

# The singular $H_\infty$ control problem with dynamic measurement feedback

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## THE SINGULAR $H_\infty$ CONTROL PROBLEM WITH DYNAMIC MEASUREMENT FEEDBACK\*

A. A. STOORVOGEL†

**Abstract.** This paper is concerned with the  $H_\infty$  problem with measurement feedback. The problem is to find a dynamic feedback from the measured output to the control input such that the closed-loop system has an  $H_\infty$  norm strictly less than some a priori given bound  $\gamma$  and such that the closed-loop system is internally stable. Necessary and sufficient conditions are given under which such a feedback exists. The only assumption that must be made is that there are no invariant zeros on the imaginary axis for two subsystems. Contrary to recent publications no assumptions are made on the direct feedthrough matrices of the plant. It turns out that this problem can be reduced to an almost disturbance decoupling problem with measurement feedback and internal stability, i.e., the problem in which we can make the  $H_\infty$  norm arbitrarily small.

**Key words.** quadratic matrix inequality, Riccati equation, almost disturbance decoupling, measurement feedback, internal stability

**AMS(MOS) subject classifications.** 93B27, 93B50, 93C05, 93C35, 93C45, 93C60

**1. Introduction.** After the original formulation of the  $H_\infty$  problem in [22] much work has been done on the solution of this problem. Initially almost all the work was done in a mixture of time-domain and frequency-domain techniques (see [1], [4], [5]). In the last few years two new methods have evolved: the polynomial approach (see [9]) and a time-domain approach (see [2], [8], [12], [13], [20]).

This paper handles the problem in the time domain. This has the advantage that we directly obtain an upper bound on the necessary dynamic order of the controller, namely, the dynamic order of the original plant. A similar result was obtained in [10] and [11] using frequency domain techniques. Moreover, in our opinion, the results are more intuitive.

In the above-mentioned literature it was assumed that there are no invariant zeros on the imaginary axis and that the direct feedthrough matrices of the plant are nonsingular. In literature two methods have been proposed to tackle the  $H_\infty$  problem without these assumptions:

- Apply a small perturbation on the output matrices such that these assumptions are satisfied for the perturbed system. Then solve the  $H_\infty$  problem for the perturbed system. If the perturbation satisfies some prerequisites then a controller works for the original system if it works for the perturbed system. However, we do not know a priori how large the perturbations are allowed to be. Hence if for a certain perturbation no suitable controller exists, then we are not sure whether or not a suitable controller exists for a smaller perturbation (see [19]).
- Apply a transformation in the frequency domain:

$$G(s) \rightarrow \tilde{G}(s) := G\left(\frac{s+\varepsilon}{1+\varepsilon s}\right) \quad (\varepsilon > 0).$$

If we can find a suitable controller for the original system, then we can find a controller for the transformed plant for  $\varepsilon$  small enough. Vice versa, if for some  $\varepsilon$  there exists a suitable controller for the transformed system then the same

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controller is a suitable controller for the original system. This approach has the same disadvantage as the previous one since it is not clear how small we should choose  $\varepsilon$ . Another problem is that we still must make the assumption that the transfer matrix from control input to output is left invertible as a rational matrix (see [15]).

Recently, in the case of state feedback, a method of handling the singularity of the direct feedthrough matrix (see [18]) without the above-mentioned disadvantages was proposed. In the present paper we shall develop a method of handling these singularities in the case of measurement feedback. Our results reduce to the known results in [2] and [20] in case these singularities do not occur.

The necessary and sufficient conditions under which there exists an internally stabilizing dynamic compensator which makes the  $H_\infty$  norm strictly less than some a priori given bound  $\gamma$  are formulated in a way that differs from those found in recent publications [2], [20]. In these papers the results are formulated in terms of two Riccati equations. However in the case where there are singularities of the direct feedthrough matrices, these Riccati equations do not exist. We have two quadratic matrix inequalities that replace the role of these Riccati equations. The solution of each of these quadratic matrix inequalities must satisfy rank conditions. Moreover, we have a condition which couples these two matrix inequalities. The spectral radius of the product of the two solutions of these matrix inequalities should be smaller than a certain a priori given upper bound. In the regular case the first rank condition together with the quadratic matrix inequality reduces to a Riccati equation and the second rank condition guarantees that it is a stabilizing solution of the Riccati equation.

The proof of our main result only uses the result for the state feedback  $H_\infty$  control problem. Our proof will use ideas used in [2] to solve the regular  $H_\infty$  problem with measurement feedback but is independent of the results in [2] and is entirely self-contained.

The outline of the paper is as follows. In § 2 we formulate the problem and present the main result. Moreover, we show that in the regular case and the state feedback case this result reduces to the known results in [2] and [18], respectively. In § 3 it is shown that the conditions for the existence of a suitable compensator as given in our main theorem are necessary. It is also shown that the problem of finding such a compensator is equivalent to finding such a compensator for another system, i.e., it is shown that any compensator which internally stabilizes this new system and makes the closed-loop  $H_\infty$  norm less than  $\gamma$  has the same properties when applied to the original system and vice versa. This new system has some desirable properties and using these properties in § 4, it is shown that for this new system we can even make the  $H_\infty$  norm arbitrarily small. In § 5 a method for finding the desired compensator is discussed. We finish in § 6 with some concluding remarks. The proofs of § 3 are given in Appendix B since they are rather technical. Appendix A introduces a number of suitably chosen bases and some of the properties the system matrices have in these new bases. These will be needed in Appendix B.

**2. Problem formulation and main results.** We consider the linear, time-invariant, finite-dimensional system:

$$(2.1) \quad \Sigma: \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1x + D_1w, \\ z = C_2x + D_2u, \end{cases}$$

where  $x \in \mathcal{R}^n$  is the state,  $u \in \mathcal{R}^m$  the control input,  $w \in \mathcal{R}^l$  the unknown disturbance,

$y \in \mathcal{R}^p$  the measured output, and  $z \in \mathcal{R}^q$  the unknown output to be controlled.  $A, B, E, C_1, C_2, D_1,$  and  $D_2$  are matrices of appropriate dimensions. We would like to minimize the effect of the disturbance  $w$  on the output  $z$ , using the measured output  $y$ , by finding an appropriate control input  $u$ . More precisely, we seek a *dynamic* compensator  $F$  described by the following equations:

$$(2.2) \quad \Sigma_F: \begin{cases} \dot{p} = Kp + Ly, \\ u = Mp + Ny, \end{cases}$$

such that after applying the feedback  $u = Fy$  in the system (2.1), the resulting closed-loop system, whose transfer matrix is denoted by  $G_F$ , is internally stable and has minimal  $H_\infty$  norm, i.e., such that

$$(2.3) \quad \|G_F\|_\infty := \sup_{\omega \in \mathcal{R}} \sigma[G_F(i\omega)]$$

is minimized over all possible dynamic feedback laws  $F$  that make the closed-loop system internally stable. Here  $\sigma[M]$  denotes the largest singular value of the matrix  $M$ . Internally stable means that when  $w \equiv 0$  then for every initial state of the system and controller the state of the system and controller in the interconnection both converge to zero as  $t \rightarrow \infty$ . If the controller is given by (2.2) and the system is given by (2.1) this is equivalent to requiring that the following matrix is asymptotically stable:

$$(2.4) \quad \begin{pmatrix} A + BNC_1 & BM \\ LC_1 & K \end{pmatrix}.$$

Although this is our ultimate goal, in this paper we shall derive necessary and sufficient conditions under which we can find a dynamic feedback law which makes the resulting  $H_\infty$  norm of the closed-loop system strictly less than some a priori given bound  $\gamma$  and such that the resulting closed-loop system is internally stable.

A central role in our study of the problem above will be played by the *quadratic matrix inequality*. For any  $\gamma > 0$  and matrix  $P \in \mathcal{R}^{n \times n}$  we define the following matrix:

$$(2.5) \quad F_\gamma(P) := \begin{pmatrix} A^T P + PA + C_2^T C_2 + \gamma^{-2} PEE^T P & PB + C_2^T D_2 \\ B^T P + D_2^T C_2 & D_2^T D_2 \end{pmatrix}.$$

If  $F_\gamma(P) \geq 0$ , we say  $P$  is a solution of the quadratic matrix inequality at  $\gamma$ . We also define a dual version of this quadratic matrix inequality. For any  $\gamma > 0$  and matrix  $Q \in \mathcal{R}^{n \times n}$  we define the following matrix:

$$(2.6) \quad G_\gamma(Q) := \begin{pmatrix} AQ + QA^T + EE^T + \gamma^{-2} QC_2^T C_2 Q & QC_1^T + ED_1^T \\ C_1 Q + D_1 E^T & D_1 D_1^T \end{pmatrix}.$$

If  $G_\gamma(Q) \geq 0$ , we say that  $Q$  is a solution of the dual quadratic matrix inequality at  $\gamma$ . In addition to these two matrices we define two polynomial matrices, whose role is again completely dual:

$$(2.7) \quad L_\gamma(P, s) := [sI - A - \gamma^2 EE^T P \quad -B],$$

$$(2.8) \quad M_\gamma(Q, s) := \begin{bmatrix} sI - A - \gamma^{-2} QC_2^T C_2 \\ -C_1 \end{bmatrix}.$$

We note that  $L_\gamma(P, s)$  is the controllability pencil associated with the system:

$$(2.9) \quad \dot{x} = (A + \gamma^{-2} EE^T P)x + Bu,$$

while  $M_\gamma(Q, s)$  is the observability pencil associated with the system:

$$(2.10) \quad \begin{aligned} \dot{x} &= (A + \gamma^{-2}QC_2^T C_2)x, \\ y &= -C_1x. \end{aligned}$$

We define the following two transfer matrices which again play a dual role:

$$(2.11) \quad G(s) := C_2(sI - A)^{-1}B + D_2,$$

$$(2.12) \quad H(s) := C_1(sI - A)^{-1}E + D_1.$$

In the formulation of our main result we also require the concept of *invariant zero* of the system  $\Sigma = (A, B, C, D)$ . These are all  $s \in \mathcal{C}$  such that

$$(2.13) \quad \text{rank} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} < \text{normrank} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}.$$

Here ‘‘normrank’’ denotes the rank of a matrix as a matrix with entries in the field of rational functions. Moreover let  $\mathcal{C}^+(\mathcal{C}^0, \mathcal{C}^-)$  denote all  $s \in \mathcal{C}$  such that  $\text{Re } s > 0$  ( $\text{Re } s = 0$ ,  $\text{Re } s < 0$ ). Finally, let  $\rho(M)$  denote the spectral radius of the matrix  $M$ . We are now in the position to formulate our main result.

