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The Sieve of Eratosthenes-Legendre.

HENRYK IWANIEC (*)

By the sieve of Eratosthenes-Legendre we mean that described by Halberstam and Richert in their beautiful book [1], Chapter 1.

Let be given a finite sequence \mathcal{A} of integers and a set \mathfrak{P} of primes. For each real number $z \geq 2$ let

$$P(z) = \prod_{p < z, p \in \mathfrak{P}} p$$

and

$$S(\mathcal{A}, \mathfrak{P}, z) = \sum_{\substack{a \in \mathcal{A} \\ (a, P(z)) = 1}} 1,$$

the number of elements in the sequence \mathcal{A} which are not divisible by any prime number $p < z$ from \mathfrak{P} . The sieve method deals with estimates of $S(\mathcal{A}, \mathfrak{P}, z)$ by linear forms in

$$(1) \quad |\mathcal{A}_d| = \sum_{\substack{a \in \mathcal{A} \\ a \equiv 0 \pmod{d}}} 1,$$

(the number of elements in \mathcal{A} which are divisible by $d|P(z)$)

$$(2) \quad \sum_{d|P(z)} \lambda_d^- |\mathcal{A}_d| \leq S(\mathcal{A}, \mathfrak{P}, z) \leq \sum_{d|P(z)} \lambda_d^+ |\mathcal{A}_d|.$$

The multipliers λ_d^+ , usually called « weights », are real numbers satisfying the following conditions

$$(3) \quad \lambda_1^- = \lambda_1^+ = 1,$$

$$(4) \quad \sum_{d|D} \lambda_d^- \leq 0 \leq \sum_{d|D} \lambda_d^+ \quad \text{for all } D > 1, D|P(z).$$

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Next, for each squarefree integer d we choose $\omega(d)$ so that $(\omega(d)/d)X$, where X is a suitable number, approximates $|\mathcal{A}_d|$, and we write the remainder as

$$r_d = |\mathcal{A}_d| - \frac{\omega(d)}{d} X.$$

Inserting this into (2) we get

$$(5) \quad S(\mathcal{A}, \mathfrak{F}, z) \leq X \sum_{d|P(z)} \lambda_d^+ \frac{\omega(d)}{d} + \sum_{d|P(z)} |\lambda_d^+ r_d|,$$

$$(6) \quad S(\mathcal{A}, \mathfrak{F}, z) \geq X \sum_{d|P(z)} \lambda_d^- \frac{\omega(d)}{d} - \sum_{d|P(z)} |\lambda_d^- r_d|.$$

We want to make these estimates optimal. This requirement determines the parameter X and the function $\omega(d)$ almost uniquely. It appears in practice that $\omega(d)$ is multiplicative and for some $\kappa \geq 0$ satisfies the condition

$$(7) \quad -L < \sum_{w \leq p < z} \frac{\omega(p) - \kappa}{p} \log p < A_2,$$

for all $2 \leq w < z$, where L and A_2 are some constants ≥ 1 . The parameter κ is called the « dimension » of the sieve.

The method of Eratosthenes-Legendre rests on the use of Möbius function $\mu(d)$ as common value of the weights

$$\lambda_d^- = \lambda_d^+ = \mu(d).$$

It turns formula (2) into the Legendre identity

$$S(\mathcal{A}, \mathfrak{F}, z) = \sum_{d|P(z)} \mu(d) |\mathcal{A}_d|$$

and this identity usually leads to a bad result because, unless z is very small, the remainder sum

$$\sum_{d|P(z)} |r_d|$$

has too many terms. It was Viggo Brun who first showed how to construct the sieving weights λ_d^\pm more effectively. For details see [1], Chapter 1.

The aim of this paper is to show that the Eratosthenes-Legendre sieve yields an asymptotic formula for the sifting function $S(\mathcal{A}, \mathfrak{F}, z)$ in the case of the dimension $\kappa < \frac{1}{2}$ and the sequence \mathcal{A} with elements not too large:

$$(8) \quad x = \max_{a \in \mathcal{A}} |a| < A_3 X$$

for some $A_3 \geq 1$. It is assumed that the remainders r_d are also not too large:

$$(9) \quad |r_d| < A_4 \omega(d)$$

for some $A_4 \geq 1$.

