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THE SIMILARITY PROBLEM FOR INDEFINITE STURM—LIOUVILLE OPERATORS WITH PERIODIC COEFFICIENTS

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Abstract. We investigate the problem of similarity to a self-adjoint operator for J-positive Sturm–Liouville operators $L=\frac{1}{\omega}\left(-\frac{d^2}{dx^2}+q\right)$ with 2π-periodic coefficients q and ω. It is shown that if 0 is a critical point of the operator L, then it is a singular critical point. This gives us a new class of J-positive differential operators with the singular critical point 0. Also, we extend the Beals and Parfenov regularity conditions for the critical point ∞ to the case of operators with periodic coefficients.

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REFERENCES

[1] R. BEALS, *Indefinite Sturm–Liouville problems and half-range completeness*, J. Differential Equations **56** (1985), 391–407.
[2] J. BEHRNDT AND C. TRUNK, *On the negative squares of indefinite Sturm–Liouville operators*, J. Differential Equations **238** (2007), 491–519.
[3] P. BINDING AND B. ĆURGUS, *A counterexample in Sturm–Liouville completeness theory*, Proc. Roy. Soc. Edinburgh Sect. A **134**, 2 (2004), 244–248.
[4] M.S. BIRMAN AND M.Z. SOLOMJAK, *Spectral theory of self-adjoint operators in Hilbert spaces*, D. Reidel Publ. Company, Dordrecht, 1987.
[5] B. ĆURGUS, *On the regularity of the critical point infinity of definitizable operators*, Integr. Equ. Oper. Theory **8** (1985), 462–488.
[6] B. ĆURGUS, *Positive operators in Krein spaces similar to self-adjoint operators in Hilbert spaces*, (2006), `http://www.math.tu-berlin.de/~trunk/abstracts6.html`.
[7] B. ĆURGUS AND H. LANGER, *A Krein space approach to symmetric ordinary differential operators with an indefinite weight function*, J. Differential Equations **79** (1989), 31–61.
[8] B. ĆURGUS AND B. NAJMAN, *The operator* $(\operatorname{sgn} x)\frac{d^2}{dx^2}$ *is similar to a selfadjoint operator in* $L^2(\mathbb{R})$, Proc. Amer. Math. Soc. **123** (1995), 1125–1128.
[9] K. DAHO AND H. LANGER, *Sturm–Liouville operators with an indefinite weight function: the periodic case*, Radovi Mat. **2** (1986), 165–188.
[10] A. FLEIGE, *Spectral theory of indefinite Krein-Feller differential operators*, Mathematical Research **98**, Berlin, Akademie Verlag, 1996.
[11] A. FLEIGE, *A necessary aspect of the generalized Beals condition for the Riesz basis property of indefinite Sturm–Liouville problems*, Oper. Theory: Adv. Appl. **175** (2007), 89–94.
[12] A. FLEIGE, *The Riesz basis property of indefinite Sturm–Liouville problems with non-odd weight function*, Integr. Equ. Oper. Theory **60** (2008), 237–246.
[13] I.M. GELFAND, *Expansion in characteristic functions of an equation with periodic coefficients*, Doklady Akad. Nauk SSSR **73** (1950), 1117–1120. (Russian)
[14] F. GESZTESY AND V. TKACHENKO, *A criterion for Hill operators to be spectral operators of scalar type*, J. Anal. Math. **107** (2009), 287–353.
[15] I.M. KARABASH, *Abstract kinetic equations with positive collision operators*, Oper. Theory: Adv. Appl. **188** (2008), 175–195.

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[16] I.M. KARABASH, *A functional model, eigenvalues, and finite singular critical points for indefinite Sturm–Liouville operators*, Oper. Theory: Adv. Appl. **203** (2009), 247–287.
[17] I.M. KARABASH AND A.S. KOSTENKO, *Indefinite Sturm–Liouville operators with the singular critical point zero*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), 801–820.
[18] I.M. KARABASH AND A.S. KOSTENKO, *On the similarity of a J-nonnegative Sturm–Liouville operator to a selfadjoint operator*, Funct. Anal. Appl. **43** (2009), 65–68.
[19] I.M. KARABASH, A.S. KOSTENKO, AND M.M. MALAMUD, *The similarity problem for J-nonnegative Sturm–Liouville operators*, J. Differential Equations **246** (2009), 964–997.
[20] I.M. KARABASH AND M.M. MALAMUD, *Indefinite Sturm–Liouville operators* $(\operatorname{sgn} x)(-\frac{d^2}{dx^2}+q)$ *with finite-zone potentials*, Oper. Matrices **1**, 3 (2007), 301–368.
[21] T. KATO, *Perturbation theory for linear operators*, Springer-Verlag, Berlin-Heidelberg, New York, 1966.
[22] M. G. KREIN, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications, II*, Mat. Sb. (N.S.) **21**, 62 (1947), 365–404. (Russian)
[23] H. LANGER, *Spectral functions of definitizable operators in Krein space*, Lecture Notes in Math. **948** (1982), 1–46.
[24] M.M. MALAMUD, *A criterion for the similarity of a closed operator to a self-adjoint operator*, Ukrainian Math. J. **37** (1985), 41–48.
[25] D. MCGARVEY, *Operators commuting with translation by one. I. Representation theorems*, J. Math. Anal. Appl. **4** (1965), 366–410.
[26] S.N. NABOKO, *Conditions for similarity to unitary and self-adjoint operators*, Funct. Anal. Appl. **18**, 1 (1984), 13–22.
[27] M.A. NAIMARK, *Linear differential operators*, Fizmatgiz, Moscow, 1969. (Russian)
[28] A.I. PARFENOV, *On an embedding criterion for interpolation spaces and application to indefinite spectral problem*, Siberian Math. J. **44**, 4 (2003), 638–644.
[29] S.G. PYATKOV, *Some properties of eigenfunctions of linear pencils*, Math. Notes **51** (1992), 90–95.
[30] S.G. PYATKOV, *Some properties of eigenfunctions and associated functions of indefinite Sturm–Liouville problems*, in: *Nonclassical Problems of Mathematical Physics*, Sobolev Institute of Mathematics, Novosibirsk (2005), 240–251.
[31] M. REED AND B. SIMON, *Methods of modern mathematical physics, IV: Analysis of operators*, Academic Press, New York, 1978.
[32] F.S. ROFE-BEKETOV AND A.M. KHOLKIN, *Spectral analysis of differential operators: interplay between spectral and oscillatory properties*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
[33] E.C. TITCHMARSH, *Eigenfunction expansions associated with second-order differential equations. II*, Clarendon Press, Oxford, 1958.
[34] O.A. VELIEV, *Spectrum and spectral singularities of differential operators with complex valued periodic coefficients*, Diff. Equats. **19** (1983), 983–989.
[35] K. VESELIĆ, *On spectral properties of a class of J-self-adjoint operators. I*, Glasnik Math. Ser. III **7**, 2 (1972), 229–248.
[36] H. VOLKMER, *Sturm–Liouville problems with indefinite weights and Everitt's inequality*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 1097–1112.
[37] J. WEIDMANN, *Spectral theory of ordinary differential operators*, Lecture Notes in Math. **1258**, 1987.
[38] A. ZETTL, *Sturm–Liouville theory*, Amer. Math. Soc., Providence, RI, 2005.