**THEOREM 2.1.** *Consider the system (2.1). Assume that the systems  $(A, B, C_2, D_2)$  and  $(A, E, C_1, D_1)$  have no invariant zeros in  $\mathcal{C}^0$ . Then the following two statements are equivalent:*

- (i) *There exists a linear, time-invariant, finite-dimensional dynamic compensator  $F$  of the form (2.2) such that by applying  $u = Fy$  in (2.1) the resulting closed-loop system, with transfer matrix  $G_F$ , is internally stable and has  $H_\infty$  norm less than  $\gamma$ , i.e.,  $\|G_F\|_\infty < \gamma$ .*
- (ii) *There exist positive semidefinite solutions  $P, Q$  of the quadratic matrix inequalities  $F_\gamma(P) \geq 0$  and  $G_\gamma(Q) \geq 0$  satisfying  $\rho(PQ) < \gamma^2$ , such that the following rank conditions are satisfied:*

- (1)  $\text{rank } F_\gamma(P) = \text{normrank } G$ ,
- (2)  $\text{rank } G_\gamma(Q) = \text{normrank } H$ ,
- (3)  $\text{rank} \begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} = n + \text{normrank } G \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$ ,
- (4)  $\text{rank} (M_\gamma(Q, s) G_\gamma(Q)) = n + \text{normrank } H \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$ .

*Remarks.*

(i) Note that since  $P \geq 0$  and  $Q \geq 0$  the matrix  $PQ$  has only real and nonnegative eigenvalues.

(ii) The construction of a dynamic compensator satisfying (i) can be done according to the method as described in § 5. It turns out that it is always possible to find a compensator of the same dynamic order as the original plant.

(iii) By Corollary A5 we know that a solution  $P$  of the quadratic matrix inequality  $F_\gamma(P) \geq 0$  satisfying (1) and (3) is unique. By dualizing Corollary A5 it can also be shown that a solution  $Q$  of the dual quadratic matrix inequality  $G_\gamma(Q) \geq 0$  satisfying (2) and (4) is unique. The existence of  $P$  and  $Q$  can be checked via a state-space transformation and investigating a reduced order Riccati equation.

(iv) We shall prove this theorem only for the case  $\gamma = 1$ . The general result can then be easily obtained by scaling.

Before we prove this result we look more closely at the result for two special cases.

*State feedback:*  $C_1 = I, D_1 = 0$ . In this case we have  $y = x$ , i.e., we know the state of the system. The first matrix inequality  $F_\gamma(P) \geq 0$  together with rank conditions (1)

and (3) does not depend on  $C_1$  or  $D_1$  so we cannot expect a simplification there. However  $G_\gamma(Q)$  does get a special form:

$$(2.14) \quad G_\gamma(Q) = \begin{pmatrix} AQ + QA^T + EE^T + \gamma^{-2}QC_2^TC_2Q & Q \\ Q & 0 \end{pmatrix}.$$

Using this special form it can be easily seen that  $G_\gamma(Q) \geq 0$  if and only if  $Q = 0$ . For the rank conditions it is interesting to investigate the normrank of  $H$ . We have

$$(2.15) \quad \text{normrank } H = \text{normrank } (sI - A)^{-1}E = \text{rank } E.$$

It can be easily checked, by using (2.15), that  $Q = 0$  satisfies rank conditions (2) and (4). The condition  $\rho(PQ) < \gamma^2$  is trivially satisfied when  $Q = 0$ . We find that in this case condition (ii) of Theorem 2.1 becomes:

*There exists a positive semidefinite solution  $P$  of the matrix inequality  $F_\gamma(P) \geq 0$  such that the following two rank conditions are satisfied:*

- (1)  $\text{rank } F_\gamma(P) = \text{normrank } G,$
- (2)  $\text{rank} \begin{pmatrix} L_\gamma(P, s) \\ F_\gamma(P) \end{pmatrix} = n + \text{normrank } G \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+,$

which is exactly the result obtained in [18].

*Regular case:  $D_1$  surjective and  $D_2$  injective.* In this case it can be shown, in the same way as in [18], that  $F_\gamma(P) \geq 0$  together with rank condition (1) is equivalent to the condition

$$A^TP + PA + C_2^TC_2 + \gamma^{-2}PEE^TP - (PB + C_2^TD_2)(D_2^TD_2)^{-1}(B^TP + D_2^TC_2) = 0.$$

The dual version of this proof can be applied to the dual matrix inequality  $G_\gamma(Q) \geq 0$  together with rank condition (2). These conditions turn out to be equivalent to the condition:

$$AQ + QA^T + EE^T + \gamma^{-2}QC_2^TC_2Q - (QC_1^T + ED_1^T)(D_1D_1^T)^{-1}(C_1Q + D_1E^T) = 0.$$

The two remaining rank conditions (3) and (4) turn out to be equivalent with the requirement that the following two matrices are asymptotically stable:

$$\begin{aligned} A + \gamma^{-2}EE^TP - B(D_2^TD_2)^{-1}(B^TP + D_2^TC_2), \\ A + \gamma^{-2}QC_2^TC_2 - (QC_1^T + ED_1^T)(D_1D_1^T)^{-1}C_1. \end{aligned}$$

Together with the remaining condition  $\rho(PQ) < \gamma^2$ , we thus re-obtain exactly the conditions derived in [2] and [6].

**3. Reduction of the original problem to an almost disturbance decoupling problem.** In this section the implication (i)  $\Rightarrow$  (ii) in Theorem 2.1 will be proven. Moreover, in case the conditions (ii) of Theorem 2.1 are satisfied, we shall show that the problem of finding a suitable compensator  $F$  for the system (2.1) is equivalent to finding a suitable compensator  $F$  for a new system which has some very nice structural properties. In the next section the  $H_\infty$  problem for this new system will be tackled. *In the remainder of this paper we assume  $\gamma = 1$ .* The general result can be easily obtained by scaling. Define  $F(P)$ ,  $G(Q)$ ,  $L(P, s)$ , and  $M(Q, s)$  to be equal to  $F_1(P)$ ,  $G_1(Q)$ ,  $L_1(P, s)$ , and  $M_1(Q, s)$ , respectively.

**LEMMA 3.1.** *Assume that  $(A, B, C_2, D_2)$  and  $(A, E, C_1, D_1)$  have no invariant zeros on  $\mathcal{C}^0$ . If there exists a linear, time-invariant, finite-dimensional dynamic compensator  $F$  such that the resulting closed-loop system is internally stable and has  $H_\infty$  norm less than one, then the following two conditions are satisfied:*

- (i) *There exists a solution  $P \geq 0$  of the quadratic matrix inequality  $F(P) \geq 0$  satisfying the following two rank conditions:*
  - (1)  $\text{rank } F(P) = \text{normrank } G$ ,
  - (2)  $\text{rank} \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + \text{normrank } G \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$ .
- (ii) *There exists a solution  $Q \geq 0$  of the dual quadratic matrix inequality  $G(Q) \geq 0$  satisfying the following two rank conditions:*
  - (1)  $\text{rank } G(Q) = \text{normrank } H$ ,
  - (2)  $\text{rank} (M(Q, s) \ G(Q)) = n + \text{normrank } H \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$ .

*Proof.* Since there exists an internally stabilizing feedback which makes the  $H_\infty$  norm less than one for the problem with measurement feedback there certainly also exists an internally stabilizing feedback which makes the  $H_\infty$  norm less than one in the full information case, i.e., the case where both  $x$  and  $w$  are known. This implies, according to [18], that there exists a matrix  $P$  satisfying the conditions in (i). By dualization it can be easily shown that there also exists a matrix  $Q$  satisfying the conditions in (ii).  $\square$

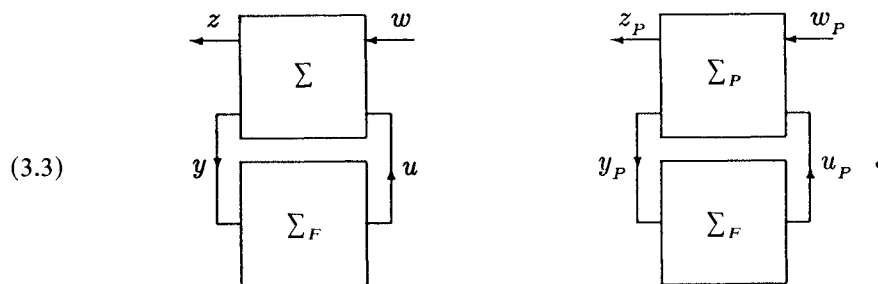
Assume there exist  $P$  and  $Q$  satisfying the conditions in parts (i) and (ii) of Lemma 3.1. We make the following factorization of  $F(P)$ :

$$(3.1) \quad F(P) = \begin{pmatrix} C_{2,P}^T \\ D_P^T \end{pmatrix} \begin{pmatrix} C_{2,P} & D_P \end{pmatrix}$$

where  $C_{2,P}$  and  $D_P$  are matrices of suitable dimensions. This can be done since  $F(P) \geq 0$ . We define the following system:

$$(3.2) \quad \Sigma_P : \begin{cases} \dot{x}_P = (A + EE^T P)x_P + Bu_P + Ew_P, \\ y_P = (C_1 + D_1 E^T P)x_P + D_1 w_P, \\ z_P = C_{2,P}x_P + D_P u_P. \end{cases}$$

LEMMA 3.2. *Let  $P$  satisfy Lemma 3.1(i). Moreover let an arbitrary linear time-invariant finite-dimensional compensator  $F$  be given, described by (2.2). Consider the following two systems, where the system on the left is the interconnection of (2.1) and (2.2) and the system on the right is the interconnection of (3.2) and (2.2):*



Then the following statements are equivalent:

- (i) *The system on the left is internally stable and its transfer matrix from  $w$  to  $z$  has  $H_\infty$  norm less than one.*
- (ii) *The system on the right is internally stable and its transfer matrix from  $w_P$  to  $z_P$  has  $H_\infty$  norm less than one.*