Suppose that

$$(10) \quad 0 < \frac{\omega(p)}{p} < 1 - \frac{1}{A_1}$$

for all $p \in \mathcal{P}$ and put

$$W(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p} \right).$$

The result reads as follows

THEOREM. *Under the assumptions (7)-(10) we have*

$$S(\mathcal{A}, \mathcal{F}, z) = \frac{e^{\gamma s}}{\Gamma(1-\kappa)} W(z) X \left\{ f(s) + O\left(\frac{(s+1)L}{1-2\kappa} (\log z)^{2\kappa-1} \right) \right\},$$

where $\gamma = 0.577 \dots$ is the Euler constant, $s = \log x / \log z$. The function $f(s)$ is defined in Section 2. For $0 < s \leq 1$ we have

$$f(s) = s^{-\kappa}.$$

The constant in the symbol O depends only on A_1, A_2, A_3 and A_4 .

In his thesis, Sullivan proved the asymptotic formula

$$S(\mathcal{A}, \mathcal{F}, z) \sim W(z) X$$

under the condition

$$\sum_{p < x} \frac{\omega(p)}{p} \log p = o(\log x)$$

instead of (7). His method is based on Halberstam-Richert's Fundamental Lemma [2] (oral communication).

A comparison of our method with Brun's method will be given elsewhere.

Keeping the notations introduced above we have

$$S(\mathcal{A}, \mathcal{F}, z) = \sum_{d|P(z)} \mu(d) |\mathcal{A}_d| = \sum_{d|P(z), d \leq x} \mu(d) |\mathcal{A}_d| = X \sum_{\substack{d \leq x \\ d|P(z)}} \mu(d) \frac{\omega(d)}{d} + \sum_{\substack{d \leq x \\ d|P(z)}} \mu(d) r_d.$$

1. – An estimate of the remainder sum.

From (9) we get

$$(12) \quad \sum_{\substack{d \leq x \\ d|P(z)}} |r_d| < A_x \sum_{\substack{d \leq x \\ d|P(z)}} \omega(d).$$

With ω we associate the generalized Mangoldt's function λ as usual by Dirichlet's convolution

$$\omega(d) \log d = \sum_{n|d} \omega(n) \lambda(d/n).$$

Since ω is multiplicative, the support of λ is contained in the set of powers of primes. It is easily seen that $\lambda(p) = \omega(p) \log p$, so

$$(13) \quad \begin{aligned} \sum_{\substack{d \leq x \\ d|P(z)}} \omega(d) \log d &= \sum_{\substack{n \leq x \\ n|P(z)}} \omega(n) \sum_{\substack{m \leq x/n \\ mn|P(z)}} \lambda(m) \\ &< \sum_{\substack{n \leq x \\ n|P(z)}} \omega(n) \sum_{p \leq x/n} \omega(p) \log p. \end{aligned}$$

Using partial summation, from the upper bound (7) we get

$$\sum_{p \leq z} \omega(p) \log p \ll z$$

and

$$\sum_{p \leq z} \omega(p)/p < \kappa \log \log z + O(1).$$

Hence

$$\begin{aligned} \sum_{\substack{d \leq x \\ d|P(z)}} \omega(d) \log d &\ll x \sum_{\substack{d \leq x \\ d|P(z)}} \frac{\omega(n)}{n} < x \prod_{p \leq x} \left(1 + \frac{\omega(p)}{p}\right) < x \exp\left(\sum_{p \leq x} \omega(p)/p\right) \\ &\ll x(\log x)^\kappa. \end{aligned}$$

Using partial summation again we get the final result

$$\sum_{d|P(z), d \leq x} \omega(d) \ll x(\log x)^{\kappa-1}$$

and thus we have the same estimate for the remainder sum (12).

2. – The function $f(s)$.

