \alpha^{|\omega|} \max\{d(y_k, y_l) \mid k, l \in \{1, 2, \dots, n\}\} \geq \alpha^{|\omega|}\varepsilon. \end{aligned}$$

It now follows from Theorem 6.1 that

$$\overline{A_\omega} = \bigcup_{i \in I} \overline{A_{\omega_i}},$$

for every $\omega \in \Lambda^*$. \square

COROLLARY 6.1. *Let (X, d) be a complete metric space, $S = (X, (f_i)_{i \in I})$ an IIFS such that the family $(f_i)_{i \in I}$ is bi-Lipschitz and $A = A(S)$ the attractor of S . Assume that there exist $\varepsilon > 0$ and $n \in \mathbb{N}^*$ such that for any different subscripts i_1, i_2, \dots, i_n and any $y_1 \in A_{i_1}, y_2 \in A_{i_2}, \dots, y_n \in A_{i_n}$ we have*

$$\max\{d(y_k, y_l) \mid k, l \in \{1, 2, \dots, n\}\} \geq \varepsilon.$$

Then $A = \bigcup_{i \in I} f_i(A)$.

Proof. This follows from Theorem 6.2, Proposition 5.2 and Proposition 5.1. \square

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