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THE SIMULTANEOUS REPRESENTATION OF INTEGERS BY PRODUCTS OF CERTAIN RATIONAL FUNCTIONS

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Abstract

It is proved that an arbitrary pair of positive integers can be simultaneously represented by products of the values at integer points of certain rational functions. Linear recurrences in $\mathbb{Z}$-modules and elliptic power sums are applied.

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Let

$$P(x) = \prod_{i=0}^{h} (x + a_i)^{b_i}$$

be a rational function with non-negative integers $a_0 < a_1 < \cdots < a_h$, and integral exponents b_i which may be positive or negative but whose highest common factor is 1.

THEOREM. *Let m_1, m_2 and t be positive integers. Then there is a (simultaneous) representation*

$$m_1 = \prod_{j=1}^{r} P(n_j)^{\varepsilon_j}, \qquad m_2 = \prod_{j=1}^{r} P(n_j + t)^{\varepsilon_j},$$

with positive integers n_j and each $\varepsilon_j = \pm 1$.

The method of proof shows that there are in fact infinitely many such representations. A bound for the n_j in terms of m_1, m_2 and t could be found at the expense of complication of detail.

The existence of a one-dimensional representation involving only m_1 was established algebraically in the author's paper Elliott (1983). The present proof applies new ideas. In particular, studies are made of linear recurrences defined over $\mathbf{Z}$-modules; and of the asymptotic behaviour of elliptic power-sums.

Let Q_1 be the abelian group of positive rational fractions with multiplication as the rule of combination, and let Q_2 be the direct sum of two copies of Q_1.

Let Γ be the subgroup of Q_2 generated by the (direct) summands $P(n) \oplus P(n+t)$, $n = 1, 2, \ldots$.

I shall establish the theorem by proving, in three steps, that the quotient group $G = Q_2/\Gamma$ is trivial.

Step one: G is finitely generated

Let H be a $\mathbf{Z}$-module, with the operation of $\mathbf{Z}$ on H written on the left. We shall study the solution-sequences $(\alpha_1, \alpha_2, \ldots)$ in H^ω of the recurrence

$$\sum_{j=0}^{k} c_j \alpha_{n+j} = 0, \tag{1}$$

where the c_j are integers with highest common factor 1.

Without loss of generality $c = c_k \neq 0$ and $k \geqslant 2$ will be assumed.

LEMMA 1. *Let M be an integer so that $M\alpha_n = 0$ for $n = 1, \ldots, k$. Then*

$$Mc^n \alpha_n = 0$$

for all $n \geqslant 1$.

PROOF. By induction on n. In fact

$$Mc\alpha_{k+1} = -\sum_{j=0}^{k-1} c_j M\alpha_{j+1} = 0,$$

and so on, to give $Mc^{n-k}\alpha_n = 0$ for $n \geqslant k+1$, from which the desired result follows.

Under the conditions of this lemma, each element $-\alpha_n$ which appears in a solution sequence of (1) has finite order. From now on we shall assume that every element of the module H has finite order.

Let p be a (positive) rational prime.

For each positive integer n let $|n|_p = p^{-r}$, where p^r is the exact power of the prime p which appears in the canonical factorisation of n in the rational integers.

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With this definition one begins the derivation of the well-known p-adic metric on the rational numbers. We shall do our best to construct a valuation on the $\mathbf{Z}$-module H.

If α is a non-zero element of H which has order m, and if p^s is the exact power of p which divides m, $s = 0$ being permissible, we define $v(\alpha) = p^s$.

We set $v(0) = 1$.

The appropriate properties of this *pre-valuation* are embodied in

LEMMA 2. (i) $v(\alpha) \geqslant 1$ *always*,
(ii) $v(n\alpha) = v(\alpha)$ *if* $(n, p) = 1$,
(iii) $v(n\alpha) \leqslant \max(|n|_p v(\alpha), 1)$,
(iv) $v(\alpha + \beta) \leqslant \max(v(\alpha), v(\beta))$.

PROOF. Assertions (i) and (ii) follow directly from the definition of the pre-valuation.

If α and β have orders u and v respectively, then the least common multiple $[u, v]$ will annihilate $\alpha + \beta$:

$$[u, v](\alpha + \beta) = 0.$$

Thus

$$v(\alpha + \beta) \leqslant |[u, v]|_p^{-1} = \max\left(|u|_p^{-1}, |v|_p^{-1}\right) = \max(v(\alpha), v(\beta)),$$

giving (iv).