*Proof.* See appendix B for the proof.  $\square$

If for the original system (2.1) there exists an internally stabilizing, linear, time-invariant, finite-dimensional compensator such that the resulting closed-loop matrix has  $H_\infty$  norm less than one then, by applying Lemma 3.2, we know that the same compensator is internally stabilizing for the new system (3.2) and yields a closed-loop

transfer matrix with  $H_\infty$  norm less than one. Hence if we consider for this new system the two quadratic matrix inequalities we know from Lemma 3.1 that there exist positive semidefinite solutions to these inequalities satisfying a number of rank conditions. We shall now formalize this in the following lemma. Define  $A_p := (A + EE^T P)$  and  $C_{1,p} := (C_1 + D_1 E^T P)$ . Then for arbitrary  $X$  and  $Y$  in  $\mathcal{R}^{n \times n}$  we define the following matrices:

$$(3.4) \quad \bar{F}(X) := \begin{pmatrix} A_p^T X + X A_p + C_{2,p}^T C_{2,p} + X E E^T X & X B + C_{2,p}^T D_p \\ B^T X + D_p^T C_{2,p} & D_p^T D_p \end{pmatrix},$$

$$(3.5) \quad \bar{G}(Y) := \begin{pmatrix} A_p Y + Y A_p^T + E E^T + Y C_{2,p}^T C_{2,p} Y & Y C_{1,p}^T + E D_1^T \\ C_{1,p} Y + D_1 E^T & D_1 D_1^T \end{pmatrix},$$

$$(3.6) \quad \bar{L}(X, s) := [sI - A_p - E E^T X \quad -B],$$

$$(3.7) \quad \bar{M}(Y, s) := \begin{bmatrix} sI - A_p - Y C_{2,p}^T C_{2,p} \\ -C_{1,p} \end{bmatrix}.$$

Moreover, we define two new transfer matrices:

$$(3.8) \quad \bar{G}(s) := C_{2,p}(sI - A_p)^{-1} B + D_p,$$

$$(3.9) \quad \bar{H}(s) := C_{1,p}(sI - A_p)^{-1} E + D_1.$$

LEMMA 3.3. *Let  $P$  and  $Q$  satisfy part (i) and part (ii) in Lemma 3.1, respectively. Assume  $(A, B, C_2, D_2)$  and  $(A, E, C_1, D_1)$  have no invariant zeros on  $\mathcal{C}^0$ . Then we have the following two results:*

(i)  $X := 0$  is a solution of the quadratic matrix inequality  $\bar{F}(X) \geq 0$  and satisfies the following two rank conditions:

(1)  $\text{rank } \bar{F}(X) = \text{normrank } \bar{G},$

(2)  $\text{rank} \begin{pmatrix} \bar{L}(X, s) \\ \bar{F}(X) \end{pmatrix} = n + \text{normrank } \bar{G} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+.$

(ii) *There exist a matrix  $Y$  satisfying the quadratic matrix inequality  $\bar{G}(Y) \geq 0$  together with the following two rank conditions:*

(1)  $\text{rank } \bar{G}(Y) = \text{normrank } \bar{H},$

(2)  $\text{rank}(\bar{M}(Y, s) \quad \bar{G}(Y)) = n + \text{normrank } \bar{H} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+,$

*if and only if  $I - QP$  is invertible. Moreover, in that case  $Y := (I - QP)^{-1} Q$  is the unique solution. This matrix  $Y$  is positive semidefinite if and only if*

$$(3.10) \quad \rho(PQ) < 1.$$

*Proof.* See Appendix B for the proof.  $\square$

*Proof of (i)  $\Rightarrow$  (ii) in Theorem 2.1.* The first part can be obtained directly from Lemma 3.1. By Lemma 3.2 we know that also for the transformed system  $\Sigma_p$  there exists a dynamic compensator which internally stabilizes the system and makes the  $H_\infty$  norm less than one. By applying Lemma 3.1 to this new system, this implies that there exists a matrix  $Y \geq 0$  satisfying Lemma 3.3(ii). Hence by Lemma 3.3 we have (3.10) and therefore all the conditions in Theorem 2.1(ii) are satisfied.  $\square$

In the remainder of this section we assume that the conditions of Theorem 2.1(ii) are satisfied.

In order to prove the implication (ii)  $\Rightarrow$  (i) in Theorem 2.1 we transform the system (3.2) once again. This time, however, we use the dualized version of the original transformation. By Lemma 3.3 we know  $Y = (I - QP)^{-1} Q \geq 0$  satisfies  $\bar{G}(Y) \geq 0$ . We factorize  $\bar{G}(Y)$ :

$$(3.11) \quad \bar{G}(Y) = \begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix} \begin{pmatrix} E_{P,Q}^T & D_{P,Q}^T \end{pmatrix}$$



where  $E_{P,Q}$  and  $D_{P,Q}$  are matrices of suitable dimensions. We define the following system:

$$(3.12) \quad \Sigma_{P,Q} : \begin{cases} \dot{x}_{P,Q} = A_{P,Q}x_{P,Q} + B_{P,Q}u_{P,Q} + E_{P,Q}w, \\ y_{P,Q} = C_{1,P}x_{P,Q} + D_{P,Q}w, \\ z_{P,Q} = C_{2,P}x_{P,Q} + D_P u_{P,Q}, \end{cases}$$

where

$$(3.13) \quad A_{P,Q} := A_P + YC_{2,P}^T C_{2,P},$$

$$(3.14) \quad B_{P,Q} := B + YC_{2,P}^T D_P.$$

By applying Lemma 3.3 to the system  $\Sigma_{P,Q}$  with the corresponding matrix inequalities we note that  $X_{P,Q} := 0$  and  $Y_{P,Q} := 0$  satisfy the matrix inequalities and the corresponding rank conditions for this new system. It can be shown that this implies that

$$(3.15) \quad \text{rank} \begin{pmatrix} sI - A_{P,Q} & -B_{P,Q} \\ C_{2,P} & D_P \end{pmatrix} = n + \text{rank} \begin{pmatrix} C_{2,P} & D_P \end{pmatrix} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$$

and

$$(3.16) \quad \text{rank} \begin{pmatrix} sI - A_{P,Q} & -E_{P,Q} \\ C_{1,P} & D_{P,Q} \end{pmatrix} = n + \text{rank} \begin{pmatrix} E_{P,Q} \\ D_{P,Q} \end{pmatrix} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+.$$

By applying Lemma 3.2 and its dualized version the following corollary can be derived.

**COROLLARY 3.4.** *Let an arbitrary compensator  $F$  of the form (2.2) be given. The following two statements are equivalent:*

(i) *The compensator  $F$  when applied to the system  $\Sigma$ , described by (2.1), is internally stabilizing and the resulting closed-loop transfer matrix has  $H_\infty$  norm less than one.*

(ii) *The compensator  $F$  when applied to the system  $\Sigma_{P,Q}$ , described by (3.12), is internally stabilizing and the resulting closed-loop transfer matrix has  $H_\infty$  norm less than one.*

In the next section we shall show how to solve the  $H_\infty$  problem for a system satisfying the extra conditions (3.15) and (3.16). It turns out that for this new system we can even make the  $H_\infty$  norm arbitrarily small.

**4. The solution of the almost disturbance decoupling problem.** Assume that the following system is given:

$$(4.1) \quad \Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = C_1x + D_1w, \\ z = C_2x + D_2u, \end{cases}$$

such that the following two conditions are satisfied:

$$(4.2) \quad \begin{pmatrix} sI - A & -B \\ C_2 & D_2 \end{pmatrix} = n + \text{rank} \begin{pmatrix} C_2 & D_2 \end{pmatrix} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$$

and

$$(4.3) \quad \begin{pmatrix} sI - A & -E \\ C_1 & D_1 \end{pmatrix} = n + \text{rank} \begin{pmatrix} E \\ D_1 \end{pmatrix} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+.$$

From the previous section we know that if the conditions in part (ii) of Theorem 2.1 are satisfied then it is always possible to transform our system into a new system that

satisfies the conditions (4.2) and (4.3). Moreover, if a compensator  $F$  given by (2.2) internally stabilizes this new system and makes the  $H_\infty$  norm of the resulting closed-loop transfer matrix smaller than one, then it does the same with the closed-loop system associated with the original system. In fact, we shall prove a stronger result.

**THEOREM 4.1.** *Assume system (4.1) is given satisfying (4.2) and (4.3). Then for all  $\varepsilon > 0$  there exists a linear, time-invariant, finite-dimensional dynamic compensator  $F$  such that the closed-loop system is internally stable and has  $H_\infty$  norm less than  $\varepsilon$ .*

*Remark.* We note that even if for this new system we can make the  $H_\infty$  norm arbitrarily small, for the original system we are only sure that the  $H_\infty$  norm will be less than one. It is very well possible that a compensator for the new system yields an  $H_\infty$  norm of say 0.0001 while the same compensator makes the  $H_\infty$  norm of the original plant only 0.9999.

Before we can prove this result we have to do some preparatory work. We first have to introduce a number of subspaces from geometric control theory as follows.

**DEFINITION 4.2.** Assume we have a system

$$(4.4) \quad \Sigma_{ci} : \begin{cases} \dot{x} = Ax + Bu, \\ y = C_2x + D_2u. \end{cases}$$

We define *the strongly controllable subspace*  $\mathcal{T}(\Sigma_{ci})$  as the smallest subspace  $\mathcal{T}$  of  $\mathcal{R}^n$  for which there exists a mapping  $G$  such that

$$(4.5) \quad (A + GC_2)\mathcal{T} \subset \mathcal{T},$$

$$(4.6) \quad \text{Im}(B + GD_2) \subset \mathcal{T}.$$

We also define the subspace  $\mathcal{T}_g(\Sigma_{ci})$  as the smallest subspace  $\mathcal{T}$  of  $\mathcal{R}^n$  for which there exists a matrix  $G$  such that (4.5) and (4.6) are satisfied and, moreover,  $A + GC_2|_{\mathcal{R}^n/\mathcal{T}}$  is asymptotically stable. It is well known that these subspaces are well defined in this way. A system is called *strongly controllable* if its strongly controllable subspace is equal to the whole state space.

We also define the dual versions of these subspaces as follows.

