The singular linear quadratic Gaussian control problem

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by

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Abstract

In this paper we discuss the standard LQG control problem for linear, finite-dimensional time-invariant systems without any assumptions on the system parameters. We give an explicit formula for the infimum over all internally stabilizing strictly proper compensators and give a characterization when the infimum is attained.

Keywords: LQG control, algebraic Riccati equation, linear matrix inequality.

1 Introduction

The linear quadratic Gaussian (LQG) control problem was one of the main research areas of the 1970's (see e.g. [4, 6, 8, 19] and the references contained therein). Recently, the LQG theory has been investigated in the form of the so-called mixed LQG/H_∞ control problems (see e.g. [1, 2, 9, 10]). However, in this extensive literature the most general LQG control problem without any assumptions on the system besides internal stabilizability has never been treated.

In this paper we want to discuss the LQG control problem with the requirement of internal stability and in continuous time without any assumptions on the system except that the system is time-invariant, linear, finite-dimensional and stabilizable. We make no assumptions on the direct feedthrough matrices (this is often referred to as singular problems) nor any assumptions on invariant zeros. Although both singular Kalman filtering (see e.g. [13]) and the singular deterministic linear quadratic control problem (see e.g. [5, 12, 20]) have been investigated in the literature, as far as we know, it was still an open problem how to apply the separation principle to combine these two to the stochastic linear quadratic problem with partial information.

For singular problems it is well known that with respect to attaining the infimum there are three possibilities:

- the infimum is attained by a strictly proper compensator

- the infimum is only attained by a compensator which is not strictly proper

- the infimum can not be attained.

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It can be shown that for any given system there exists a sequence of strictly proper compensators which is a minimizing sequence for the LQG problem. We will characterize when the infimum is attained by a strictly proper compensator. To characterize when the infimum is attained by a compensator which is not strictly proper turned out to be very hard. We will only give partial results in this paper and we will sketch, via some examples, the difficulties we encounter when trying to derive such a characterization.

The techniques of this paper are related to the derivation of the results for the singular H_∞ control problem in [16, 15].

In section 2 we formulate the problem and present our main result. In section 3 we derive an underbound for the infimum of the LQG cost-criterion. Then, in section 4, we show that this underbound is equal to the infimum by constructing a minimizing sequence. Moreover, we discuss when the infimum can be attained. We conclude with some remarks in section 5.

2   Problem formulation and main results

We consider the linear, time-invariant, finite-dimensional system:

$$
\Sigma : \begin{cases}
\dot{x} = Ax & + Ev & + Bu, \\
z = C_1 x + & & D_1 u, \\
y = C_2 x + & D_2 v,
\end{cases}
\tag{2.1}
$$

where for each t, $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}^m$ is the control input, $v(t) \in \mathcal{R}^l$ is Gaussian white noise, $y(t) \in \mathcal{R}^p$ is the measured output and $z(t) \in \mathcal{R}^q$ is the unknown output to be controlled. A, B, E, C_1, C_2, D_1 and D_2 are matrices of appropriate dimensions. We assume that (A, B) is stabilizable and (C_2, A) is detectable which are necessary and sufficient conditions to guarantee existence of a stabilizing controller. We investigate strictly proper controllers of the form:

$$
\Sigma_F : \begin{cases}
\dot{p} = Kp & + Ly, \\
u = Mp.
\end{cases}
\tag{2.2}
$$

Note that Gaussian white noise is mathematically not well-defined. Therefore we have to discuss how we define the interconnection of (2.1) and (2.5) in a mathematically correct way. The interconnection is described by the following equations:

$$
\Sigma_{cl} : \begin{cases}
\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} A & BM \\ LC_2 & K \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} + \begin{pmatrix} E \\ LD_2 \end{pmatrix} v, \\
z = \begin{pmatrix} C_1 & D_1 M \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.
\end{cases}
\tag{2.3}
$$

Denote the closed loop parameters by A_e, B_e and C_e. The solution of the differential equation in (2.3) can be defined formally via the theory of stochastic integrals (see [3]). Let w be a standard Wiener process (we view v as the derivative of w). We define the solution of the differential equation in (2.3) by:

$$
\begin{pmatrix} x \\ p \end{pmatrix} (t) = e^{A_e t} \begin{pmatrix} x \\ p \end{pmatrix} (0) + \int_0^t e^{A_e(t-\tau)} B_e \, dw(\tau)
$$

2

This is the variation of constants formula with $vd\tau$ replaced by $dw(\tau)$. The above integral is defined to be a so-called Wiener integral. In this way we can define the closed loop system in a mathematically sound way. The fact that we model noise in our system via a white noise process v which we can not even define properly shows that this method is rather arbitrary. However, it is still one of the best ways to model noise in a system.

Because the compensator is strictly proper we find that the input $u = Mp$ is a well-defined stochastic process. This is the reason for only considering strictly proper compensators.

