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THE SIGNATURE OF FIBER BUNDLES

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ABSTRACT. Let $Y \hookrightarrow Z \xrightarrow{\pi} X$ be a locally trivial fiber bundle in the category of oriented topological manifolds. It is shown that if the identity component of the structure group G has finite index, then (signature of Z) = (signature of X) $\cdot$ (signature of Y).

Let $F \to E \xrightarrow{p} B$ be a locally trivial fiber bundle such that

(1) E, F, B are closed, oriented topological manifolds.

(2) E, F, B are coherently oriented, that is, the orientation of F and B determine that of E.

In this situation, does it follow that $\sigma(E)=\sigma(B)\cdot\sigma(F)$, where $\sigma(\ )$ denotes the signature homomorphism?

That additional conditions are necessary is shown both by Kodaira [4] and Atiyah [1] when they produce a locally trivial fibering of a complex surface by a complex surface such that the total space has a nonzero signature. In fact, in the smooth case, Atiyah produces a formula computing $\sigma(E)$ and showing the dependency on the fundamental group of B.

The approach of this paper is to look at the structure group G of the bundle and determine conditions on G in order to obtain an affirmative answer to the above question. If G is any topological group, let $\Gamma = G/G_0$, where G_0 is the connected component of the identity. The main result is the

THEOREM. *Let G be a locally compact, finite dimensional topological group such that $|\Gamma|$ is finite. If $F \to E \xrightarrow{p} B$ is any oriented locally trivial topological fiber bundle with structure group G, then $\sigma E = \sigma B \cdot \sigma F$.*

REMARK 1. The theorem obviously remains valid if the structure group of the bundle is not, a fortiori, G, but can be reduced to G.

REMARK 2. The hypothesis that G be locally compact, finite dimensional only exists to insure that $G \to \Gamma$ possesses a local cross section. Any other hypothesis on G insuring this is equally valid. See [3], for instance.

PROPOSITION 1. *If $\Gamma = (e)$, then $\sigma E = \sigma B \cdot \sigma F$.*

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PROOF. This is essentially the theorem of Chern-Hirzebruch and Serre in [2]. One only needs to check that $\pi_1(B)$ acts trivially on $H^*(F)$. However this follows from the following two well-known facts.

(a) Let G be any group and B_G the classifying space for G. If F is any left G-space, then the action $\phi:\pi_1(B_G)\times H^q(F)\to H^q(F)$ in the bundle $E_G\times_G F\to B_G$ is given by $\phi(\alpha, u)=y_*(u)$ where $y\in G$ is any representative of $\partial\alpha\in\pi_0(G)$.

It follows that if G is connected, then the action is trivial.

(b) Let (B, E', G, p') be the associated principal bundle to $F\to E\xrightarrow{p} B$, so that $E=E'\times_G F$. Consider the following diagram.

$$\begin{array}{ccccc}
F & \longrightarrow & E = E'\times_G F & \xrightarrow{p} & B \\
\| & & \uparrow & & \uparrow \\
F & \longrightarrow & \tilde{E} = E'\times_{G_0} F & \xrightarrow{\tilde{p}} & E'/G_0 = \tilde{B} \\
 & & \uparrow & & \uparrow \\
 & & \Gamma & = & \Gamma
\end{array}$$

Since $G\to\Gamma$ has a local cross section, the columns are locally trivial fiber bundles with structure group G and fiber Γ, where G acts on Γ via left translation. If we extend the structure group to Γ, those columns become principal Γ bundles. The middle row $F\to\tilde{E}\xrightarrow{\tilde{p}}\tilde{B}$ is a G_0-bundle.

Since B and F are manifolds and Γ is finite, it is obvious that all the spaces involved are closed manifolds. Therefore we may apply Proposition 1 to the middle row and conclude $\sigma(\tilde{E})=\sigma(\tilde{B})\sigma(F)$.

Now since the two columns are principal Γ-bundles, i.e. finite covering spaces, it is clear that in order to prove the theorem it is sufficient to demonstrate the following result.

THEOREM. *Let X be a closed, connected, oriented topological manifold and Γ a finite group acting on X without fixed points and preserving the orientation. Then $\sigma(X)=|\Gamma|\sigma(X/\Gamma)$.*

PROOF. In [6] there are constructed, for any oriented euclidian bundle ξ over a sufficiently nice space, rational Pontrjagin classes, or equivalently Hirzebruch classes, $l(\xi)$. These classes satisfy naturality and Whitney formulas and are the rationalization of the ordinary Pontrjagin or Hirzebruch classes if ξ happens to be a vector bundle. Moreover if $l(M)$ denotes $l(\tau_M)$ for M an oriented topological manifold, then $\langle l(M), [M]\rangle = \sigma(M)$.

Now if $X\xrightarrow{\pi} X/\Gamma$ is a covering map, it is a local homeomorphism and so induces a map $d\pi:\tau_X\to\tau_{X/\Gamma}$ which is an isomorphism on fibers i.e., $\pi^*(\tau X/\Gamma)=rX$. Therefore by naturality of the l-classes, $\pi^* l(X/\Gamma)=l(X)$.

But since $X \xrightarrow{\pi} X/\Gamma$ is a finite covering, $\pi_* [X] = |\Gamma| \cdot [X/\Gamma]$. It follows that

$$\sigma(X) = \langle l(X), [X] \rangle = \langle \pi^* l(X/\Gamma), [X] \rangle$$
$$= \langle l(X/\Gamma), \pi_* [X] \rangle = |\Gamma| \, \sigma(X/\Gamma).$$

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