**DEFINITION 4.3.** Assume we have a system

$$(4.7) \quad \Sigma_{di} : \begin{cases} \dot{x} = Ax + Ew, \\ y = C_1x + D_1u. \end{cases}$$

We define *the weakly unobservable subspace*  $\mathcal{V}(\Sigma_{di})$  as the largest subspace  $\mathcal{V}$  of  $\mathcal{R}^n$  for which there exists a mapping  $F$  such that

$$(4.8) \quad (A + EF)\mathcal{V} \subset \mathcal{V},$$

$$(4.9) \quad (C_1 + D_1F)\mathcal{V} = \{0\}.$$

We also define the subspace  $\mathcal{V}_g(\Sigma_{di})$  as the largest subspace  $\mathcal{V}$  for which there exists a mapping  $F$  such that (4.8) and (4.9) are satisfied and, moreover,  $A + EF|_{\mathcal{V}}$  is asymptotically stable. It is well known that these subspaces are well defined in this way. A system is called *strongly observable* if its weakly unobservable subspace is equal to  $\{0\}$ .

In order to calculate these subspaces the following lemma will come in handy.

**LEMMA 4.4.**  $\mathcal{T}(\Sigma_{ci})$  equals the limit of the following sequence of subspaces:

$$(4.10) \quad \mathcal{T}_0(\Sigma_{ci}) := 0, \quad \mathcal{T}_{i+1}(\Sigma_{ci}) := \{x \in \mathcal{R}^n \mid \exists \tilde{x} \in \mathcal{T}_i(\Sigma_{ci}), u \in \mathcal{R}^m \text{ such that } x = A\tilde{x} + Bu \text{ and } C_2\tilde{x} + D_2u = 0\}.$$

It is well known (see [16]) that  $\mathcal{T}_i(\Sigma_{ci})$  ( $i = 1, 2, \dots$ ) is a nondecreasing sequence of subspaces that attains its limit in a finite number of steps. In the same way  $\mathcal{V}(\Sigma_{di})$  equals the limit of the following sequence of subspaces:

$$(4.11) \quad \mathcal{V}_0(\Sigma_{di}) := \mathcal{R}^n, \quad \mathcal{V}_{i+1}(\Sigma_{di}) := \{x \in \mathcal{R}^n \mid \exists \tilde{u} \in \mathcal{R}^m, \text{ such that} \\ Ax + E\tilde{u} \in \mathcal{V}_i(\Sigma_{di}) \text{ and } C_1x + D_1\tilde{u} = 0\}.$$

Moreover, if  $G$  is a mapping such that (4.5) and (4.6) are satisfied for  $\mathcal{T} = \mathcal{T}(\Sigma_{ci})$  and if  $F$  is a mapping such that (4.8) and (4.9) are satisfied for  $\mathcal{V} = \mathcal{V}(\Sigma_{di})$ , then we have the following two equalities:

$$(4.12) \quad \mathcal{T}_g(\Sigma_{ci}) = [\mathcal{T}(\Sigma_{ci}) + \mathcal{X}_b(A + GC_2)] \cap \langle \mathcal{T}(\Sigma_{ci}) + C_2^{-1} \text{im } D_2 \mid A + GC_2 \rangle,$$

$$(4.13) \quad \mathcal{V}_g(\Sigma_{di}) = \mathcal{V}(\Sigma_{di}) \cap \mathcal{X}_g(A + EF) + \langle A + EF \mid \mathcal{V}(\Sigma_{di}) \cap E \ker D_1 \rangle.$$

Here  $\mathcal{X}_b(A + GC_2)$  denotes the modal subspace of the matrix  $A + GC_2$  with respect to the closed right halfplane and  $\mathcal{X}_g(A + EF)$  denotes the modal subspace of the matrix  $A + EF$  with respect to the open left halfplane. Finally,  $\langle A + EF \mid \mathcal{V}(\Sigma_{di}) \cap E \ker D_1 \rangle$  denotes the smallest  $A + EF$  invariant subspace containing  $\mathcal{V}(\Sigma_{di}) \cap E \ker D_1$  and  $\langle \mathcal{T}(\Sigma_{ci}) + C_2^{-1} \text{im } D_2 \mid A + GC_2 \rangle$  denotes the largest  $A + GC_2$  invariant subspace contained in  $\mathcal{T}(\Sigma_{ci}) + C_2^{-1} \text{im } D_2$ .

*Proof.* The proof is almost entirely well known except possibly (4.12) and (4.13) in case the  $D$ -matrices are unequal to zero. This can be proven by first showing that there exists a  $G$  satisfying (4.5) and (4.6) for which (4.12) holds and after that, showing that the equality is independent of our particular choice of  $G$  satisfying (4.5) and (4.6). The same can be done for (4.13). Details are left to the reader.  $\square$

We can express the rank conditions (4.2) and (4.3) in terms of these subspaces (see [3], [17]) as follows.

LEMMA 4.5. *Let system (4.1) be given. The rank condition (4.2) is satisfied if and only if*

$$(4.14) \quad \mathcal{V}_g(\Sigma_{ci}) + \mathcal{T}(\Sigma_{ci}) = \mathcal{R}^n.$$

*The rank condition (4.3) is satisfied if and only if*

$$(4.15) \quad \mathcal{V}(\Sigma_{di}) \cap \mathcal{T}_g(\Sigma_{di}) = \{0\}.$$

Here  $\Sigma_{ci}$  is given by (4.4) and  $\Sigma_{di}$  is given by (4.7).

Using this we can derive the following lemma.

LEMMA 4.6. *Let system (4.1) be given satisfying (4.2) and (4.3). For all  $\varepsilon > 0$  there exist mappings  $F$  and  $G$  such that  $A + BF$  and  $A + GC_1$  are asymptotically stable and, moreover,*

$$(4.16) \quad \|(C_2 + D_2F)(sI - A - BF)^{-1}\|_\infty < \varepsilon$$

and

$$(4.17) \quad \|(sI - A - GC_1)(E + GD_1)\|_\infty < \varepsilon.$$

*Proof.* By Definition 4.3 we know there exists a mapping  $\tilde{F}$  such that

$$(4.18) \quad (A + B\tilde{F})\mathcal{V}_g(\Sigma_{ci}) \subset \mathcal{V}_g(\Sigma_{ci}),$$

$$(4.19) \quad (C_2 + D_2\tilde{F})\mathcal{V}_g(\Sigma_{ci}) = \{0\},$$

and moreover,  $A + B\tilde{F} \mid \mathcal{V}_g(\Sigma_{ci})$  is asymptotically stable. Define the canonical projection  $\Pi: \mathcal{R}^n \rightarrow \mathcal{R}^n / \mathcal{V}_g(\Sigma)$ . By (4.19) there exists a mapping  $\tilde{C}$  such that  $C_2 + D_2\tilde{F} = \tilde{C}\Pi$ .

Moreover, by (4.18) there exists a mapping  $\bar{A}$  such that  $\Pi(A + B\tilde{F}) = \bar{A}\Pi$ . Finally, define  $\bar{B} := \Pi B$  and the system:

$$(4.20) \quad \Sigma_{fs} : \begin{cases} \dot{p} = \bar{A}p + \bar{B}u, \\ z = \bar{C}p + D_2u. \end{cases}$$

It can be easily shown by induction using the algorithm (4.10) that  $\mathcal{T}_i(\Sigma_{fs}) = \Pi \mathcal{T}_i(\Sigma_{ci})$  for  $i = 0, 1, \dots$ . Hence we have

$$\mathcal{T}(\Sigma_{fs}) = \Pi \mathcal{T}(\Sigma_{ci}) = \Pi \{ \mathcal{T}(\Sigma_{ci}) + \mathcal{V}_g(\Sigma_{ci}) \} = \Pi \mathcal{R}^n = \mathcal{R}^n / \mathcal{V}_g(\Sigma_{ci}).$$

This implies that the system (4.20) is strongly controllable.

Define  $F_0$  such that  $(\bar{C} + D_2F_0)^T D_2 = 0$  and define  $M$  such that  $\ker D_2 = \text{im } M$ . It can be easily checked that  $\mathcal{T}(\Sigma_{fs}) = \mathcal{T}(\bar{A} + \bar{B}F_0, \bar{B}M, \bar{C} + D_2F_0, 0)$ . Hence by Theorem 3.36 of [21] we know there exist an  $\bar{F}$  such that

$$(4.21) \quad \|(\bar{C} + D_2F_0) e^{((\bar{A} + \bar{B}F_0) + \bar{B}M\bar{F})t}\|_1 < \varepsilon$$

and such that  $\bar{A} + \bar{B}F_0 + \bar{B}M\bar{F}$  is asymptotically stable.

Define  $F := \tilde{F} + (F_0 + M\bar{F})\Pi$ ; then

$$(4.22) \quad A + BF \mid \mathcal{V}_g(\Sigma_{ci}) = A + B\tilde{F} \mid \mathcal{V}_g(\Sigma_{ci}),$$

$$(4.23) \quad \Pi(A + BF) = (\bar{A} + \bar{B}F_0 + \bar{B}M\bar{F})\Pi.$$

It can be easily shown that this implies that  $A + BF$  is asymptotically stable. Moreover, we have

$$(4.24) \quad (C_2 + D_2F) e^{(A+BF)t} = (\bar{C} + D_2F_0) e^{((\bar{A} + \bar{B}F_0) + \bar{B}M\bar{F})t}$$

for all  $t > 0$ . Using (4.24), we find for all  $s \in i\mathcal{R}$  (use that  $|e^{st}| = 1$ ):

$$\begin{aligned} \|(C_2 + D_2F)(sI - (A + BF))^{-1}\| &= \left\| \int_0^\infty (C_2 + D_2F) e^{((A+BF) - sI)t} dt \right\| \\ &\cong \int_0^\infty \|(C_2 + D_2F) e^{((A+BF) - sI)t}\| dt \\ &= \|(C_2 + D_2F) e^{(A+BF)t}\|_1 \\ &= \|(\bar{C} + D_2F_0) e^{((\bar{A} + \bar{B}F_0) + \bar{B}M\bar{F})t}\|_1 \\ &\cong \varepsilon. \end{aligned}$$

This implies (4.16). Therefore  $F$  satisfies all the requirements of the lemma. The existence of a  $G$  such that  $A + GC$  is asymptotically stable and such that (4.17) is satisfied can be obtained by dualization.  $\square$

We can now prove Theorem 4.1.