Note that this closed loop system has the same input/output behaviour as the interconnection of the following two systems:

$$\tilde{\Sigma} : \begin{cases} \dot{x} = Ax + Ev + Bu, \\ z = C_1 x + D_1 u, \\ \dot{\tilde{y}} = C_2 x + D_2 v, \end{cases} \tag{2.4}$$

and

$$\tilde{\Sigma}_F : \begin{cases} \dot{p} = Kp + KLy, \\ u = Mp + MLy. \end{cases} \tag{2.5}$$

In case u is a well-defined stochastic process we can define the solutions of the differential equations in (2.4) by using the variations of constants formula and replacing $vd\tau$ by $dw(\tau)$. We have added an extra integration step in the measurement equation which yields a well-defined measurement $\tilde{y}$ (this was impossible in (2.1)). We compensate for this extra integration step by adding a differentiation in the compensator (2.5). Again, for a well-defined stochastic process $\tilde{y}$, we can define the solution of the differential equation in (2.5) via the variation of constants formula. The advantage of working with the interconnection $\tilde{\Sigma} \times \tilde{\Sigma}_F$ instead of $\Sigma \times \Sigma_F$ is that in the first interconnection all stochastic processes involved $(x, u, \tilde{y}, p)$ are well-defined while in the second interconnection we can not define the measurement y in a mathematically sound way. We will make no explicit use of (2.4) but this system is used in many papers on LQG. The above shows the relation between (2.1) and (2.4).

The closed loop system is called internally stable if the matrix

$$A_e = \begin{pmatrix} A & BM \\ LC_2 & K \end{pmatrix}$$

is asymptotically stable. This implies that the interconnection $\Sigma \times \Sigma_F$ is internally stable, i.e. for arbitrary initial conditions for x and p, in the closed loop system $x(t)$ and $p(t)$ converge to zero as $t \to \infty$. The interconnection $\tilde{\Sigma} \times \tilde{\Sigma}_F$ will, in general, not be internally stable (note that for this system we have initial conditions for x, p and y).

Our goal is to find an internally stabilizing compensator such that the following criterion function

$$\mathcal{J}(\Sigma \times \Sigma_F) := \lim_{s \to \infty} \mathcal{E} \left\{ \frac{1}{s} \int_0^s z(t)^{\mathsf{T}} z(t)\, dt \right\} \tag{2.6}$$

is minimized over all internally stabilizing compensators Σ_F. Here $\mathcal{E}$ denotes the expectation. z is the stochastic process defined by (2.3). Note that because of our requirement of internal stability it can be shown that the cost-criterion (2.6) is independent of the initial conditions $x(0)$ and $p(0)$ of the compensator and the system. Naturally we cannot minimize

the expectation of the $\mathcal{L}_2(0, \infty)$-norm of z. This will namely in general be infinity because the Wiener process yields a source of constant excitation. Alternative formulations for (2.6) like an exponentially weighted $\mathcal{L}_2$ norm of z have been discussed in literature (see e.g. [7]) but will not be treated in this paper. Note that

$$\mathcal{J}(\tilde{\Sigma} \times \tilde{\Sigma}_F) = \mathcal{J}(\Sigma \times \Sigma_F)$$

where $\Sigma, \Sigma_F, \tilde{\Sigma}$ and $\tilde{\Sigma}_F$ are defined by (2.1),(2.2),(2.4) and (2.5) respectively.

The LQG control problem has a strong correlation with the H_2 control problem. In the H_2 control problem we are searching for an internally stabilizing compensator which minimizes the following criterion function

$$\mathcal{J}_{H_2}(\Sigma \times \Sigma_F) := \operatorname{Trace} \frac{1}{2\pi} \int_0^\infty G_{cl}^T(-i\omega) G_{cl}(i\omega) \, d\omega, \tag{2.7}$$

where G_{cl} denotes the strictly proper, stable closed loop transfer matrix from v to z.
For any internally stabilizing compensator Σ_F we have:

$$\mathcal{J}(\Sigma \times \Sigma_F) = \mathcal{J}_{H_2}(\Sigma \times \Sigma_F).$$

Therefore the LQG control problem and the H_2 control problem, although formulated in a completely different setting, yield the same compensators. Hence by solving the LQG control problem in this paper we immediately find the results for the H_2 control problem.
We first need some definitions:

Definition 2.1 : *Consider the system*

$$\Sigma_{ci} \; : \; \left\{ \begin{array}{l} \dot{x} = Ax + Bu, \\ z = Cx + Du. \end{array} \right. \tag{2.8}$$

We define $\mathcal{T}(\Sigma_{ci})$ as the smallest subspace $\mathcal{T}$ of $\mathcal{R}^n$ for which there exists a linear mapping G such that:

$$(A + GC)\mathcal{T} \quad \subseteq \quad \mathcal{T}, \tag{2.9}$$

$$\operatorname{Im}(B + GD) \quad \subseteq \quad \mathcal{T}. \tag{2.10}$$

We also define $\mathcal{T}_g(\Sigma_{ci})$ as the smallest subspace $\mathcal{T}$ of $\mathcal{R}^n$ for which there exists a linear mapping G such that (2.9) and (2.10) are satisfied and moreover $A + GC \mid \mathcal{R}^n/\mathcal{T}$ is asymptotically stable. □

