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The Sine-Gordon model revisited I

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Abstract

We study integrable lattice regularizations of the Sine-Gordon model with the help of the Separation of Variables method of Sklyanin and the Baxter Q-operators. This allows us to characterize the spectrum (eigenvalues and eigenstates) completely in terms of polynomial solutions of the Baxter equation with certain properties. This result is analogous to the completeness of the Bethe ansatz.

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1. Introduction

1.1

The study of the Sine-Gordon model has a long history. It has in particular served as an important toy model for interacting quantum field theories. The integrability of this model gives access to detailed non-perturbative information about various characteristic quantities, which allows one to check physical ideas about quantum field theory against exact quantitative results.

It is particularly fascinating to compare the Sine-Gordon model with the Sinh-Gordon model. The Hamiltonian density h_{SG} of the Sine-Gordon model and the corresponding object h_{ShG} of the Sinh-Gordon model,

$$H = \int_0^R \frac{dx}{4\pi} h(x), \quad \begin{aligned} h_{SG} &= \Pi^2 + (\partial_x \phi)^2 + 8\pi\mu \cos(2\beta\phi), \\ h_{ShG} &= \Pi^2 + (\partial_x \phi)^2 + 8\pi\mu \cosh(2b\phi), \end{aligned} \quad (1.1)$$

are related by analytic continuation w.r.t. the parameter β and setting $\beta = ib$. The integrability of both models is governed by the same algebraic structure $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$ with $q = e^{-\pi i\beta^2}$. This leads one to expect that both models should be closely related, or at least have the same “degree of complexity”.

The physics of these two models turns out to be very different, though. Many of the key objects characteristic for the respective quantum field theories are not related by analytic continuation in the usual sense. While the Sine-Gordon model has much richer spectrum of excitations and scattering theory in the infrared (infinite R) limit, one may observe rather intricate structures in the UV-limit of the Sinh-Gordon model [Za06], which turn out to be related to the Liouville theory [ZZ95, T08a, BT09]. These differences can be traced back to the fact that the periodicity of the interaction term $8\pi\mu \cos(2\beta\phi)$ of the Sine-Gordon model allows one to treat the variable ϕ as angular variable parameterizing a compact space, while ϕ is truly non-compact in the Sinh-Gordon model.

The qualitative differences between the Sine-Gordon and the Sinh-Gordon model can be seen as a simple model for the differences between Nonlinear Sigma-Models on compact and non-compact spaces respectively. This forms part of our motivation to revisit the Sine-Gordon model in a way that makes comparison with the Sinh-Gordon model easier.

1.2

A lot of important exact results are known about the Sine-Gordon model. Well-understood are in particular the scattering theory in the infinite volume. The spectrum of elementary particle excitations and the S-matrix of the theory are known exactly [KT77, Za77, FST80, Ko80]. Relatedly, there is a wealth of information on the form-factors of local fields, see e.g. [Sm92, BFKZ, LZ01] for the state of the art and further references. In the case of finite spacial volume, the nonlinear integral equations¹ derived by Destri and De Vega [DDV92, DDV94, DDV97, FRT99] give a powerful tool for the study of the finite-size corrections to the spectrum of the Sine-Gordon model.

¹This type of equations were before introduced in a different framework in [KP91, KBP91]

However, there are several questions, some of them fairly basic, where our understanding does not seem to be satisfactory. We do not have exact results on correlation functions on the one hand, or on expectation values of local fields in the finite volume on the other hand at present.

Even the present level of understanding of the spectrum of the model does not seem to be fully satisfactory. The truth of the commonly accepted hypothesis that the equations derived by Destri and De Vega describe all of the states of the Sine-Gordon model has not been demonstrated yet. The approach of Destri and De Vega is based on the Bethe ansatz in the fermionized version of the Sine-Gordon model, the massive Thirring model [DDV87]. This approach a priori only allows one to describe the states with even topological charge, and it inherits from its roots in the algebraic Bethe ansatz some difficulties like the issue of its completeness.

This paper is intended as a first step in a program to address the above-mentioned deficiencies in our understanding of the Sine-Gordon model. Our aim in this paper is to create a new fundament for future studies the Sine-Gordon model which we hope to be solid enough to build the theory higher than before.

1.3

To this aim we will use a lattice regularization of the Sine-Gordon model that is different from the one used by Destri and De Vega. It goes back to [FST80, IK82], and it is related to formulations of the lattice Sine-Gordon model which have more recently been studied in [FV94, BBR96, Ba08]. In this class of lattice models one may introduce a certain number of “inhomogeneity” parameters. It has been observed in the past that the algebraic Bethe ansatz approach can be used for certain special values of the parameters of the model. However, we will instead focus on the values of the parameters, where the algebraic Bethe ansatz is *not* applicable due to non-existence of the reference state. Our approach will instead be based on the combination of Separation of Variables method (SOV-method) of Sklyanin [Sk85, Sk92, Sk95] with the use of the Q-operators introduced by Baxter [Ba73].

One of the main advantages of this approach is due to the fact that one directly works with the discretized Sine-Gordon degrees of freedom, which is not the case in the lattice formulation used by Destri and De Vega. Working more directly with the Sine-Gordon degrees of freedom should be useful for the problem to calculate expectation values of local fields. This in particular requires the determination of the SOV representation of local fields analogously to what has been done in the framework of the algebraic Bethe ansatz in [KMT99, MT00]. The SOV-method in principle offers a rather direct way to the construction of the expectation values, as illustrated in the case of the Sinh-Gordon model by the work [Lu01].

We will furthermore see that our approach allows us to get a very satisfactory understanding of the spectrum of the lattice model. The SOV method replaces the more traditional Bethe ansatz as a tool to construct the eigenstates of the transfer matrix corresponding to the solutions of an algebraic system of equations which is very similar to those appearing in the algebraic Bethe ansatz method.

In the present case we will be able to show that a complete set of eigenstates can be obtained in this way, an issue which is analogous to the problem of completeness of the Bethe ansatz. We furthermore prove the simplicity of the spectrum of the transfer matrix in the generic case. We will show in a future publication that the results of our approach are fully consistent with the results of Destri and De Vega.

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2. Definition of the model

2.1 Classical Sine-Gordon model

The classical counterpart of the Sine-Gordon model is a dynamical system whose degrees of freedom are described by the field $\phi(x, t)$ defined for $(x, t) \in [0, R] \times \mathbb{R}$ with periodic boundary conditions $\phi(x + R, t) = \phi(x, t)$. The dynamics of this model may be described in the Hamiltonian form in terms of variables $\phi(x, t)$, $\Pi(x, t)$, the Poisson brackets being

$$\{ \Pi(x, t), \phi(x', t) \} = 2\pi \delta(x - x').$$

The time-evolution of an arbitrary observable $O(t)$ is then given as

$$\partial_t O(t) = \{ H, O(t) \},$$

with Hamiltonian H being defined in (1.1).

The equation of motion for the Sine-Gordon model can be represented as a zero curvature condition,

$$[\partial_t - V(x, t; \lambda), \partial_x - U(x, t; \lambda)] = 0, \quad (2.2)$$

with matrices $U(x, t; \lambda)$ and $V(x, t; \lambda)$ being given by

$$\begin{aligned} U(x, t; \lambda) &= \begin{pmatrix} i\frac{\beta}{2}\Pi & -im(\lambda e^{-i\beta\phi} - \lambda^{-1}e^{i\beta\phi}) \\ -im(\lambda e^{i\beta\phi} - \lambda^{-1}e^{-i\beta\phi}) & -i\frac{\beta}{2}\Pi \end{pmatrix} \\ V(x, t; \lambda) &= \begin{pmatrix} i\frac{\beta}{2}\phi' & +im(\lambda e^{-i\beta\phi} + \lambda^{-1}e^{i\beta\phi}) \\ +im(\lambda e^{i\beta\phi} + \lambda^{-1}e^{-i\beta\phi}) & -i\frac{\beta}{2}\phi' \end{pmatrix} \end{aligned} \quad (2.3)$$

and m related to μ by $m^2 = \pi\beta^2\mu$.

2.2 Discretization and canonical quantization

In order to regularize the ultraviolet divergences that arise in the quantization of these models we will pass to integrable lattice discretizations. First discretize the field variables according to the standard

recipe

$$\phi_n \equiv \phi(n\Delta), \quad \Pi_n \equiv \Delta\Pi(n\Delta),$$

where $\Delta = R/N$ is the lattice spacing. In the canonical quantization one would replace ϕ_n, Π_n by corresponding quantum operators with commutation relations

$$[\phi_n, \Pi_m] = 2\pi i \delta_{n,m}. \quad (2.4)$$

Planck's constant can be identified with β^2 by means of a rescaling of the fields.

The scheme of quantization of the Sine-Gordon model considered in this paper will deviate from the canonical quantization by using $e^{i\frac{\beta}{2}\Pi_n}$ and $e^{-i\beta\phi_n}$ as basic variables. For technical reasons we will consider representations where both u_n and v_n have discrete spectrum. Let us therefore take a moment to explain why one may nevertheless expect that the resulting quantum theory will describe the quantum Sine-Gordon model in the continuum limit.

First note (following the discussion in [Za94]) that the periodicity of the potential $8\pi\mu \cos(2\beta\phi)$ in (1.1) implies that shifting the zero mode $\phi_0 \equiv \frac{1}{R} \int_0^R dx \phi(x)$ by the amount π/β is a symmetry. In canonical quantization one could build the unitary operator $W = e^{\frac{i}{2\beta} R p_0}$ which generates this symmetry out of the zero mode $p_0 \equiv \frac{1}{R} \int_0^R dx \Pi(x)$ of the conjugate momentum Π . W should commute with the Hamiltonian H . One may therefore diagonalize W and H simultaneously, leading to a representation for the space of states in the form

$$\mathcal{H} \simeq \int_{S_1} d\alpha \mathcal{H}_\alpha \quad \text{where} \quad W \cdot \mathcal{H}_\alpha = e^{i\alpha} \mathcal{H}_\alpha. \quad (2.5)$$

An alternative way to take this symmetry into account in the construction of the quantum theory is to construct the quantum theory separately for each α -sector. This implies that the field ϕ should be treated as periodic with periodicity π/β , and that the conjugate variables Π_n have eigenvalues quantized in units of β , with spectrum contained in $\{2\alpha\beta/N + 4\pi\beta k; k \in \mathbb{Z}\}$. The spectrum of Π_n is such that the operator $W = e^{\frac{i}{2\beta} R p_0}$, with $R p_0$ approximated by $\sum_{n=1}^N \Pi_n$, is realized as the operator of multiplication by $e^{i\alpha}$.

Let us furthermore note that it is possible, and technically useful to assume that the lattice field observable ϕ_n has discrete spectrum, which we will take to be quantized in units of β . In order to see this, note that the field $\phi(x)$ is not a well-defined observable due to short-distance singularities, whereas smeared fields like $\int_I dx \phi(x)$, $I \subset [0, R]$ may be well-defined. The observable $\int_I dx \phi(x)$ would in the lattice discretization be approximated by

$$\phi[I] \sim \sum_{n\Delta \in I} \Delta \phi_n. \quad (2.6)$$

So even if ϕ_n is discretized in units of β , say, we find that the observable $\phi[I]$ is quantized in units of $\Delta\beta$, which fills out a continuum for $\Delta \rightarrow 0$.

2.3 Non-canonical quantization

As motivated above, we will use a quantization scheme based on the quantum counterparts of the variables u_n, v_n $n = 1, \dots, N$ related to Π_n, ϕ_n as

$$u_n = e^{i\frac{\beta}{2}\Pi_n}, \quad v_n = e^{-i\beta\phi_n}. \quad (2.7)$$

The quantization of the variables u_n, v_n produces operators u_n, v_m which satisfy the relations

$$u_n v_m = q^{\delta_{nm}} v_m u_n, \quad \text{where } q = e^{-\pi i \beta^2}. \quad (2.8)$$

We are looking for representations for the commutation relations (2.8) which have discrete spectrum both for u_n and v_n . Such representations exist provided that the parameter q is a root of unity,

$$\beta^2 = \frac{p'}{p}, \quad p, p' \in \mathbb{Z}^{>0}. \quad (2.9)$$

We will restrict our attention to the case p odd and p' even so that $q^p = 1$. It will often be convenient to parameterize p as

$$p = 2l + 1, \quad l \in \mathbb{Z}^{>0}. \quad (2.10)$$

Let us consider the subset $\mathbb{S}_p = \{q^{2n}; n = 0, \dots, 2l\}$ of the unit circle. Note that $\mathbb{S}_p = \{q^n; n = 0, \dots, 2l\}$ since $q^{2l+2} = q$. This allows us to represent the operators u_n, v_n on the space of complex-valued functions $\psi : \mathbb{S}_p^N \rightarrow \mathbb{C}$ as

$$\begin{aligned} u_n \cdot \psi(z_1, \dots, z_N) &= u_n z_n \psi(z_1, \dots, z_n, \dots, z_N), \\ v_n \cdot \psi(z_1, \dots, z_N) &= v_n \psi(z_1, \dots, q^{-1} z_n, \dots, z_N). \end{aligned} \quad (2.11)$$

The representation is such that the operator u_n is represented as a multiplication operator. The parameters u_n, v_n introduced in (2.11) can be interpreted as ‘‘classical expectation values’’ of the operators u_n and v_n . The discussion in the previous subsection suggests that the v_n will be irrelevant in the continuum limit, while the average value of u_n will be related to the eigenvalue $e^{i\alpha}$ of W via $u_n = \exp(i\beta^2 \alpha/N)$.

2.4 Lattice dynamics

There is a beautiful discrete time evolution that can be defined in terms of the variables introduced above which reproduces the Sine-Gordon equation in the classical continuum limit [FV94]. It is simplest in the case where $u_n = 1, v_n = 1, n = 1, \dots, N$. We will mostly² restrict to this case in the rest of this paper. More general cases were treated in [BBR96, Ba08].