Let α be a non-zero element of order m. Let m and n be exactly divisible by p^s and p^r respectively. If $r \geqslant s$ then $|n|_p v(\alpha) = p^{-r+s} \leqslant 1$, giving the inequality of (iii). Otherwise $(p^{-r}m)n\alpha = 0$ and $v(n\alpha) \leqslant p^{s-r} = |n|_p v(\alpha)$, from which the inequality of (iii) is again obtained.

Returning to the recurrence (1) we note that not every coefficient c_j is divisible by our (arbitrary) prime p.

LEMMA 3. *Let* $(\alpha_1, \alpha_2, \ldots)$ *be a solution to the equation* (1). *Then for every* $n \geqslant k + 1$ *with* $v(\alpha_n) > 1$

either: *there is an integer* j, $1 \leqslant j \leqslant k$, *so that*

(i) $$v(\alpha_n) \leqslant v(\alpha_{n-j}),$$

or: *there is an infinite sequence*

$$v(\alpha_n) \leqslant p^{-1} v(\alpha_{n+r_1}) \leqslant p^{-2} v(\alpha_{n+r_1+r_2}) \leqslant \cdots$$

where each r_i *satisfies* $1 \leqslant r_i \leqslant k$.

PROOF. Let $\mu = c_h$ be the coefficient c_j, with the maximum j for which $(p, c_j) = 1$. Then for (each) $n \geqslant k + 1$

$$\mu \alpha_n = \sum_{j=1}^{k-h} d_j \alpha_{n+j} + \sum_{j=1}^{h} e_j \alpha_{n-j}$$

where the integers d_j are divisible by p. As usual, empty sums are deemed to be 0.

In view of Lemma 2

$$v(\alpha_n) = v(\mu \alpha_n) \leqslant \max\left\{ \max_{1 \leqslant j \leqslant k-h} v(d_j \alpha_{n+j}),\ \max_{1 \leqslant j \leqslant h} v(e_j \alpha_{n-j}) \right\}.$$

Suppose first that this upper bound is $v(e_j \alpha_{n-j})$ for some j in the range $1 \leqslant j \leqslant h$. Then since $|e_j|_p \leqslant 1$,

$$v(\alpha_n) \leqslant \max\big(v(\alpha_{n-j}), 1\big).$$

By hypothesis $v(\alpha_n) > 1$, giving $v(\alpha_n) \leqslant v(\alpha_{n-j})$, the first possibility in the lemma.

Otherwise

$$v(\alpha_n) \leqslant v(d_j \alpha_{n+j}) \leqslant \max\big(|d_j|_p v(\alpha_{n+j}), 1\big)$$

for some j in the range $1 \leqslant j \leqslant k - h \leqslant k$. Once again $v(\alpha_n) > 1$, giving now

$$v(\alpha_n) \leqslant p^{-1} v(\alpha_{n+r_1}) \tag{2}$$

for some r_1 in the interval $1 \leqslant r_1 \leqslant k$.

We suppose r_1 to be the minimal integer for which this inequality is valid, and repeat the above argument with $n + r_1$ in place of n. Note that $v(\alpha_{n+r_1}) > 1$.

If in this manner we arrive at an inequality

$$v(\alpha_{n+r_1}) \leqslant v(\alpha_{n+r_1-j})$$

with $1 \leqslant j \leqslant h$, let $m = n + r_1 - j$.

For $m < n$ we get again an inequality of the form (i) in the statement of the lemma.

With $m = n$ we would have

$$v(\alpha_n) \leqslant p^{-1} v(\alpha_{n+r_1}) \leqslant p^{-1} v(\alpha_m) = p^{-1} v(\alpha_n),$$

which is impossible.

For $n < m < n + r_1$ we would get

$$v(\alpha_n) \leqslant p^{-1} v(\alpha_{n+(m-n)}),$$

contradicting the minimality of r_1.

Otherwise we shall obtain an analogue of the inequality (2):

$$v(\alpha_{n+r_1}) \leqslant p^{-1} v(\alpha_{n+r_1+r_2})$$

for some (minimal) r_2 in the interval $1 \leqslant r_2 \leqslant k$.

The proof now proceeds by induction.

REMARKS. This lemma shows that in some sense the order of α_n either remains bounded, or grows exponentially. In particular the results of Lemma 1 is not unreasonable.