LEMMA 1. Let $0 \leq \kappa < \frac{1}{2}$ and $f(s)$ be the continuous solution of

$$\begin{cases} f(s) = s^{-\kappa} & \text{for } 0 < s \leq 1 \\ sf'(s) = \kappa f(s-1) - \kappa f(s) & \text{for } s > 1. \end{cases}$$

Then, for $s \rightarrow \infty$ we have

$$f(s) = e^{-\kappa s} \Gamma(1 - \kappa) + O(e^{-s}).$$

PROOF. The derivative $f'(s)$ satisfies the equation

$$sf'(s) = -\kappa \int_{s-1}^s f'(u) du,$$

so $f'(s) = O(e^{-s})$ and thus

$$f(s) = 1 + \int_1^s f'(u) du = c + O(e^{-s}).$$

It remains to calculate the constant c . To this end, let us consider the Laplace transform

$$\begin{aligned} L(s) &= \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} (c + O(e^{-t})) dt \\ &= cs^{-1} + O(1). \end{aligned}$$

It can easily be checked that

$$\frac{d}{ds} (sL(s)) = \kappa(1 - e^{-s})L(s)$$

and thus

$$sL(s) = c \exp\left(\kappa \int_0^s (1 - e^{-t}) \frac{dt}{t}\right).$$

Now, we calculate in two ways the limit $\lim_{s \rightarrow \infty} s^{1-\kappa} L(s)$. We have

$$\lim_{s \rightarrow \infty} s^{1-\kappa} L(s) = c \lim_{s \rightarrow \infty} s^{-\kappa} \exp \left(\kappa \int_0^s (1 - e^{-t}) \frac{dt}{t} \right) = c e^{\kappa \nu}.$$

On the other hand

$$\begin{aligned} \lim_{s \rightarrow \infty} s^{1-\kappa} L(s) &= \lim_{s \rightarrow \infty} s^{1-\kappa} \int_0^\infty e^{-st} f(t) dt \\ &= \lim_{s \rightarrow \infty} s^{1-\kappa} \int_0^1 e^{-st} t^{-\kappa} dt \\ &= \lim_{s \rightarrow \infty} \int_0^s e^{-u} u^{-\kappa} du \\ &= \Gamma(1-\kappa). \end{aligned}$$

This completes the proof of the Lemma.

COROLLARY. *The function*

$$F(s) = \int_0^s f(u) du, \quad s > 0$$

is of C^1 -class and satisfies the equations

$$\begin{aligned} F(s) &= \frac{1}{1-\kappa} s^{1-\kappa}, & \text{for } 0 < s < 1, \\ sF'(s) &= (1-\kappa)F(s) + \kappa F(s-1), & \text{for } s > 1, \\ \frac{d}{ds} \left(\frac{F(s)}{s^{1-\kappa}} \right) &= \kappa \frac{F(s-1)}{s^{2-\kappa}}, & \text{for } s > 1. \end{aligned}$$

3. – An asymptotic formula for the main term.

Let us put

$$g(d) = \mu(d) \frac{\omega(d)}{d}$$

and define for all $x \geq 1, z \geq 1$

$$G(x, z) = \sum_{\substack{d < x \\ d|P(z)}} g(d).$$

To get an asymptotic formula for $G(x, z)$ we apply the first step of the Levin-Fainleib's iteration method [4]. This method works effectively for sums of positive multiplicative functions, in particular for the one which appears in the Selberg sieve (see [3]). Although our function $g(d)$ changes sign, it turns out that the method still works in the case $\kappa < \frac{1}{2}$ considered here. Imitating [3], we shall prove

PROPOSITION 1. For $x \geq 2, z \geq 2$ we have

$$G(x, z) = Cf(s)(\log z)^{-\kappa} + O\left(\frac{(s+1)L}{1-2\kappa}(\log x)^{\kappa-1}\right),$$

where

$$s = \frac{\log x}{\log z}, \quad C = \frac{1}{\Gamma(1-\kappa)} \prod_p \left(1 - \frac{\omega(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-\kappa}$$

and the constant in the symbol O depends only on A_1, A_2 .

Theorem will follow from

COROLLARY. For $x \geq 2, z \geq 2$ we have

$$G(x, z) = \frac{e^{\nu\kappa}}{\Gamma(1-\kappa)} W(z) \left\{ f(s) + O\left(\frac{(s+1)L}{1-2\kappa}(\log z)^{2\kappa-1}\right) \right\}.$$

The constant implied in the symbol O depends only on A_1 and A_2 .

