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# THE SHORT-WAVE LIMIT FOR NONLINEAR, SYMMETRIC, HYPERBOLIC SYSTEMS

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**Abstract.** For general semilinear or quasilinear symmetric, hyperbolic systems, the short-wave approximation is studied by constructing approximate solutions to the initial-value problem associated with the hyperbolic operator by means of a multiscale WKB expansion. In the diffractive optics regime, the components of the leading-order terms of the approximate solutions are shown to satisfy differential equations of KP type. Whether these equations are scalar or not depends on the polarization of the initial datum and on the asymptotic behavior of the branches of the characteristic variety of the hyperbolic operator. The asymptotic stability of the approximate solution is proved. Following J.-L. Joly, G. Métivier, and J. Rauch, the cornerstone of this study is the characteristic variety of the hyperbolic operator. It is examined here in terms of perturbation theory, which yields new proofs of the so-called algebraic lemmas of geometric optics.

## 1. Introduction

The propagation of short waves (highly oscillating approximate solutions) in nonlinear dispersive systems has been studied by M.A. Manna et al. [18], [27] under the assumption that the dispersion relation has an expansion in the limit $k \to \infty$ as

$$\omega(k) \sim \omega_0 k + \frac{\omega_1}{k} + \cdots . \tag{1.1}$$

In the context of two-dimensional water waves with a surface wind [26], Manna derived a system which satisfies (1.1) and an equation for the velocity $u(t,x)$ of the short waves that reads

$$\partial_t \, \partial_x u + \frac{3g}{c_0 h} u = -u \, \partial_x^2 u + (\partial_x u)^2, \tag{1.2}$$

where $g$ is the acceleration of gravity, $h$ is the unperturbed initial depth and $c_0$ is the velocity of the surface wind. Note that the nonlinear term cannot be written in a conservative form (i.e., is not a derivative).

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It is shown in this paper that the propagation of short waves is a general phenomenon occurring in the context of nonlinear, dispersive systems governed by conservative, symmetric, hyperbolic operators. Moreover, it is shown that short waves do generically not propagate in systems which do not satisfy (1.1). The propagation is one-dimensional, transverse perturbative effects are taken into account, and nonlinearities of semilinear nature as well as nonlinearities of quasilinear nature are considered. A nonlinear equation (3.2) similar to (1.2) is derived for the leading term of the approximation. Three different regimes are distinguished; they have simple geometric characterizations. In the case of a quasilinear system, one does *not* assume a vanishing mean (equivalently, zero mass) condition for the initial datum in order to solve the Cauchy problem associated with (3.2).

More precisely, the purpose of Section 3 is the description of approximate solutions of the initial-value problem

$$\left\{ \begin{array}{rcl} L(\partial)u & = & \text{nonlinear } (u) \\ u(0) & = & \varepsilon\, u^0(x/\varepsilon, y), \end{array} \right. \tag{1.3}$$

where $L$ is a differential operator in the space variables $x$ and $y$ which is hyperbolic with respect to time, where the nonlinear term is quasilinear (involving a derivative with respect to $x$ only) or semilinear, and where $\varepsilon$ is a small parameter. Further details are given in the introduction of Section 3. This study is carried out using the techniques of diffractive geometric optics of J.-L. Joly, G. Métivier, and J. Rauch [12], [15]. The approximate solution $u^\varepsilon$ of (1.3), where $\varepsilon$ is linked to the small wavelength of the wave, is sought in the form of a WKB expansion

$$u^\varepsilon \sim \varepsilon \sum_j \varepsilon^j \mathbf{u}_j,$$

where the $\mathbf{u}_j$ are referred to as *profiles*, i.e., functions fitting the adequate ansatz, which is here

$$\mathbf{u}(t/\varepsilon, x/\varepsilon, t, y, \varepsilon t).$$

A justification for this ansatz is given in Section 3.1. The algebraic part of this analysis consists in the derivation of the equations satisfied by the profiles. To describe the differential operators arising in these equations is the purpose of the so-called *algebraic lemmas*. The point we make in Section 2 is that there is a way to establish all these algebraic lemmas in a considerably easier fashion than in earlier papers (among which are [12], [15], [19], [6], [7], and [10]), namely by looking at the local study of the characteristic variety of the hyperbolic operator in terms of perturbation theory. The ideas in Section 2 have been known since T. Kato [17] and J.B. Butler [8] at least. Kato noticed that continuous differentiability of the eigenvalues of a family of square matrices depending on a complex parameter was linked to semisimplicity, and Butler stated a theorem dealing with the regularity of the spectrum of a certain type of perturbation of square matrices. The discussion of Section 2 is merely a matter of clarification: it shows that these ideas contain the algebraic lemmas of geometric and diffractive optics.

The analytical part which aims at stability Theorems 3.18, 3.21, 3.26, and 3.27, in particular the derivation of a scalar equation, the treatment of the nonlinearities, and the construction and control of the corrector terms of the approximate solution, rely on averaging projectors [19], low-frequency truncations [2], [3], [4], and estimates in a class of Banach spaces introduced in [21] (see Definition 3.2). Further details are given in the introduction of Section 3.

Equation (3.2) is similar to (but different from) the diffractive pulse equation of D. Alterman and J. Rauch describing ultra-short laser pulses [2], [3], [4]. They were followed by K. Barrailh and D. Lannes [5], who derived a nonlocal differential equation generalizing the diffractive pulse equation in the dispersive case for continuous oscillating spectra.

## 2. The algebraic lemmas of diffractive geometric optics via perturbation theory

Let $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, and consider

$$L(\varepsilon\partial) = \varepsilon\partial_t + \sum_{j=1}^{d} A_j \, \varepsilon\partial_{x_j} + L_0 = \varepsilon\partial_t + A(\varepsilon\partial_x) + L_0, \tag{2.1}$$

a hyperbolic differential operator with constant coefficients, the $A_j$'s being $n \times n$, complex, Hermitian matrices for all $j$. We do *not* suppose that $L_0$ is skew-Hermitian. The formal derivation of the ansatz motivates the following classical

**Definition 2.1** (characteristic variety)**.** Introduce Char $L$, the characteristic variety of a hyperbolic operator $L$, as the set of all $(\tau, \eta) \in \mathbb{C} \times \mathbb{C}^d$ solutions of the characteristic equation

$$\det\,(\tau + A(\eta) + L_0/i) = 0.$$

The polynomial defining the characteristic variety is called the characteristic polynomial. The polynomial defined by the product of the irreducible factors of the characteristic polynomial is called the reduced characteristic polynomial.

The characteristic variety is the collection of the spectra of the matrices $-A(\eta) - L_0/i$ for all $\eta$ in $\mathbb{C}^d$. Earlier papers (among which are [12], [15], and [19]) considered the characteristic variety restricted to real frequencies $\eta \in \mathbb{R}^d$ :

**Definition 2.2.** Denote by $\mathrm{Char}_{\mathbb{R}}\, L$ the restriction of $\mathrm{Char}\, L$ to real $\eta \in \mathbb{R}^d$. $\mathrm{Char}_{\mathbb{R}}\, L$ is called the real characteristic variety.

When $L_0$ is skew-Hermitian, $\mathrm{Char}_{\mathbb{R}}\, L$ is a real analytic variety.

**Notation.** For $\beta = (\tau, \eta) \in \mathbb{C} \times \mathbb{C}^d$, set $L(\beta) = \tau + A(\eta) + L_0/i$. For $\beta \notin \mathrm{Char}\, L$, $L(\beta)^{-1}$ is the inverse of $L(\beta)$; for $\beta \in \mathrm{Char}\, L$ and such that 0 is a semisimple eigenvalue of $L(\beta)$, one denotes by $\pi(\beta)$ the linear projection onto the kernel of $L(\beta)$ along its range and by $L(\beta)^{-1}$ the partial inverse of $L(\beta)$ defined by $\pi(\beta)L(\beta)^{-1} = 0$, $L(\beta)L(\beta)^{-1} = 1 - \pi(\beta)$.

It is a well-known fact that the characteristic variety cannot in general be parametrized by smooth functions. In this respect, recall the usual

**Definition 2.3** (regular and singular points)**.** A point $\beta$ on $\mathrm{Char}L$ is said to be regular when $\mathrm{Char}L$ is a smooth manifold in a neighborhood of $\beta$. A point which is not regular is called singular.

Similarly, a point $\beta$ on $\mathrm{Char}_{\mathbb{R}}\, L$ is said to be regular when $\mathrm{Char}_{\mathbb{R}}\, L$ is a smooth manifold in a real neighborhood of $\beta$, and singular otherwise.

Around a regular point, $\mathrm{Char}L$ is a graph. It is natural to ask whether the frequency $\eta$ can parametrize the variety at a regular point. In this respect, consider the projection

$$p : \left\{ \begin{array}{ccc} \mathrm{Char}\ L & \to & \mathbb{C}^d \\ (\tau, \eta) & \to & \eta, \end{array} \right.$$

and set the

**Definition 2.4** (critical and noncritical points)**.** $\beta = (\tau, \eta) \in \mathrm{C}har\, L$ is called noncritical when $p$ is a local homeomorphism around $\beta$. The points around which $p$ is not a local homeomorphism are called critical points.

One defines similarly the (non)critical points of $\mathrm{Char}_{\mathbb{R}}\, L$ using $p_{\mathbb{R}}$, the projection over the space of frequencies $\eta \in \mathbb{R}^d$.

A regular point may be critical, as shown by the example $X = \{(x, y) \in \mathbb{C}^2, y^2 - x = 0\}$. Note that any projection $(x, y) \mapsto ax + by$ restricted to $X$ is injective around $(0, 0)$ if $b \neq 0$. In this example, the critical point $(0, 0)$ is thus critical only for the natural projection $(x, y) \mapsto x$.

The link between regularity and criticality is established for hyperbolic polynomials (in our context, for skew-Hermitian $L_0$): J. Rauch proved [32] that the regular points of $\mathrm{Char}_{\mathbb{R}}\, L$ are the noncritical points of $\mathrm{Char}_{\mathbb{R}}\, L$.

Let $\beta = (\tau, \eta) \in \mathrm{Char}\ L$ be a zero of order $s$ with respect to $\tau$ of the reduced characteristic polynomial. Then Rouché's theorem implies that for sufficiently small neighborhoods $U$ of $\tau$ in $\mathbb{C}$, there exists a neighborhood $V$ of $\eta$ in $\mathbb{C}^d$ such that for all $\eta' \in V$, there exist $s$ eigenvalues (counted with their multiplicities) of $-A(\eta') - L_0/i$ in $U$, and the sum of these eigenvalues is holomorphic. Rouché's theorem therefore implies the following facts:

- At a noncritical point $\beta$ the mapping $\eta \mapsto \tau(\eta)$ given by definition 2.4 is holomorphic. The projector $\eta \mapsto \pi(\tau(\eta), \eta)$ is holomorphic as well.
- If $(\tau, \eta)$ is a zero of multiplicity $s > 1$ with respect to $\tau$ of the reduced characteristic polynomial, then $(\tau, \eta)$ is critical.

**Remark 2.5.** 1. Algebraic lemmas such as Propositions 3.1 and 3.2 of [12] are expressed at noncritical points of the real characteristic variety, whereas Proposition 2.6 below concerns noncritical points of the complex characteristic variety. For homogeneous hyperbolic polynomials, G. Métivier and J. Rauch proved [29] that these notions are equivalent for the real points of the variety. Precisely, [29] implies that when $L_0$ is skew-Hermitian, a real point $(\tau, \eta) \in \mathbb{R}^{1+d}$ on Char $L$ is noncritical in the real sense (that is, for $\mathrm{Char}_{\mathbb{R}}\, L$) if and only if it is noncritical in the complex sense (that is, for Char $L$).

2. Regular points of the characteristic variety in the sense of Definition 2.3 are often called smooth points in the literature, and the adjective "regular" is reserved for the points at which the gradient of the reduced characteristic polynomial does not vanish. These two notions are equivalent for complex algebraic varieties but not for real varieties [30]. It is a consequence of [29] that these notions are equivalent for reduced homogeneous hyperbolic polynomials as well.

3. In many physical examples—the Maxwell-Lorentz model, the anharmonic oscillator model, the Maxwell-Bloch equations, the equations of ferromagnetism—see Section 2.3—the critical points of the real characteristic varieties are isolated and located above the zero frequency.

We now briefly recall the context of linear, diffractive, geometric optics for one plane phase [12], [15]. The purpose is the description of high-frequency solutions of the linear partial differential equation

$$\begin{cases} L(\varepsilon\,\partial)u = 0, \\ u(t=0) = a(x)\, e^{i(\eta\cdot x)/\varepsilon}, \end{cases} \tag{2.2}$$

where $\varepsilon$ is a small parameter. The initial condition is supposed to oscillate with a phase $\eta$. Set $\beta = (\tau, \eta) \in$ Char $L$ and $L_1(\partial_t, \partial_x) = \partial_t + A(\partial_x)$.

The solutions are sought in the form

$$u^\varepsilon = \sum_j \varepsilon^j \, e^{i\beta\cdot(t,x)/\varepsilon} \, [\mathbf{u}_j(t,x,T,X)]_{T=\varepsilon\, t,\, X=\varepsilon\, x}, \tag{2.3}$$

where the *profiles* $\mathbf{u}_j$ depend on the variables $t$ and $x$, and $T = \varepsilon\, t$ and $X = \varepsilon\, x$. The profile equations for the leading term $\mathbf{u}_0$ are

$$\left\{ \begin{array}{l} \pi(\beta)L_1(\partial_t, \partial_x)\pi(\beta)\mathbf{u}_0 = 0 \\ \pi(\beta)L_1(\partial_T, \partial_X)\pi(\beta)\mathbf{u}_0 + i\,\pi(\beta)A(\partial_x)L(\beta)^{-1}A(\partial_x)\pi(\beta)\mathbf{u}_0 = 0. \end{array} \right. \tag{2.4}$$

The first equation describes the leading term of the approximate solution for times $O(1)$ (geometric optics). The second equation describes its behavior for long times $O(1/\varepsilon)$ (diffractive optics). Lax [22] proved that in the noncritical case (that is, for $\beta$ a noncritical point of Char $L$) the geometric-optics approximation of (2.2) is a transport equation; Donnat, Joly, Métivier, and Rauch proved [12] that the diffractive-optics approximation of (2.2) is a Schrödinger equation. The transport in the variable $t$, $x$ occurs at the *group* velocity, whence the denomination of *tangent* operator. Lannes proved in [19] that such a denomination was still relevant in the critical case.

We propose here new proofs of these facts in Sections 2.1 and 2.2. Applications to physical systems follow in Section 2.3.

2.1. **The noncritical case.** Let $\beta = (\tau, \eta)$ be a noncritical point on Char $L$ such that 0 is a semisimple eigenvalue of $L(\beta)$. There are holomorphic maps $\eta' \mapsto \tau(\eta')$, $\eta' \mapsto N(\tau(\eta'), \eta')$, $\eta' \mapsto \pi(\tau(\eta'), \eta')$, and $\eta' \mapsto M(\eta')$ defined on a neighborhood of $\eta$ so that the spectral representation of $A(\eta') + L_0/i$ reads

$$A(\eta') + \frac{L_0}{i} = -(\tau(\eta') + N(\tau(\eta'), \eta'))\, \pi(\tau(\eta'), \eta') - M(\eta')(1 - \pi(\tau(\eta'), \eta')), \tag{2.5}$$

where $N(\tau(\eta'), \eta')$ is nilpotent with $N(\beta) = 0$, and where $\pi(\tau(\eta'), \eta')$ is a projector. One sets $\beta' = (\tau(\eta'), \eta')$ for $\eta'$ in a neighborhood of $\eta$.

**Proposition 2.6** (Donnat-Joly-Métivier-Rauch [12], Propositions 3.1 and 3.2)**.** *Let $\beta = (\tau, \eta)$ be a noncritical point on Char$L$. Suppose that $L(\beta) \neq 0$ and that 0 is a semisimple eigenvalue of $L(\beta)$. Then the spectrum of $\pi(\beta)A(h)\pi(\beta)$ is $\{0, -\tau'(\eta)\cdot h\}$. If moreover $-\tau'(\eta)\cdot h$ is a semisimple eigenvalue of $\pi(\beta)A(h)\pi(\beta)$, then*

$$\pi(\beta)L_1(\partial_t, \partial_x)\pi(\beta) = (\partial_t - \tau'(\eta)\cdot\partial_x)\pi(\beta), \tag{2.6}$$

*and the unique nonvanishing eigenvalue of*

$$\pi(\beta)A(h)L(\beta)^{-1}A(h)\pi(\beta)$$

*is* $\tau''(\eta)(h,h)/2$. *If moreover this eigenvalue is semisimple, then*

$$\pi(\beta)A(\partial_x)L(\beta)^{-1}A(\partial_x)\pi(\beta) = \frac{1}{2}\tau''(\eta)(\partial_x,\partial_x)\,\pi(\beta). \tag{2.7}$$

**New Proof.** This is a local study in a neighborhood of $\eta \in \mathbb{C}^d$. The proof consists in the computation by two different means of the integral

$$\frac{1}{2i\pi}\int_{\mathcal{C}}(z-\tau)\;L(z,\eta')^{-1}dz,$$

where $\eta'$ is in a neighborhood of $\eta$ and where $\mathcal{C}$ is a small circle around $\tau$, containing in its interior no other eigenvalue of $-A(\eta) - L_0/i$ than $\tau$. In the light of (2.5), the Laurent series for $L(z,\eta')^{-1}$ is

$$\begin{aligned} L(z,\eta')^{-1} &= \frac{F(z,\eta')}{(z-\tau(\eta'))^2} + \frac{N(\beta')}{(z-\tau(\eta'))^2} + \frac{\pi(\beta')}{z-\tau(\eta')} \\ &+ L(\beta')^{-1} - L(\beta')^{-2}\,(z-\tau(\eta')) + O(|z-\tau(\eta')|^2), \end{aligned}$$

where for all $\eta'$ in a neighborhood of $\eta$, $\tau(\eta')$ is a pole of the meromorphic map $F(\cdot,\eta')$. On the one hand, compute the residues to find

$$\begin{aligned} &\frac{1}{2i\pi}\int_{C(\tau,r)}(z-\tau)\;L(z,\eta+h)^{-1}dz \\ &= (\tau(\eta+h)-\tau+N(\tau(\eta+h),\eta+h))\,\pi(\tau(\eta+h),\eta+h). \end{aligned} \tag{2.8}$$

On the other hand, for small $h$, expand $L(z,\eta+h)^{-1}$ in powers of $h$:

$$\begin{aligned} L(z,\eta+h)^{-1} &= L(z,\eta)^{-1} - L(z,\eta)^{-1}\,A(h)\,L(z,\eta)^{-1} \\ &+ L(z,\eta)^{-1}A(h)\,L(z,\eta)^{-1}A(h)\,L(z,\eta)^{-1} + O(|h|^3). \end{aligned}$$

Again, compute the residues to find

$$\frac{1}{2i\pi}\int_{\mathcal{C}}(z-\tau)\;L(z,\eta)^{-1}dz = N(\beta)\pi(\beta) = 0,$$

$$\frac{1}{2i\pi}\int_{\mathcal{C}}(z-\tau)\;L(z,\eta)^{-1}\,A(h)\,L(z,\eta)^{-1}dz = \pi(\beta)A(h)\pi(\beta),$$

and

$$\begin{aligned} &\frac{1}{2i\pi}\int_{\mathcal{C}}(z-\tau)L(z,\eta)^{-1}A(h)L(z,\eta)^{-1}A(h)L(z,\eta)^{-1}dz \\ &= \pi(\beta)\,A(h)\,L(\beta)^{-1}\,A(h)\,\pi(\beta) + \pi(\beta)\,A(h)\,\pi(\beta)\,A(h)\,L(\beta)^{-1} \\ &\quad + L(\beta)^{-1}A(h)\,\pi(\beta)\,A(h)\,\pi(\beta). \end{aligned}$$

Equate the expansions in powers of $h$ in both sides of (2.8) to get at order one:

$$(\tau'(\eta)\cdot h + N'(\beta)\cdot h)\pi(\beta) = \pi(\beta)A(h)\pi(\beta).$$

The semisimplicity assumption at order 1 implies $N'(\beta) = 0$, and (2.6) is proved. At order two,

$$\begin{aligned}&\frac{1}{2}(\tau''(\eta)(h,h) + N''(\beta)(h,h))\pi(\beta) + (\tau'(\eta)\cdot h)\pi'(\beta)\cdot h\\ &= \pi(\beta)A(h)L(\beta)^{-1}A(h)\pi(\beta) + \pi(\beta)A(h)\pi(\beta)A(h)L(\beta)^{-1}\\ &+ L(\beta)^{-1}A(h)\pi(\beta)A(h)\pi(\beta).\end{aligned} \tag{2.9}$$

$\pi$ being a projector, $\pi(\beta)\pi'(\beta)\pi(\beta)$ vanishes identically. Multiply then both sides of the previous equality by $\pi(\beta)$ (an operation that will be referred to hereafter as a *polarization*) to get, under the semisimplicity assumption at order 2,

$$\frac{1}{2}\tau''(\eta)(h,h)\pi(\beta) = \pi(\beta)A(h)L^{-1}(\beta)A(h)\pi(\beta).$$

**Remark 2.7.** a) This proof also yields the expression of the derivative of $\pi$ at $\eta$. With (2.6), (2.7), and (2.9), one sees indeed that

$$\pi'(\beta)\,h = -L^{-1}(\beta)A(h)\pi(\beta) - \pi(\beta)A(h)L^{-1}(\beta).$$

This could have been found by writing the Cauchy integral representation for the projector alone,

$$\pi(\tau(\eta+h),\eta+h) = \frac{1}{2i\pi}\int_{\mathcal{C}} L(z,\eta+h)^{-1}dz, \tag{2.10}$$

and by computing the right-hand side in the same fashion.

b) In the above proof, the expansion of both sides of (2.8) can be carried to arbitrary order. Keeping only polarized terms, the expansion at order 3 reads, for a conservative system (i.e., such that $L_0$ is skew-Hermitian),

$$\begin{aligned}\frac{1}{6}\tau'''(\eta)\pi(\beta) + \frac{1}{2}\tau'(\eta)\pi(\beta)\pi''(\beta)\pi(\beta) &= -\pi(\beta)AL^{-1}(\beta)AL^{-1}(\beta)A\pi(\beta)\\ + \pi(\beta)A\pi(\beta)AL^{-2}(\beta)A\pi(\beta) &+ \pi(\beta)AL^{-2}(\beta)A\pi(\beta)A\pi(\beta).\end{aligned} \tag{2.11}$$

Computing the second derivative of the projector by (2.10), one finds

$$\frac{1}{2}\pi(\beta)\pi''(\beta)\pi(\beta) = -\pi(\beta)AL^{-2}(\beta)A\pi(\beta), \tag{2.12}$$

and (2.11) becomes

$$\frac{1}{6}\tau'''(\eta)\pi(\beta) - \frac{1}{2}\tau'(\eta)\pi(\beta)\pi''(\beta)\pi(\beta) = -\pi(\beta)AL^{-1}(\beta)AL^{-1}(\beta)A\pi(\beta).$$

In the study of long waves, the derivations of the Korteweg-de Vries and Kadomtsev-Petviashvili equations involve profile equations with third-order terms at the critical point $(0,0)$. The above computations combined with the next proposition contains the algebraic lemmas that lead to the KdV and KP equations [6], [7].