²Except for Section 3.

2.4.1 Parameterization of the initial values

As a convenient set of variables let us introduce the observables f_k defined as

$$f_{2n} \equiv e^{-2i\beta\phi_n}, \quad f_{2n-1} \equiv e^{i\frac{\beta}{2}(\Pi_n + \Pi_{n-1} - 2\phi_n - 2\phi_{n-1})}. \quad (2.12)$$

These observables turn out to represent the initial data for time evolution in a particularly convenient way. The quantum operators f_n which correspond to the classical observables f_n satisfy the algebraic relations

$$f_{2n\pm 1} f_{2n} = q^2 f_{2n} f_{2n\pm 1}, \quad q = e^{-\pi i\beta^2}, \quad f_n f_{n+m} = f_{n+m} f_n \text{ for } m \geq 2. \quad (2.13)$$

There exist simple representations of the algebra (2.13) which may be constructed out of the operators u_n, v_n , given by

$$f_{2n} = v_n^2, \quad f_{2n-1} = u_n u_{n-1}. \quad (2.14)$$

The change of variables defined in (2.14) is invertible if N is odd.

2.4.2 Discrete evolution law

Let us now describe the discrete time evolution proposed by Faddeev and Volkov [FV94]. Space-time is replaced by the cylindric lattice

$$\mathcal{L} \equiv \{(\nu, \tau), \nu \in \mathbb{Z}/N\mathbb{Z}, \tau \in \mathbb{Z}, \nu + \tau = \text{even}\}.$$

The condition that $\nu + \tau$ is even means that the lattice is rhombic: The lattice points closest to (ν, τ) are $(\nu \pm 1, \tau + 1)$ and $(\nu \pm 1, \tau - 1)$. We identify the variables f_n with the initial values of a discrete "field" $f_{\nu, \tau}$ as

$$f_{2r,0} \equiv f_{2r}, \quad f_{2r-1,1} \equiv f_{2r-1}.$$

One may then extend the definition recursively to all $(\nu, \tau) \in \mathcal{L}$ by means of the evolution law

$$f_{\nu, \tau+1} \equiv g_\kappa(qf_{\nu-1, \tau}) \cdot f_{\nu, \tau-1}^{-1} \cdot g_\kappa(qf_{\nu+1, \tau}), \quad (2.15)$$

with function g defined as

$$g_\kappa(z) = \frac{\kappa^2 + z}{1 + \kappa^2 z} \quad (2.16)$$

where κ plays the role of a scale-parameter of the theory. We refer to [FV94] for a nice discussion of the relation between the lattice evolution equation (2.15) and the classical Hirota equation, explaining in particular how to recover the Sine-Gordon equation in the classical continuum limit.

2.4.3 Construction of the evolution operator

In order to construct the unitary operators U that generate the time evolution (2.15) let us introduce the function

$$W_\lambda(q^{2n}) = \prod_{r=1}^n \frac{1 + \lambda q^{2r-1}}{\lambda + q^{2r-1}}, \quad (2.17)$$

which is cyclic, i.e. defined on \mathbb{Z}_p . The function $W_\lambda(z)$ is a solution to the functional equation

$$(z + \lambda)W_\lambda(qz) = (1 + \lambda z)W_\lambda(q^{-1}z), \quad (2.18)$$

which satisfies the unitarity relation

$$(W_\lambda(z))^* = (W_{\lambda^*}(z))^{-1}. \quad (2.19)$$

Note in particular that $W_\lambda(z)$ is "even", i.e. $W_\lambda(z) = W_\lambda(1/z)$. Further properties of this function are collected in Appendix A.

Let us then consider the operator U , defined as

$$U = \prod_{n=1}^N W_{\kappa^{-2}}(f_{2n}) \cdot U_0 \cdot \prod_{r=1}^N W_{\kappa^{-2}}(f_{2n-1}), \quad (2.20)$$

where U_0 is the parity operator that acts as $U_0 \cdot f_k = f_k^{-1} \cdot U_0$. It easily follows from (2.18) that U is indeed the generator of the time-evolution (2.15),

$$f_{\nu, \tau+1} = U^{-1} \cdot f_{\nu, \tau-1} \cdot U. \quad (2.21)$$

One of our tasks is to exhibit the integrability of this discrete time evolution.

3. Separation of variables

In order to exhibit the integrability one needs to construct a mutually commutative family \mathcal{Q} of self-adjoint operators T such that

$$\begin{aligned} \text{(A)} \quad & [T, T'] = 0, \quad \forall T, T' \in \mathcal{Q}, \\ \text{(B)} \quad & [T, U] = 0, \quad \forall T \in \mathcal{Q}, \\ \text{(C)} \quad & \text{if } [T, O] = 0, \quad \forall T \in \mathcal{Q}, \text{ then } O = O(\mathcal{Q}), \end{aligned} \quad (3.1)$$

where U is the unitary time-evolution operator. We will use the quantum inverse scattering method [FST80] to first construct the family \mathcal{Q} . The Separation of Variables (SOV) method of Sklyanin as developed for lattice Sine-Gordon model in this section will then allow us to take an important step towards the simultaneous diagonalization of the family \mathcal{Q} .

3.1 T-operators

As usual in the quantum inverse scattering method, we will represent the family \mathcal{Q} by means of a Laurent-polynomial $T(\lambda)$ which depends on the spectral parameter λ . The definition of operators $T(\lambda)$ for the models in question is standard. It is of the general form

$$T(\lambda) = \text{tr}_{\mathbb{C}^2} M(\lambda), \quad M(\lambda) \equiv L_N(\lambda/\xi_N) \dots L_1(\lambda/\xi_1), \quad (3.2)$$

where we have introduced inhomogeneity parameters ξ_1, \dots, ξ_N as a useful technical device. The L-matrix may be chosen as

$$L_n^{\text{SG}}(\lambda) = \frac{\kappa_n}{i} \begin{pmatrix} i u_n (q^{-\frac{1}{2}} \kappa_n v_n + q^{+\frac{1}{2}} \kappa_n^{-1} v_n^{-1}) & \lambda_n v_n - \lambda_n^{-1} v_n^{-1} \\ \lambda_n v_n^{-1} - \lambda_n^{-1} v_n & i u_n^{-1} (q^{+\frac{1}{2}} \kappa_n^{-1} v_n + q^{-\frac{1}{2}} \kappa_n v_n^{-1}) \end{pmatrix}. \quad (3.3)$$

An important motivation for the definitions (3.2), (3.3) comes from the fact that the Lax-matrix $L_n^{\text{SG}}(\lambda)$ reproduces the Lax-connection $U(x)$ in the continuum limit (2.2).

The elements of the matrix $M(\lambda)$ will be denoted by

$$M(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \quad (3.4)$$

They satisfy commutation relations that may be summarized in the form

$$R(\lambda/\mu) (M(\lambda) \otimes 1) (1 \otimes M(\mu)) = (1 \otimes M(\mu)) (M(\lambda) \otimes 1) R(\lambda/\mu), \quad (3.5)$$

where the auxiliary R-matrix is given by

$$R(u) = \begin{pmatrix} q\lambda - q^{-1}\lambda^{-1} & & & \\ & \lambda - \lambda^{-1} & q - q^{-1} & \\ & q - q^{-1} & \lambda - \lambda^{-1} & \\ & & & q\lambda - q^{-1}\lambda^{-1} \end{pmatrix}. \quad (3.6)$$

It will be useful for us to regard the definition (3.2) as the construction of operators which generate a representation \mathcal{R}_N of the so-called Yang-Baxter algebra defined by the quadratic relations (3.5). The representation \mathcal{R}_N is characterized by the $4N$ parameters $\kappa = (\kappa_1, \dots, \kappa_N)$, $\xi = (\xi_1, \dots, \xi_N)$, $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$.

The fact that the elements of $M(\lambda)$ satisfy the commutation relations (3.5) forms the basis for the application of the quantum inverse scattering method. The mutual commutativity of the T-operators,

$$[T(\lambda), T(\mu)] = 0, \quad (3.7)$$

follows from (3.5) by standard arguments. The expansion of $T(\lambda)$ into powers of λ produces N algebraically independent operators T_1, \dots, T_N . Our main objective in the following will be the study of the spectral problem for $T(\lambda)$. The importance of this spectral problem follows from the fact that the time-evolution operator U of the lattice Sine-Gordon model will be shown to commute with $T(\lambda)$ in the next section.

3.2 The SOV-representation

The separation of variables method of Sklyanin [Sk85]-[Sk95] is based on the observation that the spectral problem for $T(\lambda)$ simplifies considerably if one works in an auxiliary representation where the commutative family $B(\lambda)$ of operators is diagonal. In the following section we will discuss a family of representations of the Yang-Baxter algebra (3.5) that has this property. We will refer to this class of representations as the SOV-representations. We will subsequently show that our original representation introduced in (3.2), (3.3) is indeed equivalent to a certain SOV-representation.

The operators representing (3.5) in the SOV-representation relevant for the case of a lattice with N sites will be denoted as

$$M^{\text{SOV}}(\lambda) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix}. \quad (3.8)$$

We will now describe the representation of the algebra (3.5) in which $B_N(\lambda)$ acts diagonally.

3.2.1 The spectrum of $B_N(\lambda)$

By definition, we require that $B_N(\lambda)$ is represented by a diagonal matrix. In order to parameterize the eigenvalues, let us fix a tuple $\zeta = (\zeta_1, \dots, \zeta_N)$ of complex numbers such that $\zeta_a^p \neq \zeta_b^p$ for $a \neq b$. The vector space \mathbb{C}^{p^N} underlying the SOV representation will be identified with the space of functions $\Psi(\eta)$ defined for η taken from the discrete set

$$\mathbb{B}_N \equiv \{ (q^{k_1} \zeta_1, \dots, q^{k_N} \zeta_N); (k_1, \dots, k_N) \in \mathbb{Z}_p^N \}. \quad (3.9)$$

The SOV-representation is characterized by the property that $B(\lambda)$ acts on the functions $\Psi(\eta)$, $\eta = (\eta_1, \dots, \eta_N) \in \mathbb{B}_N$ as a multiplication operator,

$$B_N(\lambda) \Psi(\eta) = \eta_N^{e_N} b_\eta(\lambda) \Psi(\eta), \quad b_\eta(\lambda) \equiv \prod_{n=1}^N \frac{\kappa_n}{i} \prod_{a=1}^{[N]} (\lambda/\eta_a - \eta_a/\lambda); \quad (3.10)$$

we have used the notation $e_N = 1$ for even N , $e_N = 0$ otherwise, and $[N] \equiv N - e_N$. We see that $\eta_1, \dots, \eta_{[N]}$ represent the zeros of $b_\eta(\lambda)$. In the case of even N it turns out that we need a supplementary variable η_N in order to be able to parameterize the spectrum of $B(\lambda)$.

3.2.2 Representation of the remaining operators

Given that $B_N(\lambda)$ is represented as in (3.10), it can be shown [Sk85]-[Sk95]³ that the representation of the remaining operators $A_N(\lambda)$, $C_N(\lambda)$ $D_N(\lambda)$ is to a large extent determined by the algebra (3.5). First note (see e.g. [BT09, Appendix C.2] for a proof) that the so-called quantum determinant

$$\det_q(M(\lambda)) \equiv A(\lambda)D(q^{-1}\lambda) - B(\lambda)C(q^{-1}\lambda) \quad (3.11)$$

³See [BT09] for the case of the Sinh-Gordon model which is very similar to the case at hand.

generates central elements of the algebra (3.5). In the representation defined by (3.2), (3.3) we find that $\lambda^{2N} \det_q(M(\lambda))$ is a polynomial in λ^2 of order $2N$. We therefore require that

$$A_N(\lambda)D_N(q^{-1}\lambda) - B_N(\lambda)C_N(q^{-1}\lambda) = \Delta_N(\lambda) \cdot \text{id}, \quad (3.12)$$

with $\lambda^{2N} \Delta_N(\lambda)$ being a polynomial in λ^2 of order $2N$.

The algebra (3.5) furthermore implies that $A_N(\lambda)$ and $D_N(\lambda)$ can be represented in the form

$$A_N(\lambda) = e_N b_\eta(\lambda) \left[\frac{\lambda}{\eta_A} T_N^+ - \frac{\eta_A}{\lambda} T_N^- \right] + \sum_{a=1}^{[N]} \prod_{b \neq a} \frac{\lambda/\eta_b - \eta_b/\lambda}{\eta_a/\eta_b - \eta_b/\eta_a} a_N(\eta_a) T_a^-, \quad (3.13)$$

$$D_N(\lambda) = e_N b_\eta(\lambda) \left[\frac{\lambda}{\eta_D} T_N^- - \frac{\eta_D}{\lambda} T_N^+ \right] + \sum_{a=1}^{[N]} \prod_{b \neq a} \frac{\lambda/\eta_b - \eta_b/\lambda}{\eta_a/\eta_b - \eta_b/\eta_a} d_N(\eta_a) T_a^+, \quad (3.14)$$

where T_a^\pm are the operators defined by

$$T_a^\pm \Psi(\eta_1, \dots, \eta_N) = \Psi(\eta_1, \dots, q^{\pm 1} \eta_a, \dots, \eta_N).$$

The expressions (3.13) and (3.14) contain complex-valued coefficients η_A , η_D , $a_N(\eta_r)$ and $d_N(\eta_r)$. The coefficients $a_N(\eta_r)$ and $d_N(\eta_r)$ are restricted by the condition

$$\Delta_N(\eta_r) = a_N(\eta_r) d_N(q^{-1} \eta_r), \quad \forall r = 1, \dots, N, \quad (3.15)$$

as follows from the consistency of (3.12), (3.10), (3.13) and (3.14). This leaves some freedom in the choice of $a_N(\eta_r)$ and $d_N(\eta_r)$ that will be further discussed later.