To derive Corollary from Proposition 1 we have to show

$$W(z) = e^{-\nu\kappa} \Gamma(1-\kappa) C (\log z)^{-\kappa} \left\{ 1 + O\left(\frac{L}{\log z}\right) \right\}.$$

But this is a simple consequence of the Mertens formula

$$\prod_{p < z} \left(1 - \frac{1}{p}\right) = \frac{e^{-\nu}}{\log z} \left(1 + O\left(\frac{1}{\log z}\right)\right)$$

and of the estimate

$$\Gamma(1-\kappa) C = \prod_{p < z} \left(1 - \frac{\omega(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-\kappa} \left\{ 1 + O\left(\frac{L}{\log z}\right) \right\}.$$

For details see [1], Lemma 5.3.

Before we prove Proposition 1 we shall show a few auxiliary lemmas. Let

$$T(x, z) = \int_1^x \frac{G(t, z)}{t} dt = \sum_{\substack{d < x \\ d|P(z)}} g(d) \log \frac{x}{d}$$

for $x \geq 1$ and $z \geq 1$. For $0 < x \leq 1$ we put $T(x, z) = 0$.

LEMMA 2. For $x \geq 2$ and $z \geq 2$ we have

$$(14) \quad G(x, z) \log x = (1 - \kappa) T(x, z) + \kappa T\left(\frac{x}{z}, z\right) + O(L \log^{\kappa} x).$$

The constant in the symbol O depends only on A_1 and A_2 .

PROOF. We start with the definition of generalized Mangoldt's function χ associated with g :

$$g(d) \log d = \sum_{n|d} g(n) \chi(d/n).$$

Since g is multiplicative, the support of χ is contained in the set of powers of primes. It is easily seen that $\chi(p) = -(\omega(p)/p) \log p$, so

$$\sum_{\substack{d < x \\ d|P(z)}} g(d) \log d = \sum_{\substack{d < x \\ d|P(z)}} g(d) \sum_{\substack{n < x/d \\ dn|P(z)}} \chi(n) = - \sum_{\substack{d < x \\ d|P(z)}} g(d) \sum_{\substack{p < x/d \\ p < z, p \nmid d}} \frac{\omega(p)}{p} \log p.$$

Since

$$\sum_{p < y} \frac{\omega(p)}{p} \log p = \kappa \log y + O(L), \quad \sum_{d < x} |g(d)| < \prod_{p < x} \left(1 + \frac{\omega(p)}{p}\right) \ll (\log x)^{\kappa},$$

$$\sum_{d < x} |g(d)| \sum_{p|d} \frac{\omega(p)}{p} \log p < \sum_{d < x} \frac{\omega(d)}{d} \sum_{p < x} \left(\frac{\omega(p)}{p}\right)^2 \log p \ll \sum_{d < x} \frac{\omega(d)}{d} \ll (\log x)^{\kappa}$$

we obtain

$$\sum_{\substack{d < x \\ d|P(z)}} g(d) \log d = -\kappa \sum_{\substack{d < x \\ d|P(z)}} g(d) \log \left(\min\left(z, \frac{x}{d}\right)\right) + O(L \log^{\kappa} x).$$

If we add the sum $\sum_{d|P(z), d < x} g(d) \log x/d$ to both sides, we arrive at (14).

LEMMA 3. For $x > y \geq 2$ and $z \geq 2$ we have

$$(15) \quad \frac{T(x, z)}{(\log x)^{1-\kappa}} = \frac{T(y, z)}{(\log y)^{1-\kappa}} + \kappa \int_y^x \frac{T(t/z, z)}{(\log t)^{1-\kappa}} d(\log \log t) + O\left(\frac{L}{1-2\kappa}\right) (\log y)^{2\kappa-1}.$$

The constant in the symbol O depends only on A_1 and A_2 .