2.2. **The critical case.** The study of critical points naturally occurs in the context of nonlinear dispersive optics [19]. The harmonics $n\beta, n \in \mathbb{Z}$, of the fundamental phase $\beta$ have to be taken into account due to the nonlinearity. For dispersive systems only a finite number of these harmonics are characteristic. Among them, the origin $(0,0)$ is a critical point. It is not restrictive to suppose that $(0,0)$ is the unique point of Char $L$ above $\eta = 0$.

We study directional derivatives of the eigenvalues: this is a one-dimensional study. Fix a direction $\eta \in \mathbb{C}^d$, and consider the algebraic curve

$$X(\eta) = \{(x,z) \in \mathbb{C}^2,\ \det\,(z + A(x\eta) + L_0/i) = 0\}, \tag{2.13}$$

describing the spectrum of a perturbation of $L_0/i$ along the complex line directed by $\eta$. For all $(x,z) \in X(\eta)$, one has $(x\eta, z) \in$ Char $L$. The critical points of $X(\eta)$ are isolated. Let then $D$ be a sufficiently small disc of center 0 in the complex plane such every point of the punctured disc $D^*$ is a noncritical value. The projection

$$\begin{array}{rcl} p^{-1}(D) \cap X(\eta) & \rightarrow & D \\ (x,z) & \overset{p}{\mapsto} & x, \end{array}$$

is a finite (ramified) covering of $D$ [31]. For all $x$, the fiber is $p^{-1}(x) = \{z_1(x), \ldots, z_s(x)\}$. The $z_i$ are precisely the eigenvalues of $-A(x\eta) - L_0/i$ that vanish at $x = 0$; they are defined as multivalued functions around 0. Following Kato [17], call them the 0-group. They are single-valued and holomorphic in a neighborhood of every point of $D^*$; by the monodromy theorem, for all $i$, for all simply connected, open subsets $D_i$ of $\mathbb{C} - \{0\}$, $z_i$ is single-valued and holomorphic on $D_i$.

Now consider the restriction of $p$ to $p^{-1}(D^*) \cap X(\eta)$. It is a standard fact in the theory of covering spaces ([28], Theorem 6.6) that the number of sheets determines a connected covering of $D^*$ up to isomorphism. Let then $U$ be a connected component of $p^{-1}(D^*) \cap X(\eta)$, and let $\{z_1, \ldots, z_q\}$, $q \leq s$, be the elements of the 0-group with values in $U$. Let $p_q$ be the $q$-sheeted covering of $D^*$ defined by $p_q(x) = x^q$. There exists $\psi_U$ a holomorphic homeomorphism such that the following diagram is commutative:

$$\begin{tikzcd} U \arrow[dr, "p"'] & & p_q^{-1}(D^*) \arrow[ll, "\psi_U"'] \arrow[dl, "p_q"] \\ & D^* & \end{tikzcd}$$

The elements of the 0-group are sections of $p$ : $p \circ z_i(x) \equiv x$ on $D_i$. Consequently, for all $1 \leq i \leq q$, $\psi_U \circ z_i$ is a section of $p_q$. This means that for all

$i$, $D_i$ is a domain of determinacy of a complex $n$-th root, so that one can set $D_i = (\mathbb{C} - \mathbb{R}_-) \cap D^*$, and for all $x \in D_i$,

$$(x, z_i(x)) = \psi_U(\omega^i x^{1/q}), \tag{2.14}$$

where $\omega$ is a given primitive $q$-th root of 1. (2.14) is the standard Puiseux expansion. Via the lifting of paths, $\pi^1(D^*, x)$ acts transitively for all $x$ on the fiber $\{z_1(x), \ldots, z_q(x)\}$; hence, for all indices $1 \leq i, j \leq q$ and for all $x \in D_i$ there is a closed path $\gamma(x)$ based at $x$ such that analytic continuation of $z_i$ along $\gamma(x)$ leads to $z_j$.

**Proposition 2.8** (Lannes [19], Proposition 2)**.** *Suppose that* $(0,0)$ *is a critical point of Char* $L$ *and that* $0$ *is a semisimple eigenvalue of* $L_0$*. Denote by* $\tau_1(\eta'), \ldots, \tau_s(\eta')$ *the eigenvalues of* $-A(\eta') - L_0/i$ *that vanish at* $0$*. The* $\tau_j$*'s admit directional derivatives* $d\tau_j(0) \cdot \eta$ *at* $0$ *in all directions* $\eta \in \mathbb{C}^d$*, and the following equality holds:*

$$\operatorname{Char} \pi(0) L_1(\partial_t, \partial_x) \pi(0) = \{(d\tau_j(0) \cdot \eta, \eta) \in \mathbb{C} \times \mathbb{C}^d, 1 \leq j \leq s\}.$$

**New Proof.** Fix a vector $\eta$ in $\mathbb{C}^d$. With the notation of the above discussion, $z_j(x) = \tau_j(x\eta)$, for all $x \in D_j$, $1 \leq j \leq s$. Let $\pi_j$ be the associated eigenprojector. The residue theorem gives

$$\int_{\mathcal{C}} z\,(z - A(x\,\eta) - L_0/i)^{-1} dz = \sum_1^s (\tau_j(x\eta) + N_j(x\eta))\,\pi_j(x\eta). \tag{2.15}$$

The left-hand side of (2.15) is the same as in (2.8), whence the expansion

$$-x\,\pi(0) A(\eta) \pi(0) + o(|x|) = \sum_1^s (\tau_j(x\eta) + N_j(x\eta))\,\pi_j(x\eta).$$

Note that the constant term in the left-hand side, namely the nilpotent relative to 0 in the spectral decomposition of $L_0$, vanishes thanks to the semisimplicity hypothesis. Dividing by $x$ and equating spectra, one finds

$$\operatorname{sp}\,(-\pi(0) A(\eta) \pi(0) + o(1)) = \Big\{\frac{\tau_j(x\,\eta)}{x}, 1 \leq j \leq s\Big\}. \tag{2.16}$$

By (2.16), the application $x \mapsto \tau_i(x\,\eta)/x$ is bounded in $D_i$; hence, by (2.14), it converges as $x \to 0$, $x \in D_i$, towards the $q$-th coefficient of the power series expansion of the second component of $\psi_U$ at 0. This gives the existence of the directional derivative. Finally, (2.16) yields both inclusions of the proposition.

**Remark 2.9.** When $p^{-1}(\{0\})$ is composed of several phases $0, \beta_1, \ldots, \beta_m$, one may have to change $D^*$ to a smaller disc so that any connected component $V$ of $p^{-1}(D) \cap X(\eta)$ contains a unique point $\beta$ of $p^{-1}(\{0\})$. The closure

$\bar{U}$ in $p^{-1}(D) \cap X(\eta)$ of a connected component $U$ of $p^{-1}(D^*) \cap X(\eta)$ included in $V$ is simply $U \cup \{\beta\}$ and is called a local irreducible component of $X(\eta)$. By the multiplicity of $\bar{U}$ at $\beta$, one means the degree of the first nonvanishing homogeneous term in the Taylor expansion of its parametrization $\psi_U$ at 0. The above proof shows that in the semisimple case, the multiplicity $m$ of a local irreducible component equals the number of sheets $q$ of the projection restricted to this component. (2.14) shows that one has $q \geq m$ in general, and the inequality can be strict.

We now turn to the case of a normal perturbation, in the one-dimensional case. Consider the differential operator

$$L(\varepsilon\partial) = \varepsilon\partial_t + A\,\varepsilon\partial_x + E,$$

where $E$ is skew-Hermitian: $E + E^* = 0$. Under these hypotheses, the following theorem holds:

**Theorem 2.10** (Butler [8]). *The eigenvalues and the eigenprojectors are holomorphic at* 0.

**Proof.** We formulate Butler's proof in terms of the above discussion. Consider an orbit of the 0-group $\{z_1, \ldots, z_q\}$ and $\{\pi_1, \ldots, \pi_q\}$ the associated eigenprojectors, and suppose $q > 1$. The projectors of the 0-group are defined on Char $L$; hence, for all $x \in D^*$, $\pi_1(D^*, x)$ operates on $\{\pi_1(x), \ldots, \pi_s(x)\}$, and $\{\pi_1(x), \ldots, \pi_q(x)\}$ is an orbit. The $\pi_i$'s are meromorphic, as shown by their expressions as Lagrange interpolation polynomials,

$$\pi_i(x) = \frac{\prod_{j \neq i}(A(x) + E/i - z_j(x))}{\prod_{j \neq i}(z_i(x) - z_j(x))}\, P_i(x), \tag{2.17}$$

a matrix-valued application defined in $D_i$, for all $i$, where $P_i$ involves the eigenvalues of $A(x) + E/i$ outside the 0-group and is holomorphic around 0. For real $x$, the projectors are Hermitian by the assumption on $E$, hence bounded. It follows that 0 is not a pole and that one can define $\pi_i$ at 0 so that it is continuous in $D_i \cup \{0\}$. For $1 \leq i, j \leq q$ and $i \neq j$, let $x \in D_j$ and let $\gamma$ be a class path in $\pi^1(D^*, x)$ such that $\pi_i(x) \cdot \gamma = \pi_j(x)$. At $x = 0$, with (2.17), it yields $\pi_i(0) = \pi_j(0)$. One also has $\pi_i(x)\,\pi_j(x) = 0$ in $D_j$, and by continuity $\pi_i(0)\pi_j(0) = 0$. Hence $\pi_i(0)^2 = \pi_i(0) = 0$. But by continuity, for small $x$, $0 <$ dim Ran $\pi_i(x) =$ dim Ran $\pi_i(0)$ ([13], Lemma VII.6.7), a contradiction. Hence, $q = 1$ and by (2.14), the eigenvalues are single-valued and holomorphic. By (2.17), the projectors are single-valued and holomorphic as well.

2.3. **Applications: Maxwell equations.** For the model system of Maxwell-Lorentz and for the equations of ferromagnetism, one considers the Cauchy problems (2.2), and using the results of this section one describes their geometric and diffractive approximations (2.4).

**Maxwell-Lorentz equations.** The propagation of light in dense material can be modeled by dispersive symmetric hyperbolic systems. The Lorentz model for linear dispersion is (see [11] for instance) in dimensionless units

$$\left\{\begin{array}{rcl} \partial_t \mathcal{B} + \operatorname{curl} \mathcal{E} & = & 0 \\ \partial_t \mathcal{E} - \operatorname{curl} \mathcal{B} & = & \partial_t \mathcal{P} \\ \varepsilon^2 \partial_t^2 \mathcal{P}_L + \mathcal{P}_L & = & \gamma \mathcal{E} \end{array}\right. \tag{2.18}$$

together with the divergence equations div $(\mathcal{E}+\mathcal{P}_L) = 0$ and div $\mathcal{B} = 0$ which are propagated by (2.18). The unknowns are the electric and magnetic fields $\mathcal{E}$ and $\mathcal{B}$ and the polarization per unit volume $\mathcal{P}$, which is supposed to be linear here $\mathcal{P} = \mathcal{P}_L$. $\gamma$ is a constant of size $O(1)$ with respect to the small parameter $\varepsilon$ whose size is about one wavelength. Set $u = (\mathcal{E}, \mathcal{B}, \frac{1}{\sqrt{\gamma}}\mathcal{P}_L, \frac{\varepsilon}{\sqrt{\gamma}}\partial_t \mathcal{P}_L)$. $u$ satisfies

$$\partial_t u + \begin{pmatrix} 0 & -\operatorname{curl} & 0 & 0 \\ \operatorname{curl} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u + \frac{1}{\varepsilon} \begin{pmatrix} 0 & 0 & 0 & \sqrt{\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -\sqrt{\gamma} & 0 & 1 & 0 \end{pmatrix} u = 0. \tag{2.19}$$

Examine the Cauchy problem (2.2) for the system of Maxwell-Lorentz in the form (2.19). The approximate solution $\mathbf{u}_0$ follows the ansatz (2.3). Denote by $T = \varepsilon\, t$ and $X = \varepsilon\, x$ the diffractive variables. The equation defining the characteristic variety is

$$\tau^2(\tau^2 - 1 - \gamma)((\tau^2 - 1)(\tau^2 - |\eta|^2) - \gamma \tau^2)^2 = 0.$$

The critical points are located above the 0 frequency. We call unbounded branches the branches of the variety parametrized by a map $\omega$ which is unbounded in the limit $\xi \to \infty$. By Proposition 2.8, the branches of the characteristic variety (Fig. 1) admit directional derivatives at $(0,0)$. By Theorem 2.10, in one space dimension the characteristic variety is smooth. At $\beta = (\tau, \eta)$, a noncritical point of the characteristic variety, equations (2.4) take the form

$$\left\{\begin{array}{l} (\partial_t + \frac{\eta(\tau^2-1)}{\tau(\tau^2-1)+\gamma} \cdot \partial_x)\pi(\tau,\eta)\mathbf{u}_0 = 0 \\ (\partial_T + \frac{\eta(\tau^2-1)}{\tau(\tau^2-1)+\gamma} \cdot \partial_X)\pi(\tau,\eta)\mathbf{u}_0 + \frac{1}{2}\, i\, \tau''(\eta)(\partial_x, \partial_x)\pi(\tau,\eta)\mathbf{u}_0 = 0 \end{array}\right.$$

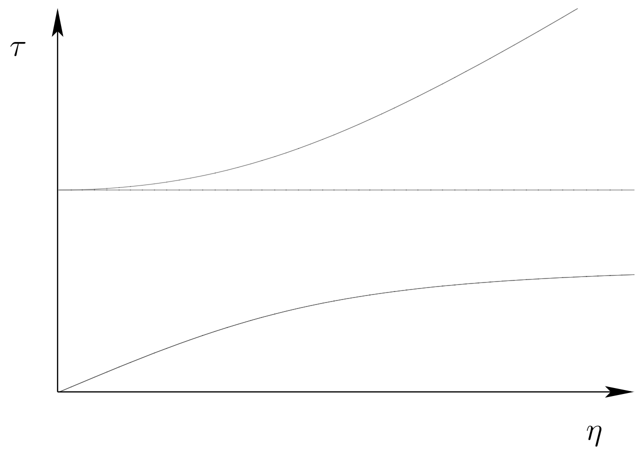


FIGURE 1. The characteristic variety for the Maxwell-Lorentz model.

by Proposition 2.8. In one space dimension, the derivatives at 0 of the unbounded branches vanish, so that the components $\pi(\pm\sqrt{1+\gamma},0)\mathbf{u}_0$ are standing waves, whereas $\pi(0,0)\mathbf{u}_0$ travels in $t,x$ at the velocity $\frac{1}{\sqrt{1+\gamma}}$.

**Ferromagnetism.** The Maxwell equations in a ferromagnetic medium read

$$\begin{cases} \partial_t\mathcal{E} - \operatorname{curl}\mathcal{H} & = & 0 \\ \partial_t\mathcal{H} + \operatorname{curl}\mathcal{E} + \partial_t\mathcal{M} & = & 0, \end{cases} \tag{2.20}$$

where $\mathcal{E}$ and $\mathcal{H}$ are the electric and magnetic fields. The magnetization vector $\mathcal{M}$ follows the Landau-Lifschitz equation

$$\partial_t\mathcal{M} = -(\mathcal{M}\times\mathcal{H}) - \frac{g}{|\mathcal{M}|}(\mathcal{M}\times(\mathcal{M}\times\mathcal{H})), \tag{2.21}$$

where $g$ is a dimensionless damping coefficient. One can describe small perturbations of an equilibrium state $(\mathcal{E}_0, \alpha\mathcal{M}_0, \mathcal{M}_0)$ with the results of this section, with $\mathcal{M}_0 = (\cos\theta, \sin\theta, 0)$, considering for simplicity waves travelling in one space dimension $x$ only.

If we neglect in (2.21) the damping term, the system satisfied by the perturbation $u = (u_1, u_2, \alpha^{1/2}u_3)$ where $(\mathcal{E},\mathcal{H},\mathcal{M}) = (\mathcal{E}_0, \alpha\mathcal{M}_0, \mathcal{M}_0) + \varepsilon$ $(u_1,u_2,u_3)(\tilde{t} = \varepsilon t, \tilde{x} = \varepsilon x)$ is (see [24]), dropping the tildes for convenience,

$$\partial_t u + \begin{pmatrix} 0 & -k\times & 0 \\ k\times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_x u + \frac{1}{\varepsilon}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathcal{M}_0\times & -\alpha^{1/2}\mathcal{M}_0\times \\ 0 & -\alpha^{1/2}\mathcal{M}_0\times & \alpha\mathcal{M}_0\times \end{pmatrix} u = f(u), \tag{2.22}$$

where $k = (1,0,0)$ and $f(u) := (0, -\alpha^{-1/2} u_2 \times u_3, u_2 \times u_3)$ is quadratic. At the noncritical point $(1+\alpha, 0,0,0)$ of the characteristic variety (Figure 2), the tangent operator vanishes. System (2.4) becomes

$$\begin{cases} \partial_t \, \pi(1+\alpha, 0) \, \mathbf{u}_0 = 0 \\ (\partial_T - i \, \frac{1+\cos^2\theta}{2(1+\alpha)^2} \, \partial_X^2) \, \pi(1+\alpha,0) \, \mathbf{u}_0 = 0, \end{cases}$$

where the diffractive variables are $T = \varepsilon t$ and $X = \varepsilon x$. Consider now the only critical point $(0,0,0)$ of the characteristic variety (Figure 2). If $\sin\theta \neq 0$, the tangent operator has four distinct, simple, nonvanishing eigenvalues $v_{1,\pm} = \pm\sqrt{\frac{\alpha+\sin^2\theta}{\alpha+1}}$, $v_{2,\pm} = \pm\sqrt{\frac{\alpha}{\alpha+1}}$. The spectral decomposition of the tangent operator takes the form

$$\pi(0)A\pi(0) = 0 \cdot \pi_0^1 + v_{1,+}(\pi_{1,+} - \pi_{1,-}) + v_{2,+}(\pi_{2,+} - \pi_{2,-}). \tag{2.23}$$

The transport equations in $t$ and $x$ (geometric optics approximation) are

$$(\partial_t - v_{j,\pm}\partial_x)\pi_{j,\pm} \, \mathbf{u}_0 = 0. \tag{2.24}$$

In a spectral basis for the tangent operator over its range, the Schrödinger equation for the linear system is

$$\begin{aligned} \Bigg( \partial_T + \begin{pmatrix} v_{1,+} & 0 & 0 & 0 \\ 0 & v_{1,-} & 0 & 0 \\ 0 & 0 & v_{2,+} & 0 \\ 0 & 0 & 0 & v_{2,-} \end{pmatrix} \partial_X \\ - \frac{i\cos\theta}{2(1+\alpha)^2} \begin{pmatrix} 0 & 0 & i & i \\ 0 & 0 & -i & -i \\ -i & i & 0 & 0 \\ -i & i & 0 & 0 \end{pmatrix} \partial_x^2 \Bigg) \pi(0)\mathbf{u}_0 = 0. \end{aligned} \tag{2.25}$$

The component of $\mathbf{u}_0$ polarized along Ker $\pi(0)A\pi(0)$ (the standing-wave component) does not interact with the other components.

If $\sin\theta = 0$, then the origin is a critical point for the characteristic variety of the tangent operator. The velocities are $\pm v = \pm\sqrt{\frac{\alpha}{\alpha+1}}$. The spectral decomposition of the tangent operator takes the form

$$\pi(0)A\pi(0) = 0 \cdot \pi_0^1 + v(\pi_+ - \pi_-),$$

where $\pm v$ have multiplicity two. The matrices $\pi_\pm A L(0)^{-1} A \pi_\pm$ are therefore not scalar. In a spectral basis for the tangent operator over its range, the

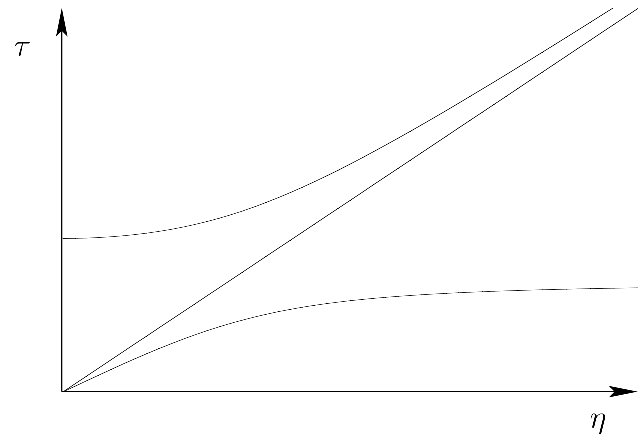


FIGURE 2. The characteristic variety for the equations of ferromagnetism.