We also define the dual version of these subspaces:

Definition 2.2 : *Consider the system (2.8). We define $\mathcal{V}(\Sigma_{ci})$ as the largest subspace $\mathcal{V}$ of $\mathcal{R}^n$ for which there exists a mapping F such that:*

$$(A + BF)\mathcal{V} \quad \subseteq \quad \mathcal{V}, \tag{2.11}$$

$$(C + DF)\mathcal{V} \quad = \quad \{0\}. \tag{2.12}$$

We also define $\mathcal{V}_g(\Sigma_{ci})$ as the largest subspace $\mathcal{V}$ for which there exists a mapping F such that (2.11) and (2.12) are satisfied and moreover $A + BF \mid \mathcal{V}$ is asymptotically stable. □

To calculate these subspaces algorithms are available in literature (see e.g. [14]).

A central role in our study of the LQG control problem will be played by the *linear matrix inequality*. For any matrix $P \in \mathcal{R}^{n \times n}$ we define the following matrix:

$$F(P) := \begin{pmatrix} A^{\mathrm{T}}P + PA + C_1^{\mathrm{T}}C_1 & PB + C_1^{\mathrm{T}}D_1 \\ B^{\mathrm{T}}P + D_1^{\mathrm{T}}C_1 & D_1^{\mathrm{T}}D_1 \end{pmatrix}. \tag{2.13}$$

If $F(P) \geq 0$, we say that P is a solution of the linear matrix inequality. We also define a dual version of this linear matrix inequality. For any matrix $Q \in \mathcal{R}^{n \times n}$ we define the following matrix:

$$G(Q) := \begin{pmatrix} AQ + QA^{\mathrm{T}} + EE^{\mathrm{T}} & QC_2^{\mathrm{T}} + ED_2^{\mathrm{T}} \\ C_2Q + D_2E^{\mathrm{T}} & D_2D_2^{\mathrm{T}} \end{pmatrix}. \tag{2.14}$$

If $G(Q) \geq 0$, we say that Q is a solution of the dual linear matrix inequality. In addition to these two matrices we define two polynomial matrices, whose role is again completely dual.

$$L(s) := \begin{pmatrix} sI - A & -B \end{pmatrix} \tag{2.15}$$

$$M(s) := \begin{pmatrix} sI - A \\ -C_2 \end{pmatrix} \tag{2.16}$$

We note that $L(s)$ is the controllability pencil associated with the system:

$$\dot{x} = Ax + Bu, \tag{2.17}$$

while $M(s)$ is the observability pencil associated with the system:

$$\begin{cases} \dot{x} = Ax, \\ y = -C_2x. \end{cases} \tag{2.18}$$

Finally, we define the following two transfer matrices:

$$G(s) := C_1(sI - A)^{-1}B + D_1, \tag{2.19}$$

$$H(s) := C_2(sI - A)^{-1}E + D_2. \tag{2.20}$$

We also require the concept of *invariant zero* of the system $\Sigma = (A, B, C, D)$. These are all $s \in \mathcal{C}$ such that

$$\mathrm{rank} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} < \mathrm{normrank} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}. \tag{2.21}$$

Here *normrank* denotes the rank of a matrix as a matrix with entries in the field of rational functions. Finally let $\mathcal{C}^+$ ($\mathcal{C}^0, \mathcal{C}^-$) denote all $s \in \mathcal{C}$ such that $\mathrm{Re}\ s > 0$ ($\mathrm{Re}\ s = 0$, $\mathrm{Re}\ s < 0$). Next, we give a key lemma which was already essentially known in literature:

Lemma 2.3 : *Consider the system (2.1). Assume that (A, B) is stabilizable and (C_2, A) is detectable. Under the above assumptions there exist matrices P and Q satisfying:*

5

(i) $F(P) \geq 0$,

(ii) rank $F(P) =$ normrank G,

(iii) rank $\begin{pmatrix} L(s) \\ F(P) \end{pmatrix} = n+$ normrank G $\forall s \in C^+$,

(iv) $G(Q) \geq 0$,

(v) rank $G(Q) =$ normrank H,

(vi) rank $\begin{pmatrix} M(s) & G(Q) \end{pmatrix} = n+$ normrank H $\forall s \in C^+$,

Moreover, both P and Q are uniquely defined by the above equations and are positive semi-definite.   □

Proof : Under the condition that (A, B) is stabilizable, the existence of a matrix P satisfying conditions (i) and (ii) has been shown in [5, 20]. Moreover, it was shown that there exists a unique maximal solution to (i) and (ii) which is positive semi-definite. Using the techniques in [15], it can be shown that P is equal to the maximal solution if and only if P satisfies condition (iii).

The existence and uniqueness of Q can be obtained via dualization.   ■

We are now in the position to formulate our main result.