The operator $C_N(\lambda)$ is finally defined such that the quantum determinant condition (3.12) is satisfied.

3.2.3 Central elements

For the representations in question, the algebra (3.5) has a large center. For its description let us, following [Ta91], define the average value \mathcal{O} of the elements of the monodromy matrix $M^{\text{SOV}}(\lambda)$ as

$$\mathcal{O}(\Lambda) = \prod_{k=1}^p \mathcal{O}(q^k \lambda), \quad \Lambda = \lambda^p, \quad (3.16)$$

where \mathcal{O} can be A_N , B_N , C_N or D_N .

Proposition 1. *The average values $\mathcal{A}_N(\Lambda)$, $\mathcal{B}_N(\Lambda)$, $\mathcal{C}_N(\Lambda)$, $\mathcal{D}_N(\Lambda)$ of the monodromy matrix $M(\lambda)$ elements are central elements.*

The proposition is proven in [Ta91], see Appendix C.2 for an alternative proof. The average values are of course unchanged by similarity transformations. They therefore represent parameters of the

representation. Let us briefly discuss how these parameters are related to the parameters of the SOV representations introduced above.

First, let us note that $\mathcal{B}_N(\Lambda)$ is easily found from (3.10) to be given by the formula

$$\mathcal{B}_N(\Lambda) = Z_N^{\text{eN}} \prod_{n=1}^N \frac{K_n}{i^p} \prod_{a=1}^{[N]} (\Lambda/Z_a - Z_a/\Lambda), \quad \begin{aligned} Z_a &\equiv \eta_a^p, \\ K_a &\equiv \kappa_a^p. \end{aligned} \quad (3.17)$$

The values $\mathcal{A}_N(Z_r)$ and $\mathcal{D}_N(Z_r)$ are related to the coefficients $a_N(q^k \eta_r)$ and $d_N(q^k \eta_r)$ by

$$\mathcal{A}_N(Z_r) \equiv \prod_{k=1}^p a_N(q^k \eta_r), \quad \mathcal{D}_N(Z_r) \equiv \prod_{k=1}^p d_N(q^k \eta_r). \quad (3.18)$$

Note that the condition (3.15) leaves some remaining arbitrariness in the choice of the coefficients $a_N(\eta)$, $d_N(\eta)$. The gauge transformations

$$\Psi(\eta) \equiv \prod_{r=1}^N f(\eta_r) \Psi'(\eta), \quad (3.19)$$

induce a change of coefficients

$$a'_N(\eta_r) = a_N(\eta_r) \frac{f(q^{-1} \eta_r)}{f(\eta_r)}, \quad d'_N(\eta_r) = d_N(\eta_r) \frac{f(q^{+1} \eta_r)}{f(\eta_r)}, \quad (3.20)$$

but clearly leave $\mathcal{A}_N(Z_r)$ and $\mathcal{D}_N(Z_r)$ unchanged. The data $\mathcal{A}_N(Z_r)$ and $\mathcal{D}_N(Z_r)$ therefore characterize gauge-equivalence classes of representations for $A_N(\lambda)$ and $D_N(\lambda)$ in the form (3.13).

3.3 Existence of SOV-representation for the lattice Sine-Gordon model

We are looking for an invertible transformation W^{SOV} that maps the lattice Sine-Gordon model defined in the previous sections to a SOV-representation,

$$(W^{\text{SOV}})^{-1} \cdot M^{\text{SOV}}(\lambda) \cdot W^{\text{SOV}} = M(\lambda). \quad (3.21)$$

Constructing $M^{\text{SOV}}(\lambda)$ is of course equivalent to the construction of a basis for \mathcal{H} consisting of eigenvectors $\langle \eta |$ of $B(\lambda)$,

$$\langle \eta | B(\lambda) = \eta_N^{\text{eN}} b_\eta(\lambda) \langle \eta |. \quad (3.22)$$

The transformation W^{SOV} is then described in terms of $\langle \eta | z \rangle$ as

$$(W^{\text{SOV}} \psi)(\eta) = \sum_{z \in (\mathbb{S}_p)^N} \langle \eta | z \rangle \psi(z). \quad (3.23)$$

The existence of an eigenbasis for $B(\lambda)$ is not trivial since $B(\lambda)$ is not a normal operator. It turns out that such a similarity transformation exists for generic values of the parameters u , v , ξ and κ .

Theorem 1. — **Existence of SOV-representation for the lattice Sine-Gordon model** —

For generic values of the parameters u, v, ξ and κ there exists an invertible operator $W^{\text{SOV}} : \mathcal{H} \rightarrow \mathcal{H}^{\text{SOV}}$ which satisfies (3.21).

The proof is given in Appendix C. It follows from (3.13), (3.14) that the wave-functions $\Psi_t(\eta) = \langle \eta | t \rangle$ of eigenstates $|t\rangle$ must satisfy the discrete Baxter equations

$$t(\eta_r)\Psi(\eta) = a(\eta_r)\mathsf{T}_a^-\Psi(\eta) + d(\eta_r)\mathsf{T}_a^+\Psi(\eta), \quad (3.24)$$

where $n = 1, \dots, N$. Equation (3.24) represents a system of p^N linear equations for the p^N different components $\Psi(\eta)$ of the vector Ψ . It may be written in the form $D_t \cdot \Psi = 0$, where D_t is a $p^N \times p^N$ -matrix that depends on $t = t(\lambda)$. The condition for existence of solutions $\det D_t = 0$ is a polynomial equation of order p^N on $t(\lambda)$. We therefore expect to find p^N different solutions, just enough to get a basis for \mathcal{H} .

We will return to the analysis of the spectral problem of $\mathsf{T}(\lambda)$ in Section 5. Let us now describe more precisely the set of values of the parameters for which a SOV representation exists.

3.4 Calculation of the average values

Necessary condition for the existence of W^{SOV} is of course the equality

$$\mathcal{M}(\Lambda) = \mathcal{M}^{\text{SOV}}(\Lambda), \quad (3.25)$$

of the matrices formed out of the average values of $M(\lambda)$ and $M^{\text{SOV}}(\lambda)$, respectively. It turns out that $\mathcal{M}(\Lambda)$ can be calculated recursively from the average values of the elements of the Lax matrices $L_n^{\text{SG}}(\lambda)$, which are explicitly given by

$$\mathcal{L}_n(\Lambda) = \frac{1}{i^p} \begin{pmatrix} U_n(K_n^2 V_n + V_n^{-1}) & K_n(\Lambda V_n / X_n - X_n / V_n \Lambda) \\ K_n(\Lambda / X_n V_n - X_n V_n / \Lambda) & U_n^{-1}(K_n^2 V_n^{-1} + V_n) \end{pmatrix}, \quad (3.26)$$

where we have used the notations $K_n = \kappa_n^p$, $X_n = \xi_n^p$, $U_n = u_n^p$ and $V_n = v_n^p$. Indeed, we have:

Proposition 2. *We have*

$$\mathcal{M}_N(\Lambda) = \mathcal{L}_N(\Lambda) \mathcal{L}_{N-1}(\Lambda) \dots \mathcal{L}_1(\Lambda). \quad (3.27)$$

This has been proven in [Ta91], see Appendix C.2 for an alternative proof.

The equality (3.25) defines the mapping between the parameters u, v, κ and ξ of the representation defined in Subsection 3.1 and the parameters of the SOV-representation. Formula (3.27) in particular allows us to calculate $\mathcal{B}(\Lambda)$ in terms of u, v, κ and ξ . Equation (3.17) then defines the numbers $Z_a \equiv \eta_a^p$ uniquely up to permutations of $a = 1, \dots, [N]$.

Existence of a SOV-representation in particular requires that $Z_a \neq Z_b$ for all $a \neq b$, $a, b = 1, \dots, [N]$. It can be shown (see Appendix C.3) that the subspace of the space of parameters u, v, κ and ξ for which this is not the case has codimension at least one. Sufficient for the existence of a SOV-representation is the condition that the representations \mathcal{R}_M exist for all $M = 1, \dots, N - 1$.

4. Integrability

Let us now introduce the Baxter Q-operators $Q(\mu)$. These operators are mutually commuting for arbitrary values of the spectral parameters λ and μ , and satisfy a functional relation of the form

$$T(\lambda)Q(\lambda) = a(\lambda)Q(q^{-1}\lambda) + d(\lambda)Q(q\lambda), \quad (4.28)$$

with $a(\lambda)$ and $d(\lambda)$ being certain model-dependent coefficient functions. The generator of lattice time evolution will be constructed from the specialization of the Q-operators to certain values of the spectral parameter λ , making the integrability of the evolution manifest.

In the rest of the paper we will restrict to the case $u_n = 1, v_n = 1, n = 1, \dots, N$ for the simplicity of exposition, leaving the general case to a forthcoming publication.

4.1 Q-operators

4.1.1 Construction

In order to construct the Q-operators let us introduce the following renormalized version of the function $W_\lambda(z)$,

$$w_\lambda(q^{2n}) = \prod_{r=1}^n \frac{1 + \lambda q^{2r-1}}{\lambda + q^{2r-1}} \prod_{r=1}^l \frac{\lambda + q^{2r-1}}{1 + q^{2r-1}}, \quad (4.29)$$

The function $w_\lambda(z)$ is the unique solution to the functional equation (2.18) which is a polynomial of order l in λ and which satisfies the normalization condition $w_1(q^{2n}) = 1$.

The Q-operators can then be constructed in the form

$$Q(\lambda, \mu) = Y(\lambda) \cdot (Y(\mu^*))^\dagger, \quad (4.30)$$

where $Y(\lambda)$ is defined by its matrix elements with

$$Y_\lambda(\mathbf{z}, \mathbf{z}') \equiv \langle \mathbf{z} | Y(\lambda) | \mathbf{z}' \rangle = \prod_{n=1}^N \bar{w}_{\epsilon\lambda/\kappa_n\xi_n}(z_n/z'_n) w_{\epsilon\lambda\kappa_n/\xi_n}(z_n z'_{n+1}), \quad (4.31)$$

where $\epsilon = -iq^{-\frac{1}{2}}$, and $\bar{w}_\lambda(z)$ is the discrete Fourier transformation of $w(z)$,

$$\bar{w}_\lambda(z) = \frac{1}{p} \sum_{r=-l}^l z^r w_\lambda(q^r), \quad w_\lambda(y) = \sum_{r=-l}^l y^{-r} \bar{w}_\lambda(q^r). \quad (4.32)$$

Note in particular the normalization condition $\bar{w}_1(q^r) = \delta_{r,0}$.

Despite the fact that $Q(\lambda, \mu)$ is symmetric in λ and μ , $Q(\lambda, \mu) = Q(\mu, \lambda)$ as follows from the identity (B.6) proven in Appendix B, will mostly consider μ as a fixed parameter which will later be chosen conveniently. This being understood we will henceforth write $Q(\lambda)$ whenever the dependence of $Q(\lambda, \mu)$ on μ is not of interest.

4.1.2 Properties

Theorem 2. — Properties of T- and Q-operators —

(A) ANALYTICITY

The operator $\lambda^{\tilde{N}}T(\lambda)$ is a polynomial in λ^2 of degree $\tilde{N} := N + e_N - 1$ while the operator $Q(\lambda)$ is a polynomial in λ of maximal degree $2lN$. In the case N odd the operators $Q_{2lN} := \lim_{\lambda \rightarrow \infty} \lambda^{-2lN}Q(\lambda)$ and $Q_0 := Q(0)$ are invertible operators and the normalization of the Q operator can be fixed by $Q_{2lN} = \text{id}$.

(B) BAXTER EQUATION

The operators $T(\lambda)$ and $Q(\lambda)$ are related by the Baxter equation

$$T(\lambda)Q(\lambda) = \mathbf{a}_N(\lambda)Q(q^{-1}\lambda) + \mathbf{d}_N(\lambda)Q(q\lambda), \quad (4.33)$$

with coefficient functions

$$\begin{aligned} \mathbf{a}_N(\lambda) &= (-i)^N \prod_{r=1}^N \kappa_r / \lambda_r (1 + iq^{-\frac{1}{2}} \lambda_r \kappa_r) (1 + iq^{-\frac{1}{2}} \lambda_r / \kappa_r), \\ \mathbf{d}_N(\lambda) &= (+i)^N \prod_{r=1}^N \kappa_r / \lambda_r (1 - iq^{+\frac{1}{2}} \lambda_r \kappa_r) (1 - iq^{+\frac{1}{2}} \lambda_r / \kappa_r). \end{aligned} \quad (4.34)$$

(C) COMMUTATIVITY

$$\begin{aligned} [Q(\lambda), Q(\mu)] &= 0, \\ [T(\lambda), Q(\mu)] &= 0, \end{aligned} \quad \forall \lambda, \mu. \quad (4.35)$$

(S) SELF-ADJOINTNESS

Under the assumption ξ_r and κ_r real or imaginary numbers, the following holds:

$$(T(\lambda))^\dagger = T(\lambda^*), \quad (Q(\lambda))^\dagger = Q(\lambda^*). \quad (4.36)$$

It follows from these properties that $T(\lambda)$ and $Q(\mu)$ can be diagonalized simultaneously for all λ, μ . The eigenvalues $Q(\lambda)$ of $Q(\lambda)$ must satisfy

$$t(\lambda)Q(\lambda) = \mathbf{a}_N(\lambda)Q(q^{-1}\lambda) + \mathbf{d}_N(\lambda)Q(q\lambda). \quad (4.37)$$

The proof of Theorem 1 is given in Appendix B.