We come now to our applications to the theorem. We shall apply the following result from the author's paper Elliott (1983).

Let Δ be a subgroup of Q_2.

LEMMA 4. *In order that the quotient group Q_2/Δ be trivial, it is necessary and sufficient that every homomorphism of it into the additive $\mathbf{Z}$-module $Q/\mathbf{Z}$ be trivial.*

REMARK. $Q/\mathbf{Z}$ is the well-known additive group of the rationals (mod 1), and it is not a field.

Any homomorphism of a group Q_2/Δ into $\mathbf{Q}/\mathbf{Z}$ will have the form

$$y \oplus z \mapsto f_1(y) + f_2(z)$$

where the $f_i(\ )$ are, in the usual notation of analytic number theory, *completely additive arithmetic functions* with values in $Q/\mathbf{Z}$.

In our present circumstances we take for Δ the group generated by Γ (see earlier) and a finite collection

$$l \oplus 1, \qquad 1 \oplus l, \qquad 1 \leqslant l \leqslant T,$$

and show that for a suitably chosen T, Q_2/Δ is trivial. It will suffice to establish

LEMMA 5. *With a suitably chosen (finite) T, any pair (f_1, f_2) of additive functions which take values in $Q/\mathbf{Z}$ and satisfies*

$$(3) \qquad f_1(P(n)) + f_2(P(n+t)) = 0$$

for all $n \geqslant 1$, together with

$$(4) \qquad f_i(l) = 0, \qquad i = 1,2,, 1 \leqslant l \leqslant T,$$

is necessarily trivial.

During the proof of this lemma we shall apply (perhaps surprisingly) the following sieve result.

Lemma 6. *Let d be a positive integer. Then there is a constant g so that the number of integers m in the interval $n < m < n + y$ which have no prime factor q in the range $d < q < \sqrt{y}$ is at most*

$$\frac{gy}{\log y}$$

uniformly for all integers $n \geqslant 1$ and real $y \geqslant 2$.

Proof. See Chapter 2 of the author's book Elliott (1979b) or the account of sieve theory given by Halberstam and Richert (1974).

Proof of Lemma 5. In view of the additive nature of the f_i

$$f_i(P(n)) = \sum_{j=0}^{h} b_j f_i(n + a_j).$$

The hypothesis (3) of Lemma 5 may thus be expressed in the form $\sum_{j=0}^{k} c_j \alpha_{n+j} = 0$ for all $n \geqslant 1$, where $k = a_h$, $\alpha_n = f_1(n) + f_2(n + t)$, and the integers c_j, not all zero, have highest common factor 1.

We aim to prove that $f_i(n) = 0$, $i = 1, 2$, and so $\alpha_n = 0$, for all n. By hypothesis this assertion is valid for $1 \leqslant n \leqslant T - t$.

Let $c = c_k$, which is without loss of generality positive. If $T \geqslant k + t$ then $\alpha_n = 0$ for $1 \leqslant n \leqslant k$, and by Lemma 1, $c^n \alpha_n = 0$ for all positive integers n.

If $c = 1$ then $\alpha_n = 0$ for all n, and this already leads to the complete result. Indeed, replacing n by nt we obtain

$$\begin{aligned} 0 = \alpha_{nt} &= f_1(nt) + f_2(t\{n + 1\}) \\ &= f_1(n) + f_2(n + 1) \end{aligned}$$

since $T \geqslant t$ and $f_1(t) = 0 = f_2(t)$.

Writing β_n for $f_1(n) + f_2(n + 1)$ we see that for $s \geqslant 2$

$$\begin{aligned} f_2(s) &= \beta_{s-1} - f_1(s - 1), \\ f_1(s) &= f_1(s/2) \quad \textit{if } s \textit{ is even}, \\ f_1(s/2) &= \beta_s - f_2((s + 1)/2) \quad \textit{if } s \textit{ is odd}, \end{aligned} \tag{5}$$

since $T \geqslant 2$ and $f_1(2) = 0 = f_2(2)$.

Together with $\beta_s = 0$ for $s \geqslant 1$ these relations clearly demonstrate (inductively) the triviality of the functions f_i.

Suppose now that $c > 1$. Choose a prime divisor p of c and define a pre-valuation $v(\ )$ on $Q/\mathbf{Z}$ in terms of p. We shall prove that if T is fixed at a large enough value, *independent of the definition of the* f_i, then $v(\alpha_n) = 1$ holds for all n.

