# The Signed Chi-square Measure for Mapping 

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## INTRODUCTION

Ratios and numerical differences are among the most common forms of expression for the ordering and classification of data for bivariate mapping. They are considered to remove the effects of variations in the size of the base population and are therefore widely employed for interarea, inter-group and temporal comparisons in demography and population geography. ${ }^{1}$ Ratios, including such measures as proportions and percentages, are frequently encountered in both statistical and non-statistical work.

However, several writers have found the above standardisation procedures inadequate. ${ }^{2-4}$ In general, numerical differences produce more extreme values in large populations while ratios have the opposite effect of tending to give the most extreme values in small populations. They exaggerate small differences in small populations; in addition, ratio measures may conceal large absolute differences in urban areas-differences which may be much more important because of the large numbers of people involved.

This paper suggests a different scheme for standardisation, namely the signed chi-square ( $X_{\mathrm{S}}^{2}$ ) measure which has been successfully used for the mapping of a wide range of themes. ${ }^{8}$ It is a compromise measure which takes into account both the absolute and relative deviations from the norm, and is represented by a single cartographic symbol.

## LIMITATIONS OF TRADITIONAL SCHEMES FOR ORDERING SAMPLES

Ratios and numerical differences have constituted the conventional means of ordering and scaling data for spatial units. Both these are influenced by variations in sample size. The sample size may be defined by the number
of residents, of households or even of some other base population such as the number of males of a particular age group within the spatial unit. It conventionally forms the denominator of the ratio measure. That part of the sample which is of substantive interest, for example the number of children or of unemployed males, often (but not always) forms the numerator of the ratio measure. The difficulties associated with the use of ratios and numerical differences stems from two features: firstly, the distribution of sample sizes is often not constant. The variation in sample sizes is a marked feature in high resolution grid-square data, but it is also present in data for irregular areas such as Local Authority Areas. This is illustrated by the Lorenz curves in Figure 1, in which the cumulative percentage of total population is plotted against the cumulative percentage of data units, which had been ranked by population size. Moreover, often the section of the sample of substantive interest is constrained by the sample size and is closely related to it. When there is a high correlation between the size of the sample (denominator) and the group of interest (the numerator), the spatial distribution of the latter closely reflects the distribution of the former and hence is of little interest.

A second complication is that the bivariate relationship between denominator and numerator-or for that matter between the residual population (if applicable) and the numerator-lacks homoscedasticity. Numerical differences, being constrained by sample size, tend to be more extreme in large samples than in small ones. On the other hand, in small samples, very small differences can produce greatly different and often extreme ratios. In the extreme example of a kilometre grid square with one person, ${ }^{5}$ only values of 0 and 100 per cent are possible; and in general, important variations in large populations tend to produce nearaverage ratios. The inherent assumption in the use of


Figure 1. Lorenz curves showing the distribution of population among data units at four levels of aggregation.
ratio cut-offs is that the magnitude of deviation from expectation is a linear function of sample size as shown in Table 1. Moreover, the absence of a particular group is often more unusual in a large sample than in a small one, but the spatial units concerned are all given the same status by a ratio value of 0 per cent.

TABLE 1
Some implications of using a cut-off value of 60 per cent males

| Total population <br> $(N)$ | Males | Deviation from an <br> expectation of <br> $48.5 \%$ males | Numerical <br> differences <br> between sexes |
| :---: | ---: | :---: | ---: |
| 10 | 6 | 1.15 | 2 |
| 10 | 60 | 11.50 | 20 |
| 1,000 | 600 | 115.00 | 200 |
| 10,000 | 6,000 | $1,150.00$ | 2,000 |
| 100,000 | 60,000 | $11,500.00$ | 20,000 |

## ALTERNATIVE SCHEMES

Several alternative or corrective measures have been suggested and tried. Dewdney and Rhind ${ }^{3}$ considered the suppression of data for populations below a threshold value. In some of their maps, they removed the worst anomalies by considering only those one-kilometre squares with more than ten people-but the discussion so far makes it clear that a distorted ranking still exists. An alternative-the separate mapping and publication of maps of both absolute number and ratio measures of selected variables is not a satisfactory solution. Apart from doubling the number of maps, it produces a complex visual cor-
relation problem for the map reader especially if highly misleading ratio values still exist. The tagging on of significance levels to ratio values ${ }^{2,6,10}$ is again not a viable solution for the mapping of high resolution data and does not substantially aid the recognition of spatial patterns on ratio maps.