4.2 Integrability

In order to recover the light-cone dynamics discussed in subsection 2.4, let us temporarily return to the homogeneous case where $\xi_n = 1$ and $\kappa_n = \kappa$ for $n = 1, \dots, N$. Let us note that the operators

$Y(\lambda)$ simplify when λ is sent to 0 or ∞ . Multiplying by suitable normalization factors one find the *unitary* operators

$$Y_0 \equiv \gamma_0^N Y(0) \quad \text{and} \quad Y_\infty \equiv \lim_{\mu \rightarrow \infty} \gamma_\infty^N \mu^{-2lN} Y(\mu),$$

where $\gamma_0 = \prod_{r=1}^l (1 - q^{4r})$ and $\gamma_\infty = (-1)^l q^l \prod_{r=1}^l (1 - q^{4r-2})$. The operators Y_0 and Y_∞ have the simple matrix elements

$$\begin{aligned} \langle \mathbf{z} | Y_0 | \mathbf{z}' \rangle &= \prod_{n=1}^N q^{-2k_n(k'_n + k'_{n+1})}, \\ \langle \mathbf{z} | Y_\infty | \mathbf{z}' \rangle &= \prod_{n=1}^N q^{+2k_n(k'_n + k'_{n+1})}, \end{aligned} \quad \text{if} \quad \left\{ \begin{array}{l} \mathbf{z} = (q^{2k_1}, \dots, q^{2k_N}), \\ \mathbf{z}' = (q^{2k'_1}, \dots, q^{2k'_N}), \end{array} \right\} \quad (4.38)$$

and

$$Q^+(\lambda) = Y(\lambda) \cdot Y_\infty^\dagger, \quad Q^-(\lambda) = (Y(\lambda) \cdot Y_0^\dagger)^{-1} \quad (4.39)$$

Integrability follows immediately from the following observation:

$$\boxed{U = \alpha_\kappa U^+ \cdot U^-, \quad U^+ = Q^+(1/\kappa\epsilon), \quad U^- = Q^-(\kappa/\epsilon),} \quad (4.40)$$

where $\alpha_\kappa \equiv \prod_{r=1}^l (1 - q^{4r-2})^{2N} / (\kappa^2 - q^{4r-2})^{2N}$. The proof can be found in Appendix B. It is very important to remark that there is of course no problem to construct time evolution operators in the inhomogeneous cases by specializing the spectral parameter of the Q-operator in a suitable way. We are just not able to represent the time evolution as simple as in (2.15). One will still have a lattice approximation to the time evolution in the continuum field theory as long as the inhomogeneity parameters are scaled to unity in the continuum limit.

5. The spectrum

Let us now return to the analysis of the spectrum of the model. For simplicity we will consider here the case of odd N , while we will discuss the case of even N in appendix D. The existence of the SOV representation allows one to reformulate the spectral problem for $T(\lambda)$ as the problem to find all solution of the discrete Baxter equations (3.24). This equation may be written in the form

$$\mathcal{D}_r \Psi(\eta) = 0, \quad \mathcal{D}_r \equiv a(\eta_r) T_a^- + d(\eta_r) T_a^+ - t(\eta_r), \quad (5.41)$$

where $r = 1, \dots, N$. Previous experience with the SOV method suggests to consider the ansatz

$$\Psi_t(\eta) = \prod_{r=1}^N Q_t(\eta_r), \quad (5.42)$$

where $Q_t(\lambda)$ is the eigenvalue of the corresponding Q-operator which satisfies the *functional* Baxter equations

$$t(\lambda) Q_t(\lambda) = \mathbf{a}_N(\lambda) Q_t(q^{-1}\lambda) + \mathbf{d}_N(\lambda) Q_t(q\lambda). \quad (5.43)$$

This approach will turn out to work, but in a way that is more subtle than in previously analyzed cases.

5.1 States from solutions of the Baxter equation

First, in the present case it is not immediately clear if the functional Baxter equation (5.43) and the discrete Baxter equation (5.41) are compatible. The question is if one can always assume that the coefficients $a(\eta_r)$ and $d(\eta_r)$ in (5.41) coincide with the coefficients $\mathfrak{a}_N(\eta_r)$, $\mathfrak{d}_N(\eta_r)$ appearing in the functional equation (5.43) satisfied by the Q-operator. The key point to observe is contained in the following Lemma.

Lemma 1. *Let $A_N(\Lambda)$ and $D_N(\Lambda)$ be the average values of the coefficients $\mathfrak{a}_N(\lambda)$ and $\mathfrak{d}_N(\lambda)$ of the Baxter equation (5.43),*

$$A_N(\Lambda) \equiv \prod_{k=1}^p \mathfrak{a}_N(q^k \eta_r), \quad D_N(\Lambda) \equiv \prod_{k=1}^p \mathfrak{d}_N(q^k \eta_r). \quad (5.44)$$

We then have

$$A_N(\Lambda) = \mathcal{A}_N(\Lambda) - \mathcal{B}_N(\Lambda), \quad D_N(\Lambda) = \mathcal{A}_N(\Lambda) + \mathcal{B}_N(\Lambda). \quad (5.45)$$

Proof. The claim is checked for $N = 1$ by straightforward computation. Let us assume now that the statement holds for $N - 1$ and let us show it for N . The average values $A_N(\Lambda)$ and $D_N(\Lambda)$ satisfy by definition the factorization:

$$A_N(\Lambda) = A_1^{(N)}(\Lambda) A_{N-1}^{(N-1, \dots, 1)}(\Lambda), \quad D_N(\Lambda) = D_1^{(N)}(\Lambda) D_{N-1}^{(N-1, \dots, 1)}(\Lambda), \quad (5.46)$$

where the upper indices are referred to the quantum sites involved while the lower indices to the total number of sites. We can use now the induction hypothesis to get the result:

$$A_N(\Lambda) = (\mathcal{A}_1^{(N)} - \mathcal{B}_1^{(N)}(\Lambda)) (\mathcal{A}_{N-1}^{(N-1, \dots, 1)}(\Lambda) - \mathcal{B}_{N-1}^{(N-1, \dots, 1)}(\Lambda)) = \mathcal{A}_N(\Lambda) - \mathcal{B}_N(\Lambda), \quad (5.47)$$

$$D_N(\Lambda) = (\mathcal{A}_1^{(N)} + \mathcal{B}_1^{(N)}(\Lambda)) (\mathcal{A}_{N-1}^{(N-1, \dots, 1)}(\Lambda) + \mathcal{B}_{N-1}^{(N-1, \dots, 1)}(\Lambda)) = \mathcal{A}_N(\Lambda) + \mathcal{B}_N(\Lambda), \quad (5.48)$$

where in the last formulae we have used (3.27) together with the fact that $\mathcal{A}_N(\Lambda) = \mathcal{D}_N(\Lambda)$ and $\mathcal{B}_N(\Lambda) = \mathcal{C}_N(\Lambda)$ for $u_n = 1, v_n = 1, n = 1, \dots, N$. \square

The Lemma implies in particular

$$A_N(Z_r) = \mathcal{A}_N(Z_r), \quad D_N(Z_r) = \mathcal{D}_N(Z_r), \quad (5.49)$$

for all $r = 1, \dots, N$. We may therefore always find a gauge transformation (3.20) such that the coefficients $a_N(\eta_r)$ and $d_N(\eta_r)$ in (5.41) become equal to

$$a_N(\eta_r) = \mathfrak{a}_N(\eta_r), \quad d_N(\eta_r) = \mathfrak{d}_N(\eta_r), \quad (5.50)$$

respectively. So from now on we will denote also the coefficients in (3.24) with \mathbf{a} and \mathbf{d} omitting the index N unless necessary. The ansatz (5.42) therefore indeed yields an eigenstate of $T(\lambda)$ for each solution $Q_t(\lambda)$ of the functional Baxter equation (4.37). We are going to show that *all* eigenstates can be obtained in this way.

5.2 Non-degeneracy of $T(\lambda)$ -eigenvalues

In order to analyze the equations (5.41), let us note that the matrix representation of the operator D_r defined in (5.41) is block diagonal with blocks labeled by $n = 1, \dots, N$. Let $\Psi_n(\eta) \in \mathbb{C}^p$ be the vector with components

$$\Psi_{rk}(\eta) = \Psi_t(\eta_1, \dots, \eta_{r-1}, \zeta_r q^k, \eta_{r+1}, \dots, \eta_N).$$

Equation (5.41) is then equivalent to the set of linear equations

$$D^{(r)} \cdot \Psi_r(\eta) = 0, \quad r = 1, \dots, N. \quad (5.51)$$

where $D^{(r)}$ is the $p \times p$ -matrix

$$\begin{pmatrix} t(\zeta_r) & -\mathbf{d}(\zeta_r) & 0 & \cdots & 0 & -\mathbf{a}(\zeta_r) \\ -\mathbf{a}(q\zeta_r) & t(q\zeta_r) & -\mathbf{d}(q\zeta_r) & 0 & \cdots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & \cdots & & & \vdots \\ \vdots & & & \cdots & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & 0 & -\mathbf{a}(q^{2l-1}\zeta_r) & t(q^{2l-1}\zeta_r) & -\mathbf{d}(q^{2l-1}\zeta_r) \\ -\mathbf{d}(q^{2l}\zeta_r) & 0 & \cdots & 0 & -\mathbf{a}(q^{2l}\zeta_r) & t(q^{2l}\zeta_r) \end{pmatrix} \quad (5.52)$$

The equation (5.51) can have solutions only if $\det(D^{(r)}) = 0$. The determinant $\det(D^{(r)})$ is a polynomial of degree p in each of the N coefficients of the polynomial $t(\lambda)$.

Proposition 3. *Given that $\det(D^{(r)}) = 0$, the dimension of the space of solutions to the equation (5.51) for any $r = 1, \dots, N$ is one for generic values of the parameters ξ and κ .*

Proof. Let us decompose the $p \times p$ matrix $D^{(r)}$ into the block form

$$D^{(r)} = \begin{pmatrix} v^{(r)} & E^{(r)} \\ d^{(r)} & w^{(r)} \end{pmatrix}, \quad (5.53)$$

where the submatrix $E^{(r)}$ is a $(p-1) \times (p-1)$ matrix, $v^{(r)}$ and $w^{(r)}$ are column and row vectors with $p-1$ components, respectively. We assume that $\det(D^{(r)}) = 0$, so existence of a solution to

$D^{(r)}\Psi = 0$ is ensured. It is easy to see that the equation $D^{(r)}\Psi = 0$ has a unique solution provided that $\det(E^{(r)}) \neq 0$.

It remains to show that $\det(E^{(r)}) \neq 0$ holds for generic values of the parameters ξ and κ . To this aim let us observe that the coefficients $\mathfrak{a}(q^k \zeta_r)$ and $\mathfrak{d}(q^k \zeta_r)$ appearing in (5.51) depend analytically on the parameters κ . If $\det(E^{(r)}) = 0$ is not identically zero, it can therefore only vanish at isolated points. It therefore suffices to prove the statement in a neighborhood of the values for the parameters κ which are such that

$$\mathfrak{a}(\zeta_r) = 0, \quad \mathfrak{d}(q^{-1} \zeta_r) = 0. \quad (5.54)$$

Such values of κ and ξ exist: Setting $\kappa_n = \pm i$ for $n = 1, \dots, N$, one finds that

$$\mathcal{B}_N(\lambda^p) = \prod_{n=1}^N (\lambda^p / \xi_n^p - \xi_n^p / \lambda^p), \quad (5.55)$$

which vanishes for $\lambda = q^{\frac{1}{2}} \xi_n$. We may therefore choose⁴ $\zeta_n = q^{\frac{1}{2}} \xi_n$. We then find (5.54) from (4.34), (5.50).

Given that (5.54) holds, it is easy to see that $\det(E^{(r)}) \neq 0$. Indeed, the submatrix $E_{kl}^{(r)}$, is lower triangular if (5.54) is valid, and it has $-\mathfrak{d}(q^k \zeta_r)$, $k = 0, \dots, p-2$ as its diagonal elements. It follows that $\det(E^{(r)}) = \prod_{k=0}^{p-2} \mathfrak{d}(q^k \zeta_r)$ which is always nonzero if (5.54) is satisfied. \square

The previous results admit the following reformulation which is central for the classification and construction of the spectrum of $\mathbb{T}(\lambda)$:

Theorem 3. *For generic values of the parameters κ and ξ the spectrum of $\mathbb{T}(\lambda)$ is simple and all the wave-functions $\Psi_t(\eta)$ can be represented in the factorized form (5.42) with Q_t being the eigenvalue of the Q-operator on the eigenstate $|t\rangle$.*

The eigenvectors $|t\rangle$ of $\mathbb{T}(\lambda)$ are in one-to-one correspondence with the polynomials $Q_t(\lambda)$ of order $2lN$, with $Q_t(0) \neq 0$, which satisfy the Baxter equation (4.37) with $t(\lambda)$ being an even Laurent polynomial in λ of degree $N - 1$.

Proof. Proposition 3 implies that the spectrum of $\mathbb{T}(\lambda)$ is simple. Let $|t\rangle$ be an eigenstate of $\mathbb{T}(\lambda)$. Self-adjointness and mutual commutativity of $\mathbb{T}(\lambda)$ and $\mathbb{Q}(\mu)$ imply that $|t\rangle$ is also eigenstate of $\mathbb{Q}(\lambda)$. Let $Q_t(\lambda)$ be the Q-eigenvalue on $|t\rangle$. The polynomial $Q_t(\lambda)$ is related to $t(\lambda)$ by the Baxter equation (4.37) which specialized to the values $\lambda = \eta_r$ yields the equations (5.51). It follows that there must exist nonzero numbers ν_r such that

$$Q_t(\zeta_r q^k) = \nu_r \Psi_{rk}(\zeta_1, \dots, \zeta_N). \quad (5.56)$$

This implies that the wave-functions $\Psi_t(\eta)$ can be represented in the form (5.42) with Q_t being the eigenvalue of the Q-operator on the eigenstate $|t\rangle$. \square

⁴Note that this choice implies that $v_n \in (-1)^{p'/2} q^{1/2} \mathbb{S}_p$.

