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The singular zero-sum differential game with stability using H_∞ control theory

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Abstract

In this paper we consider the time-invariant, finite-dimensional, infinite-horizon, linear quadratic differential game. We will derive sufficient conditions for the existence of (almost) equilibria as well as necessary conditions. Contrary to all classical references we allow for singular weighting on the minimizing player in the cost-criterion. It turns out that this problem has a strong relation with the singular H_∞ problem with state feedback, i.e. the H_∞ problem where the direct feedthrough matrix from control input to output is not necessarily injective.

1 Introduction

In this paper we will consider the zero-sum linear quadratic finite-dimensional differential game. This is an area of research which was rather popular during the seventies.

However, in the last few years, the solution of the H_∞ control problem (see [2,4,6]) turned out to contain the same kind of algebraic Riccati equation as the Riccati equation appearing in the solution of the zero-sum differential game (see [1,5,11]). This Riccati equation has the special property that the quadratic term is in general indefinite. Contrary to for instance the linear quadratic optimal control theory (see [15]) where the quadratic term in the Riccati equation is definite.

Since in H_∞ control theory the solution of the algebraic Riccati equation has no meaning in itself it is interesting to have the more intuitive explanation as an equilibrium in the theory of differential games. Recently a number of papers appeared which studied the differential game with this goal (see [7,8,14])

In a recent paper [10] about H_∞ control theory it has been shown that in case the direct feedthrough matrix from control input to output is not injective then instead of an algebraic Riccati equation we get a quadratic matrix inequality. This phenomenon also occurs in linear quadratic optimal control theory although in that case we get a *linear* matrix inequality (see [15]).

This paper is concerned with the zero-sum differential game in the case that the direct feedthrough matrix is not injective. It will be shown that, as expected, we also get a quadratic matrix inequality. Moreover by using results from H_∞ control theory we are able to derive necessary conditions for the existence of an equilibrium which, to our knowledge, has not been done in previous papers. We will study the differential game *with stability* since it turns out to give results which indeed depend on the same solution of the quadratic matrix inequality as the one we need in H_∞ control. If we assume detectability then the problems with and without stability turn out to be equivalent.

The outline of the paper is as follows:
In section 2 we will formulate the problem and give our main results. In section 3 we will introduce a system transformation which will enable us to prove our main results. In section 4 we will prove the existence of an equilibrium under some sufficient conditions. After that, in section 5 we will be concerned with necesary conditions for the existence of equilibria. In section 6 we will show that if the direct feedthrough matrix from control input to output is injective that the necessary conditions of section 5 are also sufficient. We will conclude in section 7 with some concluding remarks.

2 Problem formulation and main results

We will consider the zero sum, infinite horizon, linear quadratic differential game with cost criterion

$$\mathcal{J}(u,w) = \int_0^\infty y^T(s)y(s) - w^T(s)Qw(s)ds \tag{2.1}$$

and dynamics given by the following linear, time-invariant and finite-dimensional system

$$\Sigma: \qquad \begin{array}{ll} \dot{x} = Ax + Bu + Ew, & x(0) = x_0 \\ y = Cx + Du. & \end{array} \tag{2.2}$$

Here $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, $w \in \mathcal{R}^l$ and $y \in \mathcal{R}^p$. A,B,C,D and E are matrices of appropriate dimensions and Q is a positive definite matrix. We assume (A, B) stabilizable. We define the following class of functions,

$$\mathcal{U}_{fb}^k = \left\{ v : \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}^k \mid \forall x \in \mathcal{L}_2^n(\mathcal{R}^+) \quad v(x(.),.) \in \mathcal{L}_2^k(\mathcal{R}^+) \right\} \tag{2.3}$$

Here $\mathcal{L}_2^k(\mathcal{R}^+)$ denotes the space of square integrable functions from $\mathcal{R}^+$ to $\mathcal{R}^k$. On this space we define the standard $\mathcal{L}_2$-norm:

$$\|q\|_2 = \left(\int_0^\infty q^T(s)q(s)ds \right)^{1/2} \tag{2.4}$$

Note that we can consider $\mathcal{L}_2^k$ as a subset of $\mathcal{U}_{fb}^k$ by identifying to each function $v \in \mathcal{L}_2^k$ a function $\tilde{v} \in \mathcal{U}_{fb}^k$ as follows $\tilde{v}(x,t) = v(t)$ $\forall x \in \mathcal{R}^n, t \in \mathcal{R}^+$. We call $(\tilde{u}, \tilde{w}) \in \mathcal{U}_{fb}^m \times \mathcal{U}_{fb}^l$ *an admissible pair* if by applying $u(.) = \tilde{u}(x(.),.)$ and $w(.) = \tilde{w}(x(.),.)$ in (2.2) the resulting state, denoted by x_{u,w,x_0}, satisfies $x_{u,w,x_0} \in \mathcal{L}_2^n(\mathcal{R}^+)$. Hence by definition of $\mathcal{U}_{fb}$ we have $u \in \mathcal{L}_2^m(\mathcal{R}^+)$ and $w \in \mathcal{L}_2^l(\mathcal{R}^+)$. This implies that the resulting output, denoted by y_{u,w,x_0}, satisfies $y_{u,w,x_0} \in \mathcal{L}_2^p(\mathcal{R}^+)$ and hence $\mathcal{J}(u,w)$ as defined in (2.1) is well-defined. We will only consider inputs of this form.

Since (A, B) stabilizable the class of admissible pairs is non-empty for every initial value x_0.

We will call u a *minimizing player* and his goal is to minimize the cost criterion $\mathcal{J}(u,w)$. In the same way we will call w a *maximizing player* who would like to maximize the cost criterion $\mathcal{J}(u,w)$.

Definition 2.1 *The system (2.2) with criterion function (2.1) is said to have an* equilibrium *if for all initial values x_0 there exists an admissible pair (u_0, w_0) such that*

$$\mathcal{J}(u_0, w) \leq \mathcal{J}(u_0, w_0) \leq \mathcal{J}(u, w_0) \tag{2.5}$$

for all $u \in \mathcal{U}_{fb}^m$ and $w \in \mathcal{U}_{fb}^l$ such that (u, w_0) and (u_0, w) are admissible pairs. Here u_0 should be such that for all $w \in \mathcal{L}_2^l$, (u_0, w) is an admissible pair.

The existence of an equilibrium is, in general, a too strong condition. We will define a weaker version.