Theorem 2.4 : *Consider the system (2.1). Assume that (A, B) is stabilizable and (C_2, A) is detectable. Let P and Q be the matrices uniquely defined by lemma 2.3. The infimum over all internally stabilizing compensators Σ_F of the cost-criterion (2.6) is equal to*

$$\text{Trace } E^T P E + \text{Trace } (A^T P + PA + C_1^T C_1)Q \tag{2.22}$$

The infimum is attained by an internally stabilizing strictly proper compensator of the form (2.2) if and only if the following conditions hold:

(i) *Im* $E_Q \subset \mathcal{V}_g(\Sigma_{ci})$

(ii) *Ker* $C_P \supset \mathcal{T}_g(\Sigma_{di})$

(iii) $A\mathcal{T}_g(\Sigma_{di}) \subset \mathcal{V}_g(\Sigma_{ci})$

where C_P, D_P, E_Q and D_Q are arbitrary matrices satisfying

$$F(P) = \begin{pmatrix} C_P^T \\ D_P^T \end{pmatrix} \begin{pmatrix} C_P & D_P \end{pmatrix}, \qquad G(Q) = \begin{pmatrix} E_Q \\ D_Q \end{pmatrix} \begin{pmatrix} E_Q^T & D_Q^T \end{pmatrix}. \tag{2.23}$$

and $\Sigma_{ci} = (A, B, C_P, D_P)$, $\Sigma_{di} = (A, E_Q, C_2, D_Q)$.   □

Remarks:

6

(*i*) Note that it can be shown that conditions (a)-(c) are always satisfied under some usual assumptions made in literature:

- Normrank G = rank D_1.
- The system (A, B, C_1, D_1) has no invariant zeros on the imaginary axis.
- Normrank H = rank D_2.
- The system (A, E, C_2, D_2) has no invariant zeros on the imaginary axis.

(*ii*) Note that if we want to attain the infimum with an internally stabilizing compensator then the necessary and sufficient conditions in our theorem do not satisfy the separation principle: condition (i) is the condition we obtain in case of state feedback, condition (ii) is the condition for the existence of a strictly proper Kalman filter and condition (iii) is a coupling condition (it is related *both* to state feedback and to Kalman filtering).

(*iii*) We discuss the use of compensators which are not strictly proper in subsection 4.2. It turns out that the infimum remains (2.22) but that for a larger class of systems the infimum is attained.

3 A system transformation

In this section we are going to transform the system Σ into a new system $\Sigma_{P,Q}$ with some desirable properties. It will be shown that a compensator is internally stabilizing for Σ if and only if this compensator is internally stabilizing for $\Sigma_{P,Q}$. Moreover, the cost-criterion evaluated for some compensator Σ_F applied to Σ and the cost-criterion evaluated for the same compensator Σ_F but this time applied to $\Sigma_{P,Q}$, differ a constant which is independent of the choice for the compensator. Hence we need only investigate the system $\Sigma_{P,Q}$, which we will do extensively in the next section. The transformation will go in two steps. The first step is related to the control problem with full-information (i.e. state feedback) while the second step is related to a filtering problem. Throughout this section we assume that matrices P and Q satisfying the requirements of lemma 2.3 are given.

3.1 The first transformation from Σ to Σ_P

We define the following system:

$$\Sigma_P : \begin{cases} \dot{x}_P = Ax_P & + Ev & + Bu_P, \\ z_P = C_Px_P + & & D_Pu_P, \\ y_P = C_2x_P & + D_2v, \end{cases} \tag{3.1}$$

where C_P and D_P are such that (2.23) is satisfied. It is straightforward to derive the following lemma:

Lemma 3.1 : *A compensator of the form (2.2) is internally stabilizing for Σ if and only if the same compensator is internally stabilizing for Σ_P. Moreover, if we apply the same compensator to both systems then their respective cost-criteria satisfy the following relationship:*

7

$$\mathcal{J}(\Sigma \times \Sigma_F) = \mathcal{J}(\Sigma_P \times \Sigma_F) + Trace\ (E^{\mathrm{T}}PE). \tag{3.2}$$

$$\square$$

Proof : Clearly a compensator is internally stabilizing for Σ if and only if the same compensator is internally stabilizing for Σ_P since the matrices C_1 and D_1, or equivalently C_P and D_P, are not appearing in the requirements for internal stability.

Note that the systems Σ and Σ_P have the same state and measurement equations. Hence, after applying the same internally stabilizing compensator to both systems we have the same closed loop states for both systems (i.e. $x = x_P$). We know that $u = Mp$ (where p is defined by (2.3)) is a well-defined stochastic process. Therefore, by applying Ito differential rule (see e.g. [3]), we find the following relation for the respective closed loop systems:

$$\int_0^s z(t)^{\mathrm{T}}z(t)\,dt + x(s)^{\mathrm{T}}Px(s) = \int_0^s z_P(t)^{\mathrm{T}}z_P(t) + Trace\ (E^{\mathrm{T}}PE)dt. \tag{3.3}$$

Here we assumed zero initial conditions. We know that the closed loop system $\Sigma \times \Sigma_F$ is internally stable. Therefore $\mathcal{E}\ x(s)^{\mathrm{T}}Px(s)$ converges to some finite number as $s \to \infty$. Dividing (3.3) by s and taking the limit for $s \to \infty$ of the expectation and we find (3.2). ∎

The fact that a compensator is internally stabilizing for Σ if and only if the same compensator is internally stabilizing for Σ_P, together with equality (3.2), shows that it is sufficient to investigate Σ_P to prove all the claims in theorem 2.4.