The above measures attempt to redress a distorted ordination by what amounts to cosmetic procedures. The signed chi-square measure constitutes a more direct solution since it results in an ordering scheme which is intermediate to ratios, on the one hand, and absolute deviations on the other.

## the signed chi-square measure

The signed chi-square measure is a modified form of the standard $X^{2}$ test statistic. $X^{2}$ measures the magnitude of the (absolute $X$ relative) departure of observed values from an expected value.

$$
\begin{equation*}
X^{2}=\Sigma\left[(O-E) \times\left(\frac{O-E}{E}\right)\right]=\Sigma\left(\frac{(O-E)^{2}}{E}\right) \tag{1}
\end{equation*}
$$

where $O$ is the observed frequency and $E$ is the expected frequency.

Thus it is a compromise measure which simultaneously considers the magnitude of absolute and relative deviations from expectation. When there are two categories A and B in a sample of $N$, the above expression takes the following form:

$$
\begin{equation*}
X^{2}=\frac{\left(O_{\mathrm{A}}-E_{\mathrm{A}}\right)^{2}}{E_{\mathrm{A}}}+\frac{\left(O_{\mathrm{B}}-E_{\mathrm{B}}\right)^{2}}{E_{\mathrm{B}}} \tag{2}
\end{equation*}
$$

The $X^{2}$ measure is uni-directional. To distinguish between areas with an excess of A as opposed to B , the $X^{2}$ formula was modified as follows:

$$
\begin{equation*}
X_{\mathrm{S}}^{2}=\operatorname{sgn}\left(O_{\mathrm{A}}-E_{\mathrm{A}}\right) X^{2} \tag{3}
\end{equation*}
$$

where $\operatorname{sgn}(x)$ is the signum function whose value is 1 for $x>0,-1$ for $x<0$ and $O$ for $x=0$. Thus areas with an excess and deficit of A are distinguished by positive and negative values of $X_{S}^{2}$ respectively. The sign is only a code and the numerical values remain exactly the same as the standard $X^{2}$ sample value.

If we nominally assume that expected values for $\mathrm{A}=\mathrm{B}$, then some further implications of $X_{S}^{2}$ standardisation become apparent. In this particular case:

$$
\begin{equation*}
X^{2}=2 \frac{\left(O_{\mathrm{A}}-E_{\mathrm{A}}\right)^{2}}{E_{\mathrm{A}}} \tag{4}
\end{equation*}
$$

From the above equation it can be appreciated that the same numerical deviation is rated more important in a smaller population than in a large one since the denominator $E_{\mathrm{A}}$ is smaller. If the above expression is inverted, we can derive $O_{\mathrm{A}}$ as:

$$
\begin{equation*}
O_{\mathrm{A}}=E_{\mathrm{A}} \pm \sqrt{\frac{X^{2}}{2}} \quad E_{\mathrm{A}}=E_{\mathrm{A}} \pm \text { tolerance } \tag{5}
\end{equation*}
$$

Hence for a specified value of $X^{2}$, the tolerance is basically a square root function of the expected value and consequently of sample size. This corrects one of the difficulties associated with the use of ratio measures, namely that ratio measures assume a linear relationship between the magnitude of deviation from expectation and sample size.