*Proof of Theorem 4.1.* Let  $\varepsilon > 0$ . We first choose a mapping  $F$  such that

$$(4.25) \quad \|(C_2 + D_2F)(sI - A - BF)^{-1}\|_\infty < \varepsilon/3 \|E\|^{-1}$$

and such that  $A + BF$  is asymptotically stable. This can be done according to Lemma 4.6. Next choose a mapping  $G$  such that

$$(4.26) \quad \|(sI - A - GC_1)^{-1}(E + GD_1)\|_\infty < \min \{ \varepsilon/3 \|D_2F\|^{-1}, \|E\| \|BF\|^{-1} \}$$

and such that  $A + GC_1$  is asymptotically stable. Again Lemma 4.6 guarantees the existence of such a  $G$ . We apply the following dynamic feedback compensator to the system (4.1):

$$(4.27) \quad \Sigma_{F,G}: \begin{cases} \dot{p} = Ap + Bu + G(C_1p - y), \\ u = Fp. \end{cases}$$

The closed-loop system is given by (where  $e := x - p$ ):

$$(4.28) \quad \Sigma_{cl}: \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A + BF & -BF \\ 0 & A + GC_1 \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} + \begin{pmatrix} E \\ E + GD_1 \end{pmatrix} w, \\ z = (C_2 + D_2F \quad -D_2F) \begin{pmatrix} x \\ e \end{pmatrix}. \end{cases}$$

It is clear that this is an internally stabilizing feedback. We now calculate the transfer matrix from  $w$  to  $z$  of this system:

$$\begin{aligned} & (C_2 + D_2F)(sI - A - BF)^{-1}E \\ & - (C_2 + D_2F)(sI - A - BF)^{-1}BF(sI - A - GC_1)^{-1}(E + GD_1) \\ & - D_2F(sI - A - GC_1)^{-1}(E + GD_1). \end{aligned}$$

Using (4.25) and (4.26) it can easily be shown that this closed-loop transfer matrix has  $H_\infty$  norm less than  $\varepsilon$ .  $\square$

We are now able to complete the proof of Theorem 2.1.

*Proof of the implication (ii)  $\Rightarrow$  (i) of Theorem 2.1.* Since we can transform the original system into a system satisfying (4.2) and (4.3) we know by Lemma 4.1 that we can find an internally stabilizing dynamic compensator for this new system which is such that the closed-loop transfer matrix has  $H_\infty$  norm less than one. By applying Corollary 3.4 we know that this compensator  $F$  satisfies the requirements in Theorem 2.1(i).  $\square$

**5. The design of an admissible compensator.** In this section we shall give a method to calculate a dynamic compensator  $F$  such that the closed-loop system is internally stable and, moreover, the closed-loop transfer matrix has  $H_\infty$  norm less than one. We shall derive this  $F$  step by step, using the following conceptual algorithm.

- (i) Calculate  $P$  and  $Q$  satisfying part (ii) of Theorem 2.1. This can, for instance, be done using Lemma A4. If they do not exist or if  $\rho(PQ) \geq 1$ , then there does not exist a dynamic feedback satisfying part (i) of Theorem 2.1 and we stop.
- (ii) Perform the factorizations (3.1) and (3.11). We can now construct the system  $\Sigma_{P,Q}$  as given by (3.12).

We now start solving the almost disturbance decoupling problem for the system (3.12) we obtained in step (ii). As in § 4 we shall rename our variables and assume that we have a system in the form (4.1). We set  $\varepsilon = 1$ . We have to construct matrices  $F$  and  $G$  such that (4.25) and (4.26) are satisfied and, moreover, such that  $A + BF$  and  $A + GC_1$  are asymptotically stable. We shall only discuss the construction of  $F$ . The construction of  $G$  can be obtained by duality.

- (iii) Construct  $\mathcal{V}_g(\Sigma_{cl})$  by using Lemma 4.4.
- (iv) Construct an  $\tilde{F}$  such that (4.18) and (4.19) are satisfied and, moreover, such that  $A + B\tilde{F}| \mathcal{V}_g(\Sigma_{cl})$  is asymptotically stable.



Since  $x_{i,k}(n) \rightarrow A_F^{k-1} \tilde{B}v_i$  as  $n \rightarrow \infty$  for  $i = 1, \dots, j$  and  $k = 1, \dots, r_i + 1$  it can be easily seen that for  $n$  sufficiently large the vectors  $\{x_{i,k}(n), i = 1, \dots, j; k = 1, \dots, r_i + 1\}$  are linearly independent and hence form a basis of  $\mathcal{R}^n$  again. Let  $N$  be such that for all  $n > N$  these vectors indeed form a basis.

(x) For all  $n > N$  define a linear mapping  $\bar{F}_n$  by

$$\begin{aligned} \bar{F}_n x_{i,1}(n) &:= -nv_i \\ \bar{F}_n x_{i,2}(n) &:= -n^2 v_i \\ &\vdots \\ \bar{F}_n x_{i,r_i+1}(n) &:= -n^{r_i+1} v_i. \end{aligned}$$

This determines  $\bar{F}_n$  uniquely. Define  $F_n := F + \bar{F}_n$ . It is shown in [21] that the spectrum of  $\tilde{A} + \tilde{B}F_n$  is the set  $\{-n\}$ . Moreover, we have

$$\lim_{n \rightarrow \infty} \|\tilde{C} e^{(A + BF_n)t}\|_1 = 0.$$

Choose  $n$  such that the impulse response satisfies the required  $\mathcal{L}_1$  bound  $\varepsilon/3 \|E\|^{-1}$ . This  $F_n$  is internally stabilizing and satisfies the  $\mathcal{L}_1$  bound. Now we can construct the  $F$  we were looking for:

(xi) Define  $F = \tilde{F} + (F_0 + MF_n)\Pi$ . This  $F$  is internally stabilizing and is such that (4.25) is satisfied.

We construct  $G$  by dualizing the construction of  $F$  and the required dynamic compensator is finally given by (4.27).

**6. Conclusion.** In this paper we have given a complete treatment of the  $H_\infty$  problem with measurement feedback without restrictions on the direct feedthrough matrices. It remains however an open problem how we can treat invariant zeros on the imaginary axis. Other open problems are the minimally required dynamic order of the controller and the behaviour of the feedbacks and closed-loop system if we make the bound  $\gamma$  tighter. The latter problem has been investigated previously. It is possible that the infimum can only be attained by a nonproper controller (see [4]). But using the ideas of this paper it is perhaps possible to characterize whether or not this problem arises.

Finally, it would be interesting to characterize all solutions. In our opinion it is, however, in general not possible to obtain a characterization similar to the one obtained in [2]. This is due to the fact that the so-called central controller can be nonproper.

In our opinion this paper gives support to our claim that the approach to solve the  $H_\infty$  problem in the time-domain is a much more intuitive and appealing approach than the other methods used in recent papers.

**Appendix A. A preliminary system transformation.** In this section we shall choose bases in input, output, and state space that will give us much more insight into the structure of our problem. Although these decompositions are not necessary in the formulation of the main steps of the proof of Theorem 2.1, the details of the proof are very much concerned with these decompositions. It will be shown that the matrices defining our systems in these bases have a very particular structure. For details we refer to [18]. We shall display this structure by writing down the matrices with respect to these suitably chosen bases for the input, state, and output spaces.

Our basic tool is the strongly controllable subspace. This subspace has already been defined in Definition 4.2.

We shall give one property of the strongly controllable subspace at this point which will come in handy in the sequel (see [7], [16]).

LEMMA A1. Consider the system (4.4). The system is strongly controllable if and only if

$$(A1) \quad \begin{pmatrix} sI - A & -B \\ C_2 & D_2 \end{pmatrix}$$

has rank  $n + \text{rank}(C_2 \ D_2)$  for all  $s \in \mathbb{C}$ .

We can now define the bases for the system (2.1) which will be used in the sequel. It is also possible to define a dual version of this decomposition but we will only need this one. First choose a basis of the control input space  $\mathcal{R}^m$ . Decompose  $\mathcal{R}^m = \mathcal{U}_1 \oplus \mathcal{U}_2$  such that  $\mathcal{U}_2 = \ker D_2$  and  $\mathcal{U}_1$  arbitrary. Choose a basis  $u_1, u_2, \dots, u_m$  of  $\mathcal{R}^m$  such that  $u_1, u_2, \dots, u_i$  is a basis of  $\mathcal{U}_1$  and  $u_{i+1}, \dots, u_m$  is a basis of  $\mathcal{U}_2$ .

Next choose an orthonormal basis  $z_1, z_2, \dots, z_p$  of the output space  $\mathcal{R}^p$  such that  $z_1, \dots, z_j$  is a basis of  $\text{im } D_2$  and  $z_{j+1}, \dots, z_p$  is a basis of  $(\text{im } D_2)^\perp$ . Because this is an orthonormal basis this basis transformation does not change the norm  $\|z\|$ .

Finally, we choose a decomposition of the state space  $\mathcal{R}^n = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$  such that  $\mathcal{X}_2 = \mathcal{T}(\Sigma_{ci}) \cap C_2^{-1} \text{im } D_2$ ,  $\mathcal{X}_2 \oplus \mathcal{X}_3 = \mathcal{T}(\Sigma_{ci})$  and  $\mathcal{X}_1$  arbitrary. We choose a corresponding basis  $x_1, x_2, \dots, x_n$  such that  $x_1, \dots, x_r$  is a basis of  $\mathcal{X}_1$ ,  $x_{r+1}, \dots, x_s$  is a basis of  $\mathcal{X}_2$  and  $x_{s+1}, \dots, x_n$  is a basis of  $\mathcal{X}_3$ .

With respect to these bases the maps  $B, C_2$ , and  $D_2$  have the following form:

$$(A2) \quad B = (B_1 \ B_2), \quad C_2 = \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \end{pmatrix}, \quad D_2 = \begin{pmatrix} \hat{D}_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where  $\hat{D}_2$  is invertible. Next, we define a linear mapping  $F_0: \mathcal{R}^n \rightarrow \mathcal{R}^m$  by

$$(A3) \quad F_0 := \begin{pmatrix} -\hat{D}_2^{-1} \hat{C}_1 \\ 0 \end{pmatrix} \quad \text{and hence } C_2 + D_2 F_0 = \begin{pmatrix} 0 \\ \hat{C}_2 \end{pmatrix}.$$

We have the following properties of this decomposition which are proven in [18].