Schrödinger equation for the linear system is

$$\left(\partial_T + \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & -v & 0 \\ 0 & 0 & 0 & -v \end{pmatrix} \partial_X - i\frac{1}{2(1+\alpha)^2} \begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & i \\ i & 0 & 0 & i \\ 0 & -i & -i & 0 \end{pmatrix} \partial_x^2 \right) \pi(0)\mathbf{u}_0 = 0. \tag{2.26}$$

Asymptotic expansions of system (2.20), (2.21) with the damping term are carried out in [23] and [9]. In the variables $T = \varepsilon^2 t$, $\tilde{t} = \varepsilon t$, and $\tilde{x} = \varepsilon x$, the system is, dropping the tildes,

$$\varepsilon \partial_T u + \partial_t u + \begin{pmatrix} 0 & -k\times & 0 \\ k\times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_x u + \frac{1}{\varepsilon} L_0 u = F(u) + \varepsilon G(u), \tag{2.27}$$

where $F$ is quadratic and $G$ cubic and

$$L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathcal{M}_0\times & -\alpha^{1/2}\mathcal{M}_0\times \\ 0 & -\alpha^{1/2}\mathcal{M}_0\times & \alpha\mathcal{M}_0\times \end{pmatrix} + g \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathcal{M}_0\times(\mathcal{M}_0\times) & -\alpha^{1/2}\mathcal{M}_0\times(\mathcal{M}_0\times) \\ 0 & -\alpha^{1/2}\mathcal{M}_0\times(\mathcal{M}_0\times) & \alpha\mathcal{M}_0\times(\mathcal{M}_0\times) \end{pmatrix}.$$

The factor $|\mathcal{M}|^{-1}$ in the damping term is supposed to be constant, $|\mathcal{M}|^{-1} = |\mathcal{M}_0|^{-1}$. One wants to describe the geometric and diffractive optics approximations of (2.27) with the nonlinearity [9].

The dispersion matrix in (2.27) is not skew-symmetric, so that theorem 2.10 does not hold. The equation for the leading term $u_0$ of the approximation is

$$(\partial_t + \pi(0)A\pi(0)\partial_x)\,\pi(0)\mathbf{u}_0 = 0. \tag{2.28}$$

The semisimplicity of 0 in the spectrum of $L_0$ implies that the projector $\pi(0)$ onto the Kernel of $L_0$ is Hermitian, so that the tangent operator $\pi(0)A\pi(0)$ is Hermitian, too. Actually, the projector onto Ker $L_0$ is the same as when $g = 0$; hence, (2.23) still holds. Consequently, the profile equations at first order are scalar transport equations, the same equations as for the simplified model (2.24). $\pi(0)A\pi(0)$ being Hermitian, all its eigenvalues are semisimple. Therefore $\pm\tau_1$ and $\pm\tau_2$ are twice continuously differentiable.

If $\sin\theta \neq 0$, in the spectrum of the tangent operator only 0 has algebraic multiplicity $> 1$. Therefore, for an index $j$ corresponding to a branch $\pm\tau_j$ such that $\tau_j'(0) \neq 0$, the matrix $\pi_{j,\pm}AL_0^{-1}A\pi_{j,\pm}$ is scalar, and one has $\pi_{j,\pm}AL(0)^{-1}A\pi_{j,\pm} = \frac{1}{2}\tau_{j,\pm}''(0)\pi_j$. The use of averaging projectors (see [19]) as in Section 3 shows that the equations governing the components $\pi_{j,\pm}\mathbf{u}_0$ for diffractive times is

$$\partial_T \pi_{j,\pm}\mathbf{u}_0 + \frac{1}{2}\, i\,\tau_{j,\pm}''(0)\,\partial_x^2\,\pi_{j,\pm}\mathbf{u}_0 = \text{nonlinear }(\pi_{j,\pm}\mathbf{u}_0). \tag{2.29}$$

The point is that $\tau_{j,\pm}''(0) \in i\mathbb{R}$, so that (2.29) are scalar heat equations. Two different nonvanishing diffusion coefficients are found; the equations are

$$\partial_T \pi_{1,\pm}\mathbf{u}_0 - \frac{g\cos^2\theta}{2(1+\alpha)^2(1+g^2)}\,\partial_x^2\,\pi_{1,\pm}\mathbf{u}_0 = \text{ nonlinear }(\pi_{1,\pm}\mathbf{u}_0), \tag{2.30}$$

and

$$\partial_T \pi_{2,\pm}\mathbf{u}_0 - \frac{g}{2(1+\alpha)^2(1+g^2)}\,\partial_x^2\,\pi_{2,\pm}\mathbf{u}_0 = \text{nonlinear }(\pi_{2,\pm}\mathbf{u}_0). \tag{2.31}$$

The diffusion matrix for $\pi_0^1\mathbf{u}_0$ vanishes.

If $\sin\theta = 0$, the equations for diffractive times are no longer scalar, as in the previous case. In an orthonormal basis over the range of $\pi_+$ (respectively $\pi_-$) the spectral projector of $\pi(0)A\pi(0)$ related to the positive (respectively negative) eigenvalue,

$$\partial_T \pi_\pm\mathbf{u}_0 - \frac{1}{2(1+\alpha)^2(1+g^2)}\begin{pmatrix} g & i \\ -i & g \end{pmatrix}\partial_x^2\pi_\pm\mathbf{u}_0 = \text{nonlinear }(\pi_\pm\mathbf{u}_0). \tag{2.32}$$

**Remark 2.11.** The use of average operators [19] in nonlinear diffractive optics induces a decoupling of the components of the main profile. This explains why (2.30) and (2.31) are scalar and not (2.25). At a critical point on the characteristic variety the profile equations are not scalar, as seen in (2.26) and (2.32).

## 3. The short-wave limit

In this section, the short-waves approximation for general symmetric hyperbolic systems is studied. *Short waves* stands here for short-wavelength approximate solutions, or equivalently approximate solutions with initial data whose oscillatory frequencies are large compared to the parameters of the system. The space variable will be denoted by $(x, y) \in \mathbb{R} \times \mathbb{R}$, while the time variable will still be denoted by $t \in \mathbb{R}_+$.

Precisely, one considers initial data of the form

$$\begin{cases} \mathbb{R} \times \mathbb{R} & \to \quad \mathbb{R}^n \\ (x, y) & \to \quad u^0(\frac{x}{\varepsilon}, y) \end{cases}$$

and a hyperbolic operator

$$L(\partial) = \partial_t + A\partial_x + B\partial_y + E.$$

$A$ and $B$ are assumed to be real, symmetric, $n \times n$ matrices. One assumes that the spectrum of $A$ consists of three different eigenvalues: $c$, $-c$, and $0$, with $\dim \operatorname{Ker} A > 1$. Such an assumption is satisfied by physical systems based on the Maxwell equations. Dissipative effects are taken into account as the dispersion matrix $E$ is supposed to satisfy $E + E^* \geq 0$. $E$ is also supposed to have real entries. It will be shown that short waves do generically not propagate for strictly dissipative systems (i.e., $E + E^* > 0$).

We are interested in the description of approximate solutions of the quasilinear (respectively semilinear) initial-value problem

$$L(\partial)u = \tilde{F}(u)\partial_x u \ \ (\text{respectively } \tilde{f}(u)), \quad u(0) = \varepsilon\, u^0(x/\varepsilon, y). \tag{3.1}$$

This is a one-dimensional asymptotic study ($x \in \mathbb{R}$). The perturbative term $B\partial_y$ is also chosen to be one-dimensional ($y \in \mathbb{R}$). Setting $y \in \mathbb{R}^{d-1}$, $d > 2$, would induce changes in the notation only in Sections 3.4 and 3.5, and only minor changes in Section 3.6—see Remark 3.25.

Nonlinearities of quasilinear nature and of semilinear nature are handled without a vanishing mean condition for the initial datum, thanks to a technique of low-frequency truncation built up by D. Alterman and J. Rauch [2], [3], [4].

Following [15], the approximate solution is sought in the form of a WKB expansion:

$$u^\varepsilon = \varepsilon^p \sum_j \varepsilon^j \mathbf{u}_j,$$

where $\mathbf{u}_j$ are profiles depending on a certain set of variables to be defined (see Section 3.1). $p$ is chosen in order that nonlinear effects occur at a diffractive time scale. It depends on the size of the nonlinear terms, for which we make the following hypothesis, as in [12].

**Assumption 3.1** (orders of the nonlinearities)**.** *The quasilinear term is* $\tilde{F}(u)\partial_X u$, *where* $\tilde{F}$ *is smooth and is of order* 2; *that is,*

$$\partial^\alpha \tilde{F}(0) = 0, \text{ for all } |\alpha| \le 1.$$

*The semilinear term* $\tilde{f}$ *is supposed to be smooth and of order* 2:

$$\partial^\alpha \tilde{f}(0) = 0, \text{ for all } |\alpha| \le 1.$$

With this assumption and Taylor's theorem, one has, for all $u \in \mathbb{C}^n$, $\tilde{F}(\varepsilon u) = \varepsilon^2 F(u) + \varepsilon^3 F_1(\varepsilon, u)$, where $F$ is a homogeneous polynomial of order 2 and where $F_1$ is smooth, and similarly $\tilde{f}(\varepsilon u) = \varepsilon^2 f(u) + \varepsilon^3 f_1(\varepsilon, u)$, where $f$ is a homogeneous polynomial of order 2 and where $f_1$ is smooth. It follows ([12] and [15]) that the scaling for which diffractive and nonlinear effects occur at the same time scale is $p = 1$.

One could more generally describe quasilinear (respectively semilinear) terms of order $K \ge 1$ (respectively $J \ge 2$), changing the amplitude $\varepsilon^p$ of the solution accordingly (precisely, one would set $p = \frac{2}{K-1}$ (respectively $p = \frac{1}{J-1}$) as in [12]).

Three different regimes have to be distinguished. The components of $u^0$ polarized along noncritical asymptotic branches of Char $L$ (defined in Section 3.2) give rise to a dynamics described as the *noncritical* case (Sections 3.4 and 3.5). The components of $u^0$ polarized along critical asymptotic branches of Char $L$ correspond to the *critical* dynamics (Section 3.6). There are two different critical dynamics: one for the bounded branches, and one for the unbounded branches. In order to study these regimes separately, one is led to make additional assumptions on Char $L$ and on the polarization of the initial datum $u^0$.

The features of the critical dynamics related to the unbounded branches of the characteristic variety are quite close to those of the noncritical and of the first critical dynamics. It will therefore not be studied here.

For technical reasons, we will make use in the semilinear case of the class of Banach spaces introduced by Lannes in [21]:

**Definition 3.2.** For all $\sigma, s, p \in \mathbb{R}$, $p \geq 1$, one defines

$$E^{\sigma,s,p}(\mathbb{R}^2) = \{f \in \mathcal{S}'(\mathbb{R}^2),\ (1+\eta^2)^{\sigma/2}||(1+|\xi|^2)^{s/2}\hat{f}(\cdot,\eta)||_{L^p(\mathbb{R})} \in L^2(\mathbb{R})\}.$$

Equipped with the norm

$$\|f\|_{E^{\sigma,s,p}(\mathbb{R}^2)} = \|(1+|\eta|^2)^{\sigma/2}\|(1+|\xi|^2)^{s/2}\hat{f}(\xi,\eta)\|_{L^p(\mathbb{R}_\xi)}\|_{L^2(\mathbb{R}_\eta)},$$

$E^{\sigma,s,p}$ is a Banach space. Note that $E^{\sigma,s,p}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, if $\sigma, s > 1/2$ and $\frac{sp}{p-1} > 1$. Under these conditions for $\sigma$, $s$, and $p$, $E^{\sigma,s,p}$ is also an algebra, and nonlinear estimates hold. For more details, see [21].

One denotes by $\partial_X H^s$ the vector space

$$\partial_X H^s = \{u \in H^s(\mathbb{R}^2),\ \exists\ v \in H^s(\mathbb{R}^2),\ u = \partial_X v\}.$$

The following set of assumptions is valid throughout this study.

**Assumption 3.3** (characteristic variety)**.** *The critical points of* $\mathit{Char}_{\mathbb{R}}\, L$ *are isolated and located on* $\mathbb{C}_\tau \times \{0\}_{\mathbb{R}^2_{\xi,\eta}}$. *Char* $L$ *is locally parametrized by functions depending on* $\xi^2+\eta^2$, *where* $\xi$ *is the Fourier variable of* $x$ *and* $\eta$ *the Fourier variable of* $y$.

The hypothesis regarding the critical locus of the real characteristic variety is satisfied by the physical examples, among which those given in Section 2.3. [29] implies that under assumption 3.3 and when $E$ is skew-symmetric, a point $\beta = (\tau, \eta) \in \mathbb{R} \times \mathbb{R}^d$ with $\eta \neq 0$ is noncritical for Char $L$ as well. The axisymmetry hypothesis in assumption 3.3 is generally not satisfied by the equations of ferromagnetism. It is shown in the example in Section 3.7 how the equations are modified in this case.

**Assumption 3.4** (quasilinear systems)**.**

- $E + E^* \geq 0$.
- $\tilde{F}$ *satisfies assumption* 3.1, *and for all* $u \in \mathbb{C}^n$, $\tilde{F}(u)$ *and* $F(u)$ *are Hermitian,* $n \times n$ *matrices.*
- *The initial datum* $u^0$ *satisfies* $u^0(x,y) = [\mathbf{u}^0(X,y)]_{X=x/\varepsilon}$, *where the profile* $\mathbf{u}^0$ *lies in* $H^s(\mathbb{R}^2)$ *with* $s > 5$.

**Assumption 3.5** (semilinear systems)**.**

- $E + E^* \geq 0$.
- $\tilde{f}$ *satisfies Assumption* 3.1.
- *The initial datum* $u^0$ *satisfies* $u^0(x,y) = [\mathbf{u}^0(X,y)]_{X=x/\varepsilon}$, *where the profile* $\mathbf{u}^0$ *lies in* $E^{\sigma,s,p}(\mathbb{R}^2) \cap E^{\sigma,s,pr}(\mathbb{R}^2)$ *for some* $\sigma > 5$, $s > \max(\frac{p}{p-1}, \frac{pr}{pr-1})$, $1 < p, r < \infty$, *and* $\frac{pr}{r-1} < 2$.

The condition $s > 5$ (respectively $\sigma > 5$) is here to ensure that the bounds in $H^s$ (respectively $E^{\sigma,s,p}$) in the profile variables give bounds in $L^\infty$ in the physical variables.

Under Assumptions 3.3 and 3.4 (respectively Assumptions 3.3 and 3.5), one undertakes the description of approximate solutions of the initial-value problem (3.1).

The derivation of profile equations follows the classical scheme [15]. The computations for the profile equations are spelled out as in the previous section. The nonlinearities are simplified by the techniques of averaging from [19]. The control of the corrector terms of the ansatz is the major difficulty of this study. This is true first, because, in the noncritical case at least, the ansatz involves three different time scales that differ by a factor $O(1/\varepsilon^2)$, so that errors in the rapid time scale $T$ may accumulate dramatically in the slow time scale; and second, because the corrector terms of the ansatz satisfy transport equations with source terms that are expressed in terms of integro-differential operators of the main profile. Three different arguments are used to overcome these difficulties:

- Low-frequency truncations (see Definition 3.15) as in [2], [3], and [4], whose effect is a smoothing with respect to $\partial_X^{-1}$ and whose speed of convergence can be estimated in $E^{\sigma,s,p}$.
- Secular bounds as in [6], i.e., estimates of the time growth of a profile solution of a transport operator, depending on the source term.
- Sharper secular bounds in $E^{\sigma,s,p}$, as in [21].

Let us briefly state the results of this section. For different polarizations of the initial datum $u^0$, related to the geometry of the hyperbolic operator, a family of maps $u_0^\varepsilon(t,x,y) = \varepsilon \sum_k [\mathbf{u}_{k,0}(T,X,y,t,\varepsilon t)]_{T=t/\varepsilon, X=x/\varepsilon, \tau=\varepsilon t}$ is constructed, such that

- $u_0^\varepsilon(0) = \varepsilon u^0$.
- $\mathbf{u}_{k,0}$ satisfies transport equations in $T, X$ and $t, y$.
- $\mathbf{u}_{k,0}$ satisfies a differential equation in $\tau$ of the form

$$\partial_\tau \partial_X \mathbf{u}_{k,0} + M_k(D_y)\mathbf{u}_{k,0} = \text{ nonlinear } (\mathbf{u}_{k,0}). \tag{3.2}$$

Under a polarization condition for $u^0$, one has $M_k(D_y) = \alpha_k + \beta_k\, \partial_y^2$ with $\alpha_k,\, \beta_k \in \mathbb{R}$. It is shown that for diffractive times $O(1/\varepsilon)$, $u_0^\varepsilon$ is asymptotically close, as $\varepsilon \to 0$, to the exact solution $v^\varepsilon$ of (3.1). The rate of convergence depends on the assumption on $u^0$ and on the nature of the nonlinearity. In the quasilinear case, the need to use Hilbert spaces techniques to solve the profile equations prevents us from using the Banach spaces $E^{\sigma,s,p}$, and it follows that the rate of convergence is $o(1)$ only.

3.1. **The ansatz.** One goes through a rapid formal computation using a spatial Fourier transform to motivate the choice of the ansatz (3.5) and (3.6). Transverse effects are not taken into account in the following lines and $E$ is supposed to be skew-symmetric. Let

$$A = 0\,\pi_0 + c\,(\pi_+ - \pi_-), \quad A\xi + E/i = \sum_j \omega_j(\xi)\pi_j(\xi),$$

be the spectral representation of $A$ and $A\xi + E/i$ respectively. A spatial Fourier transform applied to (3.1) with a null nonlinearity leads to

$$(\partial_t + i(A\xi + E/i))\hat{u}(t,\xi) = 0,$$

with $\hat{u}(0) = \varepsilon^2 \hat{u}^0(\varepsilon\,\xi)$, and so

$$u(t,x) = \varepsilon \int_{\mathbb{R}} e^{ix\xi} e^{-it(A\xi+E/i)} \varepsilon \hat{u}^0(\varepsilon\xi)d\xi = \varepsilon \int_{\mathbb{R}} \sum_j e^{i(x\xi - t\omega(\xi))}\pi_j(\xi)\varepsilon\hat{u}^0(\varepsilon\xi)d\xi.$$

The change of variables $\xi' = \varepsilon\xi$ leads to

$$u(t,x) = \varepsilon \sum_j \int_{\mathbb{R}} e^{i(x\xi/\varepsilon - t\omega_j(\xi/\varepsilon))}\pi_j(\xi/\varepsilon)\,\hat{u}^0(\xi)d\xi.$$

If we now suppose that $\omega_j$ has an expansion in a neighborhood of $\infty$ as

$$\omega_j(\xi) = c_{j,1}\,\xi + c_{j,0} + \frac{c_{j,-1}}{\xi} + o(\frac{1}{\xi}), \tag{3.3}$$

we get

$$u(t,x) \simeq \varepsilon \sum_j \int_{\mathbb{R}} e^{i\xi(x - c_{j,1}\,t)/\varepsilon}\, e^{-i\,c_{j,0}\,t}\, e^{-it\varepsilon\, c_{j,-1}/\xi}\pi_j(\xi/\varepsilon)\hat{u}^0(\xi)d\xi. \tag{3.4}$$

This suggests looking for approximate solutions in the variables $T = \frac{t}{\varepsilon}$, $X = \frac{x}{\varepsilon}$, $t, y, \tau = \varepsilon t$ if $c_{j,1} \neq 0$, and in the variables $X = \frac{x}{\varepsilon}$, $t, y, \tau = \varepsilon t$ if $c_{j,1} = 0$. The approximate solutions will be sought in the form

$$u^\varepsilon(t,x,y) = \varepsilon\,(\mathbf{u}_0 + \varepsilon\,\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2)(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, t, y, \varepsilon\,t) \tag{3.5}$$

in the first case, and in the form

$$u^\varepsilon(t,x,y) = \varepsilon\,(\mathbf{u}_0 + \varepsilon\,\mathbf{u}_1 + \varepsilon^2\,\mathbf{u}_2)(\frac{x}{\varepsilon}, t, y, \varepsilon\,t) \tag{3.6}$$

in the second case.

There is no slow $x$ in (3.4), nor in the initial datum $u^0$. This is why there is no slow $x$ in the ansatz. There is no fast $y$ in the initial datum either, and this is why there is no fast $y$ in the ansatz.

**Notation.** The following notation will be used throughout this paper.

- Profiles (that is, functions fitting the ansatz) are written in a boldface font; for a profile $\mathbf{a}$, one denotes by $a^\varepsilon$ or by $(\mathbf{a})^\varepsilon$ the function $a^\varepsilon(t,x,y) = (\mathbf{a})^\varepsilon(t,x,y) = \mathbf{a}(t/\varepsilon, x/\varepsilon, t, y, \varepsilon t)$. For example, the function associated with the profile $\partial_\tau \mathbf{a}$ will be written $(\partial_\tau \mathbf{a})^\varepsilon$.
- The Fourier transform of a function $u$ is written $\mathcal{F}(u)$, or $\hat{u}$ anytime when there cannot be any confusion regarding the space variable with respect to which the transform is performed.
- Constants are often denoted by $C$.

3.2. **The long-wave operator associated with the short-wave limit.** Introduce the family of matrices

$$M(\tau,\xi,\eta) := \tau + (E/i + B\,\eta)\xi + A, \quad (\tau,\xi,\eta) \in \mathbb{C}^3.$$

One has the identity, for $\xi \neq 0$,

$$\det\, M(\tau,\xi,\eta) = 0 \Leftrightarrow \det\, L(\tau/\xi, 1/\xi, \eta) = 0.$$

In this paper, $M$ is called the long-wave operator associated with $L$.

Each (local) noncritical branch of Char $L$ parametrized by $\omega$ is associated with a (local) noncritical branch of Char $M$ parametrized by $\tau$ via the identity

$$\tau(\xi,\eta) = \xi\omega(\frac{1}{\xi}, \eta), \quad \xi \neq 0. \tag{3.7}$$

An asymptotic branch $\omega$ of $\mathrm{Char}_{\mathbb{R}}\, L$ is called (non)critical when the corresponding point $(\tau(0,\eta), 0, \eta) \in \mathbb{C} \times \mathbb{R}^2$ given by (3.7) is (non)critical on $\mathrm{Char}_{\mathbb{R}}\, M$.