Remark 1. Our proof of existence of a polynomial $Q_t(\lambda)$ which allows to represent the wavefunctions $\Psi_t(\eta)$ in the factorized form (5.42) is based on the existence of the Q-operator. It would be interesting if one could bypass the construction of the Q-operator by *defining* $Q_t(\lambda)$ via (5.56); we plan to comeback to this interesting issue in a next publication.

5.3 Connection with the Bethe ansatz

It follows from the property (A) of $Q(\lambda)$ that $Q(\lambda)$ must be a polynomial of order $2lN$ normalized by the condition $Q_{2lN} = 1$. Such a polynomial is fully characterized by its zeros $\lambda_1, \dots, \lambda_{2lN}$,

$$Q(\lambda) = \prod_{k=1}^{2lN} (\lambda - \lambda_k). \quad (5.57)$$

It follows from the Baxter equation (4.37) that the zeros must satisfy the Bethe ansatz equations

$$\frac{\mathbf{a}(\lambda_r)}{\mathbf{d}(\lambda_r)} = - \prod_{s=1}^{2lN} \frac{\lambda_s - \lambda_r q}{\lambda_s - \lambda_r/q}. \quad (5.58)$$

Conversely, given a solution $(\lambda_1, \dots, \lambda_{2lN})$ of the Bethe ansatz equations (5.58), we could try to construct the polynomial $Q(\lambda)$ as in equation (5.57). Let

$$t(\lambda) = \frac{\mathbf{a}(\lambda)Q(q^{-1}\lambda) + \mathbf{d}(\lambda)Q(q\lambda)}{Q(\lambda)}. \quad (5.59)$$

$t(\lambda)$ is nonsingular for $\lambda = \lambda_k, k = 1, \dots, M$ thanks to the Bethe ansatz equations (5.58). However, in general it is not clear that the function $t(\lambda)$ will satisfy $t(\lambda) = t(-\lambda)$.

What we need is therefore a characterization of the subset of the solutions of the Bethe ansatz equations (5.58) that yield even functions $t(\lambda)$ via (5.59). To this aim let us consider the logarithmic version of the Bethe ansatz equations (5.58),

$$(2k_r + 1)\pi = \log \frac{\mathbf{d}(\lambda_r)}{\mathbf{a}(\lambda_r)} + \sum_{s=1}^{2lN} \log \frac{1 - \lambda_r q/\lambda_s}{1 - \lambda_r/q\lambda_s}. \quad (5.60)$$

The integers k_r are called the Bethe quantum numbers. We are going to argue that $t(\lambda)$ will satisfy $t(\lambda) = t(-\lambda)$ if the Bethe quantum numbers k_r are equal in pairs, i.e. if for each index r there exists an index s such that $k_r = k_s$.

In order to see this, let us first introduce the following Lemma which is of interest in its own right.

Lemma 2. *For a given $t(\lambda)$, there is at most one polynomial of degree $2lN$ which satisfies the Baxter equation (4.37).*

Proof. Let us define the q-Wronskian:

$$W(\lambda) = Q_1(\lambda)Q_2(q^{-1}\lambda) - Q_2(\lambda)Q_1(q^{-1}\lambda). \quad (5.61)$$

written in terms of two solutions $Q_1(\lambda)$ and $Q_2(\lambda)$ of the Baxter equation; then $W(\lambda)$ satisfies the equation

$$\mathbf{a}(\lambda) W(\lambda) = \mathbf{d}(\lambda) \mathbb{T}^+ W(\lambda). \quad (5.62)$$

Note now that Lemma 1 implies:

$$\prod_{k=0}^{2l} \mathbf{a}(\lambda q^k) \neq \prod_{k=0}^{2l} \mathbf{d}(\lambda q^k), \quad \forall \lambda \notin \mathbb{B}_N, \quad (5.63)$$

so for any $\lambda \notin \mathbb{B}_N$ the only solution consistent with cyclicity $(\mathbb{T}^+)^p = 1$ is $W(\lambda) \equiv 0$. It is then easy to see that this implies that $Q_1(\lambda) = Q_2(\lambda)$. \square

Let us consider now the case N even and $\kappa_{2n+1} = -\kappa_{2n}$. In this case we have $\mathbf{a}(-\lambda) = \mathbf{a}(\lambda)$ and $\mathbf{d}(-\lambda) = \mathbf{d}(\lambda)$. From the previous Lemma follows in particular that the polynomials $Q(\lambda)$ of degree $2lN$ which solve the Baxter equation (4.37) with $t(\lambda) = t(-\lambda)$ must be even, $Q(-\lambda) = Q(\lambda)$, which means that for each index r there exists an index r' such that $\lambda_r = -\lambda_{r'}$. A quick inspection of (5.60) shows that this implies that $k_r = k_{r'}$ for this case. Note that the solutions λ_r of the Bethe ansatz equations (5.60) depend on the parameters $\kappa = (\kappa_1, \dots, \kappa_N)$. If the dependence of λ_r on κ is smooth in the neighborhood of the submanifold defined by the condition $\kappa_{2n+1} = -\kappa_{2n}$, we may conclude that $k_r = k_{r'}$ holds for generic values of κ .

A. Cyclic solutions of the star-triangle relation

It will sometimes be convenient for us to identify $\mathbb{Z}_p \equiv \mathbb{Z}/p\mathbb{Z}$ with the subset $\mathbb{S}_p = \{q^{2n}; n = -l, \dots, l\}$ of the unit circle since $q^{2l+1} = 1$.

A.1 Definition and elementary properties

A.2 The function $w_\lambda(z)$

Let us define a function $w_\lambda : \mathbb{S}_p \rightarrow \mathbb{C}$ by

$$w_\lambda(q^{2n}) = \prod_{r=1}^n \frac{1 + \lambda q^{2r-1}}{\lambda + q^{2r-1}} \prod_{r=1}^l \frac{\lambda + q^{2r-1}}{1 + q^{2r-1}}, \quad n = 0, \dots, p-1. \quad (A.1)$$

This function is indeed cyclic (defined on \mathbb{S}_p) since $\prod_{k=1}^p (1 - xq^{2k}) = 1 - x^p$ implies

$$w_\lambda(q^{2p}) = w_\lambda(q^{4l+2}) = w_\lambda(1), \quad (A.2)$$

The function $w_\lambda(z)$ is the unique solution to the functional equation

$$(z + \lambda)w_\lambda(qz) = (1 + \lambda z)w_\lambda(q^{-1}z), \quad (A.3)$$

which is a polynomial of order l in λ and which satisfies the normalization condition

$$w_1(q^n) = 1 \quad \forall n \in \mathbb{Z}_p. \quad (\text{A.4})$$

The function $w_\lambda(z)$ satisfies the inversion relation

$$w_\lambda(z)w_{1/\lambda}(z) = \chi_\lambda, \quad \chi_\lambda = \lambda^{-l} \prod_{r=1}^l \frac{(\lambda + q^{2r-1})(1 + \lambda q^{2r-1})}{(q^{2r-1} + 1)^2}. \quad (\text{A.5})$$

A.2.1 The function $\bar{w}_\lambda(z)$

Let us also introduce the function $\bar{w}_\lambda(z)$ as the discrete Fourier transformation of w_λ ,

$$\bar{w}_\lambda(z) = \frac{1}{p} \sum_{k=-l}^l z^k w_\lambda(q^k) \quad (\text{A.6})$$

$\bar{w}_\lambda(z)$ can be characterized as the unique solution to the functional relation

$$(1 - \lambda qz)\bar{w}_\lambda(qz) = (z - q\lambda)\bar{w}_\lambda(q^{-1}z), \quad (\text{A.7})$$

which is a polynomial of order l in λ and which satisfies the normalization condition $\bar{w}_1(q^n) = \delta_{n,0}$. It may therefore be represented by the product

$$\bar{w}_\lambda(q^{2n}) = \prod_{r=1}^n \frac{q\lambda - q^{2r-1}}{\lambda q^{2r} - 1} \prod_{s=1}^l \frac{\lambda q^{2s} - 1}{q^{2s} - 1}. \quad (\text{A.8})$$

It is also useful to observe that \bar{w}_λ and w_λ are related by complex conjugation as follows:

$$(w_{\epsilon\lambda}(z))^* = \bar{w}_{\epsilon\lambda^*}(z) \prod_{s=1}^l \frac{1 - q^{2s}}{1 + q^{2s-1}}. \quad (\text{A.9})$$

This relation makes it easy to deduce properties of \bar{w}_λ from those of w_λ .

A.2.2 Further functional relations

Let us list further functional relations satisfied by the function $w_\lambda(z)$.

$$\begin{aligned} (\lambda + z) w_\lambda(qz) &= q^{l^2+l} z^{+\frac{1}{2}} (1 + q\lambda) w_{q\lambda}(z), \\ (1 + \lambda) w_\lambda(qz) &= q^{l^2+l} z^{-\frac{1}{2}} (1 + \lambda z) w_{\lambda/q}(z), \\ (1 - q\lambda) \bar{w}_\lambda(qz) &= q^{-l^2-l} z^{-\frac{1}{2}} (z - q\lambda) \bar{w}_{q\lambda}(z), \\ (1 - q\lambda z) \bar{w}_\lambda(qz) &= q^{-l^2-l} z^{+\frac{1}{2}} (1 - \lambda) \bar{w}_{\lambda/q}(z). \end{aligned} \quad (\text{A.10})$$

These relations play a key role in the derivation of the Baxter equation (4.33).

A.3 Star-triangle relation

One of the most important properties of the function $w_\lambda(x)$ is the star-triangle relation [FZ82]

$$\sum_{x \in \mathbb{S}_p} \bar{w}_\alpha(x/u) w_{\alpha\beta}(x/v) \bar{w}_\beta(x/w) = w_\alpha(w/v) \bar{w}_{\alpha\beta}(u/w) w_\beta(v/u), \quad (\text{A.11})$$

see [Ba08] for an elegant proof and references to related work. We are mainly going to use the following consequence of (A.11) called the exchange relation

$$\begin{aligned} \sum_{y \in \mathbb{S}_p} \bar{w}_\alpha(y/u) w_\beta(y/v) \bar{w}_\gamma(y/w) w_\delta(y/x) &= \\ &= \frac{w_{\beta/\alpha}(u/v)}{w_{\beta/\alpha}(x/w)} \sum_{y \in \mathbb{S}_p} \bar{w}_\beta(y/u) w_\alpha(y/v) \bar{w}_\delta(y/w) w_\gamma(y/x). \end{aligned} \quad (\text{A.12})$$

for $\alpha\gamma/\beta\delta = 1$. In order to prove (A.12) let us note the relation

$$\sum_{z \in \mathbb{S}_p} \bar{w}_\alpha(u/z) \bar{w}_{1/\alpha}(z/v) = \frac{1}{p} \sum_{k=-l}^l (u/v)^k w_\alpha(q^k) w_{1/\alpha}(q^k) = \delta_{u,v} \chi_\alpha, \quad (\text{A.13})$$

since $\chi_\alpha \equiv w_\alpha(z) w_{1/\alpha}(z)$ is independent of z . By inserting (A.13) into the left hand side of (A.12) we may therefore calculate

$$\begin{aligned} \sum_{y \in \mathbb{S}_p} \bar{w}_\alpha(u/y) w_\beta(y/v) \bar{w}_\gamma(y/w) w_\delta(x/y) &= \\ &= \chi_\alpha^{-1} \sum_{y \in \mathbb{S}_p} \sum_{z \in \mathbb{S}_p} \sum_{y' \in \mathbb{S}_p} \bar{w}_\alpha(y/u) w_\beta(y/v) \bar{w}_{\beta/\alpha}(y/z) \bar{w}_{\delta/\gamma}(z/y') w_\delta(y'/x) \bar{w}_\gamma(y'/w) \\ &= \chi_\alpha^{-1} \sum_{z \in \mathbb{S}_p} w_\alpha(v/z) \bar{w}_\beta(z/u) w_{\beta/\alpha}(u/v) w_{\delta/\gamma}(w/x) \bar{w}_\delta(z/w) w_\gamma(x/z). \end{aligned}$$

The integrals over y and y' have been carried out with the help of the star-triangle relation (A.11). It remains to recall that $\chi_\alpha^{-1} w_{\delta/\gamma}(w/x) = (w_{\beta/\alpha}(w/x))^{-1}$ to complete the proof of (A.12).