The second transformation is exactly dual to the first. Therefore the required results can be derived via dualization. First we define the system $\Sigma_{P,Q}$:

$$\Sigma_{P,Q} : \begin{cases} \dot{x}_{P,Q} = Ax_{P,Q} + E_Q v + Bu_{P,Q}, \\ z_{P,Q} = C_P x_{P,Q} + D_P u_{P,Q}, \\ y_{P,Q} = C_2 x_{P,Q} + D_Q v, \end{cases} \tag{3.4}$$

where E_Q and D_Q are such that (2.23) is satisfied.

Theorem 3.2 : *A compensator Σ_F is internally stabilizing for the system Σ if and only if the same compensator Σ_F is internally stabilizing for $\Sigma_{P,Q}$. Moreover the cost-criterion for the two respective closed-loop systems are related in the following way:*

$$\mathcal{J}(\Sigma \times \Sigma_F) = \mathcal{J}(\Sigma_{P,Q} \times \Sigma_F) + Trace\ E^{\mathrm{T}}PE + Trace\ C_P QC_P^{\mathrm{T}}. \tag{3.5}$$

$$\square$$

Remark: The above theorem is still valid if we investigate finite-dimensional, time-invariant compensators which are not necessarily proper as long as these compensators yield a well-posed, internally stable closed loop system with a closed loop transfer matrix from v to z which is strictly proper. Here well-posed means that for given v in the closed loop system x and p are uniquely defined. The requirement that the closed loop transfer matrix from v

to z is strictly proper is needed to have a well-defined cost-criterion. For more details see subsection 4.2.

Proof : Assume a compensator Σ_F described by (2.5) is internally stabilizing for Σ. We know from lemma 3.1 that this implies that Σ_F is also internally stabilizing for Σ_P. Moreover we know (3.2). Since the transformation from Σ_P to $\Sigma_{P,Q}$ is completely dual to the transformation from Σ to Σ_P we can dualize the results from lemma 3.1. Thus we find that a compensator is internally stabilizing when applied to Σ_P if and only if this compensator is internally stabilizing when applied to $\Sigma_{P,Q}$. Combining the above yields the first part of the above theorem. Moreover by dualizing lemma 3.1 we also find:

$$\mathcal{J}(\Sigma_P \times \Sigma_F) = \mathcal{J}(\Sigma_{P,Q} \times \Sigma_F) + \text{Trace } C_P Q C_P^{\mathsf{T}}. \tag{3.6}$$

Combining (3.2) with (3.6) yields (3.5). ∎

The above theorem enables us to concentrate all efforts on $\Sigma_{P,Q}$. Note that (3.5) immediately implies that the infimum is always larger than or equal to (2.22) since $\mathcal{J}(\Sigma_{P,Q} \times \Sigma_F) \geq 0$. To prove that the infimum is equal to (2.22) we will construct a minimizing sequence.

4   The solution of the LQG control problem

We have to investigate three problems in this section:

- When does there exist a strictly proper, internally stabilizing compensator which attains the infimum ?

- When does there exists an internally stabilizing compensator, not necessarily (strictly) proper, which attains the infimum ?

- We have to construct a minimizing sequence of strictly proper admissible compensators.

These problems will be discussed in the next three subsections.

4.1   The existence of an internally stabilizing compensator which attains the infimum

In this subsection we will investigate when there exists an internally stabilizing compensator for the system $\Sigma_{P,Q}$ which makes the LQG cost criterion equal to zero. By (3.5) this implies that the same compensator, applied to Σ, attains the infimum (2.22) for the LQG control problem. This subsection will complete the proof of the second part of theorem 2.4.
We will make use of the following theorem. This theorem is an extension, to include direct feedthrough matrices, of known results in [20] and is worked out in [17]. Note that we require a *strictly* proper compensator.

Theorem 4.1 : *Let Σ be given of the form (2.1). There exists a strictly proper compensator of the form (2.2) such that the closed loop system is internally stable and the closed loop*

transfer matrix is equal to 0 *if and only if* (A, B) *is stabilizable,* (C_2, A) *is detectable and*

$$
\begin{aligned}
Im\ E &\subset \mathcal{V}_g(A, B, C_1, D_1) \\
Ker\ C_1 &\supset \mathcal{T}_g(A, E, C_2, D_2) \\
A\mathcal{T}_g(A, E, C_2, D_2) &\subset \mathcal{V}_g(A, B, C_1, D_1)
\end{aligned}
$$

$\square$

The LQG-cost criterion is equal to zero if and only if the closed loop transfer matrix is equal to zero. By (3.5), there exists an internally stabilizing compensator for Σ which makes the LQG-cost criterion equal to (2.22) if and only if there exists an internally stabilizing compensator for $\Sigma_{P,Q}$ which makes the closed loop transfer matrix equal to zero. This is possible if the conditions of theorem 4.1 are satisfied for $\Sigma_{P,Q}$. This completes the proof of the second part of theorem 2.4.