In a neighborhood of a noncritical point $(\tau, 0, 0) \in \mathrm{Char}_{\mathbb{R}}\, M$, the parametrization $\tau$ has the expansion for $\xi$ in a neighborhood of 0

$$\tau(\xi, 0) = \tau_0 + \tau_1 \xi + O(|\xi|^2).$$

The corresponding noncritical branch of $\mathrm{Char}_{\mathbb{R}}\, L$ has the asymptotic expansion for $\xi$ in a neighborhood of $\infty$

$$\omega(\xi, 0) = \tau_0 \xi + \tau_1 + O(\frac{1}{|\xi|}).$$

This justifies (3.3).

By Rouché's theorem, for all $(\tau_0, 0, \eta) \in$ Char $M$, $\tau_0 \in$ sp $A$. For $\omega$ an asymptotic branch of $\mathrm{Char}_{\mathbb{R}}\, L$, two cases arise:

- $\omega$ is unbounded ($\tau_0 = \pm c$). Then $\omega \sim \pm c\,\xi$. These eigenvalues correspond to nonpolarized initial values $(1 - \pi_0)u^0$. When $c$ (or $-c$) is a simple eigenvalue of $A$, the asymptotic branch is generically noncritical; it is generically critical when the multiplicity of $c$ (or $-c$) in the spectrum of $A$ is $> 1$.

- $\omega$ is bounded ($\tau_0 = 0$). These eigenvalues correspond to polarized initial values $\pi_0 u^0$. These branches are generically critical branches, for 0 is supposed to be an eigenvalue of $A$ of multiplicity $> 1$.

We now derive useful identities for the parametrizations of Char $M$.

First, under Assumption 3.3, in a neighborhood of a noncritical point $(\tau, \xi, \eta)$ on Char $M$, $\xi$ being far from 0, there is a smooth function $\Omega$ such that (3.7) takes the form $\tau(\xi, \eta) = \xi\Omega(1/\xi^2 + \eta^2)$. At $(\tau, \xi, 0)$ a noncritical point on Char $M$, such that $\xi \neq 0$, one has therefore $\partial_\eta \tau(\xi, 0) = 0$, and by continuity, $\partial_\eta \tau(0, 0) = 0$.

Second, suppose for a moment that the system is conservative: $E+E^* = 0$. Let then $(\tau, \xi, \eta) \in \mathrm{Char}_{\mathbb{R}}\, M$. The conservativity implies that $\tau \in \mathbb{R}$. Then, $E$ being real, conjugating and multiplying the matrices by $-1$ in the definition of Char $M$ yields the equivalence

$$(\tau, \xi, \eta) \in \mathrm{Char}_{\mathbb{R}}\, M \;\Leftrightarrow (\tau, -\xi, -\eta) \in \mathrm{Char}_{\mathbb{R}}\, M, \quad \text{for } \xi \neq 0.$$

In a neighborhood of a noncritical point $(\tau_0, 0, 0)$ it is thus possible in the conservative case to parametrize $\mathrm{Char}_{\mathbb{R}}\, M$ by an even function.

**Remark 3.6.** For conservative systems, the evenness of the noncritical branches of $\mathrm{Char}_{\mathbb{R}}\, M$ implies that an unbounded asymptotic branch of $\mathrm{Char}_{\mathbb{R}} L$ parametrized by a smooth function $\omega$ satisfies in the section plane $\eta = 0$

$$\omega(\xi, 0) = \lambda\xi + O(\frac{1}{\xi}), \tag{3.8}$$

where $\lambda \in \mathrm{sp}\ A - \{0\}$. This particular form of asymptotic expansion for the frequency was taken as an assumption in [27] in order to study the propagation of short waves. It is shown below that generically, short waves do propagate only for conservative systems, so that (3.8) appears here as a necessary condition, under Assumptions 3.3 and 3.4 (or 3.3 and 3.5).

3.3. **The asymptotic branches: a look at the examples.** Let us have a look at the examples which (3.1) aims at describing.

**Maxwell-Lorentz equations.** The equation defining the characteristic variety associated with system (2.19) is in one space dimension, with the notation of Section 3.1,

$$\omega^2\,(\omega^2 - \,\gamma)\,(\,(\omega^2 - 1)\,(\omega^2 - \xi^2) - \gamma\,\omega^2)^2 = 0.$$

There is one *double* eigenvalue $\omega_+$ such that $\omega_+ = \xi + o(\xi)$. Thus $(1, 0) \in$ Char $M$ is noncritical. Similarly, $(-1, 0)$ is noncritical. The origin $(0, 0)$ is critical. See Figures 1 and 3.

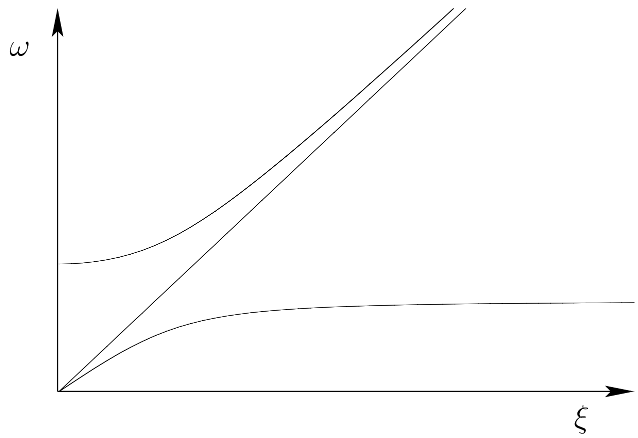


FIGURE 3. The characteristic variety for the long-wave operator associated with the Maxwell-Lorentz model.

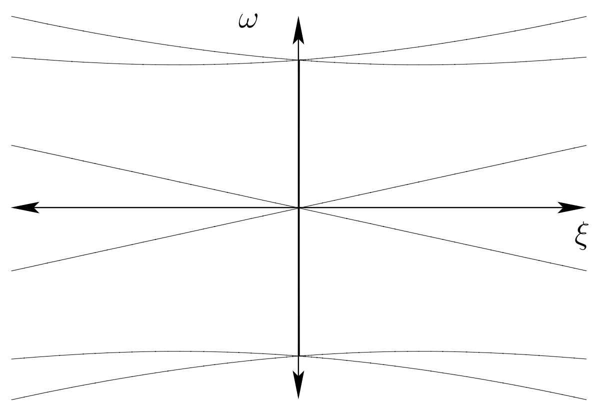


FIGURE 4. The characteristic variety for the long-wave operator associated with the equations of ferromagnetism.

**Ferromagnetism.** The equation defining the characteristic variety associated with system (2.22) is in one space dimension

$$\omega^3(-\omega^6+(2\xi^2+(1+\alpha)^2)\omega^4-(\xi^4+(1+\alpha)(2\alpha+\sin^2\theta)\xi^2)\omega^2+(\alpha+\sin^2\theta)\xi^4)=0.$$

There are two single eigenvalues $\omega_1^+$ and $\omega_2^+$ tending to $\infty$, and such that $\omega_i^+ \sim \xi$. The point $(1,0) \in \mathrm{Char}\, M$ is therefore critical. $(0,0) \in \mathrm{Char}\, M$ is also critical. See Figures 2 and 4.

These examples and the previous section suggest that we focus on three cases

- $(\tau_0, 0, 0)$, $\tau_0 \in sp(A) - \{0\}$, a noncritical point of Char $M$.
- $(0, 0, \eta)$, $\eta \neq 0$, a critical point of Char $M$.
- $(\tau_0, 0, \eta)$, $\tau_0 \in sp(A) - \{0\}$, $\eta \neq 0$, a critical point of Char $M$.

The first case is examined in Sections 3.4 and 3.5, the second case in Section 3.6. The third case follows from the two previous ones.

3.4. **The noncritical case for a quasilinear system.** Assumptions 3.3 and 3.4 are made throughout this section. Moreover, one considers polarized initial conditions $(1-\pi_0)u^0$, and to put in evidence the noncritical dynamics, one makes the additional assumption,

**Assumption 3.7.** $(c, 0, 0)$ *and* $(-c, 0, 0)$ *are noncritical points of* $Char_{\mathbb{R}} M$.

Figure 4 shows that this assumption is not satisfied by the equations of ferromagnetism. One can easily adapt the profile equations to this case. See Section 3.7.

3.4.1. *Profile equations.* The approximate solution of (3.1) is here sought in the form (3.5). Compute

$$\begin{aligned} L(\partial)u^\varepsilon - \tilde{F}(u^\varepsilon)\partial_x u^\varepsilon &= \Big[\sum \varepsilon^k \mathbf{r}_k(T, X, y, \tau)\Big]_{T=\frac{t}{\varepsilon}, X=\frac{x}{\varepsilon}, \tau=\varepsilon t} \qquad (3.9)\\ &= [((\partial_T + A\partial_X) + \varepsilon(E + B\partial_y) + \varepsilon^2\partial_\tau)(\mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2)\\ &\quad \times (X, y, t, \tau))]_{T=\frac{t}{\varepsilon}, X=\frac{x}{\varepsilon}, \tau=\varepsilon t} - \frac{1}{\varepsilon}[\tilde{F}(\varepsilon(\mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2))\\ &\quad \times \partial_X(\varepsilon(\mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2)))(X, y, t, \tau))]_{T=\frac{t}{\varepsilon}, X=\frac{x}{\varepsilon}, \tau=\varepsilon t}. \end{aligned}$$

Following [12], the strategy is to choose profiles that annihilate the first terms of the right-hand side, called the residual. The conditions $\mathbf{r}_1 = 0$, $\mathbf{r}_2 = 0$, and $\mathbf{r}_3 = 0$ read

$$(\partial_T + A\partial_X)\mathbf{u}_0 = 0 \qquad (3.10)$$

$$(\partial_T + A\partial_X)\mathbf{u}_1 + (\partial_t + E + B\partial_y)\mathbf{u}_0 = 0 \qquad (3.11)$$

$$(\partial_T + A\partial_X)\mathbf{u}_2 + (\partial_t + E + B\partial_y)\mathbf{u}_1 + \partial_\tau\mathbf{u}_0 = F(\mathbf{u}_0)\partial_X\mathbf{u}_0). \qquad (3.12)$$

With (3.10), (3.11), and (3.12), the residual $\mathbf{r}^\varepsilon$ becomes

$$\mathbf{r}^\varepsilon = \frac{1}{\varepsilon}\tilde{F}(\mathbf{u}^\varepsilon)\partial_X\mathbf{u}^\varepsilon - \varepsilon^2 F(\mathbf{u}_0)\partial_X\mathbf{u}_0 + \varepsilon^3((\partial_t + E + B\partial_y)\mathbf{u}_2 + \partial_\tau\mathbf{u}_1) + \varepsilon^4\partial_\tau\mathbf{u}_2.$$

Note that with the chosen ansatz, dispersive and transverse effects occur at the same time, as seen in (3.11). Projecting the profile equations over the eigendirections of $A$ leads to

$$(\partial_T + c_j\partial_X)\pi_j\mathbf{u}_0 = 0 \qquad (3.13)$$

$$(\partial_T + c_j\partial_X)\pi_j\mathbf{u}_1 + \pi_j(\partial_t + E + B\partial_y)\mathbf{u}_0 = 0 \tag{3.14}$$

$$(\partial_T + c_j\partial_X)\pi_j\mathbf{u}_2 + \pi_j(\partial_t + E + B\partial_y)\mathbf{u}_1 + \partial_\tau\pi_j\mathbf{u}_0 = \pi_j F(\mathbf{u}_0)\partial_X\mathbf{u}_0 \tag{3.15}$$

with $j = +, -$. The component of $\mathbf{u}_0$ polarized along Ker $A$ is identically zero by the polarization condition on $u^0$. We see on (3.13) that different components of $\mathbf{u}_0$ propagate at different speeds. To sort out the components of the profiles and of the nonlinearities in (3.14) and in (3.15), we will make use of another tool of the paraphernalia of geometric optics, the *average projectors*. They were initially introduced in [19]. We cite below the results of [19] that this study requires.

A smooth, real function $\lambda$ being given, denote by $T$ the pseudo-differential operator $T(\partial) = \partial_t + i\lambda(D_X)$.

**Definition 3.8** (average projectors [19])**.** For all $h > 0$ and $w \in C^0([0, \tau_0] \times \mathbb{R}_T, L^2(\mathbb{R}^2))$, let

$$G_T^h(w)(t,x) = \frac{1}{h}\int_0^h (\int e^{i(x\xi + s\lambda(\xi))}\, \hat{w}(t+s,\xi)\, d\xi)\ ds.$$

Introduce also

$$G_T w = \lim_{h\to\infty} G_T^h w,$$

when this limit exists in $C^0([0, \tau_0] \times \mathbb{R}_T, L^2(\mathbb{R}^2))$. Call $G_T$ the average operator related to the transport operator $T$.

The following proposition states the properties of the average projectors in $C^0([0, \tau_0] \times \mathbb{R}_T, L^2(\mathbb{R}^2))$:

**Proposition 3.9** (properties of the average projectors [19])**.**

a) *If $w$ satisfies $T(\partial)w = 0$, then $G_T w$ is well defined and satisfies $G_T w = w$.*
b) *If $w$ satisfies $(\partial_t + i\lambda_1(D_y))w = 0$, with $\lambda_1 \neq \lambda$ almost everywhere in $\mathbb{R}^2$, then $G_T w = 0$.*
c) *$w$ has a sublinear growth; i.e.,* $\lim_{T\to\infty} \frac{1}{T}\|w(T)\|_{L^\infty([0,\tau_0],L^2(\mathbb{R}^2))} = 0$, *if and only if $G_T T w = 0$.*
d) *Let $\{T_i\}_{1\le i\le m}$ be $m$ (not necessarily distinct) scalar operators ($m \in \mathbb{N}^*$). Let $\{w_i\}_{1\le i\le m}$ be $m$ functions annihilating these operators: $T_i(\partial)w_i = 0$. Then if $T$ is another transport operator and if $T = T_1 = \cdots = T_m$,*

$$G_T(w_1 w_2 \cdots w_m) = w_1 w_2 \cdots w_m.$$

*In every other case, $G_T(w_1 w_2 \cdots w_m) = 0$.*

**Proof.** See [19].

Let us now define $T_j$ as the transport operator $\partial_T + c_j\partial_X$, and apply $G_{T_j}$ to equations (3.13), (3.14), and (3.15), $j \in \{+,-\}$. We obtain

$$G_{T_j}\pi_j\mathbf{u}_0 = \pi_j\mathbf{u}_0, \tag{3.16}$$

$$(\partial_t + \pi_j(E + B\partial_y))\pi_j\mathbf{u}_0 = 0, \tag{3.17}$$

$$(c_{j'} - c_j)\partial_X G_{T_j}\pi_{j'}\mathbf{u}_1 + \pi_{j'}(E + B\partial_y)\pi_j\mathbf{u}_0 = 0, \quad \text{for all } j' \neq j, \tag{3.18}$$

$$\sum_{j'} \pi_j(\partial_t + E + B\partial_y)G_{T_j}\pi_{j'}\mathbf{u}_1 + \partial_\tau\pi_j\mathbf{u}_0 = G_{T_j}\pi_j F(\mathbf{u}_0)\partial_X\mathbf{u}_0. \tag{3.19}$$

By the last point of Proposition 3.9, the nonlinear term in (3.19) involves only $\pi_j\mathbf{u}_0$. The following lemma investigates the nature of the first-order operator occurring in the profile equations. As seen in Section 2, algebraic lemmas reduce to residue computations. The proofs are similar to the ones of the previous section; therefore, we omit the details.

**Lemma 3.10** (the noncritical case: first-order operators)**.** *If the system is conservative* (*i.e., if* $E + E^* = 0$), *then the following relations hold for all* $j \in \{+,-\}$ :

$$\pi_j E\pi_j = -i\,\partial_\xi\tau_j(0,0)\pi_j = 0, \quad \pi_j B\pi_j = -i\,\partial_\eta\tau_j(0,0)\pi_j = 0.$$

**Proof.** With [29], the conservativity implies that the point $(\tau_j(0,0),0,0)$, which by assumption is noncritical for $\operatorname{Char}_{\mathbb{R}} M$, is noncritical for $\operatorname{Char} M$ as well. Then the first relation plainly follows from a residues computation at first-order in $\xi$ at the point $(\tau_j(0,0),0,0)$. As discussed in Section 3.2, if the system is conservative, then $\tau_j$ is even, so that $\partial_\xi\tau_j(0,0) = 0$. This justifies the first relation. At $(\tau_j(\xi,0),\xi,0)$, the expansion at first order in $\eta$ is

$$\partial_\eta\tau_j(\xi,0)\pi_j(\xi,0) = \xi\pi_j(\xi,0)B\pi_j(\xi,0). \tag{3.20}$$

As explained in Section 3.2, it follows from Assumption 3.4 that $\tau_j(\xi,\eta) = \Omega_j(1/\xi^2 + \eta^2)$, for $\xi \neq 0$; hence, $\partial_\eta\tau_j(\xi,0) = 0$ for $\xi \neq 0$, and also for $\xi = 0$ by the noncriticality assumption on $(c_j,0,0)$. The right-hand side of (3.20) therefore vanishes identically. The projector being smooth at $(c_j,0,0)$, deriving the right-hand side at $(c_j,0,0)$ yields the lemma. $\square$

In the case of a strictly dissipative system: $E + E^* > 0$, the point $(\tau_j,0,0)$ might be critical for $\operatorname{Char} L$. Thus one expects (3.17) not to be scalar. What's more, $-i\pi_j E\pi_j$ generically has eigenvalues with nonvanishing imaginary parts.

**Example.** For the nonconservative equations of ferromagnetism, with the notation of Section 2.3,

$$\mathrm{sp}\ \pi_1(L_0+gL_1)\pi_1 = \big\{0, -\frac{1}{4}g(1+\cos^2\theta)\mp\frac{1}{2}\sqrt{\cos^2\theta - g^2\sin^4\theta}\big\}, \quad \pi_1 B\pi_1 = 0.$$

If $g \neq 0$, then the waves that are not polarized along the Kernel of $\pi_1(L_0 + gL_1)$ are exponentially damped in $t$.

To observe short waves over long times $O(1/\varepsilon)$ without any further polarization condition, it is thus natural to make the following assumption:

**Assumption 3.11.** *The system is conservative:* $E + E^* = 0$.

With Assumption 3.11 and Lemma 3.10, equation (3.17) becomes

$$\partial_t \pi_j \mathbf{u}_0 = 0, \quad \text{for all } j.$$

In the following, one pulls the variable $t$ out of the ansatz.

Plugging (3.18) into (3.19) and applying $\partial_X$ leads to

$$\partial_\tau\partial_X\pi_j\mathbf{u}_0 + \pi_j(E+B\partial_y)(c_j - A)^{-1}(E+B\partial_y)\pi_j\mathbf{u}_0 = \partial_X G_{T_j}\pi_j(F(\mathbf{u}_0)\partial_X\mathbf{u}_0). \tag{3.21}$$

The following lemma investigates the nature of the second-order operators present in equation (3.21).

**Lemma 3.12** (the noncritical case: second-order operators)**.** *The following relations hold:*

$$-\partial_\xi^2\tau_j(0,0)\,\pi_j = 2\,\pi_j E(c_j - A)^{-1}E\pi_j,$$

$$0 = \pi_j E(c_j - A)^{-1}B\pi_j + \pi_j B(c_j - A)^{-1}E\pi_j,$$

$$\partial_\xi^2\,\partial_\eta^2\,\tau_j(0,0)\,\pi_j = 4\,\pi_j B(c_j - A)^{-1}B\pi_j.$$

**Proof.** The residue formula at second order in $\xi$ at the noncritical point $(\tau_j(0,\eta), 0, \eta)$ gives, writing only the polarized terms,

$$\frac{1}{2}\partial_\xi^2\tau_j(0,\eta)\,\pi_j(0,\eta) = \pi_j(0,\eta)(E/i + B\eta)(c_j - A)^{-1}(E/i + B\eta)\pi_j(0,\eta). \tag{3.22}$$

In the above equation, $\eta = 0$ leads to the first relation. To derive the other two equations, one could carry the residues computations to third and fourth order. It is more convenient to remark that with (2.10), the first derivative of $\pi_j$ is seen to be

$$\partial_\eta\,\pi_j(\xi,\eta) = -\xi\big(\,\pi_j(\xi,\eta)BL(\xi,\eta)^{-1} + L(\xi,\eta)^{-1}B\pi_j(\xi,\eta)\,\big).$$

Hence, $\partial_\eta \pi_j(0,\eta) = 0$. By Assumption 3.3, $\partial_\xi^2 \partial_\eta \tau_j(\xi, 0) = 0$ for $\xi \neq 0$, and also for $\xi = 0$ by continuity (everything is smooth by Assumption 3.7). Differentiate now (3.22) with respect to $\eta$ and set $\eta = 0$:

$$0 = \pi_j E(c_j - A)^{-1} B\pi_j + \pi_j B(c_j - A)^{-1} E\pi_j.$$

This is the second relation. Differentiating (3.22) twice with respect to $\eta$ and letting $\eta = 0$ yields

$$\frac{1}{2}\partial_\xi^2\, \partial_\eta^2\, \tau_j(0,0)\, \pi_j = 2\, \pi_j B(c_j - A)^{-1} B\pi_j,$$

keeping again only the polarized terms. This is the third relation. □

With Lemma 3.12 and the last point of Proposition 3.9, the component $\pi_j \mathbf{u}_0$ of the main profile satisfies

$$\partial_\tau \partial_X u + (-\frac{1}{2}\partial_\xi^2 \tau_j(0,0) + \frac{1}{4}\partial_\xi^2 \partial_\eta^2 \tau_j(0,0)\, \partial_y^2) u = \partial_X G_{T_j} \pi_j (F(u)\partial_X u). \quad (3.23)$$

One sets $\alpha_j := -\frac{1}{2}\partial_\xi^2 \tau_j(0,0)$, $\beta_j := \frac{1}{4}\partial_\xi^2 \partial_\eta^2 \tau_j(0,0)$, and $F_j := G_{T_j}\pi_j F$ in the following. $\alpha_j$ and $\beta_j$ are real numbers.