PROOF. Let us write (14) with x replaced by t and divide throughout by $t(\log t)^{2-\kappa}$. Integrating with respect to t from y to x , we obtain

$$\int_y^x \frac{G(t, z)}{t(\log t)^{1-\kappa}} dt = (1-\kappa) \int_y^x \frac{T(t, z)}{t(\log t)^{2-\kappa}} dt + \kappa \int_y^x \frac{T(t/z, z)}{t(\log t)^{2-\kappa}} dt + O\left(\frac{L}{1-2\kappa} (\log y)^{2\kappa-1}\right).$$

If we integrate the identity

$$\frac{\partial}{\partial t} \left(\frac{T(t, z)}{t(\log t)^{1-\kappa}} \right) = \frac{G(t, z)}{t(\log t)^{1-\kappa}} - (1-\kappa) \frac{T(t, z)}{t(\log t)^{2-\kappa}}$$

and add the result to the formula above, we arrive at (14).

4. - An asymptotic formula for $T(x, z)$.

LEMMA 4. For $z \geq x \geq 2$, we have

$$T(x, z) = \frac{C}{1-\kappa} (\log x)^{1-\kappa} + O\left(\frac{L}{1-2\kappa} \log^\kappa x\right).$$

The constant in the symbol O depends only on A_1 and A_2 .

PROOF. For $z \geq t \geq 2$ we have

$$\begin{aligned} G(t, z) &= G(t, t) = G(t), \\ T(t, z) &= T(t, t) = T(t) \end{aligned}$$

and by Lemma 2

$$(16) \quad r(t) = G(t) \log t - (1-\kappa) T(t) \ll L \log^\kappa t.$$

If we divide (16) throughout by $t(\log t)^{2-\kappa}$ and integrate with respect to t

from 2 to x , we obtain

$$\int_2^x \frac{G(t)}{t(\log t)^{1-\kappa}} dt - (1-\kappa) \int_2^x \frac{T(t)}{t(\log t)^{2-\kappa}} dt = \int_2^x \frac{r(t)}{t(\log t)^{2-\kappa}} dt = c_1 + O\left(\frac{L}{1-2\kappa} (\log x)^{2\kappa-1}\right).$$

But

$$\frac{d}{dt} \left(\frac{T(t)}{(\log t)^{1-\kappa}} \right) = \frac{G(t)}{t(\log t)^{1-\kappa}} - (1-\kappa) \frac{T(t)}{t(\log t)^{2-\kappa}},$$

so integration by parts leads to

$$\frac{T(x)}{(\log x)^{1-\kappa}} - \frac{T(2)}{(\log 2)^{1-\kappa}} = c_1 + O\left(\frac{L}{1-2\kappa} (\log x)^{2\kappa-1}\right)$$

and finally

$$T(x) = c_2 (\log x)^{1-\kappa} + O\left(\frac{L}{1-2\kappa} \log^\kappa x\right).$$

It remains to calculate the constant $c_2 = c_1 + \log^\kappa 2$.

From (16) we get

$$(17) \quad G(x) = (1-\kappa)c_2 (\log x)^{-\kappa} + O\left(\frac{L}{1-2\kappa} (\log x)^{\kappa-1}\right)$$

for $x \geq 2$. Hence, for $s > 0$ we have

$$\begin{aligned} \prod_p \left(1 - \frac{\omega(p)}{p^{s+1}}\right) &= \sum_{n=1}^{\infty} \frac{g(n)}{n^{s+1}} = s \int_1^{\infty} \frac{G(x)}{x^{s+1}} dx = s \int_1^2 + s \int_2^{\infty} = O(s) + (1-\kappa)c_2 s \\ &\cdot \int_2^{\infty} \frac{\log^{-\kappa} x}{x^{s+1}} dx + O\left(\frac{Ls}{1-2\kappa} \int_2^{\infty} \frac{(\log x)^{\kappa-1}}{x^{s+1}} dx\right) = (1-\kappa)c_2 s^\kappa \Gamma(1-\kappa) + O\left(\frac{L}{1-2\kappa} s^{1-\kappa}\right). \end{aligned}$$

Since $\lim_{s \rightarrow +0} s \zeta^s(s+1) = 1$, by the Euler product formula we obtain

$$(1-\kappa)c_2 \Gamma(1-\kappa) = \lim_{s \rightarrow +0} s^{-\kappa} \prod_p \left(1 - \frac{\omega(p)}{p^{s+1}}\right) = \prod_p \left(1 - \frac{\omega(p)}{p}\right) \left(1 - \frac{1}{p}\right)^{-\kappa}.$$

This completes the proof of Lemma 4.

