

THE SIMULTANEOUS DISTRIBUTION OF MEAN AND STANDARD DEVIATION IN SMALL SAMPLES

By ALLEN T. CRAIG

1. Introduction. If samples of n items are selected at random from a normal universe, it is well known that the arithmetic mean \bar{x} and standard deviation s computed from samples are independent in the probability sense and that the simultaneous frequency distribution is

$$F(\bar{x}, s) = C s^{n-2} e^{-\frac{ns^2 + n\bar{x}^2}{2\sigma^2}}$$

If, however, the parent population is other than the normal type, there appears to be little known regarding the form of $F(\bar{x}, s)$. In the present paper, we propose to determine the simultaneous frequency function of the arithmetic mean and standard deviation in samples of small numbers of items selected at random from a rather arbitrary universe. For convenience, we shall classify frequency distributions according as the range of the independent variable is $(-\infty, \infty)$, $(0, \infty)$ or $(0, a)$, $a > 0$. We shall further assume that the total area under the distribution function is unity.

2. The simultaneous distribution of \bar{x} and s in samples of $n=2$. Let $f(x)$, $-\infty < x < \infty$ be the frequency function of the variable x . Let x_1 and x_2 be two independent observed values of x . write

$$\begin{aligned} x_1 + x_2 &= 2\bar{x} \\ x_1^2 + x_2^2 &= 2s^2 + 2\bar{x}^2 \end{aligned}$$

We seek the function $F(\bar{x}, s)$ such that $F(\bar{x}, s)d\bar{x}ds$ is, to within infinitesimals of higher order, the probability of the simultaneous occurrence of \bar{x} in $(\bar{x}, \bar{x}+d\bar{x})$ and s in $(s, s+ds)$. For \bar{x} and s assigned, x_1 may have either value $\bar{x} - s$ or $\bar{x} + s$ and x_2 is uniquely determined by $x_2 = 2\bar{x} - x_1$.

$$F(\bar{x}, s)d\bar{x}ds = f(\bar{x}-s)f(2\bar{x}-x_1)dx_1 dx_2$$

Thus

$$+f(\bar{x}+s)f(2\bar{x}-x_1)dx_1 dx_2.$$

Since $dx_1 dx_2 = 2d\bar{x}ds$ we have

$$(1) \quad F(\bar{x}, s) = 4f(\bar{x}-s)f(\bar{x}+s).$$

If $f(x)$ is defined on the interval $(0, \infty)$, we note, for \bar{x} assigned, that $s \leq \bar{x}$. Thus (1) is valid for this type of frequency function but the surface is limited by the x -axis and the line $s = \bar{x}$.

If $f(x)$ is defined on the interval $(0, a)$, we note, for \bar{x} assigned on $(0, a/2)$, that $s \leq \bar{x}$; and, for \bar{x} assigned on $(a/2, a)$, that $s \leq a - \bar{x}$. Accordingly, for this kind of frequency function, (1) is valid but the surface is limited by the x -axis and the lines $s = \bar{x}$, $s = a - \bar{x}$.

As simple illustrations, let us find the correlation surface for the mean and standard deviation of samples of two items drawn from distributions of various types.

Example 1. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

Then

$$F(\bar{x}, S) = \frac{2}{\sigma^2 \pi} e^{-\frac{S^2 + \bar{x}^2}{\sigma^2}}$$

the well known result.

Example 2. Let

$$f(x) = e^{-x}, \quad 0 \leq x < \infty.$$

Then

$$F(\bar{x}, S) = 4 e^{-2\bar{x}}$$

over the open region of the $\bar{x}S$ -plane bounded by the \bar{x} -axis and the line $S = \bar{x}$.