Definition 2.2 *The system (2.2) with cost criterion (2.1) is said to have* an almost equilibrium *if there exists a function* $\mathcal{J}^* : \mathcal{R}^n \to \mathcal{R}$ *such that* $\forall \varepsilon > 0, x_0 \in \mathcal{R}^n \quad \exists u_0 \in \mathcal{U}_{fb}^m, w_0 \in \mathcal{U}_{fb}^l$ *such that* (u_0, w_0) *is admissible and moreover,*

$$\begin{aligned} \mathcal{J}(u_0, w) &\leq \mathcal{J}^*(x_0) + \varepsilon \\ \mathcal{J}(u, w_0) &\geq \mathcal{J}^*(x_0) - \varepsilon \end{aligned} \tag{2.6}$$

for all $u \in \mathcal{U}_{fb}^m$ *and* $w \in \mathcal{U}_{fb}^l$ *such that* (u, w_0) *and* (u_0, w) *are admissible pairs. Here* u_0 *should be such that for all* $w \in \mathcal{L}_2^l$, (u_0, w) *is an admissible pair.*

Remark Note that if either u_0 or w_0 is fixed then choosing the other input in $\mathcal{U}_{fb}$ such that we have an admissible pair, results in well-defined functions in $\mathcal{L}_2$ for state, minimizing player and maximizing player. Hence in definition 2.2 we can, without loss of generality, assume that u and w are in $\mathcal{L}_2$ instead of $\mathcal{U}_{fb}$. In that case there are no restrictions any more on w since (u_0, w) is admissible for all $w \in \mathcal{L}_2^l$. This latter condition is rather unusual in zero-sum linear quadratic differential games. Intuitively it means that we hand over the responsibility of the condition $x \in \mathcal{L}_2^n$ to the minimizing player. Without that assumption it can happen that there exists u_0, w_0 such that

$$\begin{aligned} \mathcal{J}(u_0, w) &\leq M_1 \\ \mathcal{J}(u, w_0) &\geq M_2 \end{aligned}$$

for all u, w such that (u_0, w) and (u, w_0) are admissible pairs and $M_2 > M_1$. Clearly (u_0, w_0) is not admissible but neither u_0 nor w_0 will change since that will be contrary to their objective of minimizing respectively maximizing the cost-criterion. To prevent such a deadlock we hand over the responsibility for $x \in \mathcal{L}_2^n$ to one of the players.

We will derive conditions for the existence of an almost equilibrium. Since we do not assume that the D matrix is injective it is not surprising that, as in the singular LQ problem, we find a matrix inequality instead of a Riccati equation. We define

$$F(P) := \begin{pmatrix} A^T P + PA + C^T C + PEQ^{-1}E^T P & PB + C^T D \\ B^T P + D^T C & D^T D \end{pmatrix}. \tag{2.7}$$

We call a symmetric P a solution of the *quadratic matrix inequality* if $F(P) \geq 0$. Furthermore define

$$L(P, s) := \begin{bmatrix} sI - A - EQ^{-1}E^T P & -B \end{bmatrix}. \tag{2.8}$$

This is in fact the controllability pencil associated to the system,

$$\dot{x} = (A + EQ^{-1}E^T P)\, x + Bu.$$

Define $G(s) = C(sI - A)^{-1}B + D$ and let normrank G denote the rank of G as a matrix over the field of real rational functions. We denote by $\mathcal{C}^-$ ($\mathcal{C}^0$, $\mathcal{C}^+$) the set of all $s \in \mathcal{C}$ such that $\operatorname{Re} s < 0$ ($\operatorname{Re} s = 0$, $\operatorname{Re} s > 0$). Finally we define the concept of invariant zero. The *invariant zeros* of (A, B, C, D) are all $s \in \mathcal{C}$ such that

$$\operatorname{rank}\left(\begin{array}{c|c} sI - A & -B \\ \hline C & D \end{array}\right) < \operatorname{normrank}\left(\begin{array}{c|c} sI - A & -B \\ \hline C & D \end{array}\right) \ (= n + \operatorname{normrank} G\,). \tag{2.9}$$

The following theorem is the main result of this paper.

Theorem 2.3 *Consider the system (2.2) with cost criterion (2.1). Assume (A, B) stabilizable. There exists an almost equilibrium if the following condition is satisfied,*

There exists a positive semi-definite solution P of $F(P) \geq 0$ such that

$$\begin{array}{lcl} \mathit{rank} \quad F(P) & = & \mathit{normrank}\ G \\ \mathit{rank} \begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} & = & n + \mathit{normrank}\ G \qquad \forall s \in \mathcal{C}^+ \cup \mathcal{C}^0. \end{array} \tag{2.10}$$

Moreover $\mathcal{J}^*(x_0) = x_0^T P x_0$ defines an almost equilibrium and for each bounded set of initial values we can find static state feedbacks F_u, F_w such that $u_0 = F_u x$ and $w_0 = F_w x$ satisfy (2.6) for all initial values in that set.

Remark For this specific almost equilibrium we have the following equality:

$$x_0^T P x_0 = \sup_{w \in \mathcal{U}_{fb}^l} \ \inf_{u \in \mathcal{U}_{st}^m(x_0)} \ \mathcal{J}(u, w). \tag{2.11}$$

About the necessity of the above conditions we have the following result:

Theorem 2.4 *Consider the system (2.2) with cost criterion (2.1). Assume (A, B) stabilizable and assume (A, B, C, D) has no invariant zeros in $\mathcal{C}^0$. If there exists an almost equilibrium then the following condition is satisfied:*

There exists a positive semi-definite solution P of $F(P) \geq 0$ such that

$$\begin{aligned} \text{rank} \quad F(P) &= \text{normrank } G \\ \text{rank} \begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} &= n + \text{normrank } G \quad \forall s \in \mathcal{C}^+. \end{aligned} \tag{2.12}$$

Moreover in case D is injective the above condition is also sufficient.

Remark Although, in case D injective, we can prove the existence of an equilibrium under the assumptions of theorem 2.4, we haven't been able to find static state feedback laws for u_0 and w_0 which we could find under the assumptions of theorem 2.3.

3 A preliminary system transformation

In this section we will apply a preliminary feedback transformation $u = F_0 x + v$ to the system (2.2). It will be shown that the resulting system has a very particular structure. For details we refer to [10]. In the proof of theorem 2.3 this tranformation will be our main tool to prove the result. We shall display the structure of the transformed system by writing down the matrices with respect to some suitably chosen bases for the input, state and output spaces.

Our basic tool is the strongly controllable subspace. We will first define this subspace and give an important property which will be used in the sequel.