3.4.2. *Solving the equation for the main profile.* In (3.23), the point is that $\partial_X^{-1}(\alpha_j + \beta_j \partial_y^2)$ generates a unitary group $e^{-\tau \partial_X^{-1}(\alpha_j + \beta_j \partial_y^2)}$ on $H^s(\mathbb{R}^2_{X,y})$. For a given $\varphi$ in $H^s(\mathbb{R}^2_{X,y})$, the equation

$$\partial_\tau v = \big(e^{\tau \partial_X^{-1}(\alpha_j + \beta_j \partial_y^2)} F_j (e^{-\tau \partial_X^{-1}(\alpha_j + \beta_j \partial_y^2)} v) e^{-\tau \partial_X^{-1}(\alpha_j + \beta_j \partial_y^2)}\big) \partial_X v$$

with initial datum $v(0) = \varphi$ is proved by a classical iteration scheme to have a unique solution $v$ defined on a time interval $[0, \tau^*)$, with $v \in C^0([0,\tau_0], H^s(\mathbb{R}^2))$, for all $0 \le \tau_0 < \tau^*$. Note that $v$ is not regular with respect to $\tau$ and that regularity in $\tau$ is *not* used in the iteration scheme. Then

$$u := e^{-\tau \partial_X^{-1}(\alpha_j + \beta_j \partial_y^2)} v$$

solves (3.23), with $u(0) = \varphi$ and $u \in C^0([0,\tau_0], H^s(\mathbb{R}^2))$. Again, $u$ is only continuous with respect to $\tau$.

We now state a regularity result for the solutions of (3.23) with initial data satisfying the vanishing-mean condition and under the additional assumption that the quasilinear term is conservative. This is done by smoothing the operator $\partial_X^{-1}$ and by using compactness in time. The index $j$ is dropped for convenience. We use the regularization of [14] and [10]:

**Definition 3.13.** For $\mu > 0$, define $\partial_\mu^{-1}$ as the operator given by the symbol

$$\frac{-i\xi}{\xi^2 + \mu^2}.$$

Consider (3.23) in a conservation form, that is, where $F_j = G'$ with a smooth $G$ such that $G(0) = 0$. Then a regular solution of (3.23) with initial datum $\varphi \in \partial_X H^s(\mathbb{R}^2_{X,y})$ is defined as a triple $(\tau^*, u, v)$, with $\tau^* > 0$, a map $u \in C^0([0,\tau_0], H^s(\mathbb{R}^2_{X,y})) \cap C^1([0,\tau_0], H^{s-2}(\mathbb{R}^2_{X,y}))$ and a map $v \in C^0([0,\tau_0], H^{s-2}(\mathbb{R}^2_{X,y}))$, for all $0 < \tau_0 < \tau^*$, such that $u(0) = \varphi$ and

$$\left\{ \begin{array}{l} \partial_\tau u + v = \partial_X G(u) \\ (\alpha_j + \beta_j \partial_y^2) u = \partial_X v. \end{array} \right. \tag{3.24}$$

**Theorem 3.14.** *Suppose that the quasilinear term is conservative: $F_j = G'$, with a smooth $G$ such that $G(0) = 0$. Then for $s > 2$ and for all $\varphi \in \partial_X H^s(\mathbb{R}^2)$, there exists a unique regular solution to* (3.23) *with initial condition $\varphi$.*

**Proof.** *First step: solving the regularized equation.* Look first at the quasilinear regularized equation

$$\partial_\tau u + \partial_\mu^{-1}(\alpha + \beta\, \partial_y^2) u - G'(u)\partial_X u = 0. \tag{3.25}$$

To solve (3.25) by a classical iterative scheme, an energy estimate for the linearized equation is needed. The Hermitian scalar product in $L^2(\mathbb{R}^2_{X,y})$ will be denoted by $(\cdot,\cdot)$. The symbol of $\partial_\mu^{-1}$ being purely imaginary, for all $v \in L^2(\mathbb{R}^2_{X,y})$, one has

$$\mathrm{Re}\ (\partial_\mu^{-1}\, v, v) = 0.$$

The energy estimate for the linearized operator of (3.25) follows: for $w \in C^0([0,\tau_0], L^2(\mathbb{R}^2))$ such that $w$ and $\nabla w$ are in $L^\infty([0,\tau_0] \times \mathbb{R}^2)$,

$$\begin{aligned} &\|u(\tau)\|_{L^2(\mathbb{R}^2)} \\ &\le \|u_0\|_{L^2(\mathbb{R}^2} + C \int_0^\tau \|\partial_\tau u + \partial_\mu^{-1}(\alpha + \beta\, \partial_y^2) u - G'(w)\partial_X u(\tau')\|_{L^2(\mathbb{R}^2)}\, d\tau', \end{aligned}$$

where the constant $C$ depends on $\|w, \nabla w\|_{L^\infty}$ but does *not* depend on $\mu$. The proof now goes the same as for a standard quasilinear differential operator, and the following result holds (see for instance [1], Theorem III.B.1.2):

*For $s > 2$ and for all $\varphi \in \partial_X H^s$, there exist $\tau^* > 0$ such that for all $\mu > 0$ there exists a unique solution $u^\mu$ to* (3.25)*, satisfying $u^\mu \in C^0([0,\tau_0], H^s(\mathbb{R}^2)) \cap C^1([0,\tau_0], H^{s-2}(\mathbb{R}^2))$ for all $0 \le \tau_0 < \tau^*$, and satisfying the initial condition $u^\mu(\tau = 0) = \varphi$.*

Moreover, the proof of the convergence of the iteration scheme yields the uniform estimate

$$||u^\mu||_{L^\infty([0,\tau_0], H^s(\mathbb{R}^2))} \le C(\tau_0), \quad \text{for all } \mu > 0. \tag{3.26}$$

*Second step: the limit* $\mu \to 0$. Let $\psi \in H^s(\mathbb{R}^2_{X,y})$ be such that $\partial_X \psi = \varphi$, and denote by $\{e^{\tau \partial_\mu^{-1}(\alpha+\beta\partial_y^2)}\}_\tau$ the group of operators associated with $\partial_\mu^{-1}(\alpha+\beta\partial_y^2)$. It operates in $H^s$, and its norm is 1. With (3.26) and nonlinear estimates in $H^s$ $(s > 1)$, $G(u^\mu)$ is bounded in $H^s$, uniformly in $\mu$. $u^\mu$ satisfies

$$\partial_\mu^{-1}(\alpha + \beta\partial_y^2)u^\mu(\tau) = e^{-\tau\partial_\mu^{-1}(\alpha+\beta\partial_y^2)}\partial_\mu^{-1}(\alpha + \beta\partial_y^2)\varphi + \int_0^\tau e^{-(\tau-s)\partial_\mu^{-1}(\alpha+\beta\partial_y^2)}\partial_\mu^{-1}\partial_X(\alpha + \beta\partial_y^2)G(u^\mu)\, ds. \tag{3.27}$$

Since $\partial_\mu^{-1}\partial_X$ operates in $H^s$ (and its norm is less than 1), the second term of the right-hand side of (3.27) is bounded in $H^{s-2}$ with respect to $\mu$. Thanks to the hypothesis $\varphi \in \partial_X H^s$, the first term of the right-hand side is seen to be bounded as well:

$$||e^{-t\partial_\mu^{-1}(\alpha+\beta\partial_y^2)}(\alpha + \beta\partial_y^2)\partial_\mu^{-1}\varphi||_{H^{s-2}(\mathbb{R}_X)} \le C||\partial_\mu^{-1}\varphi||_{H^s(\mathbb{R}_X)} \le C||\psi||_{H^s(\mathbb{R}_X)}.$$

Going back to (3.25), we now see that $\partial_\tau u^\mu$ is bounded in $L^\infty(H^{s-2})$, uniformly in $\mu$. By (3.26), changing $u^\mu$ into one of its subsequences if necessary, there exists $u$ such that $u^\mu \rightharpoonup u$ in $L^\infty([0,\tau_0], H^s(\mathbb{R}^2))*$ and $\partial_X u^\mu \rightharpoonup \partial_X u$ in $L^\infty([0,\tau_0], H^{s-1}(\mathbb{R}^2))*$. By Aubin's lemma ([25], Lemma 1.5.2, for instance), the convergence $u^\mu \to u$ also stands in $L^\infty([0,\tau_0], H^{s-1}_{\text{loc}}(\mathbb{R}^2))$, and nonlinear estimates show that $G'(u^\mu) \to G'(u)$ in $L^\infty([0,\tau_0], H^{s-1}_{\text{loc}}(\mathbb{R}^2))$. It follows that $G'(u^\mu)\partial_X u^\mu \to G'(u)\partial_X u$ in $\mathcal{D}'([0,\tau^*[\times\mathbb{R}^2)$. Besides, by the dominated convergence theorem,

$$\partial_X\partial_\mu^{-1}(\alpha + \beta\partial_y^2)u^\mu \to (\alpha + \beta\partial_y^2)u \quad \text{in } L^2.$$

Summing up, the existence of $v \in C^0([0,\tau_0], H^{s-2}(\mathbb{R}^2))$, for all $0 < \tau_0 < \tau^*$, satisfying (3.24) for the above $u$ is proved, and $u$ has the requested regularity.

3.4.3. *The approximate solution.* In Section 3.4.1, a certain number of necessary conditions were derived in order that $u^\varepsilon$ be an approximate solution of problem (3.1). In this section, these equations are solved and it is proved that the function thus obtained actually is an approximate solution.

The component $\pi_j \mathbf{u}_0$ of the main profile must satisfy the transport equation in $T$ and $X$ (3.13) and the equation in $\tau$ (3.21) together with the initial condition $\pi_j\mathbf{u}_0 = \pi_j\mathbf{u}^0$ at $T = \tau = 0$. Such a system is generally overdetermined. The average operators have removed the terms that would prevent us from solving it. First, there is a unique $a_j$ solution of (3.23) with initial condition $\pi_j\mathbf{u}^0$. Let $\tau^*$ be its maximal existence time. Then for all $0 \le \tau_0 < \tau^*$, $a_j \in C^0([0,\tau_0], H^s(\mathbb{R}^2))$. Second, with (3.13), the main profile is chosen to be

$$\pi_j\mathbf{u}_0 = a_j(\tau, X - c_jT, y). \tag{3.28}$$

It satisfies both the transport equation (3.13) and the slow-time scale equation (3.21). With equation (3.28) and with the condition $s > 1$, $\mathbf{u}_0$ is bounded with respect to $T$: $\mathbf{u}_0 \in L^\infty([0,\tau_0] \times \mathbb{R}_T, H^s(\mathbb{R}^2))$. Returning to the initial variables, set $u_0^\varepsilon = \mathbf{u}_0(t/\varepsilon, x/\varepsilon, y, \varepsilon t)$. Provided that $s > 1$, $u_0^\varepsilon$ is bounded with respect to $\varepsilon$ in $L^\infty([0,\tau_0/\varepsilon] \times \mathbb{R}^2_{x,y})$.

We now prove that one can determine the following terms in the expansion of $\mathbf{u}^\varepsilon$ so that they are effectively correctors. This is not obvious because this study was led over two time scales that differ by a factor $\varepsilon^2$, so that errors of size $O(1)$ in the short time scale $T$ may accumulate like $O(1/\varepsilon^2)$ in the long time scale $\tau$. In view of the stability Theorem 3.18, one also has to control the growth of the correctors and of their derivatives with respect to the slow time scale. This is done by a technique of low-frequency cut-offs as in [2].

**Definition 3.15** (low-frequency truncation)**.** Let $\chi$ be a smooth function in $\mathbb{R}_X$, such that $|\chi| \leq 1$, $\chi = 0$ for $|X| < 1$, and $\chi = 1$ for $|X| \geq 2$. Define the Fourier multiplier $\chi^\delta(D_X)$ as the operator acting on $H^s(\mathbb{R}^2)$ by

$$\chi^\delta(D_X): \quad f \to \mathcal{F}^{-1}_{\xi,\eta\to X,y}\ (\chi(\frac{\xi}{\delta})\ \hat{f}(\xi,\eta)).$$

The dominated convergence theorem shows that

$$\chi^\delta f - f \to 0 \quad \text{in } H^s. \tag{3.29}$$

Note that there is no rate of convergence in $H^s$. This will be done in the semilinear case in another class of Banach spaces (Section 3.5).

The action of $\chi^\delta(D_X)$ is a low-frequency truncation: for all $f \in H^s(\mathbb{R}^2)$, $\chi^\delta f$ lies in $\partial_X H^s(\mathbb{R}^2)$, and one has the straightforward estimate

$$||\partial_X^{-1}\chi^\delta f||_{H^s(\mathbb{R}^2)} \leq \frac{1}{\delta}||f|||_{H^s(\mathbb{R}^2)}. \tag{3.30}$$

Set now $\mathbf{u}_0^\delta = \chi^\delta \mathbf{u}_0$. $\mathbf{u}_0^\delta$ satisfies the equation

$$\partial_\tau \pi_j \mathbf{u}_0^\delta + \partial_X^{-1}(\alpha + \beta\partial_y^2)\pi_j \mathbf{u}_0^\delta = G_{T_j}\pi_j\chi^\delta(F(\mathbf{u}_0)\partial_X \mathbf{u}_0). \tag{3.31}$$

Define the first corrector $\mathbf{u}_1^\delta$ by

$$\pi_j \mathbf{u}_1^\delta := -\partial_X^{-1}\pi_j(E + B\partial_y)(c_j - A)^{-1}\mathbf{u}_0^\delta. \tag{3.32}$$

$\mathbf{u}_1^\delta$ lies in $C^0([0,\tau_0] \times \mathbb{R}_T, H^{s-1}(\mathbb{R}^2)) \cap L^\infty([0,\tau_0] \times \mathbb{R}_T, H^{s-1}(\mathbb{R}^2))$, equation (3.14) is solved exactly, and for $s - 1 > 1$, one has the uniform bounds in $\varepsilon$:

$$||\mathbf{u}_1^\delta||_{L^\infty([0,\tau_0]\times\mathbb{R}_T,H^{s-1}(\mathbb{R}^2))} \leq C/\delta \text{ and } ||(\mathbf{u}_1^\delta)^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2)} \leq C/\delta, \tag{3.33}$$

where $C$ is independent of $\varepsilon$ and $\delta$. The key is that with (3.32), $\mathbf{u}_1^\delta$ is smooth with respect to $\tau$. With estimate (3.30) and equation (3.31), one has $||\partial_X^{-1}\partial_\tau \mathbf{u}_0^\delta|| = O(||\partial_X^{-2}\mathbf{u}_0||) = O(1/\delta^2)$ in $L^\infty([0,\tau_0]\times\mathbb{R}_T, H^{s-3}(\mathbb{R}^2))$:

$$||\partial_\tau \mathbf{u}_1^\delta||_{L^\infty([0,\tau_0]\times\mathbb{R}_T,H^{s-2}(\mathbb{R}^2))} \le C/\delta^2, \tag{3.34}$$

where $C$ does not depend on $\delta$.

The second corrector is set to be the solution of

$$(\partial_T + c_j\partial_X)\pi_j\mathbf{u}_2^\delta = \tag{3.35}$$
$$(1-G_{T_j})\pi_j\Big(\chi^\delta(F(\mathbf{u}_0)\partial_X\mathbf{u}_0) - \partial_X^{-1}\sum_{k\neq j}(E+B\partial_y)(c_k-A)^{-1}(E+B\partial_y)\pi_k\mathbf{u}_0^\delta\Big),$$

together with null initial conditions. As $\mathbf{u}_0^\delta$ does not solve (3.23) but solves (3.31), the profile $\mathbf{u}_0^\delta + \varepsilon\mathbf{u}_1^\delta + \varepsilon^2\mathbf{u}_2^\delta$ is not a WKB solution; i.e., $\mathbf{r}_2 \neq 0$ for this profile. However, $\mathbf{r}_2$ will be seen to be $o(\delta)$.

(3.35) is a linear transport equation with a nonlinear source term. To estimate the time growth of $\mathbf{u}_2^\delta$, we will make use of the following proposition ([6], Propositions 3.2 and 3.3):

**Proposition 3.16. i)** *Let $a, b \in \mathbb{R}$, $a \neq b$, and let $f(T,X)$ and $g(T,X)$ be sufficiently smooth functions such that*

$$(\partial_T + a\,\partial_X)f = \partial_X g, \quad \text{and} \quad (\partial_T + b\,\partial_X)g = 0.$$

*Then*

$$||f||_{L^\infty(\mathbb{R}_T,H^s(\mathbb{R}_X))} \ \le\ 2\ ||g||_{L^\infty(\mathbb{R}_T,H^s(\mathbb{R}_X))}.$$

**ii)** *Let $a, b \in \mathbb{R}$, $a \neq b$, and let $f(T,X)$, $g(T,X)$, and $h(T,X)$ be sufficiently smooth functions such that $g, h \in L^\infty(\mathbb{R}_T, H^s(\mathbb{R}_X))$ and*

$$(\partial_T + a\,\partial_X)f = gh,\ (\partial_T + a\,\partial_X)g = 0 \quad \text{and} \quad (\partial_T + b\,\partial_X)h = 0.$$

*Then*

$$\lim_{T\to\infty}\frac{1}{\sqrt{T}}||f||_{H^s(\mathbb{R})} = 0.$$

**Proof.** See [6]. □

The linear source term in (3.35) is estimated with point i) of the proposition. The nonlinear source term in (3.35) is, with $j = +$,

$$(1-G_+)\chi^\delta(F(\mathbf{u}_0)\partial_X\mathbf{u}_0) = \chi^\delta\big(F(\pi_-\mathbf{u}_0)\partial_X\pi_-\mathbf{u}_0 + F_{+-}(\pi_+\mathbf{u}_0^\delta, \pi_-\mathbf{u}_0^\delta)\partial_X\mathbf{u}_0\big), \tag{3.36}$$

where $F_{+-}$ is polynomial. The point is that the first term in the right-hand side of (3.36) involves only $\pi_-\mathbf{u}_0$. With the truncation and point i) of the previous proposition, its contribution to the secular growth of $\mathbf{u}_2^\delta$ is thus $O(1/\delta)$. The contribution of the second term in the right-hand side of (3.36) is estimated by point ii) of the previous proposition. The key hypothesis here is that there are only two traveling components $\pi_+\mathbf{u}_0$ and $\pi_-\mathbf{u}_0$. It yields

$$||\mathbf{u}_2^\delta||_{H^{s-2}(\mathbb{R}^2)} \le C(l_1^\delta(T) + \frac{1}{\delta}), \tag{3.37}$$

uniformly in $\tau$, where for all $\delta > 0$, $\lim_{T\to\infty} \frac{1}{\sqrt{T}} l_1^\delta(T) = 0$.

Differentiating (3.35) with respect to $\tau$, one gets similarly

$$||\partial_\tau \mathbf{u}_2^\delta||_{H^{s-4}(\mathbb{R}^2)} \le C(l_2^\delta(T) + \frac{1}{\delta^2}),$$

uniformly in $\tau$, where for all $\delta$, $\lim_{T\to\infty} \frac{1}{\sqrt{T}} l_2^\delta(T) = 0$. In the original set of variables, the bounds are in $L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2)$:

$$||(\mathbf{u}_2^\delta)^\varepsilon|| \le C\big(\sup_{T\le\frac{\tau_0}{\varepsilon^2}} l_1^\delta(T) + \frac{1}{\delta}\big), \text{ and } ||(\partial_\tau\mathbf{u}_2^\delta)^\varepsilon|| \le C\big(\sup_{T\le\frac{\tau_0}{\varepsilon^2}} l_2^\delta(T) + \frac{1}{\delta^2}\big),$$

where $C$ is independent of $\varepsilon$ and $\delta$. Setting

$$\mathbf{u}^{\varepsilon,\delta} = \varepsilon(\mathbf{u}_0^\delta + \varepsilon\mathbf{u}_1^\delta + \varepsilon^2\mathbf{u}_2^\delta), \tag{3.38}$$

one now proves that $\mathbf{u}^{\varepsilon,\delta}$ is an approximate solution by estimating the residual as defined by equation (3.9). First, for all $\delta$, $\mathbf{u}_1^\delta$ and $\mathbf{u}_2^\delta$ deserve to be called "corrector terms of the ansatz," as shown by the above bounds. Second, the residual $\mathbf{r}^{\varepsilon,\delta}$ associated with $\mathbf{u}^{\varepsilon,\delta}$ is

$$\begin{aligned}\mathbf{r}^{\varepsilon,\delta} = &(\frac{1}{\varepsilon}\tilde{F}(\mathbf{u}^{\varepsilon,\delta})\partial_X\mathbf{u}^{\varepsilon,\delta} - \varepsilon^2 F(\mathbf{u}_0^\delta)\partial_X\mathbf{u}_0^\delta) + \varepsilon^2\big(F(\mathbf{u}_0^\delta)\partial_X\mathbf{u}_0^\delta - \chi^\delta(F(\mathbf{u}_0)\partial_X\mathbf{u}_0)\big)\\ &+ \varepsilon^3((E + B(\partial_y))\mathbf{u}_2^\delta + \partial_\tau\mathbf{u}_1^\delta) + \varepsilon^4\partial_\tau\mathbf{u}_2^\delta.\end{aligned} \tag{3.39}$$

With Assumption 3.1 and the bounds (3.33) and (3.37), the first nonlinear term in the residual is of size $O(\varepsilon^2)o(\delta)$ in $L^\infty([0,\tau_0]\times\mathbb{R}_T, H^{s-3}(\mathbb{R}^2))$. The second nonlinear term stems from (3.35). By (3.29) and the dominated convergence theorem, it is of size $O(\varepsilon^2)o(\delta)$ in $L^\infty([0,\tau_0]\times\mathbb{R}_T, H^{s-1}(\mathbb{R}^2))$. The last two terms are estimated via the above bounds. It yields the following proposition.