Example 3. Let

$$f(x) = \frac{1}{a}, \quad 0 \leq x \leq a.$$

Then

$$F(\bar{x}, S) = \frac{4}{a^2},$$

over the region of the $\bar{x}S$ -plane bounded by the isosceles triangle with sides $S = 0$, $S = \bar{x}$ and $S = a - \bar{x}$. With a uniform distribution proportional to $4/a^2$ over this triangle, it follows incidentally from very elementary geometry that the marginal totals of the distribution of \bar{x} are given by the known values

$$\phi(\bar{x}) = \frac{4}{a^2} \bar{x}, \quad 0 \leq \bar{x} \leq \frac{a}{2},$$

$$= \frac{4}{a^2} (a - \bar{x}), \quad \frac{a}{2} \leq \bar{x} \leq a,$$

and that the marginal totals for the distribution of \mathbf{s} are given by

$$\psi(s) = \frac{4}{a^2} (a - 2s), \quad 0 \leq s \leq \frac{a}{2},$$

which is the result given by Rider.¹

3. The simultaneous distribution of \bar{x} and \mathbf{s} in samples of $n=3$. Consider first a frequency function $f(x)$, $-\infty < x < \infty$. We have

$$x_1 + x_2 + x_3 = 3\bar{x},$$

$$x_1^2 + x_2^2 + x_3^2 = 3s^2 + 3\bar{x}^2.$$

Upon eliminating x_3 , we have

$$2x_1^2 + 2x_1x_2 + 2x_2^2 - 6\bar{x}x_1 - 6\bar{x}x_2 - 3s^2 + 6\bar{x}^2 = 0.$$

From simple properties of this ellipse, it follows, for assigned \bar{x} and \mathbf{s} that x_1 may be chosen arbitrarily from the interval $(\bar{x} - s\sqrt{2}, \bar{x} + s\sqrt{2})$. With x_1 assigned, x_2 must be selected with certainty as either

$$\frac{3\bar{x} - x_1 - [6s^2 - 3(x_1 - \bar{x})^2]^{\frac{1}{2}}}{2} \quad \text{or}$$

$$\frac{3\bar{x} - x_1 + [6s^2 - 3(x_1 - \bar{x})^2]^{\frac{1}{2}}}{2}$$

Finally we must have

$$x_3 = 3\bar{x} - x_1 - x_2.$$

¹ P. R. Rider, On the distribution of ratio of mean to standard deviation etc., *Biometrika*, vol. 21 (1929) pp. 124-141.

Thus

$$F(\bar{x}, s) d\bar{x} ds = 2 \int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} \frac{f(x_1) f(x_2) f(x_3)}{R} dx_1 dx_2 dx_3.$$

From

$$\begin{aligned} x_1 &= x_1, \\ x_2 &= \frac{3\bar{x} - x_1 \pm [9s^2 - 3(x_1 - \bar{x})^2]^{\frac{1}{2}}}{2}, \\ x_3 &= 3\bar{x} - x_1 - x_2 \end{aligned}$$

we obtain

$$\begin{aligned} dx_1 dx_2 dx_3 &= \frac{9s}{[9s^2 - 3(x_1 - \bar{x})^2]^{\frac{1}{2}}} dx_1 d\bar{x} ds. \\ &= 9s dx_1 d\bar{x} ds / R \end{aligned}$$

where

$$R \equiv [9s^2 - 3(x_1 - \bar{x})^2]^{\frac{1}{2}}$$

Thus

(2)

$$F(\bar{x}, s) = 18s \int_{\bar{x}-s\sqrt{2}}^{\bar{x}+s\sqrt{2}} \frac{1}{R} f(x_1) f\left(\frac{3\bar{x}-x_1+R}{2}\right) f\left(\frac{3\bar{x}-x_1-R}{2}\right) dx_1,$$

If $f(x)$ is defined on the interval $(0, \infty)$, we note, for \bar{x} assigned, that $0 \leq s \leq \bar{x}\sqrt{2}$. Thus the surface is limited by the \bar{x} -axis and the line $s = \bar{x}\sqrt{2}$. Moreover, since x_1, x_2, x_3 are non-negative, x , may be selected from the interval $(\bar{x} - s\sqrt{2}, \bar{x} + s\sqrt{2})$ only as long as $s \leq \bar{x}\sqrt{2}/2$. If $\bar{x}\sqrt{2}/2 \leq s \leq \bar{x}\sqrt{2}$.

then x , may be selected from the intervals

$$\left(0, \frac{3\bar{x} - [0s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2} \right)$$

and

$$\left(\frac{3\bar{x} + [0s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}, \bar{x} + s\sqrt{2} \right).$$