A further understanding of the effect of the $X_{S}^{2}$ measure can be gained through its equivalent in the two-category case namely the $Z$-score measure.

$$
\begin{equation*}
Z=\frac{p-p}{\sqrt{p q / N}} \tag{6~A}
\end{equation*}
$$

and

$$
\begin{equation*}
Z^{2}=\frac{(P-p)^{2}}{p q / N}=X^{2} \tag{6B}
\end{equation*}
$$

where $p$ is the population proportion in Category A, i.e. the expected proportion, $q$ is the population proportion in Category B.
$P$ is the proportion of Category A in the sample, i.e. the observed proportion,
$N$ is the sample size,
$p+q=1.0$, and
$\sqrt{p q / N}$ is the standard error for proportions.
Thus it can be appreciated that in both the $X^{2}$ and $Z$-score ordering schemes, the departure of the observed ratios are expressed in units of standard errors, i.e. in terms of the inherent variability which is inversely proportional to the sample size. Thus ratio values are standardised with respect to sample size and unless observed ratios coincide with expected values, the same sample value is deemed more extreme in a larger population than in a smaller one.

The $Z$-score and $X_{S}^{2}$ methods of computation are equally valid for proportions or the two-category case. The $X_{S}^{2}$ model is favoured because it is more general. The number of terms in the expression may be expanded or contracted for other applications. Only one term was used for standardising open ratios, i.e. when the numerator and denominator were two separate samples. The value of such ratios can be in excess of unity, for example in the ratio of children born to young women. The $X_{S}^{2}$ map of the fertility of young women shows marked regional contrasts. ${ }^{8}$ On the other hand, several terms could be used for the study of age composition ${ }^{8}$ or the mix of housing types or even the mix of socio-economic groups.
$X_{S}^{2}$ values may be mapped by proportional symbols. However, for maps of a small scale, it may be necessary to classify the data and adopt some sort of shading scheme. The extreme 10 per cent of cases are mapped in the examples given in this paper (see below). While a quantile system for the delimitation of class boundaries was excellent for the comparison of the ratio and $X_{S}^{2}$ ordering schemes, cut-off values are preferable in general as shown by Visvalingam and Dewdney. ${ }^{5}$ Conventionally, $X^{2}$ values are associated with some level of statistical significance and $X_{S}^{2}$ cut-off values of $\pm 3.84$ have been used successfully.

Some caution must be exercised in using the probability levels too rigidly. It has been suggested (J. Besag, personal communication) that demographic samples may not possess the necessary condition of statistical independence. Further as stated by White, 'significance may be measured only in terms of some a priori expectation'. ${ }^{9}$ The key issue in the use of $X_{S}^{2}$ is the formulation of expectation. Values for $X^{2}$ calculations are usually derived mechanically from contingency tables or related to some probability distribution such as the Binomial or Poisson ${ }^{2}$ or Normal ${ }^{10}$ distributions. The population proportion, $p$, often substitutes for
expectation in the calculation of $Z$-score. Experience with several variables in the Census Research Unit of Durham suggests that the use of national averages may be adequate for most general purpose maps.