**Proposition 3.17.** *The corrector terms of the ansatz* $u_1^{\varepsilon,\delta} = (\mathbf{u}_1^\delta)^\varepsilon$ *and* $u_2^{\varepsilon,\delta} = (\mathbf{u}_2^\delta)^\varepsilon$, *as defined by equations* (3.32) *and* (3.35) *respectively, satisfy, for all* $0 \le \tau_0 < \tau^*$,

$$||\varepsilon u_1^{\varepsilon,\delta} + \varepsilon^2 u_2^{\varepsilon,\delta}||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} \le O(\varepsilon)l(\delta) + \varepsilon^2 \sup_{T\le\frac{\tau_0}{\varepsilon^2}} l_1^\delta(T),$$

*where* $l(\delta) \to \infty$ *as* $\delta \to 0$ *and* $\lim_{T\to\infty} \frac{1}{\sqrt{T}} l_1^\delta(T) = 0$. *The residual* $\mathbf{r}^{\varepsilon,\delta}$ *satisfies, for all* $0 \le \tau_0 < \tau^*$,

$$\frac{1}{\varepsilon^2}||r^{\varepsilon,\delta}||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} \le o(\delta) + \varepsilon \sup_{T\le\frac{\tau_0}{\varepsilon^2}} (l_1^\delta + \varepsilon l_2^\delta)(T) + O(\varepsilon)\tilde{l}(\delta),$$

*where* $\tilde{l}(\delta) \to \infty$ *as* $\delta \to 0$ *and* $\lim_{T\to\infty} \frac{1}{\sqrt{T}} l_2^\delta(T) = 0$.

3.4.4. *Asymptotic stability.* The following theorem asserts that there exists an exact solution $v^\varepsilon$ valid for diffractive times $O(1/\varepsilon)$ which is asymptotically close to $u_0^\varepsilon$, as $\varepsilon \to 0$.

**Theorem 3.18.** *Under Assumptions* 3.3, 3.4, 3.7, *and* 3.11, *there exists* $\tau^* > 0$ *and a unique* $u_0^\varepsilon = [\mathbf{u}_0(T, X, y, \tau)]_{T=t/\varepsilon, X=x/\varepsilon, \tau=\varepsilon t}$, *with* $\mathbf{u}_0 \in C^0([0,\tau_0] \times \mathbb{R}_T, H^s(\mathbb{R}^2))$, *for all* $0 < \tau_0 < \tau^*$, *such that* $\mathbf{u}_0 = \sum_j \pi_j \mathbf{u}_0$, *and for all* $j$,

$$\begin{aligned} (\partial_T + c_j\partial_X)\pi_j\mathbf{u}_0 &= 0, \\ \partial_\tau\partial_X\pi_j\mathbf{u}_0 + (\alpha_j + \beta_j\,\partial_y^2)\pi_j\mathbf{u}_0 &= \pi_j\partial_X G_{T_j}(F(\mathbf{u}_0)\partial_X\mathbf{u}_0), \\ \mathbf{u}_0(T=\tau=0) &= (1-\pi_0)u^0. \end{aligned}$$

*Set* $u_0^\varepsilon = \mathbf{u}_0(t/\varepsilon, x/\varepsilon, y, \varepsilon t)$. (3.1) *with initial datum* $v^\varepsilon = \varepsilon\,((1-\pi_0)u^0 + v^0(\varepsilon))$, *with* $\mathbf{v^0}(\varepsilon) \to 0$ *in* $H^s(\mathbb{R}^2)$, *has a unique solution* $v^\varepsilon$ *defined and smooth on* $[0, \tau^*/\varepsilon) \times \mathbb{R}^2$ *for sufficiently small* $\varepsilon$. *For all* $0 < \tau_0 < \tau^*$ *and sufficiently small* $\varepsilon$, *the asymptotic estimate holds,*

$$\frac{1}{\varepsilon}||v^\varepsilon - \varepsilon u_0^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} = o(1), \tag{3.40}$$

*as* $\varepsilon \to 0$.

**Proof.** The exact solution $v^\varepsilon$ is sought as a perturbation of the family of approximate solutions $u^{\varepsilon,\delta}$ defined in (3.38). Set $u^{\varepsilon,\delta} = \varepsilon[\underline{u}^{\varepsilon,\delta}(X,y,\tau)]_{X=x/\varepsilon,\tau=\varepsilon t}$ with a "semi-profile" $\underline{u}^{\varepsilon,\delta}$ and similarly $r^{\varepsilon,\delta} = \varepsilon^2\underline{r}^{\varepsilon,\delta}$. Look for $v^\varepsilon$ in the form $v^\varepsilon = \varepsilon(\underline{u}^{\varepsilon,\delta} + \underline{w}^{\varepsilon,\delta})$, where $\underline{w}^{\varepsilon,\delta}$ depends on $X, y, \tau$. Define

$$\mathcal{L}^\varepsilon(\underline{u} + \underline{z}, \partial)\underline{z} = (\partial_\tau + \frac{1}{\varepsilon^2}A\,\partial_X + \frac{1}{\varepsilon}(B\partial_y + E))\underline{z} - \frac{1}{\varepsilon^2}\tilde{F}(\varepsilon(\underline{u} + \underline{z}))\partial_X\underline{z}.$$

The equations for $\underline{w}^{\varepsilon,\delta}$ to satisfy in order that $v^\varepsilon$ be an exact solution to (3.1) with initial condition $\varepsilon((1-\pi_0)u^0 + v^0(\varepsilon))$ are

$$\begin{cases} \mathcal{L}^\varepsilon(\underline{u}^{\varepsilon,\delta} + \underline{w}^{\varepsilon,\delta}\partial)\underline{w}^{\varepsilon,\delta} = \frac{1}{\varepsilon^2}(\tilde{F}(\varepsilon(\underline{u}^{\varepsilon,\delta} + \underline{w}^{\varepsilon,\delta})) - \tilde{F}(\varepsilon\underline{u}^{\varepsilon,\delta}))\partial_X\underline{u}^{\varepsilon,\delta} - \underline{r}^{\varepsilon,\delta} \\ \underline{w}^{\varepsilon,\delta}(0) = (1-\pi_0)(1-\chi^\delta)\mathbf{u}^0 + \mathbf{v^0}(\varepsilon) - \varepsilon\mathbf{u}_1^{\varepsilon,\delta}(0). \end{cases} \tag{3.41}$$

Local existence for the solution $\underline{w}^{\varepsilon,\delta}$ of (3.41) over a time $\tau^*(\varepsilon,\delta)$ follows from the standard local existence theory for quasilinear symmetric hyperbolic systems, thanks to the symmetry property stated in Assumption 3.4.

Set then

$$\tau(\varepsilon,\delta) := \sup\,\{0 \le \tau \le \min\,(\tau^*(\varepsilon,\delta),\tau_0),\ \sup_{0\le\tau'\le\tau} \|\underline{w}^{\varepsilon,\delta}(\tau')\|_{H^s} \le 1\}.$$

There exist $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that for $0 < \delta < \delta_1$ and $0 < \varepsilon < \varepsilon_1$, $\|\underline{w}^{\varepsilon,\delta}(0)\|_{H^{s-4}} \le 1$; hence, the definition of $\tau(\varepsilon,\delta)$ makes sense for these choices of $\delta$ and $\varepsilon$. Nonlinear and commutator estimates yield the $H^{s-4}$ estimate for the quasilinear operator $\mathcal{L}^\varepsilon$ (see [15], Lemma 6.3): for $0 \le \tau < \tau(\varepsilon,\delta)$,

$$\begin{aligned}&||\underline{w}^{\varepsilon,\delta}(\tau)||_{H^{s-4}(\mathbb{R}^2)}\\ &\le C(||\underline{w}^{\varepsilon,\delta}(0)||_{H^{s-4}(\mathbb{R}^2)} + \int_0^\tau ||\mathcal{L}^\varepsilon(\underline{u}^{\varepsilon,\delta} + \underline{w}^{\varepsilon,\delta},\partial)\underline{w}^{\varepsilon,\delta}(\tau')||_{H^{s-4}(\mathbb{R}^2)}d\tau').\end{aligned}$$

The nonlinear term in (3.41) can be estimated in $H^{s-4}$, and one has, for $0 \le \tau < \tau(\varepsilon,\delta)$,

$$\begin{aligned}&||\underline{w}^{\varepsilon,\delta}(\tau)||_{H^{s-4}(\mathbb{R}^2)}\\ &\le C(||\underline{w}^{\varepsilon,\delta}(0)||_{H^{s-4}(\mathbb{R}^2)} + ||\underline{r}^{\varepsilon,\delta}||_{L^\infty([0,\tau_0]\times\mathbb{R}^2)} + C\int_0^\tau ||\underline{w}^{\varepsilon,\delta}(\tau')||_{H^{s-4}(\mathbb{R}^2)}\,d\tau').\end{aligned}$$

Gronwall's inequality yields finally

$$||\,\underline{w}^{\varepsilon,\delta}(\tau)||_{H^{s-4}(\mathbb{R}^2)} \le e^{C\tau}(\,||\,\underline{w}^{\varepsilon,\delta}(0)||_{H^{s-4}(\mathbb{R}^2)} + ||\,\underline{r}^{\varepsilon,\delta}||_{L^\infty([0,\tau_0]\times\mathbb{R}^2)}\,).$$

Let now $\mu > 0$. By Proposition 3.17 and by (3.29), there exists $\delta_2 > 0$ such that for fixed $\delta_0$, $0 < \delta_0 < \min\,(\delta_1,\delta_2)$, there exists $\varepsilon_2 > 0$ such that for all $0 < \varepsilon < \min\,(\varepsilon_1,\varepsilon_2)$,

$$\begin{cases} e^{C\tau_0}(||\,\underline{w}^{\varepsilon,\delta_0}(0)||_{H^{s-4}(\mathbb{R}^2)} + ||\,\underline{r}^{\varepsilon,\delta_0}||_{L^\infty([0,\tau_0]\times\mathbb{R}^2)}) \le \mu/3,\\ \|\varepsilon u_1^{\varepsilon,\delta_0} + \varepsilon^2 u_2^{\varepsilon,\delta_0}\|_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} \le \mu/3,\\ \|u_0^\varepsilon - u_0^{\varepsilon,\delta_0}\|_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} \le \mu/3.\end{cases}$$

Hence, for all $0 \le \tau \le \tau(\varepsilon,\delta_0)$ and for all $0 < \varepsilon < \min\,(\varepsilon_2,\varepsilon_1)$,

$$||\underline{w}^{\varepsilon,\delta_0}(\tau)||_{H^{s-4}(\mathbb{R}^2)} \le \mu/3.$$

For $\mu < 1$ and for these choices of $\varepsilon$, by continuity of $\tau \mapsto \|\underline{w}^{\varepsilon,\delta_0}(\tau)\|_{H^{s-4}(\mathbb{R}^2)}$ and by definition of $\tau(\varepsilon,\delta_0)$ and $\tau^*(\varepsilon,\delta_0)$, one has thus $\tau(\varepsilon,\delta_0) = \tau_0$. Besides,

$$\sup_{0\le\tau\le\tau_0} ||\,\underline{w}^{\varepsilon,\delta_0}(\tau)||_{H^{s-4}(\mathbb{R}^2)} \le \mu/3.$$

Finally, for $0 \le \tau \le \tau_0$ and $0 < \varepsilon < \min\,(\varepsilon_2,\varepsilon_1)$,

$$\frac{1}{\varepsilon}||v^\varepsilon - \varepsilon u_0^\varepsilon|| \le ||\underline{w}^{\varepsilon,\delta_0}||_{L^\infty([0,\tau_0]\times\mathbb{R}^2)} + \frac{1}{\varepsilon}||u^{\varepsilon,\delta_0} - \varepsilon u_0^{\varepsilon,\delta_0}|| + ||u_0^{\varepsilon,\delta_0} - u_0^\varepsilon|| \le \mu,$$

in $L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})$.

3.5. **The noncritical case for a semilinear system.** It is shown in this section that under Assumption 3.5, that is for semilinear systems, an approximate solution can be constructed which is asymptotically close to the exact solution with a close initial datum. What's more, a rate of convergence can be obtained.

$E^{\sigma,s,p}$ is well designed for estimating the rate of convergence in (3.29). Let indeed $u^0$ satisfy the condition stated in Assumption 3.5, and let $r'$ be such that $\frac{1}{r}+\frac{1}{r'}=1$. Then

$$\begin{aligned}
&||\chi^\delta u^0 - u^0||_{E^{\sigma,s,p}(\mathbb{R}^2)}\\
&= \Big(\int_{\mathbb{R}}(1+\eta^2)^\sigma\big(\int_{\mathbb{R}}|\chi(\frac{\xi}{\delta})-1|^p(1+\xi^2)^{sp/2}|\hat{\mathbf{u}}^0|^p(\xi,\eta)\ d\xi\big)^{2/p}\ d\eta\Big)^{1/2}\\
&\le C\Big(\int_{\mathbb{R}}(1+\eta^2)^\sigma\big(\int_{\xi\le\delta}(1+\xi^2)^{sp/2}|\hat{\mathbf{u}}^0|^p(\xi,\eta)\ d\xi\big)^{2/p}\ d\eta\Big)^{1/2}\\
&\le C\Big(\int_{\mathbb{R}}(1+\eta^2)^\sigma\big(\int_{\xi\le\delta}1^{r'}d\xi\big)^{2/pr'}\big(\int_{\xi\le\delta}(1+\xi^2)^{spr/2}|\hat{\mathbf{u}}^0|^{pr}\,d\xi\big)^{2/pr}\ d\eta\Big)^{1/2}\\
&\le C\delta^{\frac{1}{pr'}}||u^0||_{E^{\sigma,s,pr}(\mathbb{R}^2)}.
\end{aligned}$$

The growth of a function transported by a linear operator with a linear source term can also be estimated in $E^{\sigma,s,p}$. Let indeed $c\neq c'$ be two real numbers, let $u^0$ be as in Assumption 3.5, and let $u$ and $v$ satisfy

$$(\partial_T + c\,\partial_X)u = v \ \text{ and } \ (\partial_T + c'\,\partial_X)v = 0,$$

together with the initial conditions $u(0)=0,\ v(0)=u^0$. Then $\hat{u}(T,\xi) = e^{-icT\xi}\int_0^t e^{ic's\xi}\hat{u}^0(\xi)ds$, and

$$\begin{aligned}
||(1+\xi^2)^{s/2}\hat{u}(T)||_{L^p(\mathbb{R})} &\le \Big(\int_{\mathbb{R}}(1+\xi^2)^{sp/2}|\frac{e^{i(c'-c)T\xi}-1}{i(c-c')\xi}|^p|\hat{u}^0(\xi)|^p d\xi\Big)^{1/p}\\
&\le \Big(\int_{\mathbb{R}}|\frac{e^{i(c'-c)T\xi}-1}{i(c-c')\xi}|^{pr'}d\xi\Big)^{1/pr'}||(1+\xi^2)^{s/2}\hat{u}^0||_{L^{pr}(\mathbb{R})} \qquad (3.42)
\end{aligned}$$

by Hölder's inequality, where $r'$ is such that $1/r'+1/r=1$. In the integral present in the last inequality, only low frequencies contribute to unbounded terms in $T$:

$$\begin{aligned}
\Big(\int_{\mathbb{R}}|\frac{e^{i(c'-c)T\xi}-1}{i(c-c')\xi}|^{pr'}d\xi\Big)^{1/pr'} &\le C\Big(1+\int_{|\xi|\le 1}|\frac{\sin T\xi}{\xi}|^{pr'}d\xi\Big)^{1/pr'}\\
&\le C(1+T^{1-\frac{1}{pr'}});
\end{aligned}$$

hence,

$$||u(T)||_{E^{\sigma,s,p}(\mathbb{R}^2)} \le C(1+T^{1-\frac{1}{pr'}})||u^0||_{E^{\sigma,s,pr}(\mathbb{R}^2)}. \qquad (3.43)$$

This estimate is a special case of a result of [21]. Low frequencies come into play here because they correspond to critical points of the characteristic variety of

$$\partial_T + \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix} \partial_X.$$

In the case of a nonlinear source term, the resonances in the characteristic variety create small divisors in the integral. This is the context of the following:

**Proposition 3.19** ([21], Proposition 3.3)**.** *Let $u^0, v_1^0, v_2^0 \in E^{\sigma,s,p} \cap E^{\sigma,s,pr}(\mathbb{R}^2)$, with $\sigma$, $s$, $p$, and $r$ as in Assumption* 3.5, *$(pr)'$ such that $1/pr + 1/(pr)' = 1$, and $c$, $c_1$, and $c_2$ three distinct real numbers. Then the unique solution in $E^{\sigma,s,p}(\mathbb{R}^2)$ of*

$$\left\{ \begin{array}{rcl} (\partial_T + c\,\partial_X)u & = & v_1 v_2 \\ u(0) & = & u^0, \end{array} \right. \quad \textit{with} \quad \left\{ \begin{array}{rcl} (\partial_T + c_i\,\partial_X)v_i & = & 0 \\ v_i(0) & = & v_i^0, \end{array} \right. \tag{3.44}$$

*satisfies the estimate*

$$||u(T)||_{E^{\sigma,s,p}} \leq C(1 + T^{1-\frac{1}{(pr)'}})||v_1^0||_{E^{\sigma,s,pr}}||v_2^0||_{E^{\sigma,s,pr}}.$$

**Proof.** See [21]. □

Note that $(pr)' \leq pr'$: as expected, the estimate in the nonlinear case is better than the estimate in the linear case.

The equations for the profiles are those of Section 3.4.1, changing the nonlinearity into a semilinear nonlinearity satisfying Assumption 3.5. The equation (3.23) becomes in this context

$$\partial_\tau \partial_X u + (\alpha_j + \beta_j \partial_y^2)u = \partial_X G_{T_j} \pi_j f(u). \tag{3.45}$$

$E^{\sigma,s,p}$ is defined by growth and integrability conditions on the Fourier side, so that the groups of operators $\{e^{\tau\partial_X^{-1}(\alpha_j + \beta_j \partial_y^2)}\}_\tau$ acts on $E^{\sigma,s,p}(\mathbb{R}^2)$, for all $j$. Under Assumption 3.5, equations (3.13) and (3.45) can thus be solved with an initial condition $\mathbf{u}^0 \in E^{\sigma,s,p}(\mathbb{R}^2) \cap E^{\sigma,s,pr}(\mathbb{R}^2)$. Let $\mathbf{u}_0 \in C^0 \cap L^\infty([0,\tau_0] \times \mathbb{R}_T, E^{\sigma,s,p} \cap E^{\sigma,s,pr}(\mathbb{R}^2))$ be the solution. One sets $\mathbf{u}_0^\delta = \chi^\delta \mathbf{u}_0$ in the following.

The components of $\mathbf{u}_1$ are defined by equation (3.14). By (3.13), there exists $a_j$ solution of (3.45) such that $\pi_j \hat{\mathbf{u}}_0^\delta(T,\xi,\eta,\tau) = e^{-ic_jT\xi}\hat{a}_j^\delta(\xi,\eta,\tau)$. Thus $\mathbf{u}_1^\delta$ is estimated as in (3.43):

$$||\mathbf{u}_1^\delta(T)||_{E^{\sigma-1,s,p}(\mathbb{R}^2)} \leq C(1 + T^{1-\frac{1}{pr'}})||a_j^\delta||_{E^{\sigma,s,pr}(\mathbb{R}^2)} \leq C(1 + T^{1-\frac{1}{pr'}}),$$

where $C$ is independent of $\delta$.

Deriving (3.14) with respect to $\tau$, one gets a similar bound for $\partial_\tau \mathbf{u}_1^\delta$:

$$||\partial_\tau \mathbf{u}_1^\delta||_{E^{\sigma-2,s,p}(\mathbb{R}^2)} \leq CT^{1-\frac{1}{pr'}}||\partial_\tau a_j^\delta||_{E^{\sigma-2,s,pr}(\mathbb{R}^2)}$$

$$\leq CT^{1-\frac{1}{pr'}}(||\partial_X^{-1}a_j^\delta||_{E^{\sigma,s,pr'}(\mathbb{R}^2)} + ||a_j^\delta||_{E^{\sigma-2,s,pr}(\mathbb{R}^2)}) \leq \frac{C}{\delta}T^{1-\frac{1}{pr'}}.$$

Set for the second corrector

$$\begin{aligned}&(\partial_T + c_j\partial_X)\pi_j\mathbf{u}_2^\delta \\ &= (1-G_{T_j})\pi_j(f(\mathbf{u}_0^\delta) + \partial_X^{-1}\sum_{k\neq j}(E+B\partial_y)(c_k-A)^{-1}(E+B\partial_y)\pi_k\mathbf{u}_0^\delta),\end{aligned} \tag{3.46}$$

together with null initial conditions. Again, the term $G_{T_j}\pi_j(f(\mathbf{u}_0^\delta) - \chi^\delta f(\mathbf{u}_0))$ was dropped. The contribution of the linear source term in (3.46) to the growth of $\mathbf{u}_2^\delta$ is estimated by (3.43), and the contribution of the nonlinear term is estimated by Proposition 3.19:

$$\begin{aligned}&||\mathbf{u}_2^\delta(T)||_{E^{\sigma-2,s,p}(\mathbb{R}^2)} \\ &\leq C(1+T^{1-\frac{1}{pr'}})||\partial_X^{-1}\mathbf{u}_0^\delta||_{E^{\sigma,s,pr}(\mathbb{R}^2)} + C(1+T^{1-\frac{1}{(pr)'}})||\mathbf{u}_0||^2_{E^{\sigma-2,s,pr}} \\ &\leq \frac{C}{\delta}(1+T^{1-\frac{1}{pr'}}).\end{aligned}$$

The estimate for $\partial_\tau\mathbf{u}_2^\delta$ follows:

$$||\partial_\tau\mathbf{u}_2^\delta(T)||_{E^{\sigma-4,s,p}(\mathbb{R}^2)} \leq \frac{C}{\delta^2}(1+T^{1-\frac{1}{pr'}}).$$

Set $\delta = \varepsilon^\alpha$ and the approximate solution

$$\mathbf{u}^\varepsilon = \varepsilon(\mathbf{u}_0^{\varepsilon^\alpha} + \varepsilon\,\mathbf{u}_1^{\varepsilon^\alpha} + \varepsilon^2\,\mathbf{u}_2^{\varepsilon^\alpha}). \tag{3.47}$$

The residual is

$$\begin{aligned}\mathbf{r}^\varepsilon = &\big(f(\mathbf{u}^\varepsilon) - \varepsilon^2 f(\mathbf{u}_0^{\varepsilon^\alpha})\big) + \varepsilon^2\sum_j \pi_j G_{T_j}\big(f(\mathbf{u}_0^{\varepsilon^\alpha}) - \chi^{\varepsilon^\alpha}f(\mathbf{u}_0)\big) \\ &+ \varepsilon^3\big((E+B(\partial_y))\mathbf{u}_2^{\varepsilon^\alpha} + \partial_\tau\mathbf{u}_1^{\varepsilon^\alpha}\big) + \varepsilon^4\partial_\tau\mathbf{u}_2^{\varepsilon^\alpha}.\end{aligned}$$

One wishes to estimate $r^\varepsilon$ in $L^\infty([0,\tau_0/\varepsilon], E^{\sigma-4,s,p}(\mathbb{R}^2))$. With Assumption 3.1 and the bounds for the correctors, the first semilinear term is of size $O(\varepsilon^{3-2(1-1/pr')}) + O(\varepsilon^{4-\alpha-2(1-1/pr')})$. The second semilinear term is of size $O(\varepsilon^{2+\alpha/pr'})$ with the bound for $\mathbf{u}_0^{\varepsilon^\alpha}$ and the estimate for the rate of convergence of the low-frequency truncation in $E^{\sigma,s,p}$. With the bounds for the correctors, the second term is of size $O(\varepsilon^{3-\alpha-2(1-1/pr')})$. The last term is of size $O(\varepsilon^{4-2\alpha-2(1-1/pr')})$. Besides, by (3.43), the error on the initial condition is of size $O(\varepsilon^{1+\alpha/pr'})$. In view of the stability theorem, the residual must be of size $o(\varepsilon^2)$ and the error on the initial datum must be of size $o(\varepsilon)$.