Definition 3.1 *Assume we have a system*

$$\Sigma_{ci}: \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.1}$$

We define the strongly controllable subspace $\mathcal{T}(\Sigma_{ci})$ *as the limit of the following sequence of subspaces,*

$$\mathcal{T}_0(\Sigma_{ci}) = 0, \qquad \mathcal{T}_{i+1}(\Sigma_{ci}) = \{x \in \mathcal{R}^n \mid \exists\, \tilde{x} \in \mathcal{T}_i(\Sigma_{ci}),\ u \in \mathcal{R}^m \text{ such that } x = A\tilde{x} + Bu \text{ and } C\tilde{x} + Du = 0\} \tag{3.2}$$

It is well known (see [9]) that $\mathcal{T}_i(\Sigma_{ci})$ $(i = 1, 2, \ldots)$ is a non-decreasing sequence of subspaces and attains its limit in a finite number of steps. A system is called strongly controllable *if*

its strongly controllable subspace is equal to the whole state space.

We will give one property of the strongly controllable subspace at this point which will come in handy in the sequel (see [3,9]),

Lemma 3.2 *Assume we have the system (3.1) with $(C\ D)$ surjective. The system is strongly controllable if and only if*

$$\left(\begin{array}{c|c} sI - A & -B \\ \hline C & D \end{array}\right) \tag{3.3}$$

has ful row rank for all $s \in \mathcal{C}$.

We can now define the bases which will be used in the sequel.

First choose a basis of the input space $\mathcal{R}^m$. Decompose $\mathcal{R}^m = \mathcal{U}_1 \oplus \mathcal{U}_2$ such that $\mathcal{U}_2 = \ker D$ and $\mathcal{U}_1$ arbitrary. Choose a basis $u_1, u_2, \ldots, u_m$ of $\mathcal{R}^m$ such that $u_1, u_2, \ldots, u_i$ is a basis of $\mathcal{U}_1$ and $u_{i+1}, \ldots, u_m$ is a basis of $\mathcal{U}_2$.

Next choose *an orthonormal basis* $y_1, y_2, \ldots, y_p$ of $\mathcal{R}^p$ such that $y_1, \ldots, y_j$ is a basis of $\operatorname{im} D$ and $y_{j+1}, \ldots, y_p$ is a basis of $(\operatorname{im} D)^\perp$. Because this is an orthonormal basis this basis transformation does not change the norm $\|y\|$.

Finally we choose a decomposition of the state space $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$ such that $\mathcal{X}_2 = \mathcal{T}(\Sigma) \cap C^{-1}\operatorname{im} D$, $\mathcal{X}_2 \oplus \mathcal{X}_3 = \mathcal{T}(\Sigma)$ and $\mathcal{X}_1$ arbitrary. We choose a corresponding basis $x_1, x_2, \ldots, x_n$ such that $x_1, \ldots, x_r$ is a basis of $\mathcal{X}_1$, $x_{r+1}, \ldots, x_s$ is a basis of $\mathcal{X}_2$ and $x_{s+1}, \ldots, x_n$ is a basis of $\mathcal{X}_3$.

With respect to these bases the maps B, C, D have the following form,

$$B = (B_1\ \ B_2)\,,\ C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},\ D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4}$$

Next, we define a linear mapping $F_0 : \mathcal{R}^n \to \mathcal{R}^m$ by

$$F_0 := \begin{pmatrix} -D_1^{-1}C_1 \\ 0 \end{pmatrix} \text{ and hence } C + DF_0 = \begin{pmatrix} 0 \\ C_2 \end{pmatrix}. \tag{3.5}$$

We have the following properties of this decomposition which are proven in [10].

Lemma 3.3 *Let F_0 be given by (3.5). Then we have*

(i) $(A+BF_0)(\mathcal{T}(\Sigma)\cap C^{-1}imD)\subseteq \mathcal{T}(\Sigma)$,
(ii) $imB_2\subseteq \mathcal{T}(\Sigma)$,
(iii) $\mathcal{T}(\Sigma)\cap C^{-1}imD\subseteq kerC_2$.

By applying this lemma we find that the matrices $A+BF_0$, B, $C+DF_0$ and D with respect to these bases have the following form.

$$\begin{cases} A+BF_0=\begin{pmatrix} A_{11} & 0 & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}, \quad B=\begin{pmatrix} B_{11} & 0\\ B_{21} & B_{22}\\ B_{31} & B_{32}\end{pmatrix},\\ C+DF_0=\begin{pmatrix} 0 & 0 & 0\\ C_{21} & 0 & C_{23}\end{pmatrix}, \quad D=\begin{pmatrix} D_1 & 0\\ 0 & 0\end{pmatrix}. \end{cases} \tag{3.6}$$

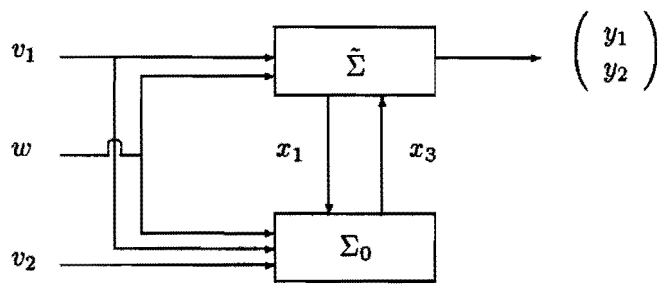
We now apply the feedback $u=F_0x+v$ to the system. Let $(v_1^T,v_2^T)^T$ be the coordinate vector of a given $v\in\mathcal{R}^m$. Likewise we use the notation $(x_1^T,x_2^T,x_3^T)^T$ and $(y_1^T,y_2^T)^T$. Finally decompose $E=(E_1^T,E_2^T,E_3^T)^T$ correspondingly. Then the system (2.2) has the following form:

$$\dot{x}_1 = A_{11}x_1+(B_{11}\ \ A_{13})\begin{pmatrix} v_1\\ x_3\end{pmatrix}+E_1w, \tag{3.7}$$

$$\begin{pmatrix}\dot{x}_2\\ \dot{x}_3\end{pmatrix}=\begin{pmatrix} A_{22} & A_{23}\\ A_{32} & A_{33}\end{pmatrix}\begin{pmatrix} x_2\\ x_3\end{pmatrix}+\begin{pmatrix} B_{22}\\ B_{32}\end{pmatrix}v_2+\begin{pmatrix} B_{21} & A_{21}\\ B_{31} & A_{31}\end{pmatrix}\begin{pmatrix} v_1\\ x_1\end{pmatrix}+\begin{pmatrix} E_2\\ E_3\end{pmatrix}w, \tag{3.8}$$

$$\begin{pmatrix} y_1\\ y_2\end{pmatrix}=\begin{pmatrix} 0\\ C_{21}\end{pmatrix}x_1+\begin{pmatrix} D_1 & 0\\ 0 & C_{23}\end{pmatrix}\begin{pmatrix} v_1\\ x_3\end{pmatrix}. \tag{3.9}$$

Note that (u,w) is an admissible pair for the system (2.2) if and only if (v,w) is an admissible pair for the system (3.7)-(3.9). As already suggested by the way we arranged these equations, the system (3.7)-(3.9) can be considered as the interconnection of two systems :



Here,

$$\tilde{\Sigma} := \left[A_{11}, (B_{11} \;\; A_{13}), \begin{pmatrix} 0 \\ C_{21} \end{pmatrix}, \begin{pmatrix} D_1 & 0 \\ 0 & C_{23} \end{pmatrix} \right] \tag{3.10}$$

is the system given by the equations (3.7) and (3.9). It has inputs v_1, w and x_3, state x_1 and output y_1, y_2. The system Σ_0 is given by equation (3.8). It has inputs v_1, v_2, w and x_1, state x_2, x_3 and output x_3.