B. Properties of the Q-operator

B.1 Proof of the Baxter equation

The strategy is similar to [Ba73, BS90]. Consider

$$\mathbf{T}(\lambda) \cdot \langle \mathbf{z} | \mathbf{Y}(\lambda) | \mathbf{z}' \rangle \equiv \langle \mathbf{z} | \mathbf{T}(\lambda) \mathbf{Y}(\lambda) | \mathbf{z}' \rangle. \quad (\text{B.1})$$

The operator $\mathbf{T}(\lambda)$ is the difference operator obtained by replacing $L_n^{\text{SG}}(\lambda) \rightarrow \mathbf{L}_n^{\text{SG}}(\lambda)$ in (3.2), with $\mathbf{L}_n^{\text{SG}}(\lambda)$ obtained from (B.2) by replacing u_n and v_n by the corresponding multiplication and shift

operators \mathbf{u}_n and \mathbf{v}_n defined in (2.11),

$$\mathbf{L}_n^{\text{SG}} = \frac{\kappa_n}{i} \begin{pmatrix} -\mathbf{u}_n(\vartheta_n^{-1}\mathbf{v}_n - \vartheta_n\mathbf{v}_n^{-1}) & \lambda_n\mathbf{v}_n - \lambda_n^{-1}\mathbf{v}_n^{-1} \\ \lambda_n\mathbf{v}_n^{-1} - \lambda_n^{-1}\mathbf{v}_n & \mathbf{u}_n^{-1}(\vartheta_n\mathbf{v}_n - \vartheta_n^{-1}\mathbf{v}_n^{-1}) \end{pmatrix}. \quad (\text{B.2})$$

In writing (B.2) we have introduced the short-hand notation $\vartheta_n = iq^{\frac{1}{2}}\kappa_n^{-1}$ and $\lambda_n \equiv \lambda/\xi_n$. Note that $\mathbf{T}(\lambda)$ acts on the argument $\mathbf{z} = (z_1, \dots, z_N)$ of $Y_\lambda(\mathbf{z}, \mathbf{z}')$, while it does not act on \mathbf{z}' . In order to simplify the expression for $\mathbf{T}(\lambda)$ we may therefore use a gauge-transformation of the form

$$\tilde{\mathbf{L}}_n^{\text{SG}}(\lambda) = g_{n+1} \mathbf{L}_n^{\text{SG}}(\lambda) g_n^{-1}, \quad g_n = \begin{pmatrix} 1 & 0 \\ z'_n & 1 \end{pmatrix}. \quad (\text{B.3})$$

The key point to observe is that

$$\begin{aligned} \frac{i}{\kappa_n} \tilde{\mathbf{L}}_n^{\text{SG}}(\lambda)_{21} \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') &= \lambda_n^{-1} (z'_n + \lambda_n \vartheta_n \mathbf{u}_n) (z'_{n+1} + \lambda_n \vartheta_n^{-1} \mathbf{u}_n^{-1}) \mathbf{v}_n^{-1} \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') \\ &\quad - \lambda_n^{-1} (1 + \lambda_n \vartheta_n z'_n \mathbf{u}_n^{-1}) (1 + \lambda_n \vartheta_n^{-1} z'_{n+1} \mathbf{u}_n) \mathbf{v}_n \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') \\ &= 0, \end{aligned} \quad (\text{B.4})$$

the last step being an easy consequence of the recursion relations (A.3), (A.7) satisfied by the functions $w_\lambda(z)$ and $\bar{w}_\lambda(z)$ which appear in the kernel $Y_\lambda(\mathbf{z}, \mathbf{z}')$.

Equation (B.4) implies that

$$\mathbf{T}(\lambda) \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') = \left(\prod_{n=1}^N \tilde{\mathbf{L}}_n^{\text{SG}}(\lambda)_{11} + \prod_{n=1}^N \tilde{\mathbf{L}}_n^{\text{SG}}(\lambda)_{22} \right) \cdot Y_\lambda(\mathbf{z}, \mathbf{z}'). \quad (\text{B.5})$$

We have

$$\begin{aligned} \tilde{\mathbf{L}}_n^{\text{SG}}(\lambda)_{11} \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') &= -\frac{\kappa_n}{i} z'_n [\vartheta_n^{-1} (z_n/z'_n + \lambda_n \vartheta_n) \mathbf{v}_n - \lambda_n^{-1} (1 + \vartheta_n \lambda_n z_n/z'_n) \mathbf{v}_n^{-1}] \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') \\ &= -\frac{\kappa_n}{i} z'_n (\lambda_n/\vartheta_n^2 - 1/\lambda_n) \frac{1 + \lambda_n \vartheta_n z_n/z'_n}{1 + z_n z'_{n+1} \lambda_n/\vartheta_n} \mathbf{v}_n^{-1} \cdot Y_\lambda(\mathbf{z}, \mathbf{z}'). \end{aligned}$$

By using the recursion relations (A.10) one may rewrite this as

$$\tilde{\mathbf{L}}_n^{\text{SG}}(\lambda)_{11} \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') = \frac{\kappa_n}{i} (z'_n/z'_{n+1})^{\frac{1}{2}} (1/\lambda_n - 1/\vartheta_n) (1 + q^{-1} \lambda_n \vartheta_n) Y_{q^{-1}\lambda}(\mathbf{z}, \mathbf{z}').$$

We may similarly calculate

$$\begin{aligned} \tilde{\mathbf{L}}_n^{\text{SG}}(\lambda)_{22} \cdot Y_\lambda^r &= \frac{\kappa_n}{i} [(\vartheta_n z_n^{-1} + z_{n+1} \lambda_n) \mathbf{v}_n - (\vartheta_n^{-1} z_n^{-1} + z_{n+1} \lambda_n^{-1}) \mathbf{v}_n^{-1}] \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') \\ &= -\frac{\kappa_n}{i} (z'_n)^{-1} (\lambda_n - 1/\lambda_n \vartheta_n^2) \frac{1 + z'_{n+1} z_n \vartheta_n/\lambda_n}{1 + z_n/z'_n \lambda_n \vartheta_n} \mathbf{v}_n^{-1} \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') \\ &= -\frac{\kappa_n}{i} (z'_{n+1}/z'_n)^{\frac{1}{2}} (1/\lambda_n + q/\vartheta_n) (1 - \lambda_n \vartheta_n) Y_{q\lambda}(\mathbf{z}, \mathbf{z}'). \end{aligned}$$

It follows that

$$\begin{aligned} \prod_{n=1}^N \tilde{\mathbf{L}}_n^{\text{SG}}(\lambda)_{11} \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') &= \prod_{n=1}^N \frac{\kappa_n}{i} (1/\lambda_n - 1/\vartheta_n) (1 + q^{-1} \lambda_n \vartheta_n) \cdot Y_{q^{-1}\lambda}(\mathbf{z}, \mathbf{z}'), \\ \prod_{n=1}^N \tilde{\mathbf{L}}_n^{\text{SG}}(\lambda)_{22} \cdot Y_\lambda(\mathbf{z}, \mathbf{z}') &= \prod_{n=1}^N i \kappa_n (1/\lambda_n + q/\vartheta_n) (1 - \lambda_n \vartheta_n) \cdot Y_{q\lambda}(\mathbf{z}, \mathbf{z}'). \end{aligned}$$

This concludes the proof.

B.2 Proof of the commutativity

The key observation to be made is the fact that the operators $Y(\lambda)$ satisfy the exchange relation

$$Y(\lambda) \cdot (Y(\mu^*))^\dagger = Y(\mu) \cdot (Y(\lambda^*))^\dagger. \quad (\text{B.6})$$

This is an easy consequence of the exchange relation (A.12). Since we have $\lambda_n/\mu_n = \lambda_m/\mu_m$ for all $n, m = 1, \dots, N$ we may calculate

$$\begin{aligned} \langle \mathbf{z} | Y(\lambda) (Y(\mu^*))^\dagger | \mathbf{z}' \rangle &= \\ &= \sum_{\mathbf{y} \in \mathbb{S}_p^N} \prod_{n=1}^N \overline{w_{\epsilon \lambda_n / \kappa_n}(z_n / y_n) w_{\epsilon \lambda_n \kappa_n}(z_n y_{n+1})} \overline{w_{\epsilon \mu_n^* / \kappa_n}(y_n / z'_n) w_{\epsilon \mu_n^* \kappa_n}(y_{n+1} z'_n)} \\ &= \phi_0 \sum_{\mathbf{y} \in \mathbb{S}_p^N} \prod_{n=1}^N \overline{w_{\epsilon \lambda_n / \kappa_n}(z_n / y_n) w_{\epsilon \mu_n / \kappa_n}(y_n / z'_n)} \overline{w_{\epsilon \mu_{n-1} \kappa_{n-1}}(y_n z'_{n-1}) w_{\epsilon \lambda_{n-1} \kappa_{n-1}}(z_{n-1} y_n)} \\ &= \phi_0 \sum_{\mathbf{y} \in \mathbb{S}_p^N} \prod_{n=1}^N \overline{w_{\epsilon \mu_n / \kappa_n}(z_n / y_n) w_{\epsilon \lambda_n / \kappa_n}(y_n / z'_n)} \overline{w_{\epsilon \lambda_{n-1} \kappa_{n-1}}(y_n z'_{n-1}) w_{\epsilon \mu_{n-1} \kappa_{n-1}}(z_{n-1} y_n)} \\ &= \sum_{\mathbf{y} \in \mathbb{S}_p^N} \prod_{n=1}^N \overline{w_{\epsilon \mu_n / \kappa_n}(z_n / y_n) w_{\epsilon \mu_n \kappa_n}(z_n y_{n+1})} \overline{w_{\epsilon \lambda_n^* / \kappa_n}(y_n / z'_n) w_{\epsilon \lambda_n^* \kappa_n}(y_{n+1} z'_n)} \\ &= \langle \mathbf{z} | Y(\mu) (Y(\lambda^*))^\dagger | \mathbf{z}' \rangle, \end{aligned}$$

where $\phi_0 \equiv (-1)^{lN} q^{2Nl(l+1)}$ as it follows by formula (A.9). The mutual commutativity of the operators Q is an easy consequence. Let us furthermore note that $(a(\lambda))^* = d(\lambda)$ and $(T(\lambda))^\dagger =$

$T(\lambda)$ for $\lambda \in \mathbb{R}$. Using (B.6) we may calculate

$$\begin{aligned}
T(\lambda) \cdot Y(\lambda) \cdot (Y(\mu))^\dagger &= [\mathbf{a}(\lambda)Y(q^{-1}\lambda) + \mathbf{d}(\lambda)Y(q\lambda)] \cdot (Y(\mu))^\dagger \\
&= \mathbf{a}(\lambda)Y(\mu) \cdot (Y(q\lambda))^\dagger + \mathbf{d}(\lambda)Y(\mu) \cdot (Y(q^{-1}\lambda))^\dagger \\
&= Y(\mu) \cdot [\mathbf{d}(\lambda)Y(q\lambda) + \mathbf{a}(\lambda)Y(q^{-1}\lambda)]^\dagger \\
&= Y(\mu) \cdot [T(\lambda) \cdot Y(\lambda)]^\dagger \\
&= Y(\mu) \cdot (Y(\lambda))^\dagger \cdot T(\lambda),
\end{aligned}$$

which obviously implies $[T(\lambda), Q(\lambda)] = 0$.

B.3 Proof of integrability

In order to prove (4.40) first note that (4.32) allows us to write

$$\langle z_n | w_\lambda(\mathbf{f}_{2n}) | z'_n \rangle = \sum_{r=-l}^l \langle z_n | \mathbf{f}_{2n}^{-r} | z'_n \rangle \bar{w}_\lambda(q^r). \quad (\text{B.7})$$

Noting that $\langle q^{2k_n} | \mathbf{f}_{2n}^{-r} | q^{2k'_n} \rangle = \langle q^{2k_n-2r} | q^{2k'_n} \rangle = \delta_{r, k'_n - k_n}$ we find that

$$\langle z_n | w_\lambda(\mathbf{f}_{2n}) | z'_n \rangle = \bar{w}_\lambda(z_n/z'_n). \quad (\text{B.8})$$

Thanks to this identity and (A.4) it is easy to see that

$$\langle \mathbf{z} | Y(1/\kappa\epsilon) | \mathbf{z}' \rangle = \prod_{n=1}^N \delta_{z'_n, z_n} \bar{w}_{\kappa^{-2}}(z_n/z'_n) = \langle \mathbf{z} | \prod_{n=1}^N w_{\kappa^{-2}}(\mathbf{f}_{2n}) | \mathbf{z}' \rangle,$$

which implies

$$\mathbf{Q}^+(1/\kappa\epsilon) = \prod_{n=1}^N w_{\kappa^{-2}}(\mathbf{f}_{2n}) \cdot Y_\infty^\dagger. \quad (\text{B.9})$$

Similarly note that

$$\langle \mathbf{z} | Y(\kappa/\epsilon) | \mathbf{z}' \rangle = \prod_{n=1}^N w_{\kappa^2}(z_n z'_{n+1}) = \langle \mathbf{z} | \prod_{n=1}^N w_{\kappa^2}(\mathbf{f}_{2n+1}) | \mathbf{z}' \rangle,$$

which implies

$$\mathbf{Q}^-(\kappa/\epsilon) = Y_0 \cdot \prod_{n=1}^N (w_{\kappa^2}(\mathbf{f}_{2n+1}))^{-1}. \quad (\text{B.10})$$

It remains to notice that $Y_\infty^\dagger \cdot Y_0 = U_0$ to conclude the proof. Indeed, using the notation $\mathbf{z} = (q^{2k_1}, \dots, q^{2k_N})$ and $\mathbf{z}'' = (q^{2k'_1}, \dots, q^{2k'_N})$, we may calculate

$$\begin{aligned} \langle \mathbf{z} | Y_\infty^\dagger \cdot Y_0 | \mathbf{z}'' \rangle &= \frac{1}{p^N} \sum_{(k'_1, \dots, k'_N) \in \mathbb{Z}_p^N} \prod_{n=1}^N q^{-2k'_n(k_n + k_{n+1})} q^{-2k'_n(k'_n + k''_{n+1})} \\ &= \prod_{n=1}^N \delta_{k_n + k_{n+1} + k'_n + k''_{n+1}, 0} = \prod_{n=1}^N \delta_{k_n, -k''_n} \\ &= \langle \mathbf{z} | U_0 | \mathbf{z}'' \rangle, \end{aligned}$$

keeping in mind that we consider the case of odd N .

C. Separation of variables

C.1 Construction of an eigenbasis for $B(\lambda)$

We will construct the eigenstates $\langle \eta |$ of $B(\lambda) \equiv B_N(\lambda)$ recursively by induction on N . The corresponding eigenvalues $B(\lambda)$ are parameterized by the tuple $\eta = (\eta_a)_{a=1, \dots, N}$ as

$$B(\lambda) = \eta_N^{e_N} b_\eta(\lambda), \quad b_\eta(\lambda) \equiv \prod_{n=1}^N \frac{\kappa_n}{i} \prod_{a=1}^{[N]} (\lambda/\eta_a - \eta_a/\lambda); \quad (\text{C.11})$$

We remind that e_N is zero for N odd and 1 for N even.