Accordingly, for this type of frequency function,

$$F(\bar{x}, s) = 18s \int_{\bar{x} - s\sqrt{2}}^{\bar{x} + s\sqrt{2}} \frac{1}{R} f(x_1) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right) dx_1,$$

$$0 \leq s \leq \frac{\bar{x}\sqrt{2}}{2},$$

(2.1)

$$= 18s \left[\int_0^{\frac{3\bar{x} - [0s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x} + [0s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}}^{\bar{x} + s\sqrt{2}} \right]$$

$$\frac{1}{R} f(x_1) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right) dx_1,$$

$$\frac{\bar{x}\sqrt{2}}{2} \leq s \leq \bar{x}\sqrt{2}.$$

If $f(x)$ is defined on the interval $(0, a)$, we note:

$$\text{for } 0 \leq \bar{x} \leq a/3, \quad 0 \leq s \leq \bar{x}\sqrt{2};$$

$$\text{for } a/3 \leq \bar{x} \leq 2a/3, \quad 0 \leq s \leq [2\bar{x}^2 - 2a\bar{x} + \frac{2a^2}{3}]^{\frac{1}{2}};$$

$$\text{for } 2a/3 \leq \bar{x} \leq a, \quad 0 \leq s \leq (a-\bar{x})\sqrt{2}.$$

Thus in this case, the surface is limited by the x -axis, the lines $s = \bar{x}\sqrt{2}$ and $s = (a-\bar{x})\sqrt{2}$ and the hyperbola

$$s = [2\bar{x}^2 - 2a\bar{x} + \frac{2a^2}{3}]^{\frac{1}{2}}.$$

(Fig. 1.). Now x , may be selected from the interval $(\bar{x} - s\sqrt{2}, \bar{x} + s\sqrt{2})$ as long as $s \leq \bar{x}\sqrt{2}/2$ and $s \leq (a-\bar{x})\sqrt{2}/2$. This holds for that part of the surface over the region bounded by OPa . For that part of the surface over the region bounded by OPU, x , may be selected from the intervals

$$\left(0, \frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2} \right)$$

and

$$\left(\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}, \bar{x} + s\sqrt{2} \right).$$

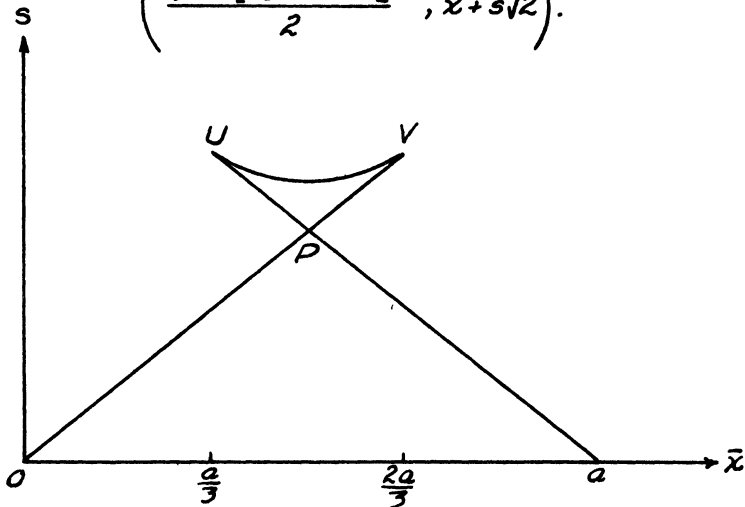


Fig. 1

It is clear that the ranges of arbitrary selection of x , for that part of the surface over the region bounded by DVa are

$$\left(\bar{x} - s\sqrt{2}, \frac{3\bar{x} - a - [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{2} \right)$$

and

$$\left(\frac{3\bar{x} - a + [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{2}, a \right)$$