The systems $\tilde{\Sigma}$ and Σ_0 turn out to have some nice structural properties, which have been shown in [10].

Lemma 3.4 *We have the following properties:*

(i) C_{23} *is injective,*

(ii) *The system,*

$$\Sigma_1 := \left[\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix}, (0 \;\; I), 0 \right] \tag{3.11}$$

is strongly controllable,

(iii) *We have,*

$$\textit{normrank}\, G = \textit{rank} \begin{pmatrix} C_{23} & 0 \\ 0 & D_1 \end{pmatrix}. \tag{3.12}$$

We need the following results from [10] which connects the conditions of theorem (2.3) to the matrices in the transformed system (3.7)-(3.9).

Lemma 3.5 *Assume a symmetric P is a solution of $F(P) \geq 0$. We have*

(i) $P\,\mathcal{T}(\Sigma) = 0$ *i.e in our decomposition P can be written as*

$$P = \begin{pmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.13}$$

(ii) *If P is in the form (3.13) then*

$$R(P) := P_{11}A_{11} + A_{11}^T P_{11} + C_{21}^T C_{21} + P_{11}\left(E_1 Q^{-1} E_1^T - B_{11}\left(D_1^T D_1\right)^{-1} B_{11}^T\right) P_{11}$$

$$- \left(A_{13}^T P_{11} + C_{23}^T C_{21}\right)^T \left(C_{23}^T C_{23}\right)^{-1} \left(A_{13}^T P_{11} + C_{23}^T C_{21}\right) \geq 0. \qquad (3.14)$$

Moreover $R(P) = 0$ if and only if rank $F(P) = normrank G$.

(iii) *If $R(P)=0$ then we have for all $s \in C$,*

$$rank \begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} = n + normrank\, G$$

if and only if

$$A_{11} + E_1 Q^{-1} E_1^T P_{11} - B_{11}\left(D_1^T D_1\right)^{-1} B_{11}^T P_{11} - A_{13}\left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right) \quad (3.15)$$

has no eigenvalue in s.

4 Solution of the quadratic differential game

In this section we assume that the condition of theorem (2.3) is satisfied. We show that there exists an *almost equilibrium*. We will use the following two lemmas which will give theorem 2.3 as an almost direct result.

Lemma 4.1 *Let P be given such that $F(P) \geq 0$. Then for all admissible pairs (v, w) we have:*

$$\mathcal{J}(u,w) = \|y\|_2^2 - \|Q^{1/2} w\|_2^2 =$$

$$= x_0^T P x_0 + \int_0^\infty x_1(\tau)^T R(P) x_1(\tau)\ d\tau + \|C_{23} q_3\|_2^2 + \|D_1 \tilde{v}_1\|_2^2 - \|Q^{1/2} w_1\|_2^2 \qquad (4.1)$$

where

$$q_3 := x_3 + \left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right) x_1, \qquad (4.2)$$

$$\tilde{v}_1 := v_1 + \left(D_1^T D_1\right)^{-1} B_{11}^T P_{11} x_1, \qquad (4.3)$$

$$w_1 := w - Q^{-1} E^T P x. \qquad (4.4)$$

Moreover the dynamics in these new coordinates are given by:

$$\dot{x}_1 = \tilde{A}_{11}x_1 + A_{13}q_3 + B_{11}\tilde{v}_1 + E_1 w_1 \tag{4.5}$$

$$\begin{pmatrix} \dot{x}_2 \\ \dot{q}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & \tilde{A}_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} B_{22} \\ B_{32} \end{pmatrix} v_2 + \begin{pmatrix} \tilde{A}_{21} \\ \tilde{A}_{31} \end{pmatrix} x_1 + \begin{pmatrix} B_{21} \\ B_{31} \end{pmatrix} \tilde{v}_1 + \begin{pmatrix} E_2 \\ \tilde{E}_3 \end{pmatrix} w_1 \tag{4.6}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_3 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ C_{23} \end{pmatrix} q_3 \tag{4.7}$$

where we used the following matrices,

$$\begin{aligned}
\tilde{A}_{11} &:= A_{11} + E_1 Q^{-1} E_1^T P_{11} - A_{13}\left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right) \\
&\qquad - B_{11}\left(D_1^T D_1\right)^{-1} B_{11}^T P_{11}, \\
\tilde{A}_{21} &:= A_{21} + E_2 Q^{-1} E_1^T P_{11} - A_{23}\left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right) \\
&\qquad - B_{21}\left(D_1^T D_1\right)^{-1} B_{11}^T P_{11}, \\
\tilde{A}_{31} &:= A_{31} + E_3 Q^{-1} E_1^T P_{11} - A_{33}\left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right) \\
&\qquad - B_{31}\left(D_1^T D_1\right)^{-1} B_{11}^T P_{11} + \left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right)\tilde{A}_{11}, \\
\tilde{A}_{33} &:= A_{33} + \left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right) A_{13}, \\
\tilde{C}_1 &:= -D_1\left(D_1^T D_1\right)^{-1} B_{11}^T P_{11}, \\
\tilde{C}_2 &:= C_{21} - C_{23}\left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right), \\
\tilde{E}_3 &:= E_3 + \left(C_{23}^T C_{23}\right)^{-1}\left(A_{13}^T P_{11} + C_{23}^T C_{21}\right) E_1.
\end{aligned}$$