Now, we are ready to prove the general result

PROPOSITION 2. For $x \geq 2$ and $z \geq 2$ we have

$$(18) \quad T(x, z) = CF(s)(\log z)^{1-\kappa} + O\left(\frac{(s+1)L}{1-2\kappa} \log^{\kappa} x\right).$$

The constant in the symbol O depends only on A_1 and A_2 .

PROOF. The proof will proceed by induction. Putting

$$(19) \quad T(x, z) = CF(s)(\log z)^{1-\kappa} + R(x, z)$$

we have to prove

$$(20) \quad R(x, z) \ll \frac{(s+1)L}{1-2\kappa} \log^{\kappa} x,$$

for all $x \geq 2$ and $z \geq 2$. This has already been proved in Lemma 4 for all $z \geq x \geq 2$. In this range $R(x, z) = R(x, x)$.

If we introduce (19) into (15) we find out that the leading terms disappear throughout and we are left with a relation between the remainder terms only, namely

$$(21) \quad \frac{R(x, z)}{(\log x)^{1-\kappa}} = \frac{R(y, z)}{(\log y)^{1-\kappa}} + \kappa \int_y^x \frac{R(t/z, z)}{(\log t)^{1-\kappa}} d(\log \log t) + O\left(\frac{L}{1-2\kappa} (\log y)^{2\kappa-1}\right)$$

for all $x > y \geq 2$ and $z \geq 2$.

Let us assume for a moment that $y = z$ and

$$(22) \quad z < x < z^2.$$

Accordingly, we can use (20) to the right-hand side of (21) and in the result we arrive again at (20) but now for the range (22). Therefore, we already have

$$(23) \quad R(x, z) < \frac{BL}{1-2\kappa} (s+1) \log^{\kappa} x,$$

for all $z \geq 2$ and $2 < x < z^2$. The constant B depends only on A_1 and A_2 : Now, we shall show by induction that if B is sufficiently large then (23) holds for all $z \geq 2$ and $x \geq 2$. For that, it is enough to prove the implication: if (23) holds for all $x < z^u$ then it holds for $x = z^{u+1}$, where u is real number ≥ 1 .

After putting

$$y = z^u, \quad x = z^{u+1}$$

in (21), we get by the inductive assumption

$$\begin{aligned} \frac{R(x, z)}{(\log x)^{1-\kappa}} &< \frac{BL}{1-2\kappa} \left\{ (u+1)(\log x)^{2\kappa-1} + \kappa \int_y^x \frac{\log^\kappa t}{\log z} \left(\log \frac{t}{z} \right)^\kappa d(\log \log t) \right\} + \\ &+ O\left(\frac{L}{1-2\kappa} (\log x)^{2\kappa-1} \right) < \frac{BL}{1-2\kappa} \left\{ u+1 + \kappa \left(\frac{u+1}{u} \right)^{1-\kappa} + \varepsilon \right\} (\log x)^{2\kappa-1}. \end{aligned}$$

If B is sufficiently large then $\varepsilon < 1 - 1/\sqrt{2}$ and thus the term in the bracket is less than $u+2$. This completes the proof of Proposition 2.

5. - Completion of the proof of Proposition 1.

Putting $F(s) = 0$ for all $s < 0$ we find out from (14) and (18) that

$$\begin{aligned} G(x, z) &= Cs^{-1}\{(1-\kappa)F(s) + \kappa F(s-1)\}(\log z)^{-\kappa} + \\ &+ O\left\{ \frac{(s+1)L}{1-2\kappa} (\log x)^{\kappa-1} + L(\log x)^{\kappa-1} \right\} = CF'(s)(\log z)^{-\kappa} + \\ &+ O\left(\frac{(s+1)L}{1-2\kappa} (\log x)^{\kappa-1} \right) \end{aligned}$$

for all $x \geq 2$ and $z \geq 2$. This completes the proof of Proposition 1.

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