We argue by contradiction, noting that $v(\alpha_n) = 1$ for $1 \leqslant n \leqslant T - t$. Assume that there is an integer $n \geqslant k + 1$ with $v(\alpha_n) > 1$. We apply Lemma 3 with the least such n. This rules out the possibility (i) given by that lemma, and we must have an infinite chain of inequalities

$$(6) \qquad v(\alpha_n) \leqslant p^{-1}v(\alpha_{n+r_1}) \leqslant p^{-2}v(\alpha_{n+r_1+r_2}) \leqslant \cdots$$

with $1 \leqslant r_i \leqslant k$.

There must be an integer J, bounded only in terms of k and t, so that each of the integers

$$n + \sum_{i=1}^{J} r_i, \qquad \left(n + \sum_{i=1}^{J} r_i\right) + t$$

has a prime factor q in the range $2t \leqslant q \leqslant n/(2t)$. For otherwise the integers

$$n + \sum_{i=1}^{w} r_i + \begin{Bmatrix} 0 \\ t \end{Bmatrix}$$

for $w = 1, 2, \ldots, z$, will between them generate at least $z/4$ numbers m which have no such factors, and which lie in the interval $n < m < n + kz + t$. According to Lemma 6, either $n \leqslant 2t(kz+t)^{1/2}$ or

$$z/4 \leqslant g(kz+t)/\log(kz+t).$$

We choose for z a value large enough that this last inequality fails, and then restrict T to exceed $2t(kz+t)^{1/2} + t$. Since $v(\alpha_n) > 1$ this will not allow the penultimate inequality.

Hence, writing δ for the sum $r_1 + \cdots + r_J$, we have

$$n + \delta = m_1 m_2, \qquad n + \delta + t = m_3 m_4$$

where $2t < m_i \leqslant (n + \delta + t)/(2t)$ for $i = 1, \ldots, 4$. Therefore

$$\alpha_{n+\delta} = \sum_{i=1}^{2} f_1(m_i) + \sum_{j=3}^{4} f_2(m_j)$$

where for all large enough values of n

$$\max_{1 \leqslant i \leqslant 4} m_i \leqslant (n + Jk + t)/(2t) < (n-1)/t.$$

According to our temporary hypothesis, $v(\alpha_u) = 1$ for $1 \leqslant u \leqslant n - 1$, so that $v(\beta_s) = 1$ for $1 \leqslant s \leqslant (n-1)/t$. The relations (5) then allow us to assert that

$$v(f_i(s)) = 1$$

for $i = 1, 2$ and all s not exceeding $(n-1)/t$. In particular we may conclude that $v(\alpha_{n+\delta}) = 1$. Our chain of inequalities (6) now gives the impossible $v(\alpha_n) \leqslant 1$.

We may carry out this argument using each of the prime divisors of c, and since the primes which divide the order of α_n also divide c, obtain that $\alpha_n = 0$ for every positive n.

Lemma 5 is now immediate, and with its proof we have completed step one.

Step two: G is finite

In this section I apply quite different ideas.

Elliptic power-sums.

LEMMA 7. *Let z_j, $j = 1,\dots,k$, be complex numbers which satisfy $|z_j| = 1$. Let ρ_j, $j = 1,\dots,k$ be further complex numbers, and assume that the function*

$$H(n) = \sum_{j=1}^{k} \rho_j z_j^{n^2}$$

is not zero for all positive integers n. Then

$$\limsup_{n\to\infty} |H(n)| > 0.$$

PROOF. We argue by induction on k. The case $k = 1$ is trivial.

Let $k \geqslant 2$. Without loss of generality $\rho_1 \neq 0$.

Suppose first that no z_j/z_1 is a root of unity. Then

$$H(n) = z_1^{n^2}\left(\rho_1 + \sum_{j=2}^{k} \rho_j\left(z_j z_1^{-1}\right)^{n^2}\right)$$

where for $j \geqslant 2$, $z_j z_1^{-1} = \exp(2\pi i\theta_j)$ for some irrational real number θ_j. Hence

$$\lim_{x\to\infty} x^{-1} \sum_{n\leqslant x} z_1^{-n^2} H(n) = \rho_1 + \sum_{j=2}^{k} \rho_j \lim_{x\to\infty} x^{-1} \sum_{n\leqslant x} e^{2\pi i n^2\theta_j} = \rho_1,$$