Proof By using the system equations (2.2) we find

$$\frac{d}{dt}\left[x^T(t)Px(t) - x_0^T P x_0 + \int_0^t y(\tau)^T y(\tau) - w(\tau)^T Q w(\tau)\ d\tau\right] \tag{4.8}$$

$$= \begin{pmatrix} x \\ u \\ w \end{pmatrix}^T \begin{pmatrix} A^T P + PA + C^T C & PB + C^T D & PE \\ B^T P + D^T C & D^T D & 0 \\ E^T P & 0 & -Q \end{pmatrix} \begin{pmatrix} x \\ u \\ w \end{pmatrix}$$

$$= \begin{pmatrix} x \\ v \\ w_1 \end{pmatrix}^T \begin{pmatrix} A_F^T P + PA_F + C^T C + PEQ^{-1}E^T P & PB + C^T D & 0 \\ B^T P + D^T C & D^T D & 0 \\ 0 & 0 & -Q \end{pmatrix} \begin{pmatrix} x \\ v \\ w_1 \end{pmatrix} \tag{4.9}$$

where we have

$$\begin{aligned} v &= u - F_0 x \\ A_F &= A + BF_0 \\ C_F &= C + DF_0 \end{aligned}$$

and w_1 as given by (4.4). We can now use the decomposition as defined in (3.6) and we find that (4.9) is equal to:

$$\begin{pmatrix} x_1 \\ x_3 \\ v_1 \\ w_1 \end{pmatrix}^T \begin{pmatrix} P_{11}A_{11} + A_{11}^T P_{11} + C_{21}^T C_{21} + P_{11}E_1 Q^{-1} E_1^T P_{11} & P_{11}A_{13} + C_{21}^T C_{23} & P_{11}B_{11} & 0 \\ A_{13}^T P_{11} + C_{23}^T C_{21} & C_{23}^T C_{23} & 0 & 0 \\ B_{11}^T P_{11} & 0 & D_1^T D_1 & 0 \\ 0 & 0 & 0 & -Q \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ v_1 \\ w_1 \end{pmatrix}$$

When we finally use the definitions (4.2) and (4.3) and integrate the equation (4.8) from 0 to ∞ we find the equation (4.1). Here we used that

$$\lim_{t \to \infty} x^T(t) P x(t) = 0$$

since the pair (v, w) is admissible. Moreover (v, w) admissible implies that the integral in (4.1) is well-defined. ■

We will use the following known result which turns out to be extremely useful for singular H_∞ control and for singular differential games.

Lemma 4.2 *Consider the following system*

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx. \end{aligned}$$

Assume the system $(A, B, C, 0)$ is strongly controllable. Then we have the following result: For all bounded sets $\mathcal{V} \subset \mathcal{R}^n$, all $\varepsilon > 0$ and all $M \in \mathcal{R}$ there exists $F \in \mathcal{R}^{m \times n}$ such that

(i) $\sigma(A + BF) \subset \{s \in \mathcal{C} \mid Re\, s < M\}$

(ii) *For all $w \in \mathcal{L}_2^l$ and all $x_0 \in \mathcal{V}$ we have $\|y\|_2 \leq \varepsilon\,(\|w\|_2 + 1)$, where y is given by*

$$\begin{aligned} \dot{x} &= (A+BF)x + Ew \quad x(0) = x_0 \in \mathcal{V} \\ y &= Cx \end{aligned} \tag{4.10}$$

Proof In [13, theorem 3.36] it has been shown that if the system $(A, B, C, 0)$ is strongly controllable then for all $M \in \mathcal{R}$ and $\varepsilon > 0$ there exists an F such that

$$\|Ce^{(A+BF)t}\|_2 \leq \varepsilon \tag{4.11}$$

$$\sigma(A+BF) \subset \{s \in \mathcal{C} \mid \mathrm{Re}\, s < M\} \tag{4.12}$$

Using the above the lemma can be shown straightforwardly. ■

By using these two lemmas we can now prove theorem (2.3):

Proof of theorem 2.3 We choose the P satisfying (2.10). We choose an arbitrary $\varepsilon > 0$. First note that when we choose $w_1 = 0$, i.e. $w_0 = Q^{-1}E^TPx$ then by lemma (4.1) we have,

$$\mathcal{J}(u, w_0) \geq x_0^T P x_0 \geq x_0^T P x_0 - \varepsilon \tag{4.13}$$

for all u such that (u, w_0) is an admissible pair. This is the second inequality in (2.6).

In order to prove the other inequality in (2.6) we have to do some preparatory work. We start by choosing $\tilde{v}_1 = 0$ i.e. $v_1 = -(D_1^T D_1)^{-1} B_{11}^T P_{11} x_1$ We know by (2.10) and lemma 3.5 (iii) that $\tilde{A}_{11}$ is asymptotically stable. Assume we have an initial value in some bounded set then the mapping from q_3 and w_1 to x_1 is bounded i.e. there are M_1, M_2, M such that for all q_3 and w_1 in $\mathcal{L}_2$ we have

$$\|x_1\|_2^2 \leq M_1 \|q_3\|_2^2 + M_2 \|w_1\|_2^2 + M \tag{4.14}$$

Consider the system given by equation (4.6) with input v_2, state x_2, q_3 and output q_3. We claim that this system is strongly controllable. We have,

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & A_{33} - \tilde{A}_{33} \\ 0 & 0 & I \end{pmatrix} \left(\begin{array}{cc|c} sI - A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI - A_{33} & -B_{32} \\ \hline 0 & I & 0 \end{array}\right) = \left(\begin{array}{cc|c} sI - A_{22} & -A_{23} & -B_{22} \\ -A_{32} & sI - \tilde{A}_{33} & -B_{32} \\ \hline 0 & I & 0 \end{array}\right) \tag{4.15}$$

Since the first matrix has full rank for all $s \in \mathcal{C}$ we find that both system matrices have the same rank for all $s \in \mathcal{C}$. By using lemmas 3.2 and 3.4 we find that the above mentioned system described by (4.6) is strongly controllable. We assumed that we have an upper bound on the initial value. Therefore by lemma 4.2 we know we can find a feedback $v_2 = F_1 \begin{pmatrix} x_2 \\ q_3 \end{pmatrix}$

such that by applying that feedback in (4.6) we find

$$\|q_3\|_2^2 \leq \|C_{23}\|^{-1}\min\left(\varepsilon\left(\varepsilon M_1 + M + 1\right)^{-1}, \|Q^{-1}\|^{-1/2}\left(\|Q^{-1}\|^{-1/2}M_1 + M_2 + 1\right)^{-1}\right) \times \left(\|x_1\|_2^2 + \|w_1\|_2^2 + 1\right)$$

Combining this with (4.14) we find

$$\|C_{23}q_3\|_2^2 \leq \|Q^{-1}\|^{-1/2}\|w_1\|_2^2 + \varepsilon \leq \|Q^{1/2}w_1\|_2^2 + \varepsilon \tag{4.16}$$

By choosing $v_2 = F_1\begin{pmatrix} x_2 \\ q_3 \end{pmatrix}$ and $u_0 = F_0 x + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we therefore find

$$\mathcal{J}(u_0, w) \leq x_0^T P x_0 + \varepsilon \tag{4.17}$$

This gives the first equality in (2.6). By noting that for each bounded set of initial values the u_0 and w_0 are given by a static feedback the proof is completed. ■