## COMPARISON OF RATIO AND SIGNED CHI-SQUARE ORDERING SCHEMES

The illustrations in this paper refer to the entire land area within the three 100 kilometre squares whose south west corners are defined by grid references 300 400, 200300 and 300300 respectively; this covers most of Lancashire, Cheshire and North Wales. This section of the country was chosen for several reasons. It is sufficiently large to display a variety of general conditions. The settlements contained have a wide range of population sizes and are known to possess a marked spatial sorting of males and females, this was an important consideration since the two schemes for ordering data are compared using these variables. The data for the Local Authority Areas (LAA) were extracted from the appropriate published census volumes by J. C. Dewdney, while the data at the onekilometre grid square level were provided on magnetic tapes by the Office of Population Censuses and Surveys.
Maps provide a convenient means for illustrating the relative performance of ratio and $X_{S}^{2}$ ordering schemes. Only the two extreme 10 per cent of cases are shown. The quantile system, was considered to offer the most objective basis for the delimitation of extreme cases for purposes of identifying the disparities between the two ordering schemes. The decile system was an arbitrary choice.
Figures 2 and 3 are based on data for $2 \times 2$ kilometre grid squares; there are 4940 grid squares of this resolution in the study area. The populations in these areas range from a single person to 46,914 persons per data unit and the frequency distribution of population sizes can be seen in Figure 1.
Figure 2 shows the extremes of masculinity as defined by the masculinity proportion, i.e. the proportion of males in the total population. Figure 3 shows the spatial distribution of masculinity as defined by $X_{S}^{2}$ (refer equations 2 and 3). The $X_{S}^{2}$ values were calculated using an expectation based on the 1971 national average, i.e. 48.5 per cent males.
It is near impossible to identify any marked spatial trends in Figure 2, in which the punctiform pattern of extreme ratio values are generally associated with small populations in inland rural Wales and Lancashire; there are extensive blanks in the south-east of the study area and in the Liverpool and Manchester conurbations. The spatial pattern of extreme $X_{S}^{2}$ values in Figure 3 exhibits contiguity and is more readily interpreted. Major blocks of contiguous squares with low masculinity are predominantly coastal and pick out holiday resorts and retirement areas. Other clusters of low masculinity squares occur in the inner suburbs of Liverpool, Manchester and several smaller towns. High masculinity occurs at both ends of the ruralurban continuum, being particularly marked in the Potteries, Wolverhampton and Walsall and in the belt extending from central Manchester, towards Liverpool. It is not only associated with urban-industrial locations and recent large-scale housing development but also with military establishments and prisons (a more extensive discussion is provided by Visvalingam and Dewdney ${ }^{5}$ ).

The disparity between the ratio and $X_{S}^{2}$ ordering schemes is even more noticeable at the one kilometre level, as shown


Figure 2. Extremes of masculinity as indicated by the masculinity proportion in $2 \times 2 \mathrm{~km}$ grid squares.


Figure 3. Extremes of masculinity as indicated by the $X_{\mathrm{S}}^{2}$ measure in $2 \times 2 \mathrm{~km}$ grid squares.


Figure 4. Ratio map showing the 10 per cent of LAAs with highest and lowest values.

$\mathrm{X}_{\mathrm{s}}^{2}$ map showing the ten per cent of LAAs with highest
and lowest masculinity
LAAs with $x_{s}^{2} \geqslant 18.820$
, proportion of total units $=10.0 \%$
LAAs with $x_{s}^{2} \leqslant-20.523$
Figure 5. $X_{\mathrm{S}}^{2}$ map showing the 10 per cent of LAAs with highest and lowest values.
by the above authors. The $X_{S}^{2}$ ordering scheme yielded consistent spatial patterns of extremes of masculinity at the $1 \times 1,2 \times 2$ and $5 \times 5$ kilometre square levels. The ratio system on the other hand is not as robust. While the spatial patterns of masculinity as portrayed by ratio and $X_{S}^{2}$ measures are markedly different at the $1 \times 1$ and $2 \times 2$ kilometre square levels, the ratio maps begin to reflect the spatial pattern portrayed by $X_{\mathrm{S}}^{2}$ values at coarser levels of aggregation, especially at the LAA level as shown in Figures 4 and 5.