In the case $N = 1$ we may simply take $\langle \eta_1 | = \langle v |$, where $\langle v |$ is an eigenstate of the operator \mathbf{v}_1 with eigenvalue v . It is useful to note that the inhomogeneity parameter determines the subset of \mathbb{C} on which the variable η_1 lives, $\eta_1 \in \xi_1 \mathbb{S}_p$.

Now assume we have constructed the eigenstates $\langle \chi |$ of $B_M(\lambda)$ for any $M < N$. The eigenstates $\langle \eta |$, $\eta = (\eta_N, \dots, \eta_1)$, of $B_N(\lambda)$ may then be constructed in the following form

$$\langle \eta | = \sum_{\chi_1} \sum_{\chi_2} K_N(\eta | \chi_2; \chi_1) \langle \chi_2 | \otimes \langle \chi_1 |, \quad (\text{C.12})$$

where $\langle \chi_2 |$ and $\langle \chi_1 |$ are eigenstates of $B_M(\lambda)$ and $B_{N-M}(\lambda)$ with eigenvalues parameterized as in (C.11) by the tuples $\chi_2 = (\chi_{2a})_{a=1, \dots, M}$ and $\chi_1 = (\chi_{1a})_{a=1, \dots, N-M}$, respectively. It suffices to consider the cases where $N - M$ is odd.

It follows from the formula

$$\begin{aligned} B_N(\lambda) &= A_M(\lambda) \otimes B_{N-M}(\lambda) + B_M(\lambda) \otimes D_{N-M}(\lambda) \\ &\equiv A_{2M}(\lambda) B_{1N-M}(\lambda) + B_{2M}(\lambda) D_{1N-M}(\lambda) \end{aligned} \quad (\text{C.13})$$

that the matrix elements $K_N(\eta | \chi_2; \chi_1)$ have to satisfy the relations

$$\begin{aligned} & (\mathbf{A}_{2M}(\lambda)\mathbf{B}_{1N-M}(\lambda) + \mathbf{B}_{2M}(\lambda)\mathbf{D}_{1N-M}(\lambda))^t K_N(\eta | \chi_2; \chi_1) \\ &= \eta_N^{eN} \prod_{n=1}^N \frac{\kappa_n}{i} \prod_{a=1}^{[N]} (\lambda/\eta_a - \eta_a/\lambda) K_N(\eta | \chi_2; \chi_1), \end{aligned} \quad (\text{C.14})$$

where we used the notation O^t for the transpose of an operator O .

Let us assume that

$$\chi_{1a}q^{h_1} \notin \Delta_1, \quad \chi_{2b}q^{h_2} \notin \Delta_2 \quad \text{and} \quad \chi_{1a}q^{h_1} \neq \chi_{2b}q^{h_2}, \quad (\text{C.15})$$

where $h_i \in \{1, \dots, p\}$, $a \in \{1, \dots, N-M\}$, $b \in \{1, \dots, M\}$ and Δ_i is the set of zeros of the quantum determinant on the subchain i , with $i = 1, 2$. Under these assumptions⁵ the previous equations yield recursion relations for the dependence of the kernels in the variables χ_{1a} and χ_{2b} simply by setting $\lambda = \chi_{1a}$ and $\lambda = \chi_{2b}$. Indeed for $\lambda = \chi_{1a}$ the first term on the left of (C.14) is killed leading to

$$\begin{aligned} & \mathbb{T}_{1a}^- K_N(\eta | \chi_2; \chi_1) d_1(q^{-1}\chi_{1a}) \chi_M^{eM} \prod_{n=1}^{N-M} \frac{i}{\kappa_n} \prod_{a=1}^{[M]} (\chi_{1a}/\chi_{2b} - \chi_{2b}/\chi_{1a}) \\ &= K_N(\eta | \chi_2; \chi_1) \eta_N^{eN} \prod_{b=1}^{[N]} (\chi_{1a}/\eta_b - \eta_b/\chi_{1a}), \end{aligned} \quad (\text{C.16})$$

while for $\lambda = \chi_{2a}$ one finds similarly

$$\begin{aligned} & \mathbb{T}_{2a}^+ K_N(\eta | \chi_2; \chi_1) a_2(q^{+1}\chi_{2a}) \prod_{n=1}^M \frac{i}{\kappa_n} \prod_{b=1}^{N-M} (\chi_{2a}/\chi_{1b} - \chi_{1b}/\chi_{2a}) \\ &= K_N(\eta | \chi_2; \eta_1) \eta_N^{eN} \prod_{b=1}^{[N]} (\chi_{2a}/\eta_b - \eta_b/\chi_{2a}). \end{aligned} \quad (\text{C.17})$$

If M is even we find the recursion relation determining the dependence on χ_{2M} by sending $\lambda \rightarrow \infty$ in (C.14), leading to

$$\mathbb{T}_{2M}^+ K_N(\eta | \chi_2; \chi_1) \frac{1}{\chi_{2A}} \prod_{a=1}^{M-1} \frac{1}{\chi_{2a}} \prod_{b=1}^{N-M} \frac{1}{\chi_{1b}} = K_N(\eta | \chi_2; \eta_1) \prod_{b=1}^N \frac{1}{\eta_b}. \quad (\text{C.18})$$

The recursion relations (C.16), (C.17) have solutions compatible with the requirement of cyclicity, $(\mathbb{T}_{1a}^-)^p = 1$ and $(\mathbb{T}_{2a}^+)^p = 1$ for all values of a , provided that the algebraic equations

$$D_1(\chi_{1a}) (\chi_{2M}^{eM})^p \prod_{n=1}^{N-M} \frac{i^p}{\kappa_n^p} \prod_{b=1}^{[M]} (\chi_{1a}^p/\chi_{2b}^p - \chi_{2b}^p/\chi_{1a}^p) = (\eta_N^{eN})^p \prod_{b=1}^{[N]} (\chi_{1a}^p/\eta_b^p - \eta_b^p/\chi_{1a}^p), \quad (\text{C.19})$$

where $D_1(\chi_{1a}) \equiv \prod_{k=1}^p d_1(q^k \chi_{1a})$,

⁵The subspace within the space of parameters where these conditions are not satisfied has codimension at least one.

and

$$A_2(\chi_{2a}) \prod_{n=1}^M \frac{i^p}{\kappa_n^p} \prod_{b=1}^{N-M} (\chi_{2a}^p / \chi_{1b}^p - \chi_{1b}^p / \chi_{2a}^p) = (\eta_N^{e_N})^p \prod_{b=1}^{[N]} (\chi_{2a}^p / \eta_b^p - \chi_b^p / \eta_{2a}^p), \quad (\text{C.20})$$

$$\text{where } A_2(\chi_{2a}) \equiv \prod_{k=1}^p a_2(q^k \chi_{2a}),$$

are satisfied. If M is even the recursion relation (C.18) yields the additional relation

$$\frac{1}{\chi_{2a}^p} \prod_{a=1}^{M-1} \frac{1}{\chi_{2a}^p} \prod_{b=1}^{N-M} \frac{1}{\chi_{1b}^p} = \prod_{b=1}^N \frac{1}{\eta_b^p}. \quad (\text{C.21})$$

We will show in the next subsection that the equations (C.19)-(C.21) completely determine η_a^p in terms of χ_{2a}^p, χ_{1a}^p .

By using (3.17) and (5.49) it is easy to see that the conditions (C.19) and (C.20) are nothing but the equations

$$\mathcal{B}_N(\lambda) = \mathcal{A}_M(\lambda) \mathcal{B}_{N-M}(\lambda) + \mathcal{B}_M(\lambda) \mathcal{D}_{N-M}(\lambda), \quad (\text{C.22})$$

evaluated at $\lambda = \chi_{1a}$ and $\lambda = \chi_{2a}$, respectively. The relation (C.21) follows from (C.22) in the limit $\lambda \rightarrow \infty$. The relations (C.22) are implied by (3.27). We conclude that our construction of $B(\lambda)$ -eigenstates will work if the representations $\mathcal{R}_N, \mathcal{R}_M$ and \mathcal{R}_{N-M} are all non-degenerate. Theorem 1 follows by induction.

C.2 On average value formulae

Proposition 4. *The average values of the Yang-Baxter generators are central elements which satisfy the following recursive equations:*

$$\mathcal{B}_N(\Lambda) = \mathcal{A}_M(\Lambda) \mathcal{B}_{N-M}(\Lambda) + \mathcal{B}_M(\Lambda) \mathcal{D}_{N-M}(\Lambda), \quad (\text{C.23})$$

$$\mathcal{C}_N(\Lambda) = \mathcal{D}_M(\Lambda) \mathcal{C}_{N-M}(\Lambda) + \mathcal{C}_M(\Lambda) \mathcal{A}_{N-M}(\Lambda), \quad (\text{C.24})$$

and

$$\mathcal{A}_N(\Lambda) = \mathcal{A}_M(\Lambda) \mathcal{A}_{N-M}(\Lambda) + \mathcal{B}_M(\Lambda) \mathcal{C}_{N-M}(\Lambda), \quad (\text{C.25})$$

$$\mathcal{D}_N(\Lambda) = \mathcal{D}_M(\Lambda) \mathcal{D}_{N-M}(\Lambda) + \mathcal{C}_M(\Lambda) \mathcal{B}_{N-M}(\Lambda), \quad (\text{C.26})$$

where $N - M$ or M is odd.

Proof. In the previous subsection we have proven the existence of SOV representations, i.e. the diagonalizability of the B -operator. First of all let us point out that $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda)$ are one parameter families of commuting operators. This implies that the corresponding average values are functions of $\Lambda = \lambda^p$.

The fact that $\mathcal{B}_N(\Lambda)$ is central trivially follows from the fact that $B_N(\lambda)$ is diagonal in the SOV-representation, while for the operators A and D we have that for N odd, $\mathcal{A}_N(\lambda)\lambda^{p(N-1)}$ and $\mathcal{D}_N(\lambda)\lambda^{p(N-1)}$ are polynomials in λ^{2p} of degree $N - 1$. It follows that the special values given by (3.18) characterize them completely,

$$\begin{aligned}\mathcal{A}_N(\Lambda) &= \sum_{a=1}^{[N]} \prod_{b \neq a} \frac{(\Lambda/Z_b - Z_b/\Lambda)}{(Z_a/Z_b - Z_b/Z_a)} A_N(\eta_a), \\ \mathcal{D}_N(\Lambda) &= \sum_{a=1}^{[N]} \prod_{b \neq a} \frac{(\Lambda/Z_b - Z_b/\Lambda)}{(Z_a/Z_b - Z_b/Z_a)} D_N(Z_a),\end{aligned}\tag{C.27}$$

where $A_N(Z_a)$ and $D_N(Z_a)$ are the average values of the coefficients of the SOV representation. In the case of N even we have just to add the asymptotic property of $\mathcal{A}_N(\lambda)$ and $\mathcal{D}_N(\lambda)$ discussed in appendix E to complete the statement. Finally, the fact that $\mathcal{C}_N(\lambda)$ is central follows by its diagonalizability in the cyclic representations.

Now the above recursive formulae (C.23-C.26) are a simple consequence of the centrality of the average values of the monodromy matrix elements. Let us consider only the case of the average value of $A_N(\lambda)$. We have the expansion:

$$A_N(\lambda) = A_{2M}(\lambda)A_{1N-M}(\lambda) + B_{2M}(\lambda)C_{1N-M}(\lambda),\tag{C.28}$$

in terms of the entries of the monodromy matrix of the subchains $\mathbf{1}$ and $\mathbf{2}$ with $(N - M)$ -sites and M -sites, respectively. It follows directly from definition (3.16) of the average value together with (C.28) that $\mathcal{A}_N(\lambda)$ can be represented in the form

$$\mathcal{A}_N(\lambda) = \mathcal{A}_{2M}(\lambda)\mathcal{A}_{1N-M}(\lambda) + \mathcal{B}_{2M}(\lambda)\mathcal{C}_{1N-M}(\lambda) + \Delta_N(\lambda)\tag{C.29}$$

where $\Delta_N(\lambda)$ is a sum over monomials which contain at least one and at most $p - 2$ factors of $A_{2M}(\lambda q^m)$. As before, we may work in a representation where the $B_{2M}(\lambda q^n)$ are diagonal, spanned by the states $\langle \chi_2 |$ introduced in the previous subsection. As the factors $A_{2M}(\lambda q^m)$ contained in $\Delta_N(\lambda)$ produce states with modified eigenvalue of $B_{2M}(\lambda q^n)$, none of the states produced by acting with $\Delta_N(\lambda)$ on $\langle \chi_2 |$ can be proportional to $\langle \chi_2 |$. This would be in contradiction to the fact that $\mathcal{A}_N(\lambda)$ is central unless $\Delta_N(\lambda) = 0$. \square

C.3 Non-degeneracy condition

Proposition 5. *The condition $Z_r = Z_s$ for certain $r \neq s$ with $r, s \in \{1, \dots, [N]\}$ defines a subspace in the space of the parameters $\{\kappa_1, \dots, \kappa_N, \xi_1, \dots, \xi_N\} \in \mathbb{C}^{2N}$ of codimension at least one.*

Proof. The parameters Z_r are related to the expectation value $\mathcal{B}_N(\Lambda)$ by means of the equation

$$\mathcal{B}_N(\Lambda) = Z_N^{e_N} \prod_{n=1}^N \frac{K_n}{i^p} \prod_{a=1}^{[N]} (\Lambda/Z_a - Z_a/\Lambda).\tag{C.30}$$

It follows from (3.27) and (3.26) that $\mathcal{B}_N(\Lambda)$ is a Laurent polynomial in X_n that depends polynomially on each of the parameters K_n . Equation (C.30) defines the tuple $Z = (Z_1, \dots, Z_{[N]})$ uniquely up to permutations of $Z_1, \dots, Z_{[N]}$ as function of the parameters $X = (X_1, \dots, X_N)$ and $K = (K_1, \dots, K_N)$. We are going to show that⁶

$$J(X; K) \equiv \det \left(\frac{\partial Z_r}{\partial X_s} \right)_{r,s=1,\dots,[N]} \neq 0. \quad (\text{C.31})$$

The dependence of $J(X; K)$ on the variables K_n is clearly polynomial. It therefore suffices to show that $J(X; K) \neq 0$ for special values of K in order to prove that $J(X; K) \neq 0$ except for values of K within a subset of \mathbb{C}^N of dimension less than N .