Corollary 4.3 *Define the following class of control inputs*

$$\mathcal{U}_{st}^m(x_0) := \left\{u \in \mathcal{U}_{fb}^m \mid \forall w \in \mathcal{L}_2^l(\mathcal{R}^+)\ (u,w) \text{ is an admissible pair for initial value } x_0\right\}.$$

We have the following equality,

$$x_0^T P x_0 = \sup_{w \in \mathcal{U}_{fb}^l} \inf_{u \in \mathcal{U}_{st}^m(x_0)} \mathcal{J}(u,w). \tag{4.18}$$

Proof For some arbitrary ε we can choose $u = u_0$ as defined in the proof of theorem 2.3. We have $\tilde{v}_1 = 0$ and $\|C_{23}q_3\| \leq \|Q^{1/2}w_1\|$ and using the equality in (4.1) we find

$$\sup_{w \in \mathcal{U}_{fb}^l} \inf_{u \in \mathcal{U}_{st}^m(x_0)} \mathcal{J}(u,w) \leq x_0^T P x_0 + \varepsilon. \tag{4.19}$$

Since this is true for all ε we find an inequality in (4.18). By choosing $w_1 = 0$ for arbitrary u we find the opposite inequality and hence equality. ■

5 Solvability of the quadratic differential game

In this section we will derive neccesary conditions for the existence of an almost equilibrium. Our main tool will be the following result from H_∞ control theory which has been proven in [10].

Theorem 5.1 *Consider the system (2.2) with $x_0 = 0$. Assume (A, B) stabilizable and assume (A, B, C, D) has no invariant zeros in C^0. Let $\gamma > 0$ be given. Define $Q := \gamma^2 I$ then the following two conditions are equivalent*

(i) *There exists a positive semi-definite solution P of $F(P) \geq 0$ such that both rank conditions in (2.10) are satisfied.*

(ii) *There exists $\delta > 0$ such that for all $w \in \mathcal{L}_2^l(\mathcal{R}^+)$*
there exists $u \in \mathcal{L}_2^m(\mathcal{R}^+)$ for which the resulting state $x_{u,w,0} \in \mathcal{L}_2^n(\mathcal{R}^+)$ and the resulting output satisfies
$\|y_{u,w,0}\|_2 \leq (\gamma - \delta)\|w\|_2$.

We define the following class of input functions,

$$\mathcal{U}(\Sigma, w, x_0) = \{u \in \mathcal{L}_2^m \mid \text{By applying } u, w \text{ in } \Sigma \text{ we have } x_{u,w,x_0} \in \mathcal{L}_2^n\} \tag{5.1}$$

We will use the following lemmas,

Lemma 5.2 *Assume for the system 2.2 with cost-criterion 2.1 there exists an almost equilibrium. Let $x_0 = 0$. Then we have*

$$\inf_{u \in \mathcal{U}(\Sigma,w,0)} \mathcal{J}(u, w) \leq 0 \qquad \forall w \in \mathcal{L}_2^l \tag{5.2}$$

Proof We know that for some arbitrary $\varepsilon > 0$ there exists $u_0 \in \mathcal{U}_{fb}$ such that (2.6) is satisfied. Choose an arbitrary $w \in \mathcal{L}_2^l$. We have

$$\inf_{u \in \mathcal{U}(\Sigma,w,0)} \mathcal{J}(u, w) \leq \mathcal{J}(u_0, w) \leq \mathcal{J}^*(0) + \varepsilon \tag{5.3}$$

This implies that for arbitrary $\lambda > 0$ we have

$$\inf_{u\in\mathcal{U}(\Sigma,w,0)} \mathcal{J}(\lambda u,\lambda w) = \inf_{u\in\mathcal{U}(\Sigma,\lambda w,0)} \mathcal{J}(u,\lambda w) \leq \mathcal{J}(u_0,\lambda w) \leq \mathcal{J}^*(0)+\varepsilon \tag{5.4}$$

But since $x_0 = 0$ we have $\mathcal{J}(\lambda u,\lambda w) = \lambda^2\mathcal{J}(u,w)$. Hence

$$\lambda^2 \inf_{u\in\mathcal{U}(\Sigma,w,0)} \mathcal{J}(u,w) \leq \mathcal{J}^*(0)+\varepsilon \tag{5.5}$$

Since this is true for all $\lambda > 0$ we find 5.2. ■

Lemma 5.3 *For all $x_0 \in \mathcal{R}^n$ we have,*

$$\sup_{w\in\mathcal{U}^l_{fb}} \inf_{u\in\mathcal{U}^m_{st}(x_0)} \mathcal{J}(u,w) \leq \mathcal{J}^*(x_0). \tag{5.6}$$

Proof Since $\mathcal{J}^*$ is an almost equilibrium we have

$$\sup_{w\in\mathcal{L}^l_2(\mathcal{R}^+)} \inf_{u\in\mathcal{U}^m_{st}(x_0)} \mathcal{J}(u,w) \leq \mathcal{J}^*(x_0). \tag{5.7}$$

We approximate an arbitrary $\bar{w} \in \mathcal{U}^l_{fb}$ by $w \in \mathcal{L}^l_2(\mathcal{R}^+)$ as follows,

$$w(t) = \begin{cases} \bar{w}\left(x_{\bar{w},u,x_0}(t),t\right) & t \leq T \\ 0 & \text{elsewhere} \end{cases} \tag{5.8}$$

We find the following inequality for all $u \in \mathcal{U}^m_{fb}$

$$\int_0^T z^T_{u,\bar{w},x_0}(t) z_{u,\bar{w},x_0}(t) - \bar{w}^T Q \bar{w} dt \leq \|z_{u,w,x_0}\|_2^2 - \|Q^{1/2}w\|_2^2 \leq \mathcal{J}^*(x_0) \tag{5.9}$$

since $w(t) = 0 \ \forall t > T$. Since T was arbitrary the desired inequality (5.6) is found by letting $T \to \infty$ ■

Proof of necessity part of theorem 2.4 We define $d := Q^{1/2}w$. Moreover we define,

$$\Sigma_0 : \begin{array}{l} \dot{x} = Ax + Bu + EQ^{-1/2}d \\ y = Cx + Du \end{array} \tag{5.10}$$

and

$$\mathcal{J}_\gamma(u,d) := \int_0^\infty y^T(t)y(t) - \gamma^2 d^T(t)d(t)\, dt = \|y\|_2^2 - \gamma^2\|d\|_2^2 \tag{5.11}$$