each right-hand limit being zero by a result of Hermann Weyl. For an account of the appropriate estimates for exponential sums see Cassels (1957) Chapter IV. Sharper bounds may be obtained by using a transcendence measure for the sum of two logarithms of algebraic integers, and then applying this to the Weyl-sum inequality given in Vaughan (1981) Lemma (2.4). In this case we deduce that

$$\limsup_{n\to\infty} |H(n)| \geqslant |\rho_1| > 0.$$

Otherwise we can write

$$z_j = \lambda_j z_1, \qquad j = 2,\dots,m,$$

where z_j/z_1 is not a root of unity for $m < j \leqslant k$. We write $H(n)$ in the form

$$z_1^{n^2}\left(\sum_{j=1}^{m} \rho_j \lambda_j^{n^2} + \sum_{j=m+1}^{k} \rho_j\left(z_j z_1^{-1}\right)^{n^2}\right) = z_1^{n^2}\left(H_1(n) + H_2(n)\right)$$

say. If $H_1(n) = 0$ for all n, then $H_2(n)$ is non-zero for at least one integer n, and we may apply our induction hypothesis to obtain the desired result. If $H_1(n) \neq 0$

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for some n, then the function

$$J(n) = \sum_{j=1}^{m} \rho_j \lambda_j^{n^2}$$

is periodic, or period q say, and there is an integer t so that $J(t) \neq 0$. Thus for all positive integers r

$$z_1^{-(t+rq)^2} H(t + rq) = J(t) + H_2(t + rq).$$

Once again $z_j/z_1 = \exp(2\pi i \theta_j)$ where θ_j is irrational for $j > m$, and by another appeal to a Weyl-sum inequality

$$\frac{1}{y} \sum_{r \leq y} e^{2\pi i (r^2 q^2 + 2rtq)\theta_j} \to 0 \quad \text{as } y \to \infty.$$

Hence $\lim_{r\to\infty} 1/y \sum_{r\leq y} H_2(t + rq) = 0$, and arguing as earlier

$$\limsup_{n\to\infty} |H(n)| \geq \limsup_{r\to\infty} |H(t + rq)| \geq |J(t)| > 0.$$

This completes the proof of Lemma 7.

REMARK. The above proof shows that if $H(n)$ vanishes for all $n \geq 1$ then either every $\rho_j = 0$, or some ratio z_i/z_j with $i \neq j$ is a root of unity.

The analogue of Lemma 4 which is relevant to this part of the proof of the theorem is the following

LEMMA 8. *In order that every element of Q_2/Γ should have a finite order, it is necessary and sufficient that there should be no non-trivial homomorphisms of Q_2/Γ into the additive group of real numbers.*

PROOF. A proof of this result may be found in the author's paper Elliott (1983), where an account is given of earlier related results.

In order to apply Lemma 8 we show that any pair (f_1, f_2) of real-valued additive arithmetic functions which satisfies

$$f_1(P(n)) + f_2(P(n + t)) = 0$$

for all $n \geq 1$ must be trivial. As in *step one*, with $\alpha_n = f_1(n) + f_2(n + t)$ we have $\sum_{j=0}^{k} c_j \alpha_{n+j} = 0$. Since the real numbers form a field this linear recurrence has a solution of the form

$$f_1(n) + f_2(n + t) = \alpha_n = \sum_{j=1}^{w} F_j(n) \delta_j^n, \qquad n = 1, 2, \ldots,$$

where the δ_j lie in some algebraic extension of the rational field Q, and the $F_j(x)$ may be taken to be polynomials defined over this same extension field.

Replacing n by tn and appealing to the additive nature of the f_i,

$$f_1(n) + f_2(n+1) = \sum_{j=2}^{w} F_j(tn)\delta_j^{tn} - \sum_{i=1}^{2} f_i(t).$$

This holds for all positive integers n, including even integers:

$$f_1(2n) + f_2(2n+1) = \sum_{j=1}^{w} F_j(2tn)\delta_j^{2tn} - \sum_{i=1}^{2} f_i(t).$$

By subtraction, writing f for f_2, we see that $f(2n+1) - f(n+1)$ and so $f(2n-1) - f(n)$ have representations of the same type:

$$f(2n-1) - f(n) = \sum_{j=1}^{s} R_j(n)\lambda_j^n.$$

Suppose now that the ratio $\lambda_1\lambda_2^{-1}$ is a root of unity, say $\lambda_1^d = \lambda_2^d$. If in this representation we replace n by dn then

$$f(2dn-1) - f(n) = \sum_{j=1}^{s} R_j(dn)\lambda_j^{dn} + f(d),$$

where the terms $R_1(dn)\lambda_1^{dn} + R_2(dn)\lambda_2^{dn}$ may be coalesced into a single term of the same form.