Finally, we find that x , may be selected from the intervals

$$\left(0, \frac{3\bar{x} - a - [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{2} \right),$$

$$\left(\frac{3\bar{x} - a + [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{2}, \frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2} \right),$$

and

$$\left(\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}, a \right)$$

for that part of the surface over the region bounded by PUV

If we adopt the notation

$$\phi = \phi(x, \bar{x}, s) \equiv \frac{1}{R} f(x) f\left(\frac{3\bar{x} - x_1 + R}{2}\right) f\left(\frac{3\bar{x} - x_1 - R}{2}\right),$$

we have

$$(2.2) \quad F(\bar{x}, s) = 18s \int_{\bar{x} - s\sqrt{2}}^{\bar{x} + s\sqrt{2}} \phi dx,$$

$$= 18s \left[\int_0^{\frac{3\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}} + \int_{\frac{3\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{2}}^{\bar{x} + s\sqrt{2}} \right] \phi dx,$$

$$= 18s \left[\int_{\bar{x} - s\sqrt{2}}^{\frac{3\bar{x} - a - [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{2}} \right.$$

$$\left. + \int_{\frac{3\bar{x} - a + [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{2}}^a \right] \phi dx,$$

$$= 18s \left[\int_0^{\frac{3\bar{x}-a-\sqrt{6s^2-3(a-\bar{x})^2}}{2}} \frac{3\bar{x}-a-\sqrt{6s^2-3(a-\bar{x})^2}}{2} + \int_{\frac{3\bar{x}-a+\sqrt{6s^2-3(a-\bar{x})^2}}{2}}^{\frac{3\bar{x}-\sqrt{6s^2-3\bar{x}^2}}{2}} \frac{3\bar{x}-\sqrt{6s^2-3\bar{x}^2}}{2} \right. \\ \left. + \int_{\frac{3\bar{x}+\sqrt{6s^2-3\bar{x}^2}}{2}}^a \right] \phi dx,$$

for the parts of the surface over the regions indicated above.

In order to illustrate the theory, we shall consider a few examples.

Example 1. Let

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

By (2),

$$F(\bar{x}, s) = \frac{3\sqrt{3}}{\sigma^3\sqrt{2\pi}} se^{-\frac{3s^2+3\bar{x}^2}{2\sigma^2}}$$

Example 2. Let

$$f(x) = e^{-x}, \quad 0 \leq x < \infty.$$

By (2.1),

$$F(\bar{x}, s) = 6\sqrt{3}\pi se^{-3\bar{x}} \quad 0 \leq s \leq \frac{\bar{x}\sqrt{2}}{2}, \\ = 6\sqrt{3} se^{-3\bar{x}} \left[\arcsin \frac{\bar{x}-\sqrt{6s^2-3\bar{x}^2}}{s2\sqrt{2}} + \arcsin \frac{\bar{x}}{s\sqrt{2}} \right. \\ \left. - \arcsin \frac{\bar{x}+\sqrt{6s^2-3\bar{x}^2}}{s2\sqrt{2}} + \frac{\pi}{2} \right], \quad \frac{\bar{x}\sqrt{2}}{2} \leq s \leq \bar{x}\sqrt{2}.$$

Example 3. Let

$$f(x) = \frac{1}{a}, \quad 0 \leq x \leq a.$$

By (2.2),

$$\begin{aligned}
 F(\bar{x}, s) &= \frac{6\sqrt{3}\pi s}{a^3}, \text{ over } OPa, \\
 &= \frac{6\sqrt{3}s}{a^3} \left[\arcsin \frac{\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \arcsin \frac{\bar{x}}{s\sqrt{2}} \right. \\
 &\quad \left. - \arcsin \frac{\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} + \frac{\pi}{2} \right], \text{ over } OPU, \\
 &= \frac{6\sqrt{3}s}{a^3} \left[\arcsin \frac{\bar{x} - a - [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}} \right. \\
 &\quad \left. + \arcsin \frac{a - \bar{x}}{s\sqrt{2}} - \arcsin \frac{\bar{x} - a + [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}} \right. \\
 &\quad \left. + \frac{\pi}{2} \right], \text{ over } PVa, \\
 &= \frac{6\sqrt{3}s}{a^3} \left[\arcsin \frac{\bar{x} - a - [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}} \right. \\
 &\quad \left. + \arcsin \frac{\bar{x}}{s\sqrt{2}} + \arcsin \frac{\bar{x} - [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} \right. \\
 &\quad \left. + \arcsin \frac{a - \bar{x}}{s\sqrt{2}} \right. \\
 &\quad \left. - \arcsin \frac{\bar{x} - a + [6s^2 - 3(a - \bar{x})^2]^{\frac{1}{2}}}{s2\sqrt{2}} \right. \\
 &\quad \left. - \arcsin \frac{\bar{x} + [6s^2 - 3\bar{x}^2]^{\frac{1}{2}}}{s2\sqrt{2}} \right], \text{ over } PVU.
 \end{aligned}$$