It is easily seen that $\mathcal{J}_\gamma(u,d) = \mathcal{J}(u,w)$ for $\gamma = 1$. Let $\gamma > 1$ be given. We have,

$$\inf_{u\in\mathcal{U}(\Sigma_0,d,0)} \mathcal{J}_\gamma(u,d) = \inf_{u\in\mathcal{U}(\Sigma,w,0)} \mathcal{J}(u,w) - (\gamma^2-1)\|d\|_2^2 \tag{5.12}$$

and hence by lemma 5.2 we have

$$\inf_{u\in\mathcal{U}(\Sigma_0,d,0)} \|y_{u,d,0}\|_2^2 - \gamma^2\|d\|_2^2 \le -(\gamma^2-1)\|d\|_2^2. \tag{5.13}$$

for all $d \in \mathcal{L}_2^l$. Hence by applying theorem 5.1 to the system Σ_0 we find that there exists a positive semi-definite matrix P_γ such that

$$F_\gamma(P) := \begin{pmatrix} A^TP + PA + C^TC + \gamma^{-2}PEQ^{-1}E^TP & PB + C^TD \\ B^TP + D^TC & D^TD \end{pmatrix} \ge 0 \tag{5.14}$$

and such that P_γ satisfies the following two rank conditions,

$$\begin{aligned} \text{rank} \quad F_\gamma(P_\gamma) &= \text{normrank } G \\ \text{rank} \begin{pmatrix} L_\gamma(P_\gamma,s) \\ F_\gamma(P_\gamma) \end{pmatrix} &= n + \text{normrank } G \qquad \forall s \in \mathcal{C}^+ \cup \mathcal{C}^0, \end{aligned} \tag{5.15}$$

where $L_\gamma(P_\gamma,s) := [sI - A - \gamma^{-2}EQ^{-1}E^TP_\gamma \quad -B]$ Since P_γ is a solution of a differential game with cost criterion $\mathcal{J}_\gamma$ it is easily seen by corollary 4.3 that if $\gamma \downarrow 1$ then P_γ increases i.e. $P_{\gamma_1} - P_{\gamma_2} \ge 0$ if $1 < \gamma_1 \le \gamma_2$. On the other hand we have by lemma 5.3

$$x_0^TP_\gamma x_0 = \sup_{w\in\mathcal{U}_{fb}^l} \inf_{u\in\mathcal{U}_{st}^m(x_0)} \mathcal{J}_\gamma(u,w) \tag{5.16}$$

$$\le \sup_{w\in\mathcal{U}_{fb}^l} \inf_{u\in\mathcal{U}_{st}^m(x_0)} \mathcal{J}_1(u,w) \tag{5.17}$$

$$\le \mathcal{J}^*(x_0) \tag{5.18}$$

Hence $\lim_{\gamma \downarrow 1} P_\gamma = P$ exists. Since rank $F_\gamma(Q) \geq$ normrank G for all symmetric matrices Q (see [10]) it can be shown that our limit P satisfies the first rank condition in (2.12) by a continuity argument. In lemma 3.5,part(iii) it has been shown that the second rank condition in (5.15) implies that a certain matrix is asymptotically stable. Therefore, again by a continuity argument, we know that in the limit this matrix has all its eigenvalues in the closed right half plane. This is equivalent with the second rank condition in (2.12) by lemma 3.5,part (iii). ■

6 The regular differential game

We will now show the last part of theorem 2.4. This has been recapitulated in the following lemma,

Theorem 6.1 *Assume we have the system (2.2) with cost-criterion (2.1). Furthermore assume D is injective and assume there exists a P such that $F(P) \geq 0$ and (2.12) is satisfied. Then there exists an almost equilibrium.*

Remark This is an extension of the results in [5]. However there is an essential difference because we require stability. In [5] one of the assumptions ((C, A) detectable) is such that the problems with and without stability are equivalent. The problem in this paper is that the set of admissible inputs is no longer a simple product space.

The proof will make use of two lemmas. The following lemma has been proven in [10].

Lemma 6.2 *Assume that D is injective. Suppose a symmetric matrix P is given. Then the following two conditions are equivalent,*

(i) $F(P) \geq 0$ *and* $\quad \operatorname{rank} F(P) = \operatorname{normrank} G.$

(ii) $$R(P) := PA + A^T P + PEQ^{-1}E^T P + C^T C - (PB + C^T D)(D^T D)^{-1}(B^T P + D^T C) = 0.$$

Moreover if P satisfies (i) (or equivalently (ii)) then the following two conditions are equivalent for all $s \in \mathcal{C}$.

(iii) $\operatorname{rank}\begin{pmatrix} L(P,s) \\ F(P) \end{pmatrix} = n + \operatorname{normrank} G.$

(iv) *the matrix $A + EQ^{-1}E^T P - B(D^T D)^{-1}(B^T P + D^T C)$ has no eigenvalue in s.*

Note that in case D is injective we have normrank $G =$ rank D. At this point we will present the following, known, result for the LQ-problem with stability (see [15]).

Lemma 6.3 *Consider the system (2.2) with cost-criterion (2.1). Let $w \equiv 0$. Assume (A, B) stabilizable, (A, B, C, D) has no invariant zeros in $\mathcal{C}^0$ and D injective. Then we have the following*

$$\inf_{u \in \mathcal{U}(\Sigma, 0, x_0)} \mathcal{J}(u, 0) = x_0^T L x_0. \tag{6.1}$$

Here L is the solution of the following algebraic Riccati equation,

$$A^T L + LA + C^T C - (PB + C^T D)(D^T D)^{-1}(B^T P + D^T C) = 0 \tag{6.2}$$

with the property that the matrix $A + B(D^T D)^{-1}(B^T P + D^T C)$ is asymptotically stable.

Proof of theorem 6.1 We know that we have a solution of $R(P) = 0$ such that the matrix $A + EQ^{-1}E^T P - B(D^T D)^{-1}(B^T P + D^T C)$ has all its eigenvalues in the *closed* left half plane. It is known from H_∞ theory (a slight extension of [12],proposition 10) that this implies

$$\sup_{d \in \mathcal{L}_2^l(\mathcal{R}^+)} \inf_{u \in \mathcal{U}_{st}^m(x_0)} \|y\|_2^2 - \|d\|_2^2 \leq 0. \tag{6.3}$$

where y is determined by the system equations (5.10). We know follow the reasoning in the proof of the necessity part of theorem 2.4 starting with formula (5.13). Hence we find there exists P_γ such that

$$x_0^T P_\gamma x_0 = \sup_{d \in \mathcal{U}_{fb}^l} \inf_{u \in \mathcal{U}_{st}^m(x_0)} \mathcal{J}_\gamma(u, d). \tag{6.4}$$