Continuing in this manner we reach a representation

$$f(D^2n-1) - f(n) = \sum_{j=1}^{r} S_j(n)\omega_j^n + \text{constant}, \tag{7}$$

with D a positive integer, and where no ratio $\omega_i\omega_j^{-1}$ with $i \neq j$ is a root of unity.

We shall prove that a representation of this type is only available to trivial additive functions f. To this end we need

LEMMA 9. *Let A ($\geqslant 2$) be a positive integer. If a completely real-valued function f has $f(An-1) - f(n)$ bounded for all $n \geqslant 1$, then it must be of the form $B\log n$ for all positive n.*

PROOF. A (somewhat) complicated proof of a similar result may be found in the author's paper Elliott (1979a). In order to obtain the present result by the same method only minor adjustments are necessary, together with a proof that if $f(An-1) - f(n)$ is bounded, then so for $n \geqslant 2$ is $f(n)/\log n$. This last we shall now supply.

Suppose that $|f(Am-1)-f(m)| \leq C$ for all $m \geq 1$. If an integer n is divisible by a prime divisor q of A, then there is an integer $n_1 = n/q < (1-(2A)^{-1})n$ so that

$$|f(n)| \leq |f(n_1)| + |f(q)|.$$

Otherwise n will have the form $Am + l$, where $1 \leq l \leq A$, $(l, A) = 1$. In this case let z be the unique integer in the interval $1 \leq z \leq A$ which satisfies $zl \equiv -1 \pmod{A}$, say with $zl = Au - 1$. Note that $A \geq 2$ and therefore $z \leq A-1$ must hold. Then

$$\begin{aligned} f(n) &= f(zn) - f(z) \\ &= f(A\{am+u\} - 1) - f(z) \end{aligned}$$

so that writing $n_1 = zm + u$ and appealing to our hypothesis

$$|f(n)| \leq |f(n_1)| + C + |f(z)|.$$

Here the integer n_1 does not exceed $(1 - A^{-1})n + 1$.

Defining $U(x) = \max_{n \leq x} |f(n)|$ we have

(8) $$U(x) \leq U((1 - 1/2A)x) + \text{constant}$$

for all sufficiently large (in terms of A only) values of x.

An easy induction proof now completes the argument.

Without loss of generality we may assume that

$$\omega = |\omega_1| = |\omega_2| = \cdots = |\omega_h| > |\omega_{h+1}| \geq \cdots \geq |\omega_r|.$$

Moreover, we may also assume that

$$d = \text{degree}\, S_1(x) \geq \text{degree}\, S_2(x) \geq \cdots \geq \text{degree}\, S_h(x).$$

Of course the polynomials $S_j(x)$ with $j > h$ (if there are any) may have degrees greater than d.

LEMMA 10. *If $d \geq 1$ or $|\omega| > 1$ then there is a constant E so that*

$$|f(n)| \leq En^d \max(\omega, 1)^n$$

for all positive integers n.

PROOF. It follows from the representation (7) that

$$|f(D^2 n - 1) - f(n)| \leq Ln^d \max(\omega, 1)^n$$

for some constant L and all $n \geq 1$.

The argument given in the above account of Lemma 9 may be applied here also. In the same notation as before (save that $A = D^2$) we obtain

$$U(x) \leq U((1 - 1/2A)x) + Mx^d \max(\omega, 1)^x$$

as an analogue of (8).

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An inductive proof of Lemma 10 is now readily arranged.

If in the representation (7) we replace n by n^2, the term $D^2n^2 - 1$ factorises into $(Dn - 1)(Dn + 1)$ and we obtain

$$\sum_{j=1}^{r} S_j(n^2)\omega_j^{n^2} = f(Dn + 1) + f(Dn - 1) - 2f(n) + N$$

for some constant N. Under the conditions of Lemma 10 this right-hand side does not exceed a constant multiple of $n^d \max(\omega, 1)^{nD}$ in size.