4.2 The existence of an internally stabilizing compensator which is not necessarily strictly proper but attains the infimum

It can be shown that if we allow the use of non-proper compensators then we still have the same infimum (2.22). Only for a larger class of systems the infimum is attained. We will give a number of partial results and conclude with some typical examples.

We will describe non-proper compensators in this section mainly by their transfer matrices. Clearly we can also find realizations for these compensators via generalized state-space realizations. We call a compensator admissible if the closed-loop transfer matrices from v and the initial state $x(0) = x_0$ to x, u and z are well-defined, strictly proper and stable (no poles in the open left half plane). We make the requirement that the transfer matrix from v to u must be strictly proper to guarantee that the input is a well-defined stochastic process. The transfer matrix from v to z should be strictly proper and stable to guarantee a well-defined cost-criterion.

Theorem 3.2 is still valid for non-proper, admissible compensators, i.e. a compensator is admissible for Σ if and only if this compensator is admissible for $\Sigma_{P,Q}$. Moreover, if we apply the same compensator Σ_F to both systems, then we have (3.5). Hence a non-proper compensator Σ_F attains the infimum (2.22) for Σ if and only if Σ_F is admissible for $\Sigma_{P,Q}$ and makes the closed loop transfer matrix equal to 0, i.e. *the disturbance decoupling problem is solvable by a non-proper compensator.* Along these lines, we can derive with considerable effort the following necessary conditions for the existence of a admissible, not necessarily (strictly) proper compensator which attains the infimum:

Theorem 4.2 : *Let Σ be given by (2.1). If there exists an admissible compensator which attains the infimum (2.22) then the following conditions are satisfied: (A, B) is stabilizable, (C_2, A) is detectable and*

(i) $Im\ E_Q \subset \mathcal{V}_g(\Sigma_{ci})$

(ii) $Ker\ C_P \supset \mathcal{T}_g(\Sigma_{di}) \cap \mathcal{V}(\Sigma_{di})$

(iii) $A\left(\mathcal{T}_g(\Sigma_{di}) \cap \mathcal{V}(\Sigma_{di})\right) \subset \mathcal{V}_g(\Sigma_{ci})$

10

where C_P, D_P, E_Q and D_Q are arbitrary matrices satisfying (2.23) and $\Sigma_{ci} = (A, B, C_P, D_P)$, $\Sigma_{di} = (A, E_Q, C_2, D_Q)$. □

We will discuss the conditions of the above theorem under a simplifying assumption: the system (A, E, C_2, D_2) has no invariant zeros on the imaginary axis. In that case it can be shown that

$$\mathcal{T}_g(\Sigma_{di}) \cap \mathcal{V}(\Sigma_{di}) = \{0\}. \tag{4.1}$$

Hence the conditions in theorem 4.2 reduce to (i). Condition (i) guarantees (see [17, 21]) the existence of a stabilizing state feedback $u = Fx$ which, when applied to $\Sigma_{P,Q}$, makes the closed loop transfer matrix equal to 0. Equation (4.1) guarantees (see [13]) the existence of a non-proper Kalman filter such that, for $u = 0$, we have

$$\mathcal{E} \ (x - \hat{x})^2 = 0$$

where $\hat{x}$ denotes the estimate for x. The question is whether we can combine these two to find a suitable admissible compensator. Assume that this Kalman filter has transfer matrix K from y to $\hat{x}$. If the following matrix is an invertible rational matrix:

$$P(s) := I - F \left(I - K(s)C_2 \right)(sI - A)^{-1} B$$

then an admissible compensator is described by the transfer matrix $P^{-1}FK$ and this compensator attains the infimum. Note that if K is proper this is always true. Clearly, since K is independent of B, C_2, D_2 which are, together with A, E, the matrices determining F, invertibility of P is a very weak condition yet not always true. Note that P invertible is not a necessary condition: we may have to choose another matrix F which is in general not unique; also it is not clear whether any admissible compensator which attains the infimum will have the structure of a Kalman filter interconnected with a state feedback. The difficulty is that the separation principle is not valid any more: there always exist a perfect state feedback and a perfect Kalman filter, yet there does *not* always exist a suitable dynamic compensator.

We end this subsection by giving a number of examples which express the difficulties we encounter when trying to characterize the existence of a non-proper compensator which attains the infimum.