The best $\alpha$ is $\alpha = \frac{2-pr'}{1+pr'} > 0$. Set $\gamma = \frac{2-pr'}{pr'(1+pr')}$. It yields a residual of size $O(\varepsilon^{2+\gamma})$ and an error on the initial condition of size $O(\varepsilon^{1+\gamma})$. The results of this section are summed up in the following:

**Proposition 3.20.** *Set* $\delta = \varepsilon^{\frac{2-pr'}{1+pr'}}$ *and* $\gamma = \frac{2-pr'}{pr'(1+pr')} < 1/2$. *Then the corrector terms of the ansatz* $u_1^\varepsilon$ *and* $u_2^\varepsilon$ *as defined by equations* (3.32) *and* (3.46) *respectively, satisfy*

$$||\varepsilon u_1^\varepsilon + \varepsilon^2 u_2^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}_t\times\mathbb{R}^2_{x,y})} = O(\varepsilon^{\frac{2}{pr'}-1}),$$

*and the residual* $\mathbf{r}^\varepsilon$ *of the approximate solution* $u^\varepsilon$ *defined in* (3.47) *satisfies*

$$\frac{1}{\varepsilon^2}||\mathbf{r}^\varepsilon||_{L^\infty([0,\tau_0]\times\mathbb{R}_T,H^{s-4}(\mathbb{R}^2))} = O(\varepsilon^\gamma), \text{ and } \frac{1}{\varepsilon^2}||r^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^d_{x,y})} = O(\varepsilon^\gamma).$$

As in the quasilinear case, the above proposition is the main step of the proof of the following:

**Theorem 3.21.** *Under Assumptions* 3.3*,* 3.5*,* 3.7*, and* 3.11*, there exists* $\tau^* > 0$ *and a unique profile* $\mathbf{u}_0 \in C^0([0,\tau_0]\times\mathbb{R}_T, E^{\sigma,s,p}\cap E^{\sigma,s,pr}(\mathbb{R}^2))$*, for all* $0 < \tau_0 < \tau^*$*, such that* $\mathbf{u}_0 = \sum_j \pi_j \mathbf{u}_0$*, and for all* $j$*,*

$$\begin{array}{rcl} (\partial_T + c_j\partial_X)\pi_j\mathbf{u}_0 &=& 0, \\ \partial_\tau\partial_X\pi_j\mathbf{u}_0 + (\alpha_j + \beta_j\,\partial_y^2)\pi_j\mathbf{u}_0 &=& \pi_j\partial_X G_{T_j} f(\mathbf{u}_0), \\ \mathbf{u}_0(T=\tau=0) &=& (1-\pi_0)u^0. \end{array}$$

*Set* $u_0^\varepsilon = \mathbf{u}_0(t/\varepsilon, x/\varepsilon, y, \varepsilon t)$. (3.1) *with initial datum* $v^\varepsilon = \varepsilon\,((1-\pi_0)u^0 + v^0(\varepsilon))$*, with* $||\mathbf{v^0}(\varepsilon)|| = O(\varepsilon^\gamma)$ *in* $E^{\sigma,s,p}$*, has a unique solution* $v^\varepsilon$ *defined and smooth on* $[0,\tau^*/\varepsilon)\times\mathbb{R}^2$ *for sufficiently small* $\varepsilon$*. For all* $0 < \tau_0 < \tau^*$ *and sufficiently small* $\varepsilon$*, the asymptotic estimate holds:*

$$\frac{1}{\varepsilon}||v^\varepsilon - \varepsilon u_0^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} = O(\varepsilon^\gamma).$$

3.6. **The critical case.** The initial data are now supposed to be polarized along the kernel of $A$ : $u^\varepsilon(t=0) = \varepsilon\,\pi_0 u^0$, and one makes the following assumption, which is satisfied by the physical examples:

**Assumption 3.22.** *For all* $\eta$*,* $(0,0,\eta)$ *is a critical point of* $Char_{\mathbb{R}}M$*.*

3.6.1. *Profile equations.* Throughout this section, the notation is that of a quasilinear system. It is easily adapted to the case of a semilinear system, considered in Section 3.6.3. Let $\mathbf{u}^\varepsilon$ satisfy (3.6). Compute

$$L(\partial)\mathbf{u}^\varepsilon - \tilde{F}(\mathbf{u}^\varepsilon)\partial_x\mathbf{u}^\varepsilon = \sum_k \varepsilon^k \mathbf{r}_k(X,y,t,\tau). \tag{3.48}$$

Again, the strategy is to annihilate the first three terms of the residual, which yields

$$A\partial_X \mathbf{u}_0 = 0 \tag{3.49}$$

$$A\partial_X \mathbf{u}_1 + (\partial_t + E + B\partial_y)\mathbf{u}_0 = 0 \tag{3.50}$$

$$A\partial_X \mathbf{u}_2 + (\partial_t + E + B\partial_y)\mathbf{u}_1 + \partial_\tau \mathbf{u}_0 = F(\mathbf{u}_0)\partial_X \mathbf{u}_0. \tag{3.51}$$

$A$ being hermitian, $(\text{Ker } A)^\perp = \text{Im } A$, and denoting by $\pi_0$ the projector onto Ker $A$, the following equivalence holds:

$$A\,x = b \Leftrightarrow (\,\pi_0 b = 0, \ \text{ and } \ (1-\pi_0)x = A^{-1}b\,),$$

where $A^{-1}$ is the partial inverse of $A$ naturally defined on Ker $A \oplus$ Ran $A$. Therefore, the above set of equations is equivalent to

$$\pi_0 \mathbf{u}_0 = \mathbf{u}_0 \tag{3.52}$$

$$\pi_0\,(\partial_t + E + B\partial_y)\,\pi_0\,\mathbf{u}_0 = 0 \tag{3.53}$$

$$(1-\pi_0)\partial_X \mathbf{u}_1 = -A^{-1}(\partial_t + E + B\partial_y)\pi_0 \mathbf{u}_0 \tag{3.54}$$

$$\begin{aligned}\pi_0\,(\partial_t + E + B\partial_y)\,\pi_0\,\mathbf{u}_1 = &-\pi_0(\partial_t + E + B\partial_y)A^{-1}(\partial_t + E + B\partial_y)\pi_0 \mathbf{u}_0\\ &- \partial_\tau \pi_0 \mathbf{u}_0 + \pi_0 F(\mathbf{u}_0)\partial_X \mathbf{u}_0\end{aligned} \tag{3.55}$$

$$(1-\pi_0)\partial_X \mathbf{u}_2 = -A^{-1}((\partial_t + E + B\partial_y)\mathbf{u}_1 + \partial_\tau \mathbf{u}_0 - F(\mathbf{u}_0)\partial_X \mathbf{u}_0). \tag{3.56}$$

The first relation expresses the polarization of the main profile $\mathbf{u}_0$. The second one is a differential equation for $\mathbf{u}_0$ in the variables $(t, y)$. (3.55) is a *compatibility* condition on $\pi_0 \mathbf{u}_1$.

**Remark 3.23.** The presence in (3.1) of a dispersion term forces one to consider different components of the main profile, each one oscillating with respect to $t$ with a possibly different frequency, as is shown below. Setting $\pi_0 \mathbf{u}_1 = 0$, one would be left with incompatible equations for the components of $\mathbf{u}_0$. The necessity of the presence of a polarized corrector at first order thus stems from the dispersive and nonlinear nature of the system.

Denote by $\tau_1, \dots, \tau_s$ the 0-group at $(0, 0, \eta)$ and by $\pi_1, \dots, \pi_s$ the associated eigenprojectors. By assumption, 0 is a semisimple eigenvalue of $A$, so that by Proposition 2.8, the spectrum of $\pi_0(E/i + B\,\eta)\pi_0$ is the collection of derivatives $\{\partial_\xi \tau_j(0, \eta),\ 1 \leq j \leq s\}$. If $E + E^* > 0$, then these derivatives generically have nonvanishing imaginary parts. Writing the spectral representation of the tangent operator

$$\pi_0(E/i\, + B\,\eta)\pi_0 = \sum_j (\partial_\xi \tau_j(0, \eta) + N_j)\tilde{\pi}_j,$$

one sees with equation (3.53) that the component of $\mathbf{u}_0$ polarized along a $\pi_j$ such that $\partial_\xi \tau_j(0,\eta) \notin \mathbb{R}$ behaves exponentially in $t$.

**Example.** For the nonconservative equations of ferromagnetism, with the notation of Section 2.3,

$$\begin{aligned}&\mathrm{sp}\ \pi_0(L_0 + gL_1)\pi_0\\ &= \Big\{0, -g(\alpha + \frac{1}{2}\sin^2\theta) \pm \frac{1}{2}\sqrt{-4\alpha(\alpha+\sin^2\theta) + g^2\sin^4\theta}\Big\}, \quad \pi_0 B\pi_0 = 0.\end{aligned}$$

If $g \neq 0$, then the waves that are not polarized along the Kernel of $\pi_0(L_0 + gL_1)\pi_0$ are exponentially damped in $t$.

To ensure that short waves with initial data $\varepsilon\,\pi_0 u^0$ do propagate for long times $O(1/\varepsilon)$ without any further polarization condition, one is thus led to make Assumption 3.11, that is, to suppose that the system is conservative, just as in the noncritical case.

Under Assumption 3.11, Butler's Theorem 2.10 applies for the symbol of the long-wave operator. In particular, the projectors of the 0-group are holomorphic in all section planes $\eta = \eta_0$. What is more, Butler's theorem applies to the tangent operator as well, so that for all $j$, $\eta \mapsto \partial_\xi\tau_j(0,\eta)$ is holomorphic. Denote by $\pi_i(0,\eta_0)$ the limit of $\pi_i$ in the section plane $\eta = \eta_0$ as $\xi \to 0$. One has

$$\pi_0 = \sum_1^s \pi_i(0,\eta) \tag{3.57}$$

for all $\eta$, and similarly, denoting by $\pi_j(0, D_y)$ the operator whose symbol is $\pi_j(0,\eta)$,

$$\mathbf{u}_0 = \pi_0\mathbf{u}_0 = \sum_1^s \pi_i(0, D_y)\mathbf{u}_0.$$

We now describe the nature of the differential operators present in the profile equations. Again, this is done by residue computations.

**Lemma 3.24** (the critical case: first-order operators)**.** *The following relations hold:*

$$\pi_i(0,\eta)(E/i + B\,\eta)\pi_i(0,\eta) = -\partial_\xi\tau_i(0,\eta)\,\pi_i(0,\eta), \quad \textit{for all}\ \ i,$$

*and*

$$\pi_i(0,\eta)(E/i + B\,\eta)\pi_j(0,\eta) = 0, \quad \textit{for all}\ \ i \neq j.$$

**Proof.** The residue formula at $(0,0,\eta)$ at first order in $\xi$ gives

$$\sum_1^s \partial_\xi\tau_j(0,\eta)\,\pi_j(0,\eta) = -\pi_0(E/i + B\,\eta)\pi_0,$$

and with (3.57), the lemma follows. □

Consequently, (3.53) becomes for all $j$

$$(\partial_t - i\partial_\xi \tau_j(0, D_y))\pi_j(0, D_y)\,\mathbf{u}_0 = 0. \tag{3.58}$$

There is no coupling in the variable $t$ for the different components of $\mathbf{u}_0$. Note that at $(0, 0, 0)$, the components $\pi_i(0, 0)\mathbf{u}_0 = \pi_i^0\mathbf{u}_0$ of the main profiles are coupled in general, for they satisfy

$$(\partial_t - i\pi_i^0 E\pi_i^0)\pi_i^0\mathbf{u}_0 + \sum_j \pi_i^0 B\pi_j^0\mathbf{u}_0 = 0$$

and the terms $\pi_i^0 B\pi_j^0$ do not vanish in general.

To obtain decoupled equations in $\tau$ as well, one sets for all $j$

$$I_j := \{k = 1, \dots s,\ \partial_\xi \tau_k(0, \eta) = \partial_\xi \tau_j(0, \eta)\}, \quad \text{and} \quad \Pi_j(0, \eta) := \sum_{k \in I_j} \pi_k(0, \eta).$$

**Remark 3.25.** The symmetry properties of $L$ and the fact that the transverse perturbation is one-dimensional ($y \in \mathbb{R}$) allowed us to apply Butler's theorem to the tangent operator, yielding smoothness for the symbol of $\partial_\xi \tau_j(0, D_y)$. If $y \in \mathbb{R}^{d-1}$ with $d > 2$, the critical points of the characteristic variety of the tangent operator might well not be isolated, so that the functions $\eta \mapsto \partial_\xi \tau_j(0, \eta)$ might be multivalued. However, the solution of (3.53) could still be decomposed in a sum of simpler waves as in (3.58), by studying the structure of the set of critical points of $\mathrm{Char}_{\mathbb{R}}(\pi_0(E/i + B(\eta))\pi_0)$, as done in [15], Proposition 3.2.

Projecting (3.55) in the direction $\Pi_j(0, D_y)$ and applying $\partial_X$, one obtains

$$\begin{aligned}(\partial_t - i\partial_\xi \tau_j(0, D_y))\partial_X \Pi_j(0, D_y)\,\mathbf{u}_1 \\ = -\partial_\tau \partial_X \Pi_j(0, D_y)\mathbf{u}_0 + \Pi_j(0, D_y)\partial_X(F(\mathbf{u}_0)\partial_X \mathbf{u}_0) \\ - \Pi_j(0, D_y)(E + B\partial_y)A^{-1}(E + B\partial_y)\mathbf{u}_0.\end{aligned} \tag{3.59}$$

To sort out the different components of $\mathbf{u}_0$ in this latter equation, averaging projectors are again used. With Proposition 3.9, averaging (3.59) with respect to $T_j(\partial) = \partial_t - i\partial_\xi \tau_j(0, D_y)$ leads to

$$\begin{aligned}\partial_\tau \partial_X \Pi_j(0, D_y)\mathbf{u}_0 - \Pi_j(0, D_y)(E + B\partial_y)A^{-1}(E + B\partial_y)\Pi_j(0, D_y)\mathbf{u}_0 \\ = \partial_X G_{T_j} \Pi_j(0, D_y)F(\mathbf{u}_0)\partial_X \mathbf{u}_0.\end{aligned} \tag{3.60}$$

With the last point of Proposition 3.9, it follows that the equations for the components $\Pi_j(0, \eta)\mathbf{u}_0$ are decoupled.

The residue formula at second order for the 0-group at $(0,0,\eta)$ gives

$$\begin{aligned}&\sum_j\big(\frac{1}{2}\partial_\xi^2\tau_j(0,\eta)\pi_j(0,\eta)+\partial_\xi\tau_j(0,\eta)\partial_\xi\pi_j(0,\eta)\big)\\&=\pi_0(E/i+B\eta)A^{-1}(E/i+B\eta)\pi_0+\pi_0(E/i+B\,\eta)\pi_0(E/i+B\,\eta)A^{-1}\\&+A^{-1}(E/i+B\eta)\pi_0(E/i+B\eta)\pi_0,\end{aligned}$$

and projecting along $\Pi_j(0,\eta)$, one gets

$$\begin{aligned}&\sum_{k\in I_j}\big(\frac{1}{2}\partial_\xi^2\tau_k\,\pi_k\ +\ \partial_\xi\tau_k\partial_\xi\pi_k\big)(0,\eta)\\&=\Pi_j(0,\eta)(E/i+B\,\eta)A^{-1}(E/i+B\,\eta)\Pi_j(0,\eta)=:M_j(\eta).\end{aligned}$$

$M_j(\eta)$ is an $|I_j|\times|I_j|$ block matrix whose block of indices $j_0$, $k_0$ has size rank $\pi_{j_0}(0,\eta)\times$ rank $\pi_{k_0}(0,\eta)$. The diagonal blocks are the scalar matrices $\frac{1}{2}\partial_\xi^2\tau_{j_0}(0,\eta)\mathrm{Id}_{j_0}$, where $\mathrm{Id}_{j_0}$ is the identity matrix over Ran $\pi_{j_0}(0,\eta)$.

The slow-time evolution equations are

$$\partial_\tau\partial_X\Pi_j(0,D_y)\mathbf{u}_0-M_j(D_y)\Pi_j(0,D_y)\mathbf{u}_0=\partial_XG_{T_j}\Pi_j(0,D_y)F(\mathbf{u}_0)\partial_X\mathbf{u}_0. \tag{3.61}$$

3.6.2. *The approximate solution and its asymptotic stability for a quasilinear system.* With $\mathbf{u}^0\in H^s$, equations (3.58) and (3.61) can be solved simultaneously, and the components of the main profile have the same regularity as in the noncritical case: there exists $\tau^*>0$ such that for all $0<\tau_0<\tau^*$, $\Pi_j(0,D_y)\mathbf{u}_0$ lie in $C^0([0,\tau_0]\times\mathbb{R}_t,H^s(\mathbb{R}^2))\cap L^\infty([0,\tau_0]\times\mathbb{R}_t,H^s(\mathbb{R}^2))$. The need for smoothness in $\tau$ for the correctors again leads us to truncate the main profile. Set $\mathbf{u}_0^\delta=\chi^\delta\mathbf{u}_0$, with the same notation as in the noncritical case. The equation satisfied by $\mathbf{u}_0^\delta$ for long times is

$$\partial_\tau\Pi_j(0,D_y)\mathbf{u}_0^\delta-\partial_X^{-1}M_j(D_y)\Pi_j(0,D_y)\mathbf{u}_0^\delta=\Pi_j(0,D_y)G_{T_j}\chi^\delta\big(F(\mathbf{u}_0)\partial_X\mathbf{u}_0\big). \tag{3.62}$$

Equation (3.54) is solved exactly with a main profile $\mathbf{u}_0^\delta$: one sets

$$(1-\pi_0)\mathbf{u}_1^\delta:=-A^{-1}\partial_X^{-1}(E+B\partial_y)\mathbf{u}_0^\delta, \tag{3.63}$$

which with (3.30) yields the bounds in $L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2)$:

$$||(1-\pi_0)(\mathbf{u}_1^\delta)^\varepsilon||=O(1/\delta),\quad\text{and}\quad||(1-\pi_0)(\partial_\tau\mathbf{u}_1^\delta)^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2)}=O(1/\delta^2).$$

Contrary to the noncritical case, the growth of the polarized components of $\mathbf{u}_1^\delta$ are estimated by a classical sublinear growth condition. From equations (3.59) and (3.60), one sets

$$(\partial_t-i\partial_\xi\tau_j(0,D_y))\Pi_j(0,D_y)\,\mathbf{u}_1^\delta=(1-G_{T_j})\Pi_j(0,D_y)\partial_X^{-1}\Big(\chi^\delta(F(\mathbf{u}_0)\partial_X\mathbf{u}_0)$$

$$+ (E + B\partial_y)A^{-1}(E + B\partial_y)(1 - \Pi_j(0, D_y))\mathbf{u}_0^\delta\Big). \tag{3.64}$$

At this stage, the same approximation as in the noncritical case was made—namely, to drop the term $\Pi_j(0, D_y)(F(\mathbf{u}_0^\delta)\partial_X \mathbf{u}_0^\delta - \chi^\delta(F(\mathbf{u}_0)\partial_X \mathbf{u}_0))$ that would appear in the right-hand side of (3.64) according to the profile equations of the previous section. This latter term will be present in the residual. With Proposition 3.9, $\mathbf{u}_1^\delta$ satisfies the sublinear growth condition

$$||(\Pi_j(0, D_y)\mathbf{u}_1^\delta)^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2)} \leq \sup_{0\leq t\leq\tau_0/\varepsilon} l_3^\delta(t),$$

where for all $\delta > 0$, $\lim_{t\to\infty} \frac{1}{t} l_3^\delta(t) = 0$. The sublinear growth condition applied to the derivative of (3.64) with respect to $\tau$ yields

$$||(\partial_\tau \Pi_j(0, D_y)\mathbf{u}_1^\delta)^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2)} \leq \sup_{0\leq t\leq\tau_0/\varepsilon} l_4^\delta(t),$$

where for all $\delta > 0$, $\lim_{t\to\infty} \frac{1}{t} l_4^\delta(t) = 0$. The functions $l_3^\delta$ and $l_4^\delta$ are unbounded in the limit $\delta \to 0$, but as in Section 3.4.3, one will consider the limit $\lim_{\varepsilon\to 0} \sup_{0\leq t\leq\tau_0/\varepsilon} \varepsilon l_i^{\delta_0}(t)$ for a fixed, carefully chosen $\delta_0$.