Suppose now that $\omega > 1$. Dividing both sides of the above equation by $n^{2d}\omega^{n^2}$ we obtain an asymptotic relation

$$\sum_{j=1}^{h} \rho_j z_j^{n^2} \to 0, \qquad n \to \infty,$$

since no matter what the values of d or D,

$$n^{-d}\max(\omega, 1)^{nD}\omega^{-n^2} \to 0$$

as n becomes unbounded. Here we have written z_j for $\omega_j\omega^{-1}$, and ρ_j is the coefficient of x^d in the polynomial $S_j(x)$.

In view of Lemma 7, the elliptic power-sum $\sum_{j=1}^{h}\rho_j z_j^{n^2}$ must be zero for all $n \geqslant 1$. But since not all the $\rho_j = 0$, and we have arranged that no ratio z_i/z_j with $i \neq j$ is a root of unity, this cannot be the case.

Thus $\omega \leqslant 1$, and every $|\omega_j| \leqslant 1$.

Suppose now, without loss of generality, that $\omega = 1$ but that $d \geqslant 1$. Then Lemma 10 yields the bound $|f(n)| \leqslant En^d$. The argument given above will once again lead to a contradiction.

We can therefore write

$$f(D^2n - 1) - f(n) = \sum_{j=1}^{h} \rho_j\omega_j^n + Y + O(c^{-n})$$

where Y and $c > 1$ are constants, and every $|\omega_j| = 1$. In particular $f(D^2n - 1) - f(n)$ is bounded for all n.

Applying Lemma 9 with $A = D^2$ we conclude that $f(n)$ has the form $B\log n$ for all positive n.

Since $\log(D^2n - 1) - \log n = 2\log D - (D^2n)^{-1} + O(n^{-2})$ we can define

$$\rho_0 = Y - 2\log D, \qquad \omega_0 = 1,$$

and write

$$\sum_{j=0}^{h} \rho_j\omega_j^n = -\frac{B}{D^2n} + O\left(\frac{1}{n^2}\right).$$

Suppose for the moment that B is non-zero. Replacing n by n^2 gives

$$V(n) = \sum_{j=0}^{h} \rho_j \omega_j^{n^2} = -\frac{B}{(Dn)^2} + O\left(\frac{1}{n^4}\right).$$

Here the expression on the right hand side (and so also $V(n)$) does not vanish for all large n.

Another application of Lemma 7 gives $\limsup_{n\to\infty} |V(n)| > 0$, which is not compatible with the bound $V(n) = O(n^{-2})$. Hence $B = 0$, and we have proved that $f_2(n) = f(n) = 0$ for all positive integers n.

Returning to our first representation for α_n we now have the simpler

$$f_1(n) = \sum_{j=1}^{w} F_j(n)\delta_j^n \tag{9}$$

valid for $n = 1, 2, \ldots$.

There are several ways to deduce that f_1 is trivial. For example, since $f_1(n)$ satisfies the linear recurrence

$$\sum_{j=0}^{h} b_j f_1(n + a_j) = 0 \tag{10}$$

we may appeal to Theorem 1 of the author's paper Elliott (1980) to deduce that $f_1(n)$ has the form $C\log n$ for some constant C. Substituting into (10) gives

$$C\sum_{j=0}^{h} b_j \log(n + a_j) = 0.$$

If $C \neq 0$ then as $n \to \infty$ there is for positive t an asymptotic estimate

$$\sum_{j=0}^{h} b_j\left(\log n + \sum_{r=1}^{t} (-1)^{r+1}\left(\frac{a_j}{n}\right)^r\right) = O(n^{-t+1}).$$

From these we deduce that $\sum_{j=0}^{h} b_j a_j^r = 0$, $r = 0, 1, \ldots$. Hence

$$\sum_{j=0}^{h} b_j \log(x + a_j)$$

vanishes as a function of complex x, first for $|x| < 1$ and then, by analytic continuation, over the half-plane $\mathrm{Re}(x) > 0$.

Thus the rational function

$$P(x) = \prod_{j=0}^{h} (x + a_j)^{b_j}$$

is identically one; a nonsense.

Alternatively, we may treat the representation (9) as we did that of (7), after arranging that the ratios δ_i/δ_j, $i \neq j$ are not roots of unity. In this way $f_1(n)$ is seen to be bounded, and a (uniformly) bounded completely additive (real-valued) function is identically zero.

We have now proved that every element of the group $G = Q_2/\Gamma$ has finite order, and since we established in part one that G is finitely distributed, it must in fact be finite.