LEMMA A2. Let  $F_0$  be given by (A3). Then we have

- (i)  $(A + BF_0)(\mathcal{T}(\Sigma) \cap C_2^{-1} \text{im } D_2) \subseteq \mathcal{T}(\Sigma)$ ,
- (ii)  $\text{im } B_2 \subseteq \mathcal{T}(\Sigma)$ ,
- (iii)  $\mathcal{T}(\Sigma) \cap C_2^{-1} \text{im } D_2 \subseteq \ker \hat{C}_2$ .

By applying this lemma we find that the matrices  $A + BF_0, B, C_2 + D_2 F_0$  and  $D_2$  with respect to these bases have the following form:

$$(A4) \quad A + BF_0 = \begin{pmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{pmatrix},$$

$$C_2 + D_2 F_0 = \begin{pmatrix} 0 & 0 & 0 \\ C_{21} & 0 & C_{23} \end{pmatrix}, \quad D_2 = \begin{pmatrix} \hat{D}_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

We decompose the matrices  $\hat{C}_1$  and  $E$  correspondingly:

$$(A5) \quad \hat{C}_1 = (C_{11} \ C_{12} \ C_{13}), \quad E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$



These matrices turn out to have some nice structural properties, which have been shown in [18].

LEMMA A3. *We have the following properties:*

- (i)  $C_{23}$  is injective,
- (ii) the system

$$(A6) \quad \Sigma_1 := \left[ \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix}, (0 \quad I), 0 \right]$$

is strongly controllable,

- (iii) we have

$$(A7) \quad \text{normrank } G = \text{rank} \begin{pmatrix} C_{23} & 0 \\ 0 & \hat{D}_2 \end{pmatrix},$$

where  $G$  is the transfer matrix defined by  $G(s) := C_2(sI - A)^{-1}B + D_2$ .

We need the following results from [18] which connects the conditions of Theorem 2.1 to the matrices as defined in [A4].

LEMMA A4. *Assume  $P \in \mathcal{R}^{n \times n}$  is symmetric and  $F(P) \geq 0$ . Then we have the following:*

- (i)  $P\mathcal{F}(\Sigma) = 0$ , i.e., in our decomposition  $P$  can be written as

$$(A8) \quad P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- (ii) If  $P$  has the form (A8), then

$$(A9) \quad \begin{aligned} R(P_1) := & P_1 A_{11} + A_{11}^T P_1 + C_{21}^T C_{21} + P_1 (E_1 E_1^T - B_{11} (\hat{D}_2^T \hat{D}_2)^{-1} B_{11}^T) P_1 \\ & - (P_1 A_{13} + C_{21}^T C_{23}) (C_{23}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{21}) \geq 0. \end{aligned}$$

Moreover,  $R(P_1) = 0$  if and only if  $\text{rank } F(P) = \text{normrank } G$ .

- (iii) If  $R(P_1) = 0$ , then we have

$$\text{rank} \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + \text{normrank } G \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$$

if and only if

$$Z(P_1) := A_{11} + E_1 E_1^T P_1 - B_{11} (\hat{D}_2^T \hat{D}_2)^{-1} B_{11}^T P_1 - A_{13} (C_{23}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{21})$$

is an asymptotically stable matrix. Moreover, in that case also the matrix

$$A_{11} - B_{11} (\hat{D}_2^T \hat{D}_2)^{-1} B_{11}^T P_1 - A_{13} (C_{23}^T C_{23})^{-1} (A_{13}^T P_1 + C_{23}^T C_{21})$$

is an asymptotically stable matrix.

COROLLARY A5. *If there exists a matrix  $P \geq 0$  such that  $F(P) \geq 0$  and moreover:*

- (i)  $\text{rank } F(P) = \text{normrank } G$ ,

$$(ii) \text{rank} \begin{pmatrix} L(P, s) \\ F(P) \end{pmatrix} = n + \text{normrank } G \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+,$$

then this matrix is uniquely defined by the above inequality and the corresponding two rank conditions.

*Proof.* By Lemma A4 a solution  $P$  must be of the form (A8) where  $P_1$  is a solution of the algebraic Riccati equation  $R(P_1) = 0$  such that  $Z(P_1)$  is asymptotically stable. Denote the Hamiltonian matrix corresponding to this algebraic Riccati equation by

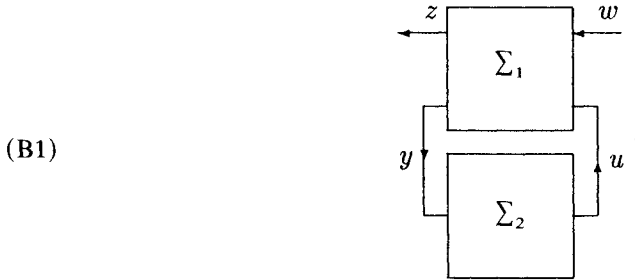
$H$ ; then we have

$$(A10) \quad H \begin{pmatrix} I \\ P_1 \end{pmatrix} = \begin{pmatrix} I \\ P_1 \end{pmatrix} Z(P_1).$$

Since a Hamiltonian matrix has the property that  $\lambda$  is an eigenvalue if and only if  $-\lambda$  is an eigenvalue of  $H$ , we know that an  $n$ -dimensional invariant subspace  $\mathcal{W}$  of  $H$  such that  $H|_{\mathcal{W}}$  is asymptotically stable must be unique. This implies that  $P_1$  is unique and hence also  $P$  is unique.  $\square$

**Appendix B. Proofs concerning the system transformations.** In order to prove Lemma 3.2 we must first do some preparatory work. We first recall the following lemma from [2] which we shall use in the sequel.

LEMMA B1. *Suppose we have the following interconnection of two systems  $\Sigma_1$  and  $\Sigma_2$ , both described by some state-space representation:*



Assume  $\Sigma_1$  is internally stable and its transfer matrix  $L$  from  $\begin{pmatrix} w \\ u \end{pmatrix}$  to  $\begin{pmatrix} z \\ y \end{pmatrix}$  satisfies  $L^- L = I$  where  $L^-(s) := L^T(-s)$ . Moreover, assume that if we decompose  $L$ :

$$(B2) \quad L =: \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

compatible with the sizes of  $w, u, z,$  and  $y$ , we have  $L_{21}^{-1} \in H_\infty$  and  $\lim_{s \rightarrow \infty} L_{22}(s) = 0$ . Then the following two statements are equivalent:

- (i) The closed-loop system (B1) is internally stable and its closed-loop transfer matrix has  $H_\infty$  norm less than one.
- (ii) The system  $\Sigma_2$  is internally stable and its transfer matrix has  $H_\infty$  norm less than one.

*Proof.* This is a well-known result although written down here in a different way. Note that if the closed-loop system (B1) is internally stable, then  $\Sigma_2$  is stabilizable and detectable. This can be shown either by writing down the closed-loop differential equation or by noting that an unstable uncontrollable mode in  $\Sigma_2$  cannot be controlled by  $y$  and hence is still unstable and uncontrollable in the closed-loop system and the same for an unstable unobservable mode. The result in this form can then be obtained by using the work in [14].  $\square$

We shall now assume that we have chosen the bases described in Appendix A. Let  $P$  satisfy the conditions of Lemma 3.1(i). Hence we know  $P$  has the form (A8). It is easily shown that it is sufficient to prove the lemma for one specific choice of  $C_{2,P}$  and  $D_p$ . We define the following matrices:

$$(B3) \quad C_{2,P} := \begin{pmatrix} \hat{D}_2(\hat{D}_2^T \hat{D}_2)^{-1} B_{11}^T P_1 + C_{11} & C_{12} & C_{13} \\ C_{23}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{21}) & 0 & C_{23} \end{pmatrix},$$

$$(B4) \quad D_P := \begin{pmatrix} \hat{D}_2 & 0 \\ 0 & 0 \end{pmatrix} (=D_2).$$

By writing down  $F(P)$  in terms of the chosen bases and by using the fact that  $P_1$  satisfies the algebraic Riccati equation  $R(P_1)=0$  where  $R(P_1)$  is defined by (A9), it can be checked after some effort that these matrices indeed satisfy (3.1). We define the following matrices:

$$(B5) \quad \tilde{A} := A_{11} - A_{13}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{21}) - B_{11}(\hat{D}_2^T \hat{D}_2)^{-1} B_{11}^T P_1,$$

$$(B6) \quad \tilde{C}_1 := -(\hat{D}_2^T)^{-1} B_{11}^T P_1,$$

$$(B7) \quad \tilde{C}_2 := C_{21} - C_{23}(C_{23}^T C_{23})^{-1}(A_{13}^T P_1 + C_{23}^T C_{21}),$$

$$(B8) \quad \tilde{B}_{11} := B_{11} \hat{D}_2^{-1},$$

$$(B9) \quad \tilde{B}_{12} := A_{13}(C_{23}^T C_{23})^{-1} C_{23}^T - P_1^\dagger C_{21}^T (I - C_{23}(C_{23}^T C_{23})^{-1} C_{23}^T),$$

where  $\dagger$  denotes the Moore-Penrose inverse. We now define the following system:

$$(B10) \quad \Sigma_U : \begin{cases} \dot{x}_U = \tilde{A}x_U + (\tilde{B}_{11} \ \tilde{B}_{12})u + E_1 w, \\ y_U = -E_1^T P_1 x_U + w, \\ z_U = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} x_U + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} u. \end{cases}$$

We have the following properties of the system  $\Sigma_U$ .

LEMMA B2. *The system  $\Sigma_U$  is internally stable. Let  $U$  denote the transfer matrix of  $\Sigma_U$  from  $\begin{pmatrix} u \\ w \end{pmatrix}$  to  $\begin{pmatrix} y_U \\ z_U \end{pmatrix}$ . We have  $U^{-1}U = I$  where  $U^{-1}(s) := U^T(-s)$ . If we decompose  $U$ :*

$$(B11) \quad U := \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

compatible with the sizes of  $u$ ,  $w$ ,  $y_U$ , and  $z_U$  then we have  $U_{21}^{-1} \in H_\infty$  and  $\lim_{s \rightarrow \infty} U_{22}(s) = 0$ .