I have succeeded in obtaining the marginal totals for S from 0 to $a\sqrt{2}/4$ by integrating $F(\bar{x}, s)$ with respect to \bar{x} from the boundary (Fig. 1) $s = \bar{x}\sqrt{2}$ to $s = (a - \bar{x})\sqrt{2}$ and obtain as a result the parabola which is known¹ to give the distribution of s from $s = 0$ to $s = a\sqrt{2}/6$.

4. The simultaneous distribution of \bar{x} and S in samples of $n = 4$. We shall consider first samples of four items drawn from a universe characterized by a law of frequency $f(x)$, $-\infty < x < \infty$. Then

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 &= 4\bar{x}, \\x_1^2 + x_2^2 + x_3^2 + x_4^2 &= 4s^2 + 4\bar{x}^2.\end{aligned}$$

The elimination of x_4 yields

$$x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 - 4\bar{x}x_1 - 4\bar{x}x_2 - 4\bar{x}x_3 - 2s^2 + 6\bar{x}^2 = 0$$

It follows from the properties of this ellipsoid that x_1 may be chosen arbitrarily from the interval $(\bar{x} - s\sqrt{3}, \bar{x} + s\sqrt{3})$. For x_1 assigned, the region of arbitrary selection of x_2 is determined by the properties of the ellipse and is

$$\left(\frac{4\bar{x} - x_1 - 2[6s^2 - 2(x_1 - \bar{x})^2]^{1/2}}{3}, \frac{4\bar{x} - x_1 + 2[6s^2 - 2(x_1 - \bar{x})^2]^{1/2}}{3} \right)$$

Upon solving for x_3 in terms of x_1 and x_2 we have

$$x_3 = \frac{4\bar{x} - x_1 - x_2 \pm [6s^2 - 8\bar{x}^2 + 8\bar{x}x_1 + 8\bar{x}x_2 - 2x_1x_2 - 3x_1^2 - 3x_2^2]^{1/2}}{2}$$

while x_4 is uniquely determined by $x_4 = 4\bar{x} - x_1 - x_2 - x_3$.

If we write

$$T \equiv [6s^2 - 8x^2 + 8\bar{x}x_1 + 8\bar{x}x_2 - 2x_1x_2 - 3x_1^2 - 3x_2^2]^{1/2}$$

and

$$\Phi \equiv f(x_1)f(x_2)f\left(\frac{4\bar{x} - x_1 - x_2 + T}{2}\right)f\left(\frac{4\bar{x} - x_1 - x_2 - T}{2}\right)$$

¹H. L. Rietz [Paper to appear presently in *Biometrika*].

then

$$(3) \quad F(\bar{x}, s) = 32s \int_{\bar{x}-s\sqrt{3}}^{\bar{x}+s\sqrt{3}} \int_{\frac{4\bar{x}-x_1-2[\frac{6s^2-2(x_1-\bar{x})^2}{3}]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x}-x_1+2[\frac{6s^2-2(x_1-\bar{x})^2}{3}]^{\frac{1}{2}}}{3}} \frac{1}{f} \phi \, dx_2 \, dx_1.$$

The integration can be carried out in an obvious manner when $f(x)$ is the normal frequency function.