Let us choose $K_n^2 = i^p$ for $n = 1, \dots, [N]$, then the average values (3.26) of the Lax operators simplify to

$$\mathcal{L}_n^{SG}(\lambda) = \begin{pmatrix} 0 & \Lambda/X_n - X_n/\Lambda \\ \Lambda/X_n - X_n/\Lambda & 0 \end{pmatrix}. \quad (\text{C.32})$$

Inserting this into (3.27) yields

$$\mathcal{B}_N(\Lambda) = (K_N^2 + 1)^{\text{en}} \prod_{n=1}^{[N]} (\Lambda/X_n - X_n/\Lambda). \quad (\text{C.33})$$

The fact that $J(X; K) \neq 0$ follows for the case under consideration easily from (C.33).

Whenever $J(X; K) \neq 0$, we have invertibility of the mapping $Z = Z(X_1, \dots, X_{[N]})$. The claim follows from this observation. \square

D. Spectrum of the even chain

D.1 The Θ -charge

In the case of a lattice with N even quantum sites, we can introduce the operator:

$$\Theta = \prod_{n=1}^N v_n^{(-1)^{1+n}}. \quad (\text{D.34})$$

Proposition 6. Θ commutes with the transfer matrix and satisfies the following commutation relations with the entries of the monodromy matrix:

$$\Theta C(\lambda) = qC(\lambda)\Theta, \quad [A(\lambda), \Theta] = 0, \quad (\text{D.35})$$

$$B(\lambda)\Theta = q\Theta B(\lambda), \quad [D(\lambda), \Theta] = 0. \quad (\text{D.36})$$

⁶It should be noted that for even N it is indeed sufficient to consider the dependence w.r.t. $X_1, \dots, X_N - 1$.

Proof. The claim can be easily verified explicitly for $N = 2$. The proof for the case of general even $N = 2M$ follows by induction. Indeed,

$$M_{2M}^2 M_{2(N-M)}^1 = \begin{pmatrix} A_{2M}^2 A_{2(N-M)}^1 + B_{2M}^2 C_{2(N-M)}^1 & A_{2M}^2 B_{2(N-M)}^1 + B_{2M}^2 D_{2(N-M)}^1 \\ C_{2M}^2 A_{2(N-M)}^1 + D_{2M}^2 C_{2(N-M)}^1 & C_{2M}^2 B_{2(N-M)}^1 + D_{2M}^2 D_{2(N-M)}^1 \end{pmatrix},$$

which easily allows one to deduce that the claim holds if it holds for all $M < N$. \square

D.2 T - Θ -spectrum simplicity

Lemma 3. *Let $k \in \{-l, \dots, l\}$ and $|t_k\rangle$ be a simultaneous eigenstate of the transfer matrix $T(\lambda)$ and of the Θ -charge with eigenvalues $t_{|k|}(\lambda)$ and q^k , respectively, then $\lambda^N t_{|k|}(\lambda)$ is a polynomial in λ^2 of degree N which is a solution of the system of equations:*

$$\det(D^{(r)}(\lambda)) = 0 \quad \forall r \in \{1, \dots, [N]\}, \quad (\text{D.37})$$

where the $p \times p$ matrices $D^{(r)}(\zeta_r)$ are defined in (5.52), with asymptotics of $t_{|k|}(\lambda)$ given by:

$$\lim_{\log(\lambda) \rightarrow \pm\infty} \lambda^{\mp N} t_{|k|}(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\mp 1}}{i} \right) (q^k + q^{-k}). \quad (\text{D.38})$$

Proof. The fact that the generic eigenvalue of the transfer matrix has to satisfy the system (D.37) has been discussed in section 5; so we have just to verify the asymptotics (D.38) for the T -eigenvalue $t_{|k|}(\lambda)$. This follows by the assumption that $|t_k\rangle$ is an eigenstate of Θ with eigenvalue q^k , and by formulae

$$\lim_{\log \lambda \rightarrow \pm\infty} \lambda^{\mp N} T(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\mp 1}}{i} \right) (\Theta + \Theta^{\mp 1}), \quad (\text{D.39})$$

derived in appendix E. \square

The previous Lemma implies in particular the following:

Theorem 4. *For generic values of the parameters κ and ξ the simultaneous spectrum of T and Θ operators is simple and the generic eigenstate $|t_k\rangle$ of the T - Θ -eigenbasis has a wave-function of the form*

$$\Psi_{t_k}(\eta) = \eta_N^{-k} \prod_{a=1}^{N-1} \psi_{|k|}(\eta_a), \quad (\text{D.40})$$

where, for any $r \in \{1, \dots, N-1\}$, the vector $(\psi_{|k|}(\zeta_r), \psi_{|k|}(\zeta_r q), \dots, \psi_{|k|}(\zeta_r q^{2l}))$ is the unique (up to normalization) solution of the linear equations (5.51) corresponding to $t_{|k|}(\lambda)$.

Proof. Let us use the SOV-construction of T-eigenstates and let us observe that an analog of Proposition 3 also holds⁷ for even N. This implies that the wave-function $\Psi_{t_k}(\eta)$ can be represented in the form

$$\Psi_{t_k}(\eta) = f_{t_k}(\eta_N) \prod_{a=1}^{N-1} \psi_{|k|}(\eta_a). \quad (\text{D.41})$$

Finally, using that $|t_k\rangle$ is eigenstate of Θ with eigenvalue q^k we get $f_{t_k}(\eta_N) \propto \eta_N^{-k}$. \square

Thanks to the explicit construction of the simultaneous T- Θ eigenstates given in (D.40), we have that the eigenstates of T(λ) with Θ -charge eigenvalue 1 are simple, while all the others are doubly degenerate with eigenspaces generated by a pair of T-eigenstates with Θ -charge eigenvalues $q^{\pm k}$.

D.3 Q-operator and Bethe ansatz

Let us point out some peculiarity of the Q-operator in the case of even chain.

Proposition 7. *The Q-operators commute with the Θ -charge and $|t_{\pm|k|}\rangle$ are Q-eigenstates with common eigenvalue $Q_{|k|}(\lambda)$ of degree $2lN - k(a_{\infty}^{\pm}p \pm 1)$ in λ and a zero of order $k(a_0^{\pm}p \pm 1)$ at $\lambda = 0$, where a_0^+ and a_{∞}^+ are non-negative integers, while a_0^- and a_{∞}^- are positive integers.*

Proof. The commutativity of T and Q-operators implies that the T-eigenspace $\mathcal{L}(|t_{\pm|k|}\rangle)$ corresponding to the eigenvalue $t_{|k|}(\lambda)$ is invariant under the action of Q and so for $k = 0$ any T-eigenstate $|t_0\rangle$ is directly a Q-eigenstate. Let us observe that the self-adjointness of Q implies that in the two-dimensional T-eigenspace $\mathcal{L}(|t_{\pm|k|}\rangle)$ with $k \neq 0$ we can always take two linear combinations of the states $|t_{|k|}\rangle$ and $|t_{-|k|}\rangle$ which are Q-eigenstates. Now thanks to the Lemma 2 for fixed T-eigenvalue $t_{|k|}(\lambda)$ the corresponding Q-eigenvalue $Q_{|k|}(\lambda)$ is unique which implies that $|t_{\pm|k|}\rangle$ are themselves Q-eigenstates. The commutativity of the Q-operator with the Θ -charge follows by observing that the $|t_{\pm|k|}\rangle$ define a basis.

Let us complete the proof showing that the conditions on the polynomial $Q_{|k|}(\lambda)$ stated in the proposition are simple consequences of the fact that $|t_{\pm|k|}\rangle$ are eigenstates of the Θ -charge with eigenvalues $q^{\pm|k|}$. Indeed, the compatibility of the asymptotics conditions (D.38) with the TQ Baxter equation implies

$$\lim_{\lambda \rightarrow 0} \frac{Q_{|k|}(\lambda q)}{Q_{|k|}(\lambda)} = q^{\pm|k|}, \quad \lim_{\lambda \rightarrow \infty} \frac{Q_{|k|}(\lambda q)}{Q_{|k|}(\lambda)} = q^{-(N \pm |k|)}, \quad (\text{D.42})$$

which are equivalent to the conditions on the polynomial $Q_{|k|}(\lambda)$ stated in the proposition. \square

Note that the uniqueness of the Q-eigenvalue $Q_{|k|}(\lambda)$ corresponding to a given T-eigenvalue $t_{|k|}(\lambda)$ implies that each vector $(\psi_{|k|}(\zeta_r), \psi_{|k|}(\zeta_r q), \dots, \psi_{|k|}(\zeta_r q^{2l}))$ appearing in (D.40) must be proportional to the vector $(Q_{|k|}(\zeta_r), Q_{|k|}(\zeta_r q), \dots, Q_{|k|}(\zeta_r q^{2l}))$ so that the previous results admit the following reformulation:

⁷The proof given previously holds for both the cases N even and odd just changing N into [N] everywhere.

Theorem 5. *The pairs of eigenvectors $|t_{|k|}\rangle$ and $|t_{-|k|}\rangle$ of $\mathbb{T}(\lambda)$ are in one-to-one correspondence with the polynomials $Q_{|k|}(\lambda)$ of maximal order $2lN$ which have the asymptotics (D.42) and satisfy the Baxter equation (4.37) with $t_{|k|}(\lambda)$ being an even Laurent polynomial in λ of degree N .*

As in the case of N odd this reformulation allows the classification and construction of the spectrum of $\mathbb{T}(\lambda)$ by the analysis of the solutions to the system of the Bethe equations.

E. Asymptotics of Yang-Baxter generators

From the known form of the Lax operator we derive the following asymptotics for $\lambda \rightarrow +\infty$ and 0 of the generators of the Yang-Baxter algebras.

N odd: The leading operators are $B_N(\lambda)$ and $C_N(\lambda)$ with asymptotics:

$$B_N(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a}{i} \right) \left(\lambda^N \prod_{a=1}^N \frac{v_a^{(-1)^{1+a}}}{\xi_a} - \lambda^{-N} \prod_{a=1}^N \xi_a v_a^{(-1)^a} \right) + \text{sub-leading}, \quad (\text{E.43})$$

$$C_N(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a}{i} \right) \left(\lambda^N \prod_{a=1}^N \frac{v_a^{(-1)^a}}{\xi_a} - \lambda^{-N} \prod_{a=1}^N \xi_a v_a^{(-1)^{1+a}} \right) + \text{sub-leading}. \quad (\text{E.44})$$

N even: The leading operators are $A_N(\lambda)$ and $D_N(\lambda)$ with asymptotics:

$$A_N(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a}{i} \right) \left(\lambda^N \prod_{a=1}^N \frac{v_a^{(-1)^{1+a}}}{\xi_a} + \lambda^{-N} \prod_{a=1}^N \xi_a v_a^{(-1)^a} \right) + \text{sub-leading}, \quad (\text{E.45})$$

$$D_N(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a}{i} \right) \left(\lambda^N \prod_{a=1}^N \frac{v_a^{(-1)^a}}{\xi_a} + \lambda^{-N} \prod_{a=1}^N \xi_a v_a^{(-1)^{1+a}} \right) + \text{sub-leading}. \quad (\text{E.46})$$

Note that these asymptotics imply for the SOV-representation of the Yang-Baxter generators the following formulae⁸:

N odd:

$$(w^{\text{SOV}})^{-1} \left(\prod_{a=1}^N v_a^{(-1)^{1+a}} \right) w^{\text{SOV}} = \prod_{a=1}^N \frac{\xi_a}{\eta_a}. \quad (\text{E.47})$$

N even:

$$\prod_{a=1}^N \xi_a (w^{\text{SOV}})^{-1} \Theta^{-1} w^{\text{SOV}} = \left(\eta_A \prod_{a=1}^{N-1} \eta_a \right) \mathbb{T}_N^-, \quad (\text{E.48})$$

$$\prod_{a=1}^N \xi_a (w^{\text{SOV}})^{-1} \Theta w^{\text{SOV}} = \left(\eta_D \prod_{a=1}^{N-1} \eta_a \right) \mathbb{T}_N^+, \quad (\text{E.49})$$

⁸Note that the transformation W^{SOV} is meant to act as a similarity transformation in the space of the representation, i.e. $W^{\text{SOV}} \equiv w^{\text{SOV}} I$ where w^{SOV} is a non-trivial operator on space of the states.

Note that taking the average value of the last two formulae we get for N odd:

$$\prod_{a=1}^N \frac{X_a}{Z_a} = \prod_{a=1}^N V_a^{(-1)^{1+a}}, \quad (\text{E.50})$$

while for N even:

$$Z_A = \langle \Theta \rangle^{-1} \prod_{a=1}^{N-1} Z_a^{-1} \prod_{a=1}^N X_a, \quad Z_D = Z_A \langle \Theta \rangle^2, \quad (\text{E.51})$$

where $\langle \Theta \rangle$ is the average value of the charge Θ .

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