It must be stressed that the difficulty associated with the use of ratio values does not vanish with an increase in the size of areal units or with the size of the sample population. For the problem is not a function of sample size (although small samples are always unreliable) so much as the incomparability of sample sizes. It could be solved if the spatial units are carved up to yield equal population samples. While irregular administrative areas are intended to produce more comparable samples, they are by no means equal or even near-equal population units; the

TABLE 2
The 10 Highest and Lowest Ratio and $X_{\mathbf{S}}^{2}$ Values for Local Authority Areas

|  | Area and location |  | Population | Males | Ratio (\% males) | $\underset{\text { value }}{X_{\mathrm{S}}^{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Ratio 10 highest values |  |  |  |  |  |  |
| 1 | Shifnal R. D. | (377 307) | 15,869 | 8,587 | 54.112 | 200.082 |
| 2 | Sedbergh R. D. | (371 491) | 3,544 | 1,912 | 53.950 | 42.150 |
| 3 | Clitheroe R. D. | (368 440) | 9,456 | 5,075 | 53.670 | 101.177 |
| 4 | Kirkham U. D. | (343 432) | 6,436 | 3,405 | 52.906 | 50.011 |
| 5 | Penllyn R. D. | $(289$ 334) | 2,313 | 1,205 | 52.097 | 11.980 |
| 6 | Clun \& Bishop's Castle R. D. | (333 301) | 8,869 | 4,577 | 51.607 | 34.272 |
| 7 | Machynlleth R. D. | (285 305) | 2,502 | 1,291 | 51.599 | 9.618 |
| 8 | Maelor R. D. | (345 341) | 4,679 | 2,393 | 51.143 | 13.090 |
| 9 | Newton and Llanidloes R. D. | (306 301) | 8,223 | 4,203 | 51.113 | 22.473 |
| 10 | Millon R. D. | (315 492) | 14,089 | 7,195 | 51.068 | 37.204 |
| Ratio 10 lowest values |  |  |  |  |  |  |
| 1 | Grange U. D. | (341 478) | 3,474 | 1,395 | 40.155 | -96.847 |
| 2 | Dolgellau U. D. | (274 317) | 2,567 | 1,062 | 41.371 | -52.228 |
| 3 | Criccieth U. D. | $(251$ 339) | 1,505 | 646 | 42.924 | -18.737 |
| 4 | Llandudno U. D. | (280 381) | 19,077 | 8,295 | 43.482 | -192.343 |
| 5 | Colwyn Bay M. B. | $(287$ 378) | 25,564 | 11,162 | 43.663 | -239.461 |
| 6 | Lytham St. Annes M. B. | (334 429) | 40,299 | 17,765 | 44.083 | -314.776 |
| 7 | Windermere U. D. | (341 497) | 8,065 | 3,557 | 44.104 | -62.394 |
| 8 | Barmouth U. D. | (263 317) | 2,106 | 929 | 44.112 | -16.234 |
| 9 | Prestatyn U. D. | (306 383) | 14,515 | 6,462 | 44.519 | -92.077 |
| 10 | Penmaenmawr U. D. | (273 376) | 3,991 | 1,787 | 44.776 | -22.162 |
| $X_{S}^{2} 10$ highest values |  |  |  |  |  |  |
| 1 | Shifnal R. D. | (377 307) | 15,869 | $8,587$ | 54.112 | 200.082 |
| 2 | Wolverhampton C. B. | (392 301) | 269,112 | 133,976 | 49.784 | 177.768 |
| 3 | Bolton C. B. | (370 410) | 154,199 | 74,709 | 50.073 | 147.876 |
| 4 | Warrington R. D. | (364 391) | 50,420 | 25,684 | 50.940 | 120.191 |
| 5 | Clitheroe R. D. | (368 440) | 9,456 | 5,075 | 53.670 | 101.177 |
| 6 | Walsall C. B. | (399 301) | 184,738 | 91,605 | 49.588 | 87.476 |
| 7 | Stafford M. B. | (393 322) | 55,001 | 27,652 | 50.275 | 69.413 |
| 8 | Cannock U. D. | (399 310) | 55,882 | 28,022 | 50.145 | 60.538 |
| 9 | Wellington R. D. | (367 317) | 30,297 | 15,360 | 50.698 | 58.606 |
| 10 | Ellesmere Pt. M. B. | (340 376) | 61,637 | 30,831 | 50.020 | 57.035 |
| $X_{\text {S }}^{2} 10$ lowest values |  |  |  |  |  |  |
| 1 | Blackpool C. B. | (330 436) | 151,860 | 69,338 | 45.659 | $-490.659$ |
| 2 | Southport C. B. | (333 416) | 84,574 | 37,921 | 44.838 | -454.155 |
| 3 | Lytham St. Annes M. B. | (334 429) | 40,299 | 17,765 | 44.083 | -314.776 |
| 4 | Colwyn Bay M. B. | (287 378) | 25,564 | 11,162 | 43.663 | -239.461 |
| 5 | Morecambe M. B. | (343 463) | 41,908 | 18,848 | 44.975 | -208.514 |
| 6 | Llandudno U. D. | $(280$ 381) | 19,077 | 8,295 | 43.482 | -192.343 |
| 7 | Grange U. D. | (341 478) | 3,474 | 1,395 | 40.155 | -96.847 |
| 8 | Prestatyn U. D. | (306 383) | 14,515 | 6,462 | 44.519 | -92.077 |
| 9 | Hoylake U. D. | (324 388) | 32,277 | 14,802 | 45.859 | -90.112 |
| 10 | Crosby M. B. | (332 400) | 57,497 | 26,756 | 46.535 | -88.919 |