In case $f(x)$ is defined on the interval $(0, \infty)$, we note, for \bar{x} assigned, that $s \leq \bar{x}/\sqrt{3}$. Thus the surface is limited by the \bar{x} -axis and the line $s = \bar{x}/\sqrt{3}$. Moreover, x_1 may be selected from the interval $(\bar{x}-s\sqrt{3}, \bar{x}+s\sqrt{3})$ with x_2 chosen as above only as long as $s \leq \bar{x}/\sqrt{3}$. If $\bar{x}/\sqrt{3} \leq s \leq \bar{x}$, then x_1 may be chosen from either of the two intervals

$$\left(0, \frac{4\bar{x}-2[\frac{6s^2-2\bar{x}^2}{3}]^{\frac{1}{2}}}{3} \right) \text{ and } \left(\frac{4\bar{x}+2[\frac{6s^2-2\bar{x}^2}{3}]^{\frac{1}{2}}}{3}, \bar{x}+s\sqrt{3} \right)$$

with x_2 chosen as above; or x_1 may be selected from the interval

$$\left(\frac{4\bar{x}-2[\frac{6s^2-2\bar{x}^2}{3}]^{\frac{1}{2}}}{3}, \frac{4\bar{x}+2[\frac{6s^2-2\bar{x}^2}{3}]^{\frac{1}{2}}}{3} \right)$$

with x_2 taken from either

$$\left(0, \frac{4\bar{x}-x_1-[\frac{6s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1}{2}]^{\frac{1}{2}}}{2} \right)$$

or

$$\left(\frac{4\bar{x}-x_1+[\frac{6s^2-8\bar{x}^2-3x_1^2+8\bar{x}x_1}{2}]^{\frac{1}{2}}}{2}, \frac{4\bar{x}-x_1+2[\frac{6s^2-2(x_1-\bar{x})^2}{3}]^{\frac{1}{2}}}{3} \right).$$

when $\bar{x} \leq s \leq \bar{x}/\sqrt{3}$ we may have

$$0 \leq x_1 \leq 2\bar{x} - [\frac{6s^2-2\bar{x}^2}{3}]^{\frac{1}{2}}$$

and

$$2\bar{x} + [2s^2 - 2\bar{x}^2]^{\frac{1}{2}} \leq x_1 \leq \frac{4\bar{x} + 2[0s^2 - 2\bar{x}^2]^{\frac{1}{2}}}{3}$$

with either

$$0 \leq x_2 \leq \frac{4\bar{x} - x_1 - [0s^2 - 8\bar{x}^2 - 3x_1^2 + 8\bar{x}x_1]^{\frac{1}{2}}}{2}$$

or

$$\frac{4\bar{x} \cdot x_1 + [0s^2 - 8\bar{x}^2 - 3x_1^2 + 8\bar{x}x_1]^{\frac{1}{2}}}{2} \leq x_2 \leq \frac{4\bar{x} - x_1 + 2[0s^2 - 2(x_1 - \bar{x})^2]^{\frac{1}{2}}}{3}$$

Or we may have

$$\frac{4\bar{x} + 2[0s^2 - 2\bar{x}^2]^{\frac{1}{2}}}{3} \leq x_1 \leq \bar{x} + s\sqrt{3}$$

with

$$\frac{4\bar{x} - x_1 - 2[0s^2 - 2(x_1 - \bar{x})^2]^{\frac{1}{2}}}{3} \leq x_2 \leq \frac{4\bar{x} - x_1 + 2[0s^2 - 2(x_1 - \bar{x})^2]^{\frac{1}{2}}}{3}$$

Accordingly, for this kind of frequency function,

$$F(\bar{x}, s) = 32s \int_{\bar{x} - s\sqrt{3}}^{\bar{x} + s\sqrt{3}} \int_{\frac{4\bar{x} - x_1 - 2[0s^2 - 2(x_1 - \bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x} - x_1 + 2[0s^2 - 2(x_1 - \bar{x})^2]^{\frac{1}{2}}}{3}} \frac{1}{T} \delta dx_2 dx_1, \tag{3.1}$$

$$= 32s \int_0^{s\sqrt{3}} \int_{\frac{4\bar{x} - 2[0s^2 - 2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x} - x_1 + 2[0s^2 - 2(x_1 - \bar{x})^2]^{\frac{1}{2}}}{3}} \frac{1}{T} \delta dx_2 dx_1, \tag{3.1}$$