By choosing $d = 0$ it is easily seen that $P_\gamma \geq L$ where L is defined by lemma 6.3. We know $P_\gamma \to \bar{P}$ as $\gamma \downarrow 1$ where $\bar{P}$ is also such that we have $R(\bar{P}) = 0$ and the matrix $A + EQ^{-1}E^T \bar{P} - B(D^T D)^{-1}(B^T \bar{P} + D^T C)$ has all its eigenvalues in the *closed* left half plane. It is well known that such a solution of the ARE is unique and hence we find $P = \bar{P}$ and therefore $P \geq L$. Consider the following Riccati differential equation (RDE),

$$\dot{K} + KA + A^T K + C^T C =$$

$$(KB + C^T D)(D^T D)^{-1}(B^T K + D^T C) - KEQ^{-1}E^T K, \qquad K(0) = L. \tag{6.5}$$

Let $T > 0$ be such that the solution of the RDE exists for all $t \leq T$. We know such a T exists. For the system (3.1) we will consider the finite horizon differential game with terminal cost. The cost-criterion is given by

$$\mathcal{J}_T(u,w) = \int_0^T y^T(s)y(s) - w^T(s)Qw(s)\,ds \; + x^T(T)Lx(T) \tag{6.6}$$

It is well known (see [5]) that the optimal strategies for w and u are given by

$$u_0(t) := -(D^TD)^{-1}(B^TK(T-t) + C^TD)\,x(t), \qquad w_0(t) := Q^{-1}E^TK(T-t)x(t) \tag{6.7}$$

and the corresponding cost $\mathcal{J}_T^* = x_0^TK(T)x_0$. It can be seen (using the interpretation of L as the cost defined in lemma 6.3) that this problem is equivalent with the original problem with cost-criterion (2.1) when we add the additional constraint $\forall t > T, \quad w(t) = 0$. This constraint is weakened for increasing T and hence it is clear that for increasing T the cost will increase since w is a maximizing player. That is, for $T \geq t_1 \geq t_2$ we have $K(t_1) \geq K(t_2)$. Moreover since P is a stationary solution of the RDE such that $P \geq L$ we know $P \geq K(t) \;\; \forall t < T$ Since $K(.)$ is an increasing solution of the RDE which is bounded from above by a stationary solution of the RDE we know that $K(.)$ exists for all t and converges to a matrix K_∞ which is a stationary solution of the RDE, i.e. $R(K_\infty) = 0$.

Next we claim that the matrix $A_1 := A - B(D^TD)^{-1}(B^TK_\infty + D^TC)$ is asymptotically stable. To this end we rewrite the ARE in the following form.

$$\begin{aligned} K_\infty A_1 + A_1^TK_\infty + K_\infty EQ^{-1}E^TK_\infty + C^TC + \\ (K_\infty B + C^TD)(D^TD)^{-1}(B^TK_\infty + D^TC) = 0 \end{aligned} \tag{6.8}$$

By applying an eigenvector x corresponding to an unstable eigenvalue λ to both sides of this equation we find $\mathrm{Re}\,\lambda\; x^TK_\infty x = 0$, $E^TK_\infty x = 0$, $Cx = 0$ and $(B^TK_\infty + D^TC)\,x = 0$. We find for $\mathrm{Re}\,\lambda > 0$ that $Ax = \lambda x$ and $K_\infty x = 0$. Since $K_\infty \geq L \geq 0$ this implies $Lx = 0$. This again implies that λ is also an unstable eigenvalue of $A - B(D^TD)^{-1}(B^TL + D^TC)$. However since by lemma 6.3 this matrix is stable we have a contradiction. If $\mathrm{Re}\,\lambda = 0$ then we have $Ax = \lambda x$ and $Cx = 0$ which contradicts the fact that we have no invariant zeros on the imaginary axis. Hence A_1 is stable.

We are now in the position to show that $\mathcal{J}^*(x_0) = x_0^TK_\infty x_0$ is an almost equilibrium of the system (3.1) with cost-criterion (2.1). Let $\varepsilon > 0$ be given. Choose $T > 0$ such that $x_0^TK_\infty x_0 - x_0^TK(T)x_0 < \varepsilon$. The following u_0, w_0 turn out to satisfy (2.6),

$$u_0(t) \;:=\; -(D^TD)^{-1}(B^TK_\infty + D^TC)\,x(t) \tag{6.9}$$

$$w_0(t) \;:=\; \begin{cases} Q^{-1}E^TK(T-t)x(t) & \text{for } t < T \\ 0 & \text{otherwise} \end{cases} \tag{6.10}$$

Indeed for admissible pairs (u, w) we can now rewrite the cost-criterion in the following way

$$\mathcal{J}(u,w) = x_0^T K_\infty x_0 + \int_0^\infty \|D\left(u(t) + (D^T D)^{-1}(B^T K_\infty + D^T C)\, x(t)\right)\|^2 dt$$

$$- \int_0^\infty \|Q^{1/2}\left(w(t) - Q^{-1}E^T K_\infty x(t)\right)\|^2 dt \tag{6.11}$$

Since u_0 is a stabilizing feedback it is easily seen from this equation that u_0 satisfies its requirements. Another way of rewriting the cost-criterion when $w(t) = 0\ \forall t > T$ is given by

$$\mathcal{J}(u,w) = x_0^T K(T) x_0 + \int_0^T \|D\left(u(t) + (D^T D)^{-1}(B^T K(T-t) + D^T C)\, x(t)\right)\|^2 dt$$

$$- \int_0^T \|Q^{1/2}\left(w(t) - Q^{-1}E^T K(T-t) x(t)\right)\|^2 dt - x^T(T) L x(T) + \int_T^\infty y^T(t) y(t) dt$$

Since the sum of the last two terms is non-negative by lemma 6.3 and the first term differs from $\mathcal{J}^*(x_0)$ less than ε, it is easily seen that w_0 satisfies the second equation in (2.6). This proves that indeed an almost equilibrium exists. ■

7 Conclusions

In this paper the linear quadratic differential game was solved. We could derive necessary conditions as well as sufficient conditions for the existence of equilibria. For the derivation of necessary conditions we made however the extra assumption that there are no invariant zeros on the imaginary axis. After that we have necessary and sufficient conditions for the existence of equilibria in cas D is injective but not in case D is not injective.

Interesting points for future research would be to find necessary and sufficient conditions in case either D is not injective or there are invariant zeros on the imaginary axis. Another point is the uniqueness of equilibria. In my opinion the equilibrium is unique but we haven't been able to prove it. The equilibria we find in theorem 2.3 can be shown to be the smallest possible.

An interesting feature is that if the differential game is solvable under the assumptions of theorem 2.3 for $Q = I_l$ then the H_∞ problem is solvable for $\gamma = 1$, i.e. there exists an internally stabilizing feedback which makes the H_∞ norm less than 1. In case D is injective then under the assumptions of theorem 2.4 we can do the same only the H_∞ norm becomes less or equal than 1. This shows the strong relationship between H_∞ control and differential games.

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