This completes step two.

Step three: G is trivial

Once again the argument takes a different turn. The argument hinges upon the following analogue of Lemma 8, a proof of which may be found in the author's paper Elliott (1983).

Let p be a prime number.

LEMMA 11. *In order that every element of the group Q_2/Γ be a pth-power, it is necessary and sufficient that there be no non-trivial homomorphism of it into the additive group of a finite field F_p of p elements.*

Let (f_1, f_2) be a pair of additive functions which take values in F_p. If

$$f_1(P(n)) + f_2(P(n+t)) = 0$$

for all positive integers n, then as in *Step two* $\alpha_n = f_1(n) + f_2(n+t)$ satisfies a linear recurrence $\sum_{j=0}^{k} c_j \alpha_{n+j} = 0$. Here the c_j are interpreted in F_p according to the map

$$c_j \to c_j \ (\text{mod}\ p) \quad \text{in } \mathbf{Z}/p\mathbf{Z},$$

and since $(c_0, \ldots, c_k) = 1$, not all the c_j vanish (mod p).

We obtain formally the same representation

$$f_1(n) + f_2(n+t) = \sum_{j=1}^{w} F_j(n)\delta_j^n$$

as in *step two*, and with f denoting f_2, reach

$$f(2n-1) - f(n) = \sum_{j=1}^{r} R_j(n)\lambda_j^n \tag{11}$$

where the λ_j and the coefficients in the polynomials R_j all belong to a finite algebraic extension of F_p, say F_q.

In particular each λ_j^n is periodic in n, of period $q-1$. The $R_j(n)$ are periodic in n, of period p, so that the whole of the expression on the right-hand side of the above equation is periodic, with a period $p(q-1)$.

The function $f(2n-1)-f(n)$ is therefore periodic, of period $d=p(q-1)$. This may not be its minimal period, but that will not matter in what follows.

Replacing n by $2n^2$ we see that the function

$$f(2n-1)+f(2n+1)-2f(n)=f(4n^2-1)-f(2n^2)+f(2)$$

is also periodic, with the same period; and by subtraction the difference

$$f(2n+1)-f(2n-1). \tag{12}$$

We shall denote their difference by $g(n)$.

Let $T=\sum_{n=1}^{d} g(n)$ be a sum over a period (mod d). Then for any positive integer s

$$\sum_{n=1}^{pds}\{f(2n+1)-f(2n-1)\}=sp\,T=0.$$

But the sum telescopes to give $f(2pds+1)=0$ for all $s\geqslant 1$.

An additive function, with values in F_p, which satisfies

$$f(Dn+1)=0$$

for some positive integer D and all $n\geqslant 1$, need not be identically zero on the integers prime to D. It will, however, be given by

$$\exp(2\pi i f(n)/p)=\chi(n)$$

for some (fixed) Dirichlet character χ (mod D).

We shall not need this last result. In fact (12) shows that $g(2pds)$ has a period 1 in s; it is constant for all $s\geqslant 1$. With what we have already established, the replacement of n in (12) by $2pds$ shows that

$$f(2pds-1)=f(2pd-1)=\text{constant}$$

for all positive s.

Equation (11) with $2pds$ in place of n allows us to assert that if $\lambda_0=1$ and $R_0(x)$ is a suitable constant (polynomial), then there is a representation

$$f(s)=-\sum_{j=0}^{r}R_j(2pds)\lambda_j^{2pds}$$

valid for all $s\geqslant 1$. The expression on the right-hand side of this equation has period 1, so that $f(n)$ is a constant, μ say.

Since

$$-\mu=f(1^2)-2f(1)=0,$$

we have proved that the additive function $f_2=f$ vanishes identically.

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In particular

$$f_1(n) = \sum_{j=1}^{w} F_j(n)\delta_j^n$$

for all $n \geqslant 1$. It is easy to obtain from this representation that $f_1(n)$ is periodic and then a constant, and so zero.

In view of Lemma 11 we see that whatever the choice of prime p, each element of the group G is a pth-power. This forces G to be trivial. For example let G have order r, so that each element g of G satisfies $g^r = 1$. If p is a prime divisor of r then there is a further element γ of G so that $g = \gamma^p$. Hence

$$g^{rp^{-1}} = \gamma^r = 1.$$

Proceeding inductively we obtain $g = 1$, and the triviality of G.

The theorem is proved.

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