Example 4.3 : *Consider the following system*

$$\Sigma : \begin{cases} \dot{x} = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v, \\ z = \begin{pmatrix} 0 & 1 \end{pmatrix} x + \qquad u, \\ y = \begin{pmatrix} 1 & 0 \end{pmatrix} x, \end{cases}$$

For this system we have $P = 0$ and $Q = 0$. Therefore the infimum (2.22) is equal to 0. However, there is only a non-proper compensator which attains this infimum: $u = \dot{y} + y$. Note that in the closed loop system $\dot{u} = v$ which yields a well-defined stochastic process u with the use of Wiener integrals. □

11

Example 4.4 : *Consider the following system*

$$\Sigma : \begin{cases} \dot{x} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\[2mm] z = \begin{pmatrix} -1 & 0 \end{pmatrix} x + \qquad\qquad u, \\[2mm] y = \begin{pmatrix} 0 & 1 \end{pmatrix} x. \end{cases}$$

For this system we have again $P = 0$ *and* $Q = 0$. *Therefore, also for this system the infimum (2.22) is equal to* 0. *Contrary to the previous example, this time there does not exist a non-proper compensator which attains the infimum. There exists a unique optimal state feedback:* $u = -x_1$. *There also exists an optimal Kalman filter (for* $u = 0$):

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \dot{y}$$

However with zero initial conditions and $z = 0$ *we have* $y = 0$ *and hence* $u = 0$. *Therefore there does not exist a dynamic compensator which makes the closed loop transfer matrix from* v *to* z *equal to* 0, *i.e. the infimum is not attained.* □

Example 4.5 : *Consider the following system*

$$\Sigma : \begin{cases} \dot{x} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\[2mm] z = \begin{pmatrix} 0 & 1 \end{pmatrix} x, \\[2mm] y = \qquad\qquad v. \end{cases}$$

This system, like in the previous two examples, has $P = 0$ *and* $Q = 0$ *and therefore the infimum (2.22) is equal to* 0. *There exists a non-proper compensator which attains this infimum:* $u = -(\dot{y} + y)$. *However, this implies that* $u = -(\dot{v} + v)$. *Since* v *is a white noise process this means that the input is ill-defined. This implies that this compensator is not acceptable.* □

We were not able to derive a characterization when the infimum is attained by a non-proper compensator because we were not able to characterize when this well-posedness problem of example 4.4 occurs.

4.3 The construction of a minimizing sequence

We will derive such a minimizing sequence directly for Σ. We will not make use of $\Sigma_{P,Q}$ in this subsection. First we state and prove our main result for a special class of systems, i.e. regular systems with no invariant zeros on the imaginary axis. This result is already known (see e.g. [4, 8]) but is added to make this derivation self-contained.

Theorem 4.6 : *Let Σ be given by (2.1) Assume that D_1 and D_2 are surjective and injective respectively. Moreover assume that (A,B,C_1,D_1) and (A,E,C_2,D_2) have no invariant zeros on the imaginary axis. Then the infimum (2.22) is attained by the following internally stabilizing compensator*

$$\Sigma_F : \left\{ \begin{array}{l} \dot{p} = Ap + Bu + K(C_2 p - y), \\ u = Fp. \end{array} \right.$$

where $F = -(D_P^T D_P)^{-1} D_P^T C_P$ and $K = -E_Q D_Q^T (D_Q D_Q^T)^{-1}$. $\qquad\qquad\square$

Proof : It can be checked straightforwardly that this compensator when applied to $\Sigma_{P,Q}$ yields an internally stable closed loop system and the closed loop transfer matrix is equal to 0. To prove internal stability we use that the condition on invariant zeros guarantees that rank conditions (iii) and (vi) in lemma 2.3 are satisfied for all s in the *closed* right half plane (see e.g. [5, 20]). The result then follows as a corollary from the previous subsection. $\qquad\blacksquare$

Remark: Note that the compensator has the usual structure of a Kalman filter attached to a state feedback. Clearly it is straightforward to write this compensator in the form (2.2).

We will use the above, to construct a minimizing sequence of strictly proper compensators for a system Σ which does not satisfy the conditions in theorem 4.6. For a given system Σ of the form (2.1) and all $\varepsilon > 0$ we define the following perturbed system:

$$\Sigma(\varepsilon) : \left\{ \begin{array}{llll} \dot{x} = & Ax + & E(\varepsilon)v + & Bu, \\ z = & C_1(\varepsilon)x + & & D_1(\varepsilon)u, \\ y = & C_2 x + & D_2(\varepsilon)v, & \end{array} \right. \qquad (4.2)$$

where

$$
\begin{aligned}
E(\varepsilon) &:= \begin{pmatrix} E & \varepsilon I & 0 \end{pmatrix} \\
D_2(\varepsilon) &:= \begin{pmatrix} D_2 & 0 & \varepsilon I \end{pmatrix} \\
C_1(\varepsilon) &:= \begin{pmatrix} C_1^T & \varepsilon I & 0 \end{pmatrix}^T \\
D_1(\varepsilon) &:= \begin{pmatrix} D_1^T & 0 & \varepsilon I \end{pmatrix}^T
\end{aligned}
$$