frequency distribution of population sizes in the LAAs is still not uniform as shown in Figure 1. The LAAs included in Figures 4 and 5 have population sizes which range from 729 to 610,113 persons. Since the population base is quite large, masculinity proportions are not as wildly variable as in the $1 \times 1$ and $2 \times 2$ grid square levels, but are confined to values within the range 40.15 to 54.11 per cent males. Despite this, the problems associated with ratio measures are present even at the LAA level. This is manifested to some extent in Figures 4 and 5 but become more apparent when the LAAs are ranked by their ratio and $X_{S}^{2}$ values, refer Table 2. The relatively smaller populations continue to dominate the extremes. Hence, the present discussion is relevant to users of all types of spatial data, except when the spatial units were derived to yield equal or near-equal population samples.
There will be situations when the mapping of absolute numbers, absolute deviations and ratio expressions are particularly relevant. However, experience at the Census Research Unit has shown that the $X_{S}^{2}$ measure can be successfully applied over a wide range of variables and has been found particularly useful for highlighting contiguity relationships.

## CONCLUSION

In conclusion, the limitations of using $X_{\mathrm{S}}^{2}$ maps must be considered along with the ramifications for other studies. While $X_{S}^{2}$ maps are excellent for picking out spatial patterns, they cannot be used in the conventional manner in which ratio and absolute number maps are used for deriving quantitative information. A $X_{S}^{2}$ value summarises a relationship between ratios and absolute numbers; and while values of $X_{S}^{2}$ may be read-off the map, the latter do not contain any clue to the particular ratio or absolute numbers involved. Moreover, no external factors are involved in the derivation of absolute number or ratio maps. Thus, while the appearance of these maps may change with alterations to the number of classes or class boundaries, the order of data units remains static and unchanged and implies a single solution to multiple problems. The value for expectation used in the calculation of $X_{S}^{2}$ values has some effect on the relative order of data units. Thus the proper use of $X_{S}^{2}$ demands that the projection of expectation corresponds to the object of the academic exercise as ranking is relative to some a priori expectation.
The use of ratios is not confined to mapping. Multivariate exercises commonly include ratio variables and there has been much discussion concerning the choice of
suitable transformations and scaling procedures. ${ }^{11}$ The problems of ordering in these exercises are even more crucial and fundamental than those of scaling. Ratios based on small populations or highly varying base populations are unsuitable for multivariate analysis since the extremes, which are the products of small populations, are bound to dominate the calculation of correlation coefficients and set the trend in regressions. Although maps have been used to illustrate the disparities between ratio and $X_{\mathbf{S}}^{2}$ ordering schemes, it must be emphasised that the implications of this work extend beyond cartographic concerns.

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