*Proof.* The fact that  $\Sigma_U$  is internally stable and that  $U_{21}^{-1} \in H_\infty$  follows directly from the fact that  $\tilde{A}$  and  $\tilde{A} + E_1 E_1^T P_1$  are asymptotically stable by Lemma A4(iii). The fact that  $\lim_{s \rightarrow \infty} U_{22}(s) = 0$  can be checked trivially. It can be easily checked using Lemma A4(ii) that  $P_1$  is the controllability gramian of  $\Sigma_U$ . Moreover, we have

$$(B12) \quad \begin{pmatrix} \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{pmatrix} \begin{pmatrix} -E_1^T P_1 \\ \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \tilde{B}_{11}^T \\ \tilde{B}_{12}^T \\ E_1^T \end{pmatrix} P_1 = 0.$$

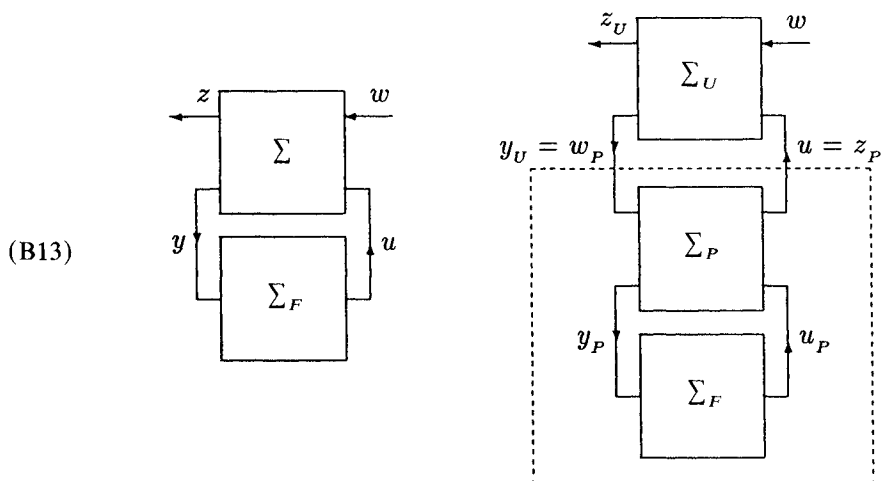
This can be checked by simply writing out and using the fact that

$$\ker P_1 \subset \ker (I - C_{23}(C_{23}^T C_{23})^{-1} C_{23}^T) C_{21}.$$

The result that  $U \sim U = I$  then follows by applying Theorem 5.1 of [5].  $\square$

*Proof of Lemma 3.2.* We have our special choice of  $C_{2,P}$  and  $D_P$  given by (B3) and (B4). As we have already noted, taking this special choice for  $C_P$  and  $D_P$  is not

essential. We shall first compare the following two systems:



The system on the left is the same as the system on the left in (3.3), and the system on the right is described by the system (B10) interconnected with the system on the right in (3.3). We decompose the state of  $\Sigma$ ,  $x$  into  $x_1, x_2$ , and  $x_3$  according to the choice of bases described in Appendix A and decompose the state of  $\Sigma_P$  into  $x_{1,P}, x_{2,P}, x_{3,P}$  of corresponding sizes. (Note that  $\Sigma$  and  $\Sigma_P$  have the same state space  $\mathcal{R}^n$ .) Writing out all the differential equations using the decompositions of the matrices given in (A3)–(A5) we find

$$\begin{cases} \begin{pmatrix} \dot{x}_U - \dot{x}_{1,P} \\ \dot{x}_P \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \tilde{A} + E_1 E_1^T P_1 & 0 & 0 \\ * & A + BNC_1 & BM \\ * & LC_1 & K \end{pmatrix} \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ E + BND_1 \\ LD_1 \end{pmatrix} w, \\ z = (* \quad C_2 + D_2 NC_1 \quad D_2 M) \begin{pmatrix} x_U - x_{1,P} \\ x_P \\ p \end{pmatrix} + D_2 ND_1 \omega. \end{cases}$$

The  $*$  denotes matrices which are unimportant for this argument. The system on the right is internally stable if and only if the system described by the above set of equations is internally stable. If we also derive the system equations for the system on the left in (B13) we immediately see that, since  $\tilde{A} + E_1 E_1^T P_1$  is asymptotically stable, the system on the left is internally stable if and only if the system on the right is internally stable. Moreover, if we take zero initial conditions and both systems have the same input  $w$ , then we have  $z = z_U$ , i.e., the input–output behaviour of both systems are equivalent. Hence the system on the left has  $H_\infty$  norm less than one if and only if the system on the right has  $H_\infty$  norm less than one.

By Lemma B2 we may apply Lemma B1 to the system on the right in (B13) and hence we find that the closed-loop system is internally stable and has  $H_\infty$  norm less than one if and only if the dashed system is internally stable and has  $H_\infty$  norm less than one.

Since the dashed system is exactly the system on the right in (3.3) and the system on the left in (B13) is exactly equal to the system on the left in (3.3), we have completed the proof.  $\square$

We will now prove Lemma 3.3. In fact, we will prove the dual version of this lemma since this is much more convenient to us. We first factorize  $G(Q)$ :

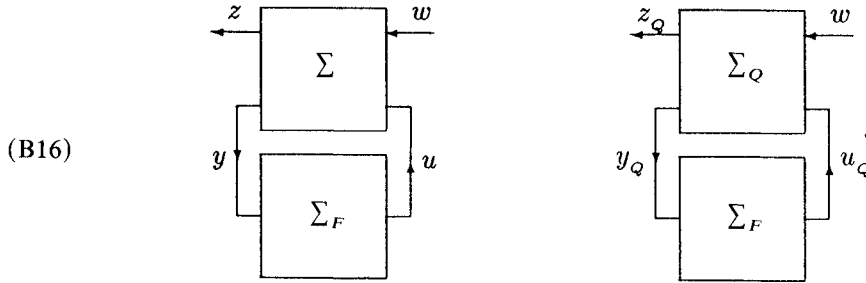
$$(B14) \quad G(Q) := \begin{pmatrix} E_Q \\ D_Q \end{pmatrix} (E_Q^T \quad D_Q^T).$$

Define  $A_Q := A + QC_2^T C_2$  and  $B_Q := B + QC_2^T D_2$  and the system:

$$(B15) \quad \Sigma_Q : \begin{cases} \dot{x}_Q = A_Q x_Q + B_Q u_Q + E_Q w, \\ y_Q = C_1 x_Q \quad \quad \quad + D_Q w \\ z_Q = C_2 x_Q + D_2 u_Q. \end{cases}$$

By using the well-known facts that  $F$  stabilizes  $\Sigma$  if and only if  $F^T$  stabilizes  $\Sigma^T$  and  $\|G\|_\infty = \|G^T\|_\infty$ , we can derive the following dualized version of Lemma 3.2 for this dual system as follows.

LEMMA B3. *Let  $Q$  satisfy Lemma 3.1(ii). Moreover, let an arbitrary linear time-invariant finite-dimensional compensator  $F$  be given, described by (2.2). Let the following two systems be given where the system on the left is the interconnection of (2.1) and (2.2) and the system on the right is the interconnection of (B15) and (2.2).*



Then the following statements are equivalent:

- (i) The system on the left is internally stable and its transfer matrix has  $H_\infty$  norm less than one.
- (ii) The system on the right is internally stable and its transfer matrix has  $H_\infty$  norm less than one.

We now investigate how the matrices appearing in the matrix inequality and the rank conditions look like for this new system  $\Sigma_Q$ :

$$(B17) \quad \tilde{F}(X) := \begin{pmatrix} A_Q^T X + X A_Q + C_2^T C_2 + X E_Q E_Q^T X & X B_Q + C_2^T D_2 \\ B_Q^T X + D_2^T C_2 & D_2^T D_2 \end{pmatrix},$$

$$(B18) \quad \tilde{G}(Y) := \begin{pmatrix} A_Q Y + Y A_Q^T + E_Q E_Q^T + Y C_2^T C_2 Y & Y C_1^T + E_Q D_Q^T \\ C_1 Y + D_Q E_Q^T & D_Q D_Q^T \end{pmatrix},$$

$$(B19) \quad \tilde{L}(X, s) := (sI - A_Q - E_Q E_Q^T X \quad -B_Q),$$

$$(B20) \quad \tilde{M}(Y, s) := \begin{pmatrix} sI - A_Q - Y C_2^T C_2 \\ -C_1 \end{pmatrix}.$$

Moreover, we define two new transfer matrices:

$$(B21) \quad \tilde{G}(s) := C_2 (sI - A_Q)^{-1} B_Q + D_2,$$

$$(B22) \quad \tilde{H}(s) := C_1 (sI - A_Q)^{-1} E_Q + D_Q.$$

Using these definitions we have the following result.

LEMMA B4. Let  $Q$  satisfy Lemma 3.1(ii). Then  $Y=0$  is the unique solution of the quadratic matrix inequality  $G(Y) \geq 0$  satisfying the following rank conditions:

- (1)  $\text{rank } \tilde{G}(Y) = \text{normrank } \tilde{H}$ ,
- (2)  $\text{rank } (\tilde{M}(Y, s) \quad \tilde{G}(Y)) = n + \text{normrank } \tilde{H} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$ .