According to (3.56), one sets

$$\mathbf{u}_2^\delta := (1-\pi_0)\mathbf{u}_2^\delta := -A^{-1}\partial_X^{-1}((\partial_t + E + B\partial_y)\mathbf{u}_1^\delta + \partial_\tau \mathbf{u}_0^\delta + F(\mathbf{u}_0^\delta)\partial_X \mathbf{u}_0^\delta). \tag{3.65}$$

As before, one gets the bounds

$$||(\mathbf{u}_2^\delta)^\varepsilon||_{L^\infty([0,\tau_0]\times\mathbb{R}^2)} \leq L(\delta) + \frac{1}{\delta} \sup_{0\leq t\leq\tau_0/\varepsilon} l_3^\delta(t),$$

and

$$||(\partial_\tau \mathbf{u}_2^\delta)^\varepsilon||_{L^\infty([0,\tau_0]\times\mathbb{R}_t, H^{s-4}(\mathbb{R}^2))} \leq \tilde{L}(\delta) + \frac{1}{\delta} \sup_{0\leq t\leq\tau_0/\varepsilon} l_4^\delta(t),$$

where $L(\delta) \to \infty$ and $\tilde{L}(\delta) \to \infty$ as $\delta \to 0$.

Define the approximate solution by

$$\mathbf{u}^\varepsilon = \varepsilon(\mathbf{u}_0^\delta + \varepsilon \mathbf{u}_1^\delta + \varepsilon^2(1 - \pi_0)\mathbf{u}_2^\delta). \tag{3.66}$$

Set $u_1^{\varepsilon,\delta} = (\mathbf{u}_1^\delta)^\varepsilon$ and $u_2^{\varepsilon,\delta} = (\mathbf{u}_2^\delta)^\varepsilon$. Consider now the residual $\mathbf{r}^{\varepsilon,\delta}$ of the approximate solution $u^\varepsilon$ defined in (3.66). The two new terms in the residual resulting from the above approximations of the profile equations for $\pi_0 \mathbf{u}_1^\delta$ and $\mathbf{u}_2^\delta$ are $O(\varepsilon^2)o(\delta)$. Then the above bounds show that given $\mu > 0$, one can choose $\delta_0$ small enough such that

$$||\varepsilon u_1^{\varepsilon,\delta_0} + \varepsilon^2 u_2^{\varepsilon,\delta_0}||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} \leq \mu,$$

and

$$\frac{1}{\varepsilon^2}||r^{\varepsilon,\delta_0}||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^d_{x,y})} \leq \mu,$$

for $\varepsilon$ small enough. As in the noncritical case, these estimates yield the following:

**Theorem 3.26.** *Under assumptions* 3.3*,* 3.4*,* 3.22*, and* 3.11*, there exists* $\tau^* > 0$ *and a unique profile* $\mathbf{u}_0 \in C^0([0,\tau_0]\times\mathbb{R}_T, \partial_X H^s(\mathbb{R}^2)) \cap C^1([0,\tau_0],$ $H^{s-2}(\mathbb{R}^2))$ *for all* $0 < \tau_0 < \tau^*$*, satisfying* $\mathbf{u}_0 = \sum_j \Pi_j(0,D_y)\mathbf{u}_0$ *and*

$$(\partial_t - i\partial_\xi\tau_j(0,D_y))\Pi_j(0,D_y)\,\mathbf{u}_0 = 0,$$
$$\partial_\tau\partial_X\Pi_j(0,D_y)\mathbf{u}_0 - M_j(D_y)\Pi_j(0,D_y)\mathbf{u}_0 = \partial_X\Pi_j(0,D_y)G_{T_j}(F(\mathbf{u}_0)\partial_X\mathbf{u}_0),$$
$$\mathbf{u}_0(T=\tau=0) = \pi_0 u^0.$$

*Set* $u_0^\varepsilon = \mathbf{u}_0(x/\varepsilon, y, t, \varepsilon t)$*.* (3.1) *with initial condition* $v^\varepsilon = \varepsilon(\pi_0 u^0 + v^0(\varepsilon))$*, with* $\mathbf{v^0}(\varepsilon) \to 0$ *in* $H^s$*, has a unique solution* $v^\varepsilon$ *defined and smooth on* $[0,\tau^*/\varepsilon)\times\mathbb{R}^2$ *for sufficiently small* $\varepsilon$*. For all* $0 < \tau_0 < \tau^*$*, the asymptotic estimate holds:*

$$\frac{1}{\varepsilon}||v^\varepsilon - \varepsilon u_0^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} = o(1).$$

To improve this convergence rate, one needs for the correctors a better estimate than the sublinear growth condition. This is achieved in the semilinear case for a certain class of systems.

3.6.3. *The approximate solution and its asymptotic stability for a semilinear system of Maxwell type.* This section addresses the semilinear critical case for a certain class of hyperbolic operators which exhibit symmetry and geometric properties. Among such systems, one finds the Maxwell equations, (2.18) and (2.20)–(2.21). A stability theorem with an explicit convergence rate is stated.

The system (2.18) takes the form (2.1) via (2.19). For $\gamma \neq 0$, the bounded branches of its characteristic variety (Figure 1) have different limits for $\xi \to \infty$. System (2.22) has the same property. This is a property of non-asymptoticity for the bounded branches. Via the definition of the long-wave operator in Section 3.2, it implies $\partial_\xi\tau_j(0,0) \neq \partial_\xi\tau_k(0,0)$ for $j \neq k$, with the notation of the previous section for the 0-group. Rouché's theorem yields continuity for the spectrum of the tangent operator, so that for $\eta$ in a neighborhood of 0 and $j \neq k$, $\partial_\xi\tau_j(0,\eta) \neq \partial_\xi\tau_k(0,\eta)$. This means that there are no critical points around $\eta = 0$ for the tangent operator—see Remark 3.25. What's more, this nonasymptoticity property implies some regularity for the projectors of the 0-group. For $\eta$ close to 0, compute with (2.10) the transverse derivative of a projector of the 0-group at the noncritical point $(\tau_i(\xi,\eta),\xi,\eta) \in \mathrm{Char}\, M$:

$$\partial_\eta \pi_i(\xi,\eta) = -\sum_{j\neq i} \frac{\pi_i(\xi,\eta)B\xi\pi_j(\xi,\eta) + \pi_j(\xi,\eta)B\xi\pi_i(\xi,\eta)}{\tau_i(\xi,\eta)-\tau_j(\xi,\eta)} - \pi_j(\xi,\eta)B\,\xi L^{-1}(\xi,\eta) - L^{-1}(\xi,\eta)B\,\xi\pi_j(\xi,\eta),$$

where the sum is carried over indices of the 0-group and where $L^{-1}(0,\eta) = A^{-1}$. In the limit $\xi \to 0$, one obtains

$$\partial_\eta \pi_i(0,\eta) = -\sum_{j\neq i} \frac{\pi_i(0,\eta)B\pi_j(0,\eta) + \pi_j(0,\eta)B\pi_i(0,\eta)}{\partial_\xi\tau_i(0,\eta)-\partial_\xi\tau_j(0,\eta)}. \tag{3.67}$$

Similarly, at order two,

$$\frac{1}{2}\pi_i(0,\eta)\,\partial_\eta^2\pi_i(0,\eta)\,\pi_i(0,\eta) = \sum_{j\neq i} \frac{\pi_i(0,\eta)B\pi_j(0,\eta)B\pi_i(0,\eta)}{(\partial_\xi\tau_i(0,\eta)-\partial_\xi\tau_j(0,\eta))^2}. \tag{3.68}$$

Note that such a nonasymptoticity condition provides sharper secular bounds for the correctors in [21]; it yields regularity in our setting.

Another feature of Maxwell equations of great interest regarding the asymptotic stability is the relation $\pi_0 B\pi_0 = 0$. It follows that the algebraic lemmas may be written at the critical point $(0,0,0) \in \mathrm{Char}\, M$ without inducing any coupling between the components of the main profile. The algebraic lemma at first order is

$$\pi_0 E\pi_0 = -i\sum_j \partial_\xi\tau_j(0,0)\pi_j(0,0).$$

With (3.67) and (3.68), the relation $\pi_0 B\pi_0 = 0$ also implies for all $j$,

$$\partial_\eta\pi_j(0,0) = 0 \quad\text{and}\quad \pi_j(0,0)\,\partial_\eta^2\pi_j(0,0)\,\pi_j(0,0) = 0.$$

Hence the algebraic lemma at second order is the same as in the noncritical case, for all $j$:

$$\pi_j(0,0)EA^{-1}E\pi_j(0,0) = -\frac{1}{2}\partial_\xi^2\tau_j(0,0)\pi_j(0,0),$$
$$\pi_j(0,0)BA^{-1}B\pi_j(0,0) = \frac{1}{4}\partial_\xi^2\partial_\eta^2\tau_j(0,0)\pi_j(0,0),$$

and

$$\pi_j(0,0)EA^{-1}B\pi_j(0,0) + \pi_j(0,0)BA^{-1}E\pi_j(0,0) = 0.$$

The first profile satisfies

$$\mathbf{u}_0 = \sum_j \pi_j(0,0)\mathbf{u}_0. \tag{3.69}$$

With the algebraic lemma at first order, the $\pi_j(0,0)\mathbf{u}_0$'s satisfy a system of linear, uncoupled ordinary differential equations in $t$:

$$(\partial_t - i\partial_\xi \tau_j(0,0))\pi_j(0,0)\mathbf{u}_0 = 0. \tag{3.70}$$

The nonlinear term in the slow-time evolution equation is $\pi_j(0,0)\partial_X G_{T_j} \times f(\mathbf{u}_0)$, where $G_{T_j}$ stands for the averaging projector with respect to the frequency $-i\partial_\xi\tau_j(0,0)$. The average with respect to $G_{T_j}$ of the product of two terms oscillating with frequencies $\partial_\xi\tau_k(0,0)$ and $\partial_\xi\tau_{k'}(0,0)$, vanishes; $f$ being a homogeneous polynomial of degree two, the nonlinear term vanishes. Changing the degree of $f$ and changing the amplitude of the solutions accordingly, the nonlinear term would not vanish because of the resonances between the $\partial_\xi\tau_j(0,0)$'s. With the algebraic lemma at second order, the equation becomes

$$\begin{aligned}\partial_\tau\partial_X\pi_j(0,0)\mathbf{u}_0 + (-\frac{1}{2}\partial_\xi^2\tau_j(0,0) + \frac{1}{4}\partial_\xi^2\partial_\eta^2\tau_j(0,0)\partial_y^2)\pi_j(0,0)\mathbf{u}_0 \\ = \pi_j(0,0)G_{T_j}\partial_X f(\mathbf{u}_0,\mathbf{u}_0).\end{aligned} \tag{3.71}$$

These equations together with the initial datum $\mathbf{u}_0 = \pi_0\mathbf{u}^0$ can be solved in $C^0 \cap L^\infty([0,\tau_0], E^{\sigma,s,p}(\mathbb{R}^2))$. Let $\tau^*$ be the maximal existence time of the solution $\mathbf{u}_0$, and let $\mathbf{u}_0^\delta = \chi^\delta\mathbf{u}_0$.

The equations for the polarized components of the first corrector are linear ordinary differential equations:

$$\begin{aligned}(\partial_t - i\partial_\xi\tau_j(0,0))\pi_j(0,0)\,\mathbf{u}_1^\delta = \pi_j(0,0)(1-G_{T_j})\big(\chi^\delta f(\mathbf{u}_0) + (E+B\partial_y) \\ \times A^{-1}(E+B\partial_y)(1-\pi_j(0,0))\mathbf{u}_0^\delta\big).\end{aligned}$$

The right-hand side is a product of terms oscillating with frequency different from $\partial_\xi\tau_j(0,0)$, because of $1-G_{T_j}$. It follows that $\pi_0\mathbf{u}_1^\delta$ is bounded. The sublinear growth condition that hampered the estimates in the previous section is replaced by a bound in $O(1)$, and one has for all $0<\tau_0<\tau^*$

$$||(\pi_0\mathbf{u}_1^\delta)^\varepsilon|| = O(1), \text{ and } ||(\pi_0\partial_\tau\mathbf{u}_1^\delta)^\varepsilon|| = O(1/\delta) \text{ in } L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y}).$$

The equations for the nonpolarized component of the first corrector is (3.63), whence follow the estimates

$$||(1-\pi_0)(\mathbf{u}_1^\delta)^\varepsilon|| = O(1/\delta), \text{ and } ||(1-\pi_0)(\partial_\tau\mathbf{u}_1^\delta)^\varepsilon|| = O(1/\delta^2)$$

in $L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})$.

The second corrector, defined by

$$\mathbf{u}_2^\delta = -A^{-1}\partial_X^{-1}\big((E+B\partial_y)\pi_0\mathbf{u}_1^\delta + \partial_X^{-1}(E+B\partial_y)A^{-1}(E+B\partial_y)\pi_0\mathbf{u}_0^\delta + \chi^\delta f(\mathbf{u}_0)\big),$$

satisfies by (3.30)

$$||(\mathbf{u}_2^\delta)^\varepsilon|| = O(1/\delta^2), \text{ and } ||(\partial_\tau\mathbf{u}_2^\delta)^\varepsilon|| = O(1/\delta^3) \text{ in } L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y}).$$

Set again $\delta = \varepsilon^\alpha$, with $0 < \alpha < 1$ and $\mathbf{u}^\varepsilon = \varepsilon\,(\,\mathbf{u}_0^{\varepsilon^\alpha} + \varepsilon \mathbf{u}_1^{\varepsilon^\alpha} + \varepsilon^2 \mathbf{u}_2^{\varepsilon^\alpha}\,)$. The corrector terms of the ansatz are $O(\varepsilon^{1-\alpha})$ in $L^\infty([0,\tau_0] \times \mathbb{R}_t, E^{\sigma-2,s,p}(\mathbb{R}^2))$. The terms created by the definitions of the correctors are like $\varepsilon^2(f(\mathbf{u}_0^{\varepsilon^\alpha}) - \chi^{\varepsilon^\alpha} f(\mathbf{u}_0))$, a term of size $O(\varepsilon^{2+\alpha/pr'})$ in $L^\infty([0,\tau_0] \times \mathbb{R}_t, E^{\sigma,s,p}(\mathbb{R}^2))$. The worst estimate for the other terms of the residual is of size $O(\varepsilon^{3-2\alpha})$. The best $\alpha$ is $\alpha = \frac{pr'}{1+pr'}$. It yields the estimate

$$\frac{1}{\varepsilon^2} ||r^\varepsilon||_{L^\infty([0,\tau_0]\times\mathbb{R}_t, E^{\sigma-4,s,p}(\mathbb{R}^2))} = O(\varepsilon^{1/(1+pr')}).$$

The stability theorem follows:

**Theorem 3.27.** *Under Assumptions* 3.3*,* 3.5*,* 3.22*, and* 3.11*, assume moreover that the elements of the 0-group of Char M satisfy* $\partial_\xi \tau_j(0,0) \neq \partial_\xi \tau_k(0,0)$ *for* $j \neq k$*, and that* $\pi_0 B \pi_0 = 0$*—these assumptions are satisfied by systems* (2.19) *and* (2.22)*. Then there exists* $\tau^* > 0$ *and a unique profile* $\mathbf{u}_0 \in C^0 \cap L^\infty([0,\tau_0], E^{\sigma,s,p} \cap E^{\sigma,s,pr}(\mathbb{R}^2))$*, for all* $0 < \tau_0 < \tau^*$*, satisfying equations* (3.69)*,* (3.70)*, and* (3.71)*, and the initial condition* $\mathbf{u}_0(0) = \pi_0 \mathbf{u}^0$*. Set* $u_0^\varepsilon = \mathbf{u}_0(x/\varepsilon, y, t, \varepsilon t)$*.* (3.1) *with initial datum* $v^\varepsilon = \varepsilon(\pi_0 u^0 + v^0(\varepsilon))$*, with* $||\mathbf{v^0}(\varepsilon)|| = O(\varepsilon^{1/(1+pr')})$ *in* $E^{\sigma,s,p}$ *has a unique solution* $v^\varepsilon$ *defined and smooth on* $[0, \tau^*/\varepsilon) \times \mathbb{R}^2$ *for all sufficiently small* $\varepsilon$*. For all* $0 < \tau_0 < \tau^*$ *and sufficiently small* $\varepsilon$*, the asymptotic estimate holds:*

$$\frac{1}{\varepsilon} ||v^\varepsilon - \varepsilon u_0^\varepsilon||_{L^\infty([0,\tau_0/\varepsilon]\times\mathbb{R}^2_{x,y})} = O(\varepsilon^{1/(1+pr')}).$$

3.7. **Examples.** For the systems (2.19) and (2.22), the equations for the leading term of the approximate solution can be made explicit by simple matrix computations.

**Maxwell-Lorentz equations.** Consider system (2.18), adding to $\mathcal{P}$ a nonlinear polarization $\mathcal{P}_{NL} =$ nonlinear $(\mathcal{E})$. In transverse magnetic mode,

$$\mathcal{E} = \begin{pmatrix} 0 \\ 0 \\ \mathcal{E} \end{pmatrix} (t,x,y), \quad \mathcal{B} = \begin{pmatrix} \mathcal{B}_x \\ \mathcal{B}_y \\ 0 \end{pmatrix} (t,x,y).$$

Set

$$\begin{pmatrix} 0 & -\mathrm{curl} & 0 & 0 \\ \mathrm{curl} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A\partial_x + B\partial_y.$$

Consider first an initial datum polarized along $\pi_1$, the projector onto Ker $(A-1)$. In normalized units $\mathcal{B}_x(0) = 0$ and $\mathcal{E}(0) = -\mathcal{B}_y(0)$. Only the electric field and the transverse component of the magnetic field are turned on. The first-order tangent operator at $(1,0,0)$ vanishes by Lemma 3.10—see Figure 3.

Computing the second-order operators, one finds that the leading term $u$ of the approximate solution satisfies

$$\left\{\begin{array}{rcl}(\partial_T + \partial_X)\pi_1 u & = & 0, \\ (\partial_\tau \partial_X + \frac{1}{2}(-\gamma + \partial_y^2))\pi_1 u & = & \text{nonlinear } (\pi_1 u),\end{array}\right.$$

where $\pi_1 u = (0, 0, \frac{1}{2}(\mathcal{E} - B_y), 0, -\frac{1}{2}(\mathcal{E} - \mathcal{B}_y), 0, 0_{\mathbb{R}^3}, 0_{\mathbb{R}^3})$.

Consider now an initial datum polarized along the Kernel of $A$. One has $\mathcal{E}(0) = 0$ and $\mathcal{B}_y(0) = 0$. All the components of $u$ are turned on except the electric field and the transverse component of the magnetic field. The equations in $t$ reduce to a system of uncoupled ordinary differential equations whose frequencies are the limits of the bounded branches of the characteristic variety for the Lorentz model (Figure 1). The second-order operators vanish. Therefore, the only nontrivial equations are

$$(\partial_t - iv_j)\pi_j(0,0)u = 0,$$

where $v_j \in \{0, \pm 1, \pm\sqrt{1+\gamma}\}$, and $\pi_0 = \sum_j \pi_j(0,0)$ is the decomposition of the 0-group. In transverse magnetic mode, one has $\pi_j(0,0)u = 0$ for all $j$: all the equations are trivial.

**Ferromagnetism.** Consider system (2.22) with an initial datum polarized along Ker $(A-1)$, with the same notation as above. For the characteristic variety pictured in Figure 4, $(1,0,0)$ is a critical point. The axisymmetry hypothesis in Assumption 3.3 is actually not satisfied by the characteristic variety of this model. The slow-time evolution equation will therefore be slightly different.

The first-order operator at $(1,0,0)$ is

$$-i\,\pi_1 E \pi_1 = \frac{1}{2}\cos\theta\,(\pi_{1,+} - \pi_{1,-}),$$

where $\pi_{1,+}$ and $\pi_{1,-}$ have rank 1 and $\pi_1 = \pi_{1,+} + \pi_{1,-}$. In transverse magnetic mode,

$$\pi_{1,\pm} u = (0, \pm\frac{i}{4}(\mathcal{E} - \mathcal{B}_y), \frac{1}{4}(\mathcal{E} - \mathcal{B}_y), 0, -\frac{1}{4}(\mathcal{E} - \mathcal{B}_y), \pm\frac{-i}{4}(\mathcal{E} - \mathcal{B}_y), 0_{\mathbb{R}^3}).$$

The nonlinearity vanishes, and it yields

$$\left\{\begin{array}{r}(\partial_T + \partial_X)\pi_{1,\pm} u = 0, \\ (\partial_t \pm \dfrac{1}{2} i\cos\theta)\pi_{1,\pm} u = 0, \\ \big(\partial_\tau \partial_X + \dfrac{1}{2}\big(\dfrac{1}{4}(1 + \sin^2\theta) + \alpha(1 - \dfrac{1}{2}\sin^2\theta) \pm i\sin\theta\partial_y + \partial_y^2\big)\big)\,\pi_{1,\pm} u = 0.\end{array}\right. \tag{3.72}$$

The fact that these equations are linear when they were expected to be nonlinear does not stem from a *transparency* property of the system as in

[16] but from the nature of the nonlinearity and the oscillations in $t$, as explained in Section 3.6.3.

The presence of a term in $\partial_y$ in the equation in $\tau$ is due to the fact that the characteristic variety is not rotation invariant with respect to $\{\xi = \eta = 0\}$. If $\sin\theta = 0$, the branches of the characteristic variety depend on $\xi^2 + \eta^2$ and $\eta^2$, and it follows from the analysis led in Section 3.4.1 that the equation in $\tau$ does not involve any term in $\partial_y$, as in (3.72).

Consider finally an initial datum polarized along Ker $A$. The focus is on the point $(0, 0, 0)$ of the characteristic variety associated with the long-wave operator (Figure 4). The tangent operator $-i\pi_0 E\pi_0$ has rank 4, and its eigenvalues are $\{0, \pm\sqrt{\alpha(\alpha + \sin^2\theta)}\}$. The second-order operators and the nonlinear term vanish. The only nontrivial equations are ordinary differential equations in $t$

$$(\partial_t - iv_j)\pi_j(0, 0)u = 0,$$

where $v_j \in \{0, \pm\sqrt{\alpha(\alpha + \sin^2\theta)}\}$ and $\pi_0 = \sum_j \pi_j(0, 0)$ is the decomposition of the 0-group. In transverse magnetic mode, these equations are trivial again.

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