For given $\varepsilon > 0$ we define $P(\varepsilon)$ and $Q(\varepsilon)$ as the matrices P and Q we obtain by applying lemma 2.3 to $\Sigma(\varepsilon)$. Since $D_1(\varepsilon)$ and $D_2(\varepsilon)$ are injective and surjective respectively, it can be checked that $P(\varepsilon)$ and $Q(\varepsilon)$ are the unique matrices $\tilde{P}$ and $\tilde{Q}$ satisfying:

- $A^T\tilde{P} + \tilde{P}A + C_1^T(\varepsilon)C_1(\varepsilon) - \left(\tilde{P}B + C_1^T(\varepsilon)D_1(\varepsilon)\right)\left(D_1^T(\varepsilon)D_1(\varepsilon)\right)^{-1}\left(B^T\tilde{P} + D_1^T(\varepsilon)C_1(\varepsilon)\right) = 0,$

- $A\tilde{Q} + \tilde{Q}A^T + E(\varepsilon)E^T(\varepsilon) - \left(\tilde{Q}C_2^T + E(\varepsilon)D_2^T(\varepsilon)\right)\left(D_2(\varepsilon)D_2^T(\varepsilon)\right)^{-1}\left(C_2\tilde{Q} + D_2(\varepsilon)E^T(\varepsilon)\right) = 0,$

- $A - B\left(D_1^T(\varepsilon)D_1(\varepsilon)\right)^{-1}\left(B^T\tilde{P} + D_1^T(\varepsilon)C_1(\varepsilon)\right)$ is asymptotically stable,

- $A - \left(\tilde{Q}C_2^T + E(\varepsilon)D_2^T(\varepsilon)\right)\left(D_2(\varepsilon)D_2^T(\varepsilon)\right)^{-1}C_2$ is asymptotically stable.

13

Note that we already have P and Q satisfying the conditions of lemma 2.3 for our original system Σ. In the following lemma we show that $P(\varepsilon)$ and $Q(\varepsilon)$ converge to P and Q respectively as $\varepsilon \downarrow 0$.

Lemma 4.7 : *Let $P(\varepsilon), Q(\varepsilon), P$ and Q be as defined before. Then we have*

$$P(\varepsilon) \to P, \qquad Q(\varepsilon) \to Q, \qquad as\ \varepsilon \downarrow 0 \qquad\qquad \square$$

Proof : The result that $P(\varepsilon) \to P$ as $\varepsilon \downarrow 0$ has been obtained in [18]. The result on Q can then be obtained by dualization. $\blacksquare$

The construction of a minimizing sequence is now straightforward. Note that for all $\varepsilon > 0$ the system $\Sigma(\varepsilon)$ satisfies all conditions of theorem 4.6. Therefore for all $\varepsilon > 0$ we have a compensator $\Sigma_F(\varepsilon)$ which is internally stabilizing and minimizes the LQG cost-criterion for $\Sigma(\varepsilon)$. However it is straightforward to check that in that case $\Sigma_F(\varepsilon)$ is also internally stabilizing for Σ. Moreover, we have

$$
\begin{aligned}
\mathcal{J}(\Sigma \times \Sigma_F(\varepsilon)) &\leq \mathcal{J}(\Sigma(\varepsilon) \times \Sigma_F(\varepsilon)) \\
&= \text{Trace } E^{\mathsf{T}} P(\varepsilon) E + \text{Trace } (A^{\mathsf{T}} P(\varepsilon) + P(\varepsilon)A + C_1^{\mathsf{T}} C_1) Q(\varepsilon) \\
&\to \text{Trace } E^{\mathsf{T}} P E + \text{Trace } (A^{\mathsf{T}} P + PA + C_1^{\mathsf{T}} C_1) Q \qquad \text{as } \varepsilon \downarrow 0
\end{aligned}
$$

However (3.5), combined with the fact that $\mathcal{J}(\Sigma_{P,Q} \times \Sigma_F) \geq 0$, shows that

$$\mathcal{J}(\Sigma \times \Sigma_F) \geq \text{Trace } E^{\mathsf{T}} P E + \text{Trace } (A^{\mathsf{T}} P + PA + C_1^{\mathsf{T}} C_1) Q$$

for all internally stabilizing compensators Σ_F. Combining the above shows that $\Sigma_F(\varepsilon)$ is a minimizing sequence for the LQG control problem for Σ. This completes the proof of our main theorem 2.4.

5 Conclusion

In this paper we solved the LQG control problem for linear, time-invariant, finite-dimensional systems without any assumptions on the system parameters. The calculation of P and Q, needed for the determination of (2.22), can be done by solving two reduced order Riccati equations (see [11, 16]). The determination of a strictly proper compensator which attains the infimum can be handled via the disturbance decoupling problem with measurement feedback and stability for $\Sigma_{P,Q}$ (see [17]). A minimizing sequence is explicitly constructed in this paper. The only result lacking is a characterization when there exists a compensator which attains the infimum but which is not strictly proper. The difficulties with obtaining such a characterization have been outlined in subsection 4.2. This subsection also contains a number of partial results.

14

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