$$+ \int_{\frac{4\bar{x} + 2[0s^2 - 2\bar{x}^2]^{\frac{1}{2}}}{3}}^{\bar{x} + s\sqrt{3}} \int_{\frac{4\bar{x} - x_1 - 2[0s^2 - 2(x_1 - \bar{x})^2]^{\frac{1}{2}}}{3}}^{\frac{4\bar{x} - x_1 + 2[0s^2 - 2(x_1 - \bar{x})^2]^{\frac{1}{2}}}{3}} \frac{1}{T} \delta dx_2 dx_1$$

$$\begin{aligned}
 & + \int \frac{4\bar{x} + 2[6s^2 - 2\bar{x}^2]^{\frac{1}{2}}}{4\bar{x} - 2[6s^2 - 2\bar{x}^2]^{\frac{1}{2}}} \int_0^{\frac{4\bar{x} - x, [6s^2 - 8\bar{x}^2 - 3x,^2 + 8\bar{x}x,]^{\frac{1}{2}}}{2}} \\
 & + \left. \int \frac{4\bar{x} + 2[6s^2 - 2\bar{x}^2]^{\frac{1}{2}}}{4\bar{x} - 2[6s^2 - 2\bar{x}^2]^{\frac{1}{2}}} \int \frac{4\bar{x} - x, + 2[6s^2 - 2(x, - \bar{x})^2]^{\frac{1}{2}}}{4\bar{x} - x, + [6s^2 - 8\bar{x}^2 - 3x,^2 + 8\bar{x}x,]^{\frac{1}{2}}} \right] \frac{1}{T} \phi dx_2 dx_1, \\
 & \qquad \qquad \qquad \frac{\bar{x}\sqrt{3}}{3} \leq s \leq \bar{x},
 \end{aligned}$$

$$\begin{aligned}
 & = 32s \left[\int_0^{2\bar{x} - [2s^2 - 2\bar{x}^2]^{\frac{1}{2}}} \int_0^{\frac{4\bar{x} - x, - [6s^2 - 8\bar{x}^2 - 3x,^2 + 8\bar{x}x,]^{\frac{1}{2}}}{2}} \right. \\
 & + \int_0^{2\bar{x} - [2s^2 - 2\bar{x}^2]^{\frac{1}{2}}} \int \frac{4\bar{x} - x, + 2[6s^2 - 2(x, - \bar{x})^2]^{\frac{1}{2}}}{4\bar{x} - x, + [6s^2 - 8\bar{x}^2 - 3x,^2 + 8\bar{x}x,]^{\frac{1}{2}}} \\
 & + \int \frac{4\bar{x} + 2[6s^2 - 2\bar{x}^2]^{\frac{1}{2}}}{2\bar{x} + [2s^2 - 2\bar{x}^2]^{\frac{1}{2}}} \int_0^{\frac{4\bar{x} - x, - [6s^2 - 8\bar{x}^2 - 3x,^2 + 8\bar{x}x,]^{\frac{1}{2}}}{2}} \\
 & + \int \frac{4\bar{x} + 2[6s^2 - 2\bar{x}^2]^{\frac{1}{2}}}{2\bar{x} + [2s^2 - 2\bar{x}^2]^{\frac{1}{2}}} \int \frac{4\bar{x} - x, + 2[6s^2 - 2(x, - \bar{x})^2]^{\frac{1}{2}}}{4\bar{x} - x, + [6s^2 - 8\bar{x}^2 - 3x,^2 + 8\bar{x}x,]^{\frac{1}{2}}} \\
 & \left. + \int \frac{\bar{x} + s\sqrt{3}}{4\bar{x} + 2[6s^2 - 2\bar{x}^2]^{\frac{1}{2}}} \int \frac{4\bar{x} - x, + 2[6s^2 - 2(x, - \bar{x})^2]^{\frac{1}{2}}}{4\bar{x} - x, - 2[6s^2 - 2(x, - \bar{x})^2]^{\frac{1}{2}}} \right] \frac{1}{T} \phi dx_2 dx_1, \\
 & \qquad \qquad \qquad \bar{x} \leq s \leq \bar{x}\sqrt{3}
 \end{aligned}$$

By similar reasoning, the writer has determined $F(\bar{x}, s)$ for $n = 4$ and $f(x)$ defined on the interval $(0, a)$. The results, however, are quite lengthy and formal and will not be presented here.

Allen T. Craig.