*Proof.* It is trivial to check that  $\tilde{G}(0) \geq 0$ . Moreover, since  $\tilde{G}(0) = G(Q)$  and  $\tilde{M}(0, s) = M(Q, s)$  it remains to show that  $\text{normrank } \tilde{H} = \text{normrank } H$ . We have

$$\begin{aligned} \text{normrank } \tilde{H} &= \text{normrank} \begin{pmatrix} sI - A_Q & E_Q \\ -C_1 & D_Q \end{pmatrix} - n \\ &= \text{normrank} \begin{pmatrix} sI - A_Q & E_Q E_Q^T & D_Q E_Q^T \\ -C_1 & D_Q E_Q^T & D_Q D_Q^T \end{pmatrix} - n \\ &= \text{normrank} (M(Q, s) \quad G(Q)) - n \\ &= \text{normrank } H. \end{aligned}$$

$Y$  is unique by Corollary A5. This is exactly what we had to prove.  $\square$

LEMMA B5. There exists a solution  $X$  of the matrix inequality  $\tilde{F}(X) \geq 0$  satisfying the following two rank conditions:

- (1)  $\text{rank } \tilde{F}(X) = \text{normrank } \tilde{G}$ ,
- (2)  $\text{rank} \begin{pmatrix} r\tilde{L}(X, s) \\ \tilde{F}(X) \end{pmatrix} = n + \text{normrank } \tilde{G} \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$ ,

if and only if  $I - PQ$  is invertible. Moreover, in that case the solution is unique and is given by  $X = (I - PQ)^{-1}P$ . We have  $X \geq 0$  if and only if

$$(B23) \quad \rho(PQ) < 1.$$

*Proof.* We first make a transformation on  $\tilde{F}(X)$ :

$$(B24) \quad F_r(X) := \begin{pmatrix} I & (I + XQ)F_0^T \\ 0 & I \end{pmatrix} \tilde{F}(X) \begin{pmatrix} I & 0 \\ F_0(I + QX) & I \end{pmatrix}$$

$$(B25) \quad = \begin{pmatrix} \bar{A}^T X + X\bar{A} + \bar{C}_2^T \bar{C}_2 + XMX & XB \\ B^T X & D_2^T D_2 \end{pmatrix},$$

where

$$(B26) \quad \bar{A} := A + BF_0 + Q(C_2 + D_2 F_0)^T (C_2 + D_2 F_0),$$

$$(B27) \quad \bar{C}_2 := C_2 + D_2 F_0,$$

$$(B28) \quad M := (A + BF_0)Q + Q(A^T + F_0^T B^T) + EE^T + Q\bar{C}_2^T \bar{C}_2 Q,$$

and  $F_0$  as defined in (A3). We also transform the second matrix appearing in the rank conditions:

$$\begin{aligned} W(X, s) &:= \begin{pmatrix} I & 0 & -QF_0^T \\ 0 & I & (I + XQ)F_0^T \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \tilde{L}(X, s) \\ \tilde{F}(X) \end{pmatrix} \begin{pmatrix} I & 0 \\ F_0(I + QX) & I \end{pmatrix} \\ &= \begin{pmatrix} sI - \bar{A} - MX & -B \\ \bar{A}^T X + X\bar{A} + \bar{C}_2^T \bar{C}_2 + XMX & XB \\ B^T X & D_2^T D_2 \end{pmatrix}. \end{aligned}$$

We have the following equality:

$$(B29) \quad \text{normrank } \tilde{G} = \text{normrank} \begin{pmatrix} sI - A_Q & -B_Q \\ C_2 & D_2 \end{pmatrix} - n$$

$$(B30) \quad = \text{normrank} \begin{pmatrix} I & QC_2^T \\ 0 & I \end{pmatrix} \begin{pmatrix} sI - A_Q & -B_Q \\ C_2 & D_2 \end{pmatrix} - n$$

$$(B31) \quad = \text{normrank} \begin{pmatrix} sI - A & -B \\ C_2 & D_2 \end{pmatrix} - n = \text{normrank } G.$$

Therefore the conditions that  $X \geq 0$  has to satisfy can be reformulated as:

- (i)  $F_r(X) \geq 0$ ,
- (ii)  $\text{rank } F_r(X) = \text{normrank } G$ ,
- (iii)  $\text{rank } W(X, s) = \text{normrank } G + n \quad \forall s \in \mathcal{C}^0 \cup \mathcal{C}^+$ .

Moreover, we note that  $T(A, B, C_2, D_2) = T(\bar{A}, B, \bar{C}_2, D_2)$ . This can be shown by using the fact that the new system is obtained by a state feedback and an output injection (note that  $B = B + Q(C_2 + D_2 F_0)^T D_2$ ) and it is well known that the strongly controllable subspace is invariant under feedback and output injection. This can easily be shown using the algorithm (4.10). We now choose the bases from Appendix A. By Lemma A4(i) we know that if  $X$  exists then it will have the form

$$(B32) \quad X = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some positive semidefinite matrix  $X_1$ . Note that there is small difference since  $M$  is not necessarily positive semidefinite, but it can be easily seen from the proof in [18] that this difference is not important. We use this decomposition for  $X$  and the corresponding decompositions for  $P$  and  $Q$ :

$$(B33) \quad P = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}.$$

Together with the decompositions for the other matrices as given in (A4)-(A5) we can decompose  $F_r(X)$  correspondingly:

$$\begin{pmatrix} X_1 \tilde{A}_{11} + \tilde{A}_{11}^T X_1 + C_{21}^T C_{21} + X_1 M_{11} X_1 & 0 & X_1 \tilde{A}_{13} + C_{21}^T C_{23} & X_1 B_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \tilde{A}_{13}^T X_1 + C_{23}^T C_{21} & 0 & C_{23}^T C_{23} & 0 & 0 \\ B_{11}^T X_1 & 0 & 0 & \hat{D}_2^T \hat{D}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$(B34) \quad \tilde{A}_{11} := A_{11} + Q_{11} C_{21}^T C_{21} + Q_{13} C_{23}^T C_{21},$$

$$(B35) \quad \tilde{A}_{13} := A_{13} + Q_{11} C_{21}^T C_{23} + Q_{13} C_{23}^T C_{23},$$

$$(B36) \quad M_{11} := A_{11} Q_{11} + A_{13} Q_{13}^T + Q_{11} A_{11}^T + Q_{13} A_{13}^T + E_1 E_1^T \\ + Q_{11} C_{21}^T (C_{21} Q_{11} + C_{23} Q_{13}) + Q_{13} C_{23}^T (C_{21} Q_{11} + C_{23} Q_{13}).$$

The rank condition  $\text{rank } F_r(X) = \text{normrank } G$  is, according to Lemma A3(iii), equivalent with the condition that the rank of the above matrix is equal to the rank of the submatrix

$$(B37) \quad \begin{pmatrix} C_{23}^T C_{23} & 0 \\ 0 & \hat{D}_2^T \hat{D}_2 \end{pmatrix}.$$

Therefore the Schur complement with respect to this submatrix should be zero. This implies that if we define

$$\begin{aligned} \tilde{R}(X_1) := & X_1 \tilde{A}_{11} + \tilde{A}_{11}^T X_1 + C_{21}^T C_{21} + X_1 (M_{11} - B_{11} (\hat{D}_2^T \hat{D}_2)^{-1} B_{11}^T) X_1 \\ & - (X_1 \tilde{A}_{13} + C_{21}^T C_{23}) (C_{23}^T C_{23})^{-1} (\tilde{A}_{13}^T X_1 + C_{23}^T C_{21}), \end{aligned}$$

then  $X_1$  should satisfy  $\tilde{R}(X_1) = 0$ . Moreover, if we decompose  $W(X, s)$  correspondingly, then we can show by using elementary row and column operations that for any matrix  $X$  in the form (B32), where  $X_1$  satisfies  $\tilde{R}(X_1) = 0$ , that for all  $s \in \mathcal{C}$ ,  $W(X, s)$  has the same rank as the following matrix:

$$(B38) \quad \begin{pmatrix} sI - \tilde{Z}(X_1) & 0 & 0 & 0 & 0 \\ * & sI - A_{22} & -A_{23} & 0 & -B_{22} \\ * & -A_{32} & sI - A_{33} & 0 & -B_{32} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$(B39) \quad \tilde{Z}(X_1) := \tilde{A}_{11} + M_{11} X_1 - B_{11} (\hat{D}_2^T \hat{D}_2)^{-1} B_{11}^T X_1 - \tilde{A}_{13} (C_{23}^T C_{23})^{-1} (\tilde{A}_{13}^T X_1 + C_{23}^T C_{21}).$$

The matrix

$$(B40) \quad \begin{pmatrix} sI - A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI - A_{33} & -B_{32} \\ 0 & I & 0 \end{pmatrix}$$

has full row rank for all  $s \in \mathcal{C}$  by Lemma A3(ii) and Lemma A1. Hence the rank of the matrix (B38) is  $n + \text{normrank } G$  for all  $s \in \mathcal{C}^+ \cup \mathcal{C}^0$  if and only if the matrix  $\tilde{Z}(X_1)$  is asymptotically stable. Using this we can now reformulate the conditions that  $X_1 \geq 0$  must satisfy:

- (i)  $\tilde{R}(X_1) = 0$ ,
- (ii)  $\tilde{Z}(X_1)$  is asymptotically stable.

That is,  $X_1$  should be the positive semidefinite stabilizing solution of the algebraic Riccati equation  $\tilde{R}(X_1) = 0$ . Denote the Hamiltonian corresponding to this ARE by  $H_{\text{new}}$ . We know that  $P_1$  is the stabilizing solution of the algebraic Riccati equation  $R(P_1) = 0$  as given by (A9). Denote the Hamiltonian corresponding to this algebraic Riccati equation by  $H_{\text{old}}$ . Then it can be checked that

$$(B41) \quad H_{\text{old}} = \begin{pmatrix} I & Q_{11} \\ 0 & I \end{pmatrix} H_{\text{new}} \begin{pmatrix} I & -Q_{11} \\ 0 & I \end{pmatrix}.$$



Since  $P_1$  is the stabilizing solution of the Riccati equation corresponding to the Hamiltonian  $H_{\text{old}}$  we know that the modal subspace of  $H_{\text{old}}$  corresponding to the open left halfplane is given by

$$(B42) \quad \mathcal{X}_g(H_{\text{old}}) = \text{Im} \begin{pmatrix} I \\ P_1 \end{pmatrix}.$$

Combining (B41) and (B42), we find

$$(B43) \quad \mathcal{X}_g(H_{\text{new}}) = \text{Im} \begin{pmatrix} I & -Q_{11} \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ P_1 \end{pmatrix} = \text{Im} \begin{pmatrix} I - Q_{11}P_1 \\ P_1 \end{pmatrix}.$$

Therefore we know that there exists a stabilizing solution to the algebraic Riccati equation  $\hat{R}(X_1) = 0$  if and only if  $I - Q_{11}P_1$  is invertible and in that case the solution is given by  $X_1 = P_1(I - Q_{11}P_1)^{-1}$ . This implies that  $X = P(I - QP)^{-1} = (I - PQ)^{-1}P$ . The requirement  $X \geq 0$  is satisfied if and only if  $\rho(PQ) < 1$ , which can be checked straightforwardly. This completes the